Annihilation of quantized vorticity in a trapped Bose-Einstein condensate via the scattering of density perturbations

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We evaluate the extraction of angular momentum from quantized vortices in trapped Bose-Einstein condensates. Within a variational approach we show that the energy barrier between the ground state and the state containing an axisymmetric vortex can be crossed in a scattering process by density perturbations. The transfer of angular momentum can be made total for suitably chosen density perturbations carrying about 10% of the ground-state energy, irrespectively of the number of particles in the condensate. A similar pattern is reproduced by the full solution of the Gross-Pitaevskii equation, although interference and diffusion effects are now seen to limit the efficiency of the process.

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I. INTRODUCTION

The study of scattering of sound by vortices in superfluids initiated with the pioneering work of Pitaevskii [1] on the mutual friction force in superfluid helium. Since then most research has focused on the momentum transfer between the acoustic field and the vortex in homogeneous media in the linear regime. More recently the problem has regained some attention due to a controversy on the vanishing of the Iordanskii force [2] (see also, e.g., Refs. [3,4], and references therein). The study of the scattering between sound-wave perturbations with vortex excitations in trapped Bose-Einstein condensates (BECs) has been also stimulated by the phenomenon of super-resonance, the acoustic-wave version of the Penrose process [5]. In classical systems, such a process, based on Zel'dovich studies [6], describes how an axially symmetric rotating body can amplify some oscillation modes in a cavity through the transfer of rotation energy. Recent calculations of sound-wave scattering from hydrodynamic vortices applicable to atomic Bose-Einstein condensates [7] have shown that at sufficiently high angular speeds and in the perturbative limit where back-reaction effects can be neglected, a sound-wave packet can extract a sizeable fraction of the vortex energy through the mechanism of super-resonance under typical experimental conditions. Since a vortex is a state of a stationary BEC with a somewhat higher energy than the ground state, its nucleation in atomic condensates requires appropriate experimental techniques to be achieved. Among these techniques we can mention stirring the condensate with a laser beam [8], exploiting interconversion between two components of the condensate with different spins [9], or directly imprinting a phase in the condensate wave function [10]. The nucleation and stability of vortices have been extensively discussed in the literature (see Fetter and Svidzinski [11], and references therein). Not only have singly quantized vortex configurations been investigated, but also more elaborate modes such as vortex rings [12], vortex clusters [13], and multiply quantized vortices [14] confined in harmonic or more complicated traps.

The possibility that substantial angular-momentum transfers from the vortex state to a sonic wave packet may persist in the nonperturbative quantum regime described by the Gross-Pitaevskii functional is what the present work focuses on. Given the quantum nature of a vortex in a BEC, one is naturally led to ask whether part or all of its energy can be extracted by the sonic wave packet. Within a variational approach to the Gross-Pitaevskii energy functional with nonrotating traps, we show that the energy barrier between the state containing an axisymmetric vortex line and the vortex-free ground state is such that a transfer of population between the two states is possible with a moderate energetic cost. A simplified dynamical model is then implemented to analyze the time evolution of the scattering events and to identify the conditions that allow a sizeable transfer of angular momentum between the vortex and the wave packet. Finally, the predictions of the variational model are compared with the full solution of the Gross-Pitaevskii equation (GPE) for the same choice of parameters. The paper is organized as follows. In Sec. II we introduce the variational model to describe steady states of the energy, and in Sec. III we analyze the dynamics of the system when a density perturbation impacts a centered vortex. Section III A highlights similarities and differences between the results of the variational model and the full solution of the GPE. Finally, in Sec. IV we offer a summary and some concluding remarks.

II. STEADY STATES

We explore the possibility of a transition between the state of a BEC containing a centered axisymmetric vortex and the vortex-free ground state by introducing a variational model that allows us to identify and estimate the steady states of the energy. This method has been used by several authors to study the BEC static and dynamical properties [15].

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The energy of the BEC at zero temperature is given by the Gross-Pitaevskii functional

$$E[\psi] = \int d^3r \left[\frac{\hbar^2}{2m} |\nabla \psi|^2 + V_{\mathrm{ext}}(\mathbf{r})|\psi|^2 + \frac{g}{2}|\psi|^4 \right], \qquad (1)$$

where $\psi(\mathbf{r})$ is the condensate wave function, $V_{\mathrm{ext}} = \frac{1}{2}m\omega^2 r^2$ is the external nonrotating isotropic trap, and $g = 4\pi\hbar^2 a_s/m$ is the strength of the interparticle interaction, with a_s the s-wave scattering length. Minimization of the functional yields the GPE. In order to locate the extrema of the functional and estimate the energy barrier between them, we parametrize ψ in the following simple form:

$$\psi(\mathbf{r}) = \mathcal{N}\left[\cos\tau + \sin\tau \frac{x+iy}{a_{\mathrm{ho}} b} \right] e^{-m\omega r^2/2\hbar b^2}, \qquad (2)$$

where $\mathcal{N} = \pi^{-3/4} N^{1/2} (a_{\mathrm{ho}} b)^{-3/2}$ is a normalization constant, with N the number of particles, and $a_{\mathrm{ho}} = \sqrt{\hbar/(m\omega)}$ is the oscillator length. This wave function is the superposition of an axisymmetric vortex line orthogonal to the (x,y) plane and a vortex-free state, and as such it is written as the sum of two terms having, respectively, angular momentum equal to zero and to $\hbar$ per particle, with populations $N_0 = N\cos^2\tau$ and $N_1 = N\sin^2\tau$. This yields total angular momentum $L_z = \hbar\sin^2\tau$ per particle. It is worth stressing that the state (2) for large N cannot be regarded as a vortex that is simply displaced, as the width of its hollow core is linked to the width of density profile and therefore increases with N, whereas normally the core of a displaced vortex is proportional to the healing length, i.e., it diminishes with N [16].

The state (2) has energy

$$E = \frac{\hbar\omega}{4}(3N_0 + 5N_1)\left(b^2 + \frac{1}{b^2}\right) + g\left(\frac{m\omega}{\hbar\pi}\right)^{3/2} \frac{1}{8\sqrt{2}b^3}(2N_0^2 + 4N_0N_1 + N_1^2). \qquad (3)$$

The steady states of the BEC are specified by the sets $\{\tau_0, b_0\}$ that extremize E, i.e., they are the solutions of the coupled equations

$$\left.\frac{\partial E}{\partial \tau}\right|_{(\tau_0,b_0)} = 0, \qquad (4a)$$

and

$$\left.\frac{\partial E}{\partial b}\right|_{(\tau_0,b_0)} = 0. \qquad (4b)$$

The results for the energy of a ^{87}Rb condensate [8] with $a_s = 5.71$ nm and $\omega = 11.7$ Hz, as a function of τ and b for $N = 10^2$, 10^3, 10^4, and 10^5, are summarized in Fig. 1. In the figure the solid and dashed curves correspond to the loci of the zeroes of the two derivatives in Eqs. (4a) and (4b), and hence the extrema of the energy are located at the intersection between these two curves.

At low N we find only two extrema located at $\tau = 0$ and $\tau = \pi/2$ for $\tau \in [0, \pi]$, corresponding to the states with $L_z = 0$ ($N_0 = N$) and $L_z = \hbar$ ($N_1 = N$), the former being an energy minimum. On increasing N a maximum develops between

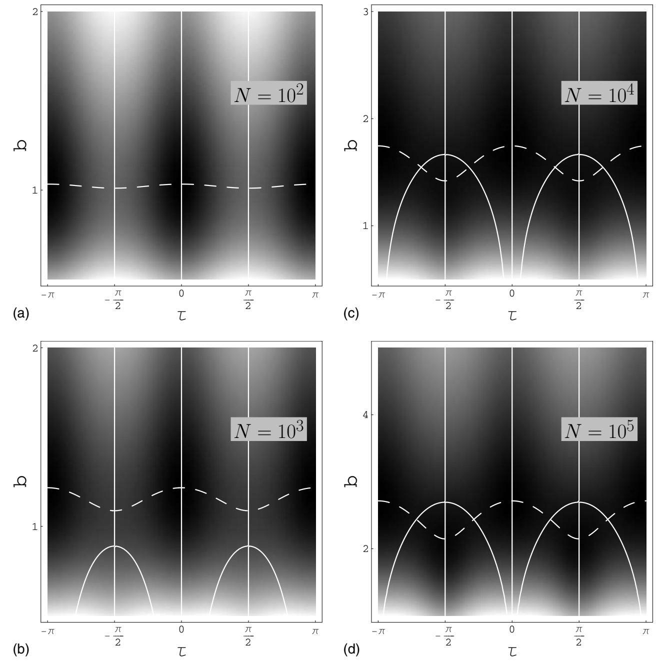

FIG. 1. Gray-level plots of the energy $E[\psi]$ as a function of the variational parameters τ and b for $N = 10^2, 10^3, 10^4, 10^5$ as indicated in the plots. Darker regions correspond to lower energies. The solid and dashed lines obey Eqs. (4a) and (4b), respectively.

$\tau = 0$ and $\pi/2$, while both extrema at $\tau = 0$ and $\pi/2$ become minima. This behavior is more clearly depicted in Fig. 2, where the energy is shown as a function of τ along the curve $b_0(\tau)$ obeying Eq. (4b). It is indeed possible to choose a path through the energy maxima that is an extremum with respect to b, as plotted in the left panel of Fig. 2. The plots show that the energy barrier W associated with the crossover from the energy minimum at $\tau = \pi/2$ to that at $\tau = 0$ is about 10% of the ground-state energy for typical values of the number of particles in a BEC. This is confirmed by approximate analytical formulas for the energy of each minimum and for the barrier as obtained from Eq. (3). At strong coupling we get $E_1 = 2^{-2/5}(5/3)^{3/5} E_0 \approx 1.03 E_0$, and $W \approx 0.101 E_0$ for the values of the parameters used in our calculations. To check the validity of our variational findings we have evaluated the GP energy for a state $\psi = \cos\tau\, \psi_0 + \sin\tau\, \psi_1$ with ψ_0 and ψ_1 exact

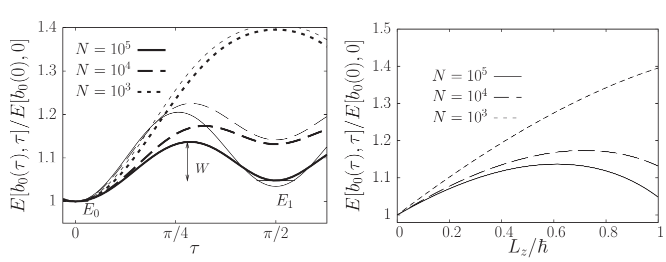

FIG. 2. Total energy E (in units of the ground-state energy) along a selected path connecting the energy extrema at $\tau = 0$ and $\pi/2$ (see text) for several values of the particle numbers N. The left panel displays the energy as a function of τ from the variational approach (thick lines) and the GPE solutions (thin lines). The right panel shows the same energy curves from the variational approach as functions of $L_z/\hbar$ per particle.

solutions of the GPE without and with vorticity, respectively. The results are shown in the left panel of Fig. 2 (thin lines) and demonstrate that while the energy differs in about 5% from the variational results, the behavior is in complete agreement; namely, the vortex state is a minimum of the energy with respect to the exchange of particles with the condensate. Furthermore, this is valid in general for large interactions and the exact states ψ_i as analytically shown in the Appendix.

The right panel of Fig. 2 illustrates how the total energy of the system varies as the vortex population diminishes, for a given number of particles. The relatively low value of the energy barrier between the two energy minima suggests that annihilation of a vortex in a trapped BEC may be affected at a moderate energy cost. Some dissipation mechanism will be needed to stabilize the transition (see, e.g., Ref. [17]), and in the lack of it one expects that the system will oscillate between the two states. In the next section we examine the scattering of density perturbations against the vortex as a possible mechanism for triggering the transition.

III. DYNAMICS OF THE VORTEX

Since the ground and vortex states are energy eigenstates, a superposition of both states is a configuration in which the two populations remain constant in time. We allow the transfer of atoms out of the vortex by a mechanism in which a compact wave packet, mimicking a classical particle with average angular momentum determined by L_z, impacts the system. Thus within the variational approach we extend the total condensate wave function to include a time dependence in the form

$$\psi(\mathbf{r},t) = A(t)\psi_0(\mathbf{r}) + B(t)\psi_1(\mathbf{r}) + C(t)\phi(\mathbf{r}-\mathbf{r}_0)\exp(i\mathbf{k}\cdot\mathbf{r}), \tag{5}$$

which will allow the states to change their amplitudes. Here ψ_0 and ψ_1 are the ground-state and vortex-state wave functions, while A, B, and C are complex time-dependent amplitudes, and $\mathbf{r}_0(t)$ and $\mathbf{k}(t)$ are real functions of time. For the last term in Eq. (5), representing a density perturbation around the position r_0 and moving with average momentum $\hbar\mathbf{k}$, we take a narrow Gaussian packet $\phi(\mathbf{r})=\exp(-r^2/b_p^2)$ of fixed width b_p. This yields an average angular momentum $\mathbf{L}=\mathbf{r}_0\times\hbar\mathbf{k}$ per particle.

The evolution of the system is dictated by the dynamics of the time-dependent parameters entering Eq. (5), which is determined from the Lagrangian

$$\mathcal{L}[\psi,t] = \int\left[\frac{i\hbar}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - \frac{\hbar^2}{2m}|\nabla\psi|^2 - V_{\text{ext}}(\mathbf{r})|\psi|^2 - \frac{g}{2}|\psi|^4\right]d\mathbf{r}; \tag{6}$$

according to the set of coupled Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\alpha}}\right) = \frac{\partial\mathcal{L}}{\partial\alpha} \tag{7}$$

for $\alpha=A$, B, C, $\mathbf{r}_0$, and $\mathbf{k}$. These dynamical equations conserve the total energy but do not conserve the angular momentum associated with the vortex, as a result of the symmetry breaking caused by a Gaussian perturbation.

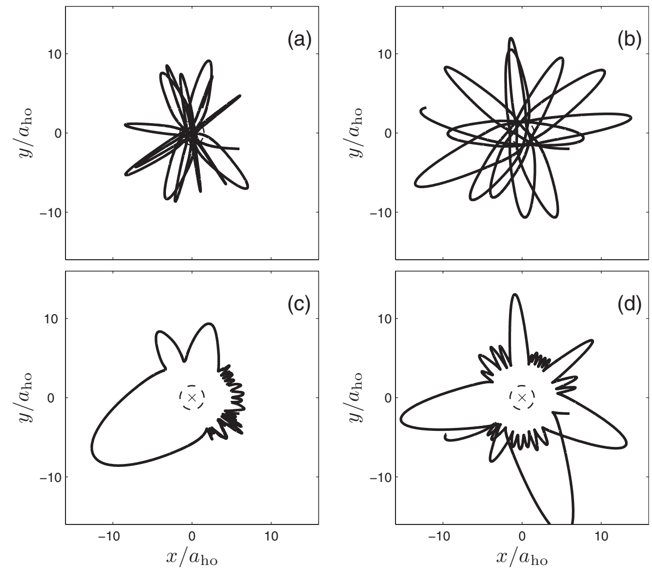

FIG. 3. Orbits of the center of the incident wave packet $\mathbf{r}_0(t)$ (in units of a_{ho}). Each panel corresponds to different values of the wave packet width b_p and of the initial velocities $\mathbf{k}(0)=-k_0\hat{x}$: (a) $b_p/a_{\text{ho}}=1$, $k_0a_{\text{ho}}=5.5$; (b) $b_p/a_{\text{ho}}=1$, $k_0a_{\text{ho}}=10$; (c) $b_p/a_{\text{ho}}=3$, $k_0a_{\text{ho}}=5.5$; (d) $b_p/a_{\text{ho}}=3$, $k_0a_{\text{ho}}=10$. The crossed circle represents the center of the condensate.

We have solved the Euler-Lagrange equations (7) for a vortex line in a BEC with $N=10^5$ particles and incoming wave packets with several values of the number of particles N_p, the initial velocities $\hbar k_0/m$, and widths b_p. The trajectories of a wave packet with $N_p/N\simeq 8\times 10^{-2}$ are displayed in Fig. 3 for several initial velocities along the x axis and widths $b_p/a_{\text{ho}}=1$ and 3. After every scattering event there is for the more energetic wave packets (smaller b_p) a modification in the area and the orientation of the orbit, indicating a change in its angular momentum that lasts for an appreciable part of the subsequent evolution. This is clearly seen in Fig. 4 where we plot the time evolution of L_z for both the BEC and the wave packet. From the results of the model we can see that the vortex exchanges angular momentum with the wave packet with an approximate periodicity determined by the sequence of scattering events. At times a total transfer occurs back and forth between the vortex and the wave packet. Of course, the trap is playing the role of a mirror reflecting back the wave packet as it approaches its walls after each scattering event against the vortex.

Although the chosen N_p in Figs. 3 and 4 is only about 8% of the number of particles in the condensate, the energy associated with the wave packet is much higher than the expected energy barrier W. Indeed, for the above parameters the energy involved is in the range of 30–100 % of the BEC energy. Therefore we proceed to present calculations for smaller N_p and consequently smaller wave-packet energy. Figure 5 shows the time evolution of L_z for $N_p/N=10^{-2}$ and 3×10^{-3}. If we focus first on the angular momentum of the vortex (solid lines) for wave-packet energies larger than 10% of the initial energy of the BEC [$e_p\geqslant 10\%$, panels (a), (b),

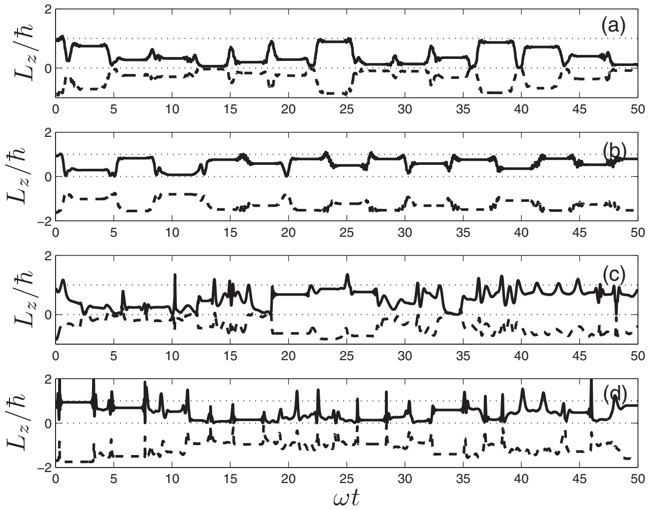

FIG. 4. Angular momentum L_z per particle (in units of $\hbar$) as a function of time t (in units of $1/\omega$). Solid and dashed lines correspond, respectively, to the angular momentum of the BEC and of the wave packet. Each panel refers to the same parameters as in Fig. 3.

and (d)], we see that the angular momentum is fully transferred and the transfer lasts for extended time intervals. For lower energies [e_p=3%, panel (c)] instead the wave packet scatters repeatedly against the condensate without long-lived exchanges of angular momentum. This behavior can also be interpreted in terms of the energy background, as is shown in Fig. 6 for $N_p/N=3\times10^{-3}$, corresponding to panel (c) and (d) of Fig. 5. Figure 6 shows the energy of the unperturbed BEC from Eq. (3) together with the BEC energy during the evolution of the system at times when the wave packet is at a distance larger than the width of the condensate, that is at times when it is meaningful to separately define the energy

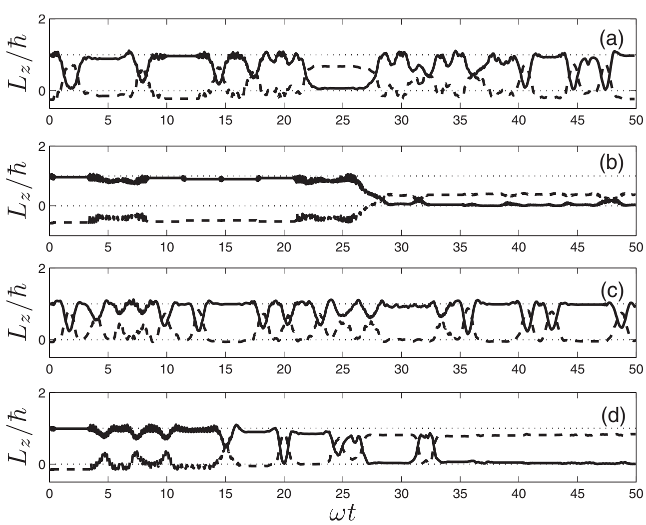

FIG. 5. Same as in Fig. 4, but for different values of the number of particles and initial velocity of the wave packet and of the ratio e_p between its energy and that of the BEC: (a) $N_p/N=10^{-2}$, $k_0a_{\rm ho}=11$, e_p=11%; (b) $N_p/N=10^{-2}$, $k_0a_{\rm ho}=25.5$, e_p=50%; (c) $N_p/N=3\times10^{-3}$, $k_0a_{\rm ho}=11$, e_p=3%; and (d) $N_p/N=3\times10^{-3}$, $k_0a_{\rm ho}=25.5$, e_p=10%.

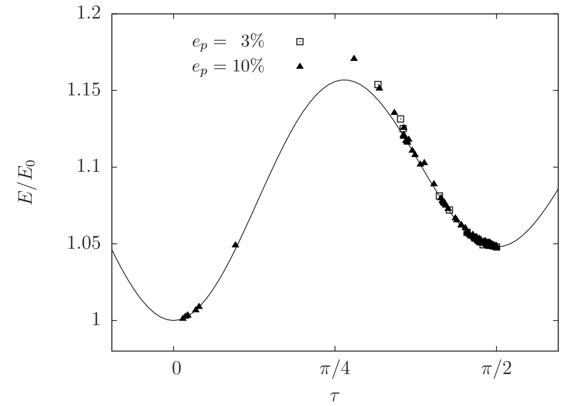

FIG. 6. Energy E of the BEC (in units of the ground-state energy E_0) for $N_p/N=3\times10^{-3}$ as a function of τ during the scattering of a wave packet. The solid line refers to the energy landscape (cf. Fig. 2) and the symbols to the BEC energy as the system evolves (see text), for two values of the wave-packet energy e_p.

of both the wave packet and the BEC. We see that the energy barrier is crossed only for e_p=10% (triangles).

The predictions of the variational model for the transfer of angular momentum in scattering events, as a function of the wave-packet energy, are summarized for the two values of N_p/N in Fig. 7, where we show the minimum value attained by L_z during the time evolution from $t=0$ to $50/\omega$. As e_p increases we find a slow decline of $\min\{L_z\}$ leading into a rather sharp transition to a situation where $\min\{L_z\}$ approximately vanishes. The location of the transition shows a residual dependence on the value N_p/N. It seems therefore possible to tune k_0, and hence e_p, so that angular momentum is extracted from the vortex to the extent of its annihilation. Of course, permanent annihilation requires that the wave packet be removed after the scattering event, to prevent its back-scatter from the trap walls into the condensate.

Although the superresonance phenomenon that was investigated in [7] concerned scattering of a sound wave against a classical hydrodynamic vortex with nonquantized vorticity, we may still compare the resonance windows observed in the two situations. From Eq. (2) we calculate the angular velocity Ω of the vortex at a distance ξ from its center as $\Omega(\xi)=\hbar/(m\xi^2)$, and taking $\xi=(8\pi\rho a_s)^{-1/2}$ as the healing length of the BEC at the trap center we estimate $\Omega\approx82\omega$. The velocity v of the wave packet is linked to its central frequency ω_0

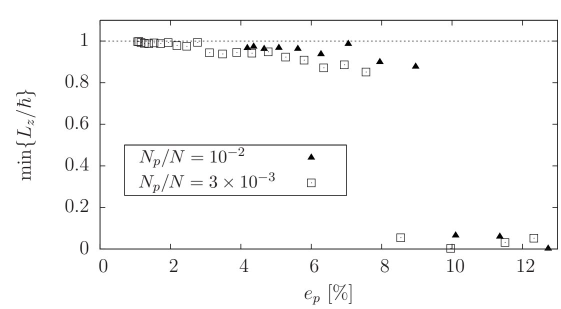

FIG. 7. Minimal values attained by L_z (in units of $\hbar$) during the time evolution of the system as functions of the wave-packet energy e_p (in units of the BEC energy) for $N_p/N=10^{-2}$ and 3×10^{-3}.

through the relation $v = \hbar\omega_0/(mc)$, where $c = \hbar/(\sqrt{2}m\xi)$ is the speed of sound. We thus obtain the superresonant limit as $\Omega/\omega_0 \approx 13/(k_0 a_{\text{ho}})$. For the lowest value of N_p/N, we see that the 10% energy barrier corresponds to $k_0 a_{\text{ho}} = 20-25$, which roughly lies within the classical superresonant regime found in [7].

On the numerical solution of the GPE

As discussed above, the variational model predicts that in a scattering event, for certain values of the system parameters, the velocity field associated with a quantized vortex in a BEC can transfer all of its angular momentum to a sonic wave packet, and that this process is followed by a series of revivals and annihilations as the outgoing wave packet is repeatedly back-scattered by the trap walls. The crucial simplification is that the model embodies, through the form that has been assumed for the order parameter $\psi(\mathbf{r},t)$ of the system, a clear separation between the angular momentum associated with the vortex and that of the wave packet. We have made some preliminary attempts to assess how much of this scenario is confirmed by a calculation of $\psi(\mathbf{r},t)$ from the solution of the time-dependent GPE in the presence of an incoming density perturbation.

The main difficulty that one meets in such an attempt is indeed posed by the fact that the total angular momentum can no longer be unambiguously separated into two components. If we assume that the angular momentum residing with the particles inside the initial volume of the condensate can be attributed to the vortex and the remainder to the wave packet, then we find that for some values of the system parameters in the range of those adopted in the variational model (e.g., for $b_p/a_{h0}=1$, $N_p/N = 2\times 10^{-2}$, and $k_0 a_{ho}=10$) there is again from the GPE total transfer of angular momentum in the first scattering event. This process appears to be followed by a single revival, but the analysis of the subsequent evolution is obscured by interference and diffusion processes of the wave packet. These processes, which are suppressed in the variational model by preserving the shape of the condensate and of the perturbation, drain energy from the wave packet. The use of narrower wave packets would limit interference, but would also foster diffusion.

Several remedies may be envisaged to overcome these difficulties, such as (i) the insertion of a central dimple in the trap potential to confine the velocity field of the vortex [18], or (ii) the adoption of wave-packet-absorbing walls in the trap. It may also prove useful to adopt a simple subtraction of the angular momentum calculated for the wave packet in the variational model from the GPE prediction of the total L_z. These possibilities are left for future study.

IV. SUMMARY AND CONCLUDING REMARKS

In summary, we have estimated the energy barrier associated with the crossover from the state of a Bose-Einstein condensate containing an axisymmetric vortex to its ground state as being about 10% of the ground-state energy. This suggests the possibility of driving a transition between the two states of angular momentum through a suitable external perturbation. A parallelism is thereby indicated with superresonance phenomena from classical hydrodynamic vortices.

We have then evaluated within a variational approach the scattering between a quantized vortex line and a sonic wave packet of varying number of particles, initial velocity, and width, by solving the Euler-Lagrange equations. Within this approach, from the trajectories of the wave packet it is clear that every scattering event against the condensate may be accompanied by a change in angular momentum that lasts for an appreciable time interval during the subsequent evolution. This behavior is directly confirmed by the time evolution of the angular momentum: the condensate exchanges angular momentum with the wave packet with an approximate periodicity which is determined by the sequence of scattering events. The transfer could be made permanent by removing the wave packet after the relevant scattering event. This can be experimentally achieved by using shallow traps where the finite depth allows the wave packet to leave the condensate after the scattering event [18]. The wave packet can be initiated by first confining some of the particles by means of an additional laser beam and then releasing the particles by switching off the additional trap.

We have achieved partial confirmation of these dynamical results by solving the time-dependent Gross-Pitaevskii equation for a condensate subject to an incoming sonic wave packet. A suitable choice of system parameters reproduces the transfer of angular momentum in a scattering event, but the correspondence with the variational model is subsequently lost because phenomena of interference and diffusion become allowed. These effects reduce the energy supplied by the wave packet in later events and limit the efficiency of the scattering process as a possible vortex-annihilation mechanism. We have indicated some directions along which the dynamical study of the induced annihilation of quantized vorticity may be further developed.

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APPENDIX

In this Appendix we show that the exact vortex state ψ_1 is a minimum of the energy respect transfering atoms to the ground state ψ_0. Therefore we consider the analog to Eq. (2) with the exact states $\psi_i(\mathbf{r})$,

$$\psi(\mathbf{r}) = \cos\tau\,\psi_0(\mathbf{r}) + \sin\tau\,\psi_1(\mathbf{r}), \tag{A1}$$

and calculate the energy $E(\tau)$ from Eq. (1) as

$$E = \cos^2\tau\,E_0^0 + \sin^2\tau\,E_1^0 + \cos^4\tau\,E_0^{\text{int}} + \sin^4\tau\,E_1^{\text{int}} + \sin^2 2\tau\frac{g}{2}\int |\psi_0|^2|\psi_1|^2, \tag{A2}$$

where the one-body terms are

$$E_i^0 = \int \left(\frac{\hbar^2}{2m} |\nabla \psi_i|^2 + V_{\text{ext}}(\mathbf{r}) |\psi_i|^2 \right) d^3 r \tag{A3}$$

and the interaction terms are

$$E_i^{\text{int}} = \frac{g}{2} \int |\psi_i|^4 d^3 r. \tag{A4}$$

From Eq. (A2) we find that $\tau = 0, \pi/2$ are extremes and that

$$\frac{\partial^2 E}{\partial \tau^2} = \begin{cases} 2\left(E_1^0 - E_0^0 - 2E_0^{\text{int}} + 2g \int |\psi_0|^2 |\psi_1|^2 d^3 r\right), & \tau = 0 \\ -2\left(E_1^0 - E_0^0 + 2E_1^{\text{int}} - 2g \int |\psi_0|^2 |\psi_1|^2 d^3 r\right), & \tau = \pi/2 \end{cases}.$$

For zero coupling we then have

$$\frac{\partial^2 E}{\partial \tau^2} = \pm 2(E_1^0 - E_0^0) \text{for } \tau = \begin{cases} 0 \\ \pi/2 \end{cases}, \tag{A5}$$

while for large couplings, the interaction energy terms are larger than the one-body one and thus we may approximate

$$\frac{\partial^2 E}{\partial \tau^2} = 4E_0^{\text{int}} > 0 \quad \text{for both } \tau = 0 \text{ and } \pi/2. \tag{A6}$$

Namely, in this limit ψ_1 is a minimum respect changing τ.

[1] L. P. Pitaevskii, J. Explt. Theoret. Phys. (U.S.S.R.) 35, 1271 (1958) [Sov. Phys. JETP **8**, 888 (1959)].

[2] S. V. Iordanskii, J. Explt. Theoret. Phys. (U.S.S.R.) 49, 225 (1965) [Sov. Phys. JETP **22**, 160 (1966)].

[3] S. G. Llewellyn Smith, J. Phys. A **35**, 3597 (2002).

[4] M. Flaig and U. R. Fischer, Phys. Rev. B **74**, 224503 (2006).

[5] H. Penrose, Gen. Relativ. Gravit. **34**, 1141 (2002).

[6] Ya. B. Zel'dovich, Zh. Eksp. Teor. Fiz. Pis'ma Red. **14**, 270 (1971) [JETP Lett. **14**, 180 (1971)]; Zh. Eksp. Teor. Fiz. **62**, 2076 (1972) [Sov. Phys. JETP **35**, 1085 (1972)].

[7] F. Federici, C. Cherubini, S. Succi, and M. P. Tosi, Phys. Rev. A **73**, 033604 (2006); C. Cherubini, F. Federici, S. Succi, and M. P. Tosi, Phys. Rev. D **72**, 084016 (2005).

[8] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys. Rev. Lett. **84**, 806 (2000).

[9] M. R. Matthews, B. P. Anderson, P. C. Haljan, D. S. Hall, C. E. Wieman, and E. A. Cornell, Phys. Rev. Lett. **83**, 2498 (1999).

[10] A. E. Leanhardt, A. Görlitz, A. P. Chikkatur, D. Kielpinski, Y. Shin, D. E. Pritchard, and W. Ketterle, Phys. Rev. Lett. **89**, 190403 (2002).

[11] A. L. Fetter and A. A. Svidzinski, J. Phys.: Condens. Matter **13**, R135 (2001).

[12] L.-C. Crasovan, V. M. Perez-Garcia, I. Danaila, D. Mihalache, and L. Torner, Phys. Rev. A **70**, 033605 (2004).

[13] L.-C. Crasovan, J. P. Torres, D. Mihalache, and L. Torner, Phys. Rev. A **68**, 063609 (2003); V. Pietilä, M. Möttönen, T. Isoshima, J. A. M. Huhtamäki, and S. M. M. Virtanen, *ibid.* **74**, 023603 (2006).

[14] J. A. M. Huhtamäki, M. Möttönen, and S. M. M. Virtanen, Phys. Rev. A **74**, 063619 (2006).

[15] G. Baym and C. J. Pethick, Phys. Rev. Lett. **76**, 6 (1996); V. M. Pérez-García, H. Michinel, J. I. Cirac, M. Lewenstein, and P. Zoller, *ibid.* **77**, 5320 (1996); Phys. Rev. A **56**, 1424 (1997).

[16] D. S. Rokhsar, Phys. Rev. Lett. **79**, 2164 (1997).

[17] J. J. García-Ripoll and V. M. Pérez-García, Phys. Rev. A **60**, 4864 (1999); J. J. García-Ripoll, G. Molina-Terriza, V. M. Pérez-García, and L. Torner, Phys. Rev. Lett. **87**, 140403 (2001); S. Komineas, N. R. Cooper, and N. Papanicolaou, Phys. Rev. A **72**, 053624 (2005).

[18] N. P. Proukakis, N. G. Parker, C. F. Barenghi, and C. S. Adams, Laser Phys. **14**, 284 (2004); C. F. Barenghi, N. P. Proukakis, and C. S. Adams, J. Low Temp. Phys. **136**, 629 (2005).