Diego Castaño

Juan Manuel Cornejo

Gentzen-Style Sequent Calculus for Semi-intuitionistic Logic

Abstract. The variety $\mathcal{SH}$ of semi-Heyting algebras was introduced by H. P. Sankappanavar (in: Proceedings of the 9th "Dr. Antonio A. R. Monteiro" Congress, Universidad Nacional del Sur, Bahía Blanca, 2008) [13] as an abstraction of the variety of Heyting algebras. Semi-Heyting algebras are the algebraic models for a logic **HsH**, known as *semi-intuitionistic logic*, which is equivalent to the one defined by a Hilbert style calculus in Cornejo (Studia Logica 98(1–2):9–25, 2011) [6]. In this article we introduce a Gentzen style sequent calculus ***GsH*** for the semi-intuitionistic logic whose associated logic **GsH** is the same as **HsH**. The advantage of this presentation of the logic is that we can prove a cut-elimination theorem for ***GsH*** that allows us to prove the decidability of the logic. As a direct consequence, we also obtain the decidability of the equational theory of semi-Heyting algebras.

Keywords: Semi-intuitionistic logic, Intuitionistic logic, Semi-Heyting algebras, Heyting algebras, Sequent calculus.

1. Introduction

The variety $\mathcal{SH}$ of semi-Heyting algebras was introduced by Sankappanavar in [13] as an abstraction of the variety of Heyting algebras. A semi-Heyting algebra is an algebra $\mathbf{A} = \langle A, \vee, \wedge, \to, \top, \bot \rangle$ that satisfies the following conditions:

$(SH1)$ $\langle A, \vee, \wedge, \top, \bot \rangle$ is a bounded lattice,
$(SH2)$ $x \wedge (x \to y) \approx x \wedge y$,
$(SH3)$ $x \wedge (y \to z) \approx x \wedge [(x \wedge y) \to (x \wedge z)]$,
$(SH4)$ $x \to x \approx \top$.

This variety contains the variety of Heyting algebras and its members share some important properties with Heyting algebras. For example, $\mathcal{SH}$ is an arithmetic variety, its algebras are pseudocomplemented distributive

Presented by **Constantine Tsinakis**; *Received* October 28, 2015

Studia Logica
DOI: 10.1007/s11225-016-9675-y

Published online: 07 June 2016

lattices with the pseudocomplement given by $x^* = x \to \bot$, and congruences are determined by lattice filters. On the contrary, semi-Heyting algebras show some remarked differences, the most important of which being that the implication on an algebra $\mathbf{A}$ is not determined by the lattice order of $\mathbf{A}$.

The variety $\mathcal{SH}$ has been recently studied in depth from an algebraic point of view. Its subvarieties and expansions have also been the subject of several articles: see, for example, [1–4,14].

It is widely known that Heyting algebras are the equivalent algebraic semantics for intuitionistic logic. In turn, semi-Heyting algebras have an associated logic, **SI**, known as *semi-intuitionistic logic*, introduced in [6] by a Hilbert style calculus over the language $\{\wedge, \vee, \to, \neg\}$. In this article all formulas are defined over the language $L = \{\bot, \top, \wedge, \vee, \to\}$, thus we use the logic **HsH** introduced in [7], which is equivalent (in the sense of [12]) to the former and complete with respect to the class of algebras $\mathcal{SH}$. The propositional calculus of the logic **HsH** is defined in a Hilbert style over the language L with the following set of axioms, where $\alpha \to_H \beta$ stands for the formula $\alpha \to (\alpha \wedge \beta)$.

(S1) $\alpha \to_H (\alpha \vee \beta)$
(S2) $\beta \to_H (\alpha \vee \beta)$
(S3) $(\alpha \to_H \gamma) \to_H [(\beta \to_H \gamma) \to_H ((\alpha \vee \beta) \to_H \gamma)]$
(S4) $(\alpha \wedge \beta) \to_H \alpha$
(S5) $(\gamma \to_H \alpha) \to_H [(\gamma \to_H \beta) \to_H (\gamma \to_H (\alpha \wedge \beta))]$
(S6) $\top$
(S7) $\bot \to_H \alpha$
(S8) $((\alpha \wedge \beta) \to_H \gamma) \to_H (\alpha \to_H (\beta \to_H \gamma))$
(S9) $(\alpha \to_H (\beta \to_H \gamma)) \to_H ((\alpha \wedge \beta) \to_H \gamma)$
(S10) $(\alpha \to_H \beta) \to_H ((\beta \to_H \alpha) \to_H ((\alpha \to \gamma) \to_H (\beta \to \gamma)))$
(S11) $(\alpha \to_H \beta) \to_H ((\beta \to_H \alpha) \to_H ((\gamma \to \beta) \to_H (\gamma \to \alpha)))$

The inference rule is *semi-modus ponens* (SMP): $\Sigma \vdash_{\mathbf{HsH}} \alpha$ and $\Sigma \vdash_{\mathbf{HsH}} \alpha \to (\alpha \wedge \beta)$ yield $\Sigma \vdash_{\mathbf{HsH}} \beta$. Notice that this is *modus ponens* for the implication $\to_H$. Some aspects of the systems **SI** and **HsH** have been studied in [7,8].

In Section 2 we introduce a Gentzen style sequent calculus for **SI**, based on the corresponding calculus for intuitionistic logic (see, for example, [10]). This calculus is denoted by ***GsH*** and is defined by setting some initial sequents, some structural rules, the cut rule and rules that introduce the connectives $\{\wedge, \vee, \to\}$. Then, in Section 3, we prove the main result of this article, namely, the cut-elimination theorem for this calculus. We also derive some important consequences of cut-elimination: the subformula property, the disjunction property and decidability. Finally, in Section 4, we prove that the logic associated with $\boldsymbol{GsH}$ is complete with respect to the variety $\mathcal{SH}$. Moreover, $\mathcal{SH}$ is the equivalent variety semantics of the logic $\boldsymbol{GsH}$. This correspondence allows us to derive some properties of the variety $\mathcal{SH}$ from the properties of the calculus $\boldsymbol{GsH}$, namely, semi-Heyting algebras have decidable equational theories and free semi-Heyting algebras are indecomposable.

2. The Calculus $\boldsymbol{GsH}$

Following [9], a *logical language* L will be a set of connectives, each with a fixed arity $n \geq 0$. For a countably infinite set Var of propositional variables, the *formulas* of the logical language L are inductively defined as usual. We denote this set by Fm. In this work we use the language $L = \{\bot, \top, \wedge, \vee, \rightarrow\}$, where $\bot$ and $\top$ have arity 0 and the rest of the connectives are binary.

In this article, a *sequent* is an expression of the form $\Gamma \Rightarrow \alpha$, where $\alpha \in Fm$ and Γ is a (possibly empty) finite multiset of elements of Fm. We use Greek capital letters, such as Γ, Δ, to denote (possibly empty) finite multisets of formulas. When writing a multiset in full, we separate its elements with commas. Thus, if $\alpha_1, \alpha_2, \ldots, \alpha_n \in Fm$ (not necessarily all different), $\Gamma = \alpha_1, \alpha_2, \ldots, \alpha_n$ is a multiset. In the sequent $\Gamma \Rightarrow \alpha$ its *antecedent* is Γ and its *succedent* is α. We denote by Γ, Δ the multiset union of Γ and Δ. We also abbreviate $\Gamma, \{\alpha\}$ as Γ, α.

In the following we introduce a Gentzen-style sequent calculus that we denote by $\boldsymbol{GsH}$. The system $\boldsymbol{GsH}$ consists of some *initial sequents* and three types of inference rules: two structural rules, the cut rule and 9 rules that introduce connectives. Note that every initial sequent and inference rule given here is a scheme that stands for any of its substitution instances.

Initial sequents:

$$\Gamma, \alpha \Rightarrow \alpha \qquad\qquad \Gamma, \bot \Rightarrow \alpha \qquad\qquad \Gamma \Rightarrow \top$$

Structural rules:

$$\frac{\Gamma \Rightarrow \alpha}{\Gamma, \beta \Rightarrow \alpha}\ (\mathrm{w}) \qquad\qquad \frac{\Gamma, \beta, \beta \Rightarrow \alpha}{\Gamma, \beta \Rightarrow \alpha}\ (\mathrm{c})$$

Cut rule:

$$\frac{\Gamma \Rightarrow \alpha \qquad \Delta, \alpha \Rightarrow \beta}{\Gamma, \Delta \Rightarrow \beta}\ (\mathrm{cut})$$

Rules for introduction of connectives:

$$\frac{\Gamma \Rightarrow \alpha \qquad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}\ (\Rightarrow\wedge) \qquad\qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta}\ (\Rightarrow\vee 1)$$

$$\frac{\Gamma, \alpha \Rightarrow \gamma}{\Gamma, \alpha \wedge \beta \Rightarrow \gamma}\ (\wedge\Rightarrow 1) \qquad\qquad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \beta \vee \alpha}\ (\Rightarrow\vee 2)$$

$$\frac{\Gamma, \alpha \Rightarrow \gamma}{\Gamma, \beta \wedge \alpha \Rightarrow \gamma}\ (\wedge\Rightarrow 2) \qquad\qquad \frac{\Gamma \Rightarrow \alpha \qquad \Gamma, \beta \Rightarrow \gamma}{\Gamma, \alpha \rightarrow \beta \Rightarrow \gamma}\ (\rightarrow\Rightarrow)$$

$$\frac{\Gamma, \alpha \Rightarrow \gamma \qquad \Gamma, \beta \Rightarrow \gamma}{\Gamma, \alpha \vee \beta \Rightarrow \gamma}\ (\vee\Rightarrow) \qquad\qquad \frac{\Gamma, \alpha \Rightarrow \beta \qquad \Gamma, \beta \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \rightarrow \beta}\ (\Rightarrow\rightarrow)$$

$$\frac{\Gamma, \alpha_1 \Rightarrow \beta_1 \qquad \Gamma, \beta_1 \Rightarrow \alpha_1 \qquad \Gamma, \alpha_2 \Rightarrow \beta_2 \qquad \Gamma, \beta_2 \Rightarrow \alpha_2}{\Gamma, \alpha_1 \rightarrow \alpha_2 \Rightarrow \beta_1 \rightarrow \beta_2}\ (\rightarrow\Rightarrow\rightarrow)$$

We define *proofs* in the system $\boldsymbol{GsH}$ and *end sequents* of proofs in the following way:

(a) Every initial sequent is a proof in itself whose end sequent is the initial sequent.

(b) Suppose that P and Q are proofs whose end sequents are s_1 and s_2 respectively and suppose that there exists an instance of a rule of $\boldsymbol{GsH}$ with upper sequents s_1 and s_2 such that the lower sequent is s. Then, the following figure is a proof whose end sequent is s.

$$\frac{P \qquad Q}{s}$$

For rules with a single or four upper sequents, we define proofs in an analogous way.

A proof with end sequent s is called a *proof of* s. If there is a proof of s, we write $\vdash^{seq}_{\boldsymbol{GsH}} s$.

We define deducibility (proof with assumptions) in this sequent calculus as usual. If $S \cup \{s\}$ is a set of sequents, we say that s is *deducible* or *provable* from assumptions S, in symbols $S \vdash^{seq}_{\boldsymbol{GsH}} s$, if there is a *proof* of s in $\boldsymbol{GsH}$ from S. This is defined in the same way as proofs in $\boldsymbol{GsH}$ adding the following:

(c) Every sequent in S is a proof whose end sequent is itself.

Associated with the sequent calculus $\boldsymbol{GsH}$ we can define a logic, that is, a consequence relation between sets of formulas and formulas. More precisely, given $\Sigma \cup \{\alpha\} \subseteq Fm$, we say that α is a consequence of Σ in the logic **GsH**, and write $\Sigma \vdash_{\mathbf{GsH}} \alpha$, if and only if $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \Rightarrow \alpha$. When $\emptyset \vdash_{\mathbf{GsH}} \alpha$ we say that α is *provable* in **GsH** or that α is a *theorem* of **GsH**. Note that α is provable in **GsH** if and only if $\Rightarrow \alpha$ is a provable sequent in the calculus $\boldsymbol{GsH}$.

In a future section we will show that the logic **GsH** and the Hilbert style logic **HsH** are the same, since both are equal to the 1-assertional logic of semi-Heyting algebras. We now state and prove some straightforward results in the calculus $\boldsymbol{GsH}$ that will be useful in future proofs.

LEMMA 2.1. *In the sequent calculus* $\boldsymbol{GsH}$ *the following derivations are valid:*

(*a*) $\vdash^{seq}_{\boldsymbol{GsH}} \Rightarrow \alpha \to \alpha$

(*b*) $\vdash^{seq}_{\boldsymbol{GsH}} \alpha \wedge (\alpha \to \beta) \Rightarrow \alpha \wedge \beta$

(*c*) $\vdash^{seq}_{\boldsymbol{GsH}} \alpha \wedge \beta \Rightarrow \alpha \wedge (\alpha \to \beta)$

(*d*) $\vdash^{seq}_{\boldsymbol{GsH}} \alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma)) \Rightarrow \alpha \wedge (\beta \to \gamma)$

(*e*) $\vdash^{seq}_{\boldsymbol{GsH}} \alpha \wedge (\beta \to \gamma) \Rightarrow \alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))$

PROOF.

(*a*)
$$\dfrac{\alpha \Rightarrow \alpha \qquad \alpha \Rightarrow \alpha}{\Rightarrow \alpha \to \alpha}\ (\Rightarrow\to)$$

(*b*)
$$\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha \wedge (\alpha \to \beta) \Rightarrow \alpha}\ (\wedge \Rightarrow 1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\alpha \Rightarrow \alpha \qquad \alpha, \beta \Rightarrow \beta}{\alpha, \alpha \to \beta \Rightarrow \beta}\ (\to\Rightarrow)}{\alpha \wedge (\alpha \to \beta), \alpha \to \beta \Rightarrow \beta}\ (\wedge \Rightarrow 1)}{\alpha \wedge (\alpha \to \beta), \alpha \wedge (\alpha \to \beta) \Rightarrow \beta}\ (\wedge \Rightarrow 2)}{\alpha \wedge (\alpha \to \beta) \Rightarrow \beta}\ (\mathrm{c})}{\alpha \wedge (\alpha \to \beta) \Rightarrow \alpha \wedge \beta}\ (\Rightarrow\wedge)$$

(*c*)
$$\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha \wedge \beta \Rightarrow \alpha}\ (\wedge \Rightarrow 1) \qquad \dfrac{\dfrac{\beta, \alpha \Rightarrow \beta}{\alpha \wedge \beta, \alpha \Rightarrow \beta}\ (\wedge \Rightarrow 2) \qquad \dfrac{\alpha, \beta \Rightarrow \alpha}{\alpha \wedge \beta, \beta \Rightarrow \alpha}\ (\wedge \Rightarrow 1)}{\alpha \wedge \beta \Rightarrow \alpha \to \beta}\ (\Rightarrow\to)}{\alpha \wedge \beta \Rightarrow \alpha \wedge (\alpha \to \beta)}\ (\Rightarrow\wedge)$$

(*d*)

$$\dfrac{\alpha, \beta \Rightarrow \beta}{\alpha, \alpha \wedge \beta \Rightarrow \beta}\ (\wedge \Rightarrow 1) \qquad (1)$$

$$\dfrac{\alpha, \beta \Rightarrow \alpha \qquad \alpha, \beta \Rightarrow \beta}{\alpha, \beta \Rightarrow \alpha \wedge \beta}\ (\Rightarrow\wedge) \qquad (2)$$

$$\dfrac{\alpha, \gamma \Rightarrow \gamma}{\alpha, \alpha \wedge \gamma \Rightarrow \gamma}\ (\wedge \Rightarrow 1) \qquad (3)$$

$$\dfrac{\alpha, \gamma \Rightarrow \alpha \qquad \alpha, \gamma \Rightarrow \gamma}{\alpha, \gamma \Rightarrow \alpha \wedge \gamma}\ (\Rightarrow\wedge) \qquad (4)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{(1) \quad (2) \quad (3) \quad (4)}{\alpha, (\alpha \wedge \beta) \to (\alpha \wedge \gamma) \Rightarrow \beta \to \gamma}\,{\scriptstyle (\to\Rightarrow\to)}}{\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma)), (\alpha \wedge \beta) \to (\alpha \wedge \gamma) \Rightarrow \beta \to \gamma}\,{\scriptstyle (\wedge \Rightarrow 1)}}{\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma)), \alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma)) \Rightarrow \beta \to \gamma}\,{\scriptstyle (\wedge \Rightarrow 2)}}{\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma)) \Rightarrow \beta \to \gamma}\,{\scriptstyle (c)} \tag{5}$$

and

$$\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma)) \Rightarrow \alpha}\,{\scriptstyle (\wedge \Rightarrow 1)} \qquad (5)}{\alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma)) \Rightarrow \alpha \wedge (\beta \to \gamma)}\,{\scriptstyle (\Rightarrow\wedge)}$$

(e) Using (1)–(4) from the previous item, we get:

$$\dfrac{\dfrac{\alpha \Rightarrow \alpha}{\alpha \wedge (\beta \to \gamma) \Rightarrow \alpha}\,{\scriptstyle (\wedge \Rightarrow 1)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{(2) \quad (1) \quad (4) \quad (3)}{\alpha, \beta \to \gamma \Rightarrow (\alpha \wedge \beta) \to (\alpha \wedge \gamma)}\,{\scriptstyle (\to\Rightarrow\to)}}{\alpha \wedge (\beta \to \gamma), \beta \to \gamma \Rightarrow (\alpha \wedge \beta) \to (\alpha \wedge \gamma)}\,{\scriptstyle (\wedge \Rightarrow 1)}}{\alpha \wedge (\beta \to \gamma), \alpha \wedge (\beta \to \gamma) \Rightarrow (\alpha \wedge \beta) \to (\alpha \wedge \gamma)}\,{\scriptstyle (\wedge \Rightarrow 2)}}{\alpha \wedge (\beta \to \gamma) \Rightarrow (\alpha \wedge \beta) \to (\alpha \wedge \gamma)}\,{\scriptstyle (c)}}{\alpha \wedge (\beta \to \gamma) \Rightarrow \alpha \wedge ((\alpha \wedge \beta) \to (\alpha \wedge \gamma))}\,{\scriptstyle (\Rightarrow\wedge)}$$

■

3. Properties of *GsH*

We would like to show a cut elimination theorem for the calculus ***GsH*** introduced in the previous section. In order to do that, we will need an auxiliary calculus $\boldsymbol{GsH}^*$ which is obtained from ***GsH*** by replacing the cut rule with the following *gmix rule*:

$$\dfrac{\Gamma \Rightarrow \alpha \qquad \Delta \Rightarrow \beta}{\Gamma, \Delta_\alpha \Rightarrow \beta}\,{\scriptstyle (\text{gmix})}$$

where $\alpha \in \Delta$ and Δ_α is the multiset obtained upon eliminating *at least* one of the occurrences of α from Δ. We call each occurrence of α in Δ a *gmix formula*.

LEMMA 3.1. *The calculi* ***GsH*** *and* $\boldsymbol{GsH}^*$ *are equivalent in the sense that for any set of sequents* $S \cup \{s\}$, $S \vdash^{seq}_{\boldsymbol{GsH}} s$ *iff* $S \vdash^{seq}_{\boldsymbol{GsH}^*} s$. *In particular, a sequent* s *is provable in* ***GsH*** *if and only if it is provable in* $\boldsymbol{GsH}^*$.

PROOF. It is enough to show that the cut rule is derivable in $\boldsymbol{GsH}^*$ (the cut rule is just a special case of the gmix rule) and the gmix rule is derivable in ***GsH*** (use the cut rule and contraction if necessary). ■

LEMMA 3.2. *If a sequent s is provable in $\boldsymbol{GsH}^*$, then it is provable without using the weakening rule.*

PROOF. Given the form of the initial sequents of the calculus, this result may be straightforwardly verified using a standard argument by induction on the length of proofs. ■

The following is the crucial lemma in order to obtain the cut elimination theorem.

LEMMA 3.3. *If a sequent s is provable in $\boldsymbol{GsH}^*$, then it is provable without using the gmix rule.*

PROOF. By the previous lemma, we can restrict ourselves to proofs that do not make use of the weakening rule. A standard argument by induction on the length of proofs reduces the statement of the present lemma to the following: any sequent s that has a proof in $\boldsymbol{GsH}^*$ that only uses the gmix rule in the last step has a proof in $\boldsymbol{GsH}^*$ that does not use the gmix rule at all.

If a proof uses the gmix rule only in the last step, this step will look like this:

$$\frac{\Gamma \Rightarrow \alpha \qquad \Delta \Rightarrow \beta}{\Gamma, \Delta_\alpha \Rightarrow \beta}\ \text{(gmix)} \tag{$*$}$$

where $\Gamma \Rightarrow \alpha$ and $\Delta \Rightarrow \beta$ are sequents whose proofs do not involve the gmix rule, α appears at least once in Δ and at least one less time in Δ_α. Here α is the gmix formula. For this kind of proofs we define the following notions:

- the *size* of the proof $s(P)$: the total number of sequents that appear in the proof.
- the *grade* of the proof $g(P)$: the length of the gmix formula α used in the last step.

The proof will proceed by double induction on the size and grade of the proof. Since it is similar to the corresponding cut-elimination proof for the intuitionistic sequent calculus and it involves the consideration of lots of different cases, we will describe an outline of all the cases that must be considered, prove in detail the ones that involve the rules that are specific to the calculus $\boldsymbol{GsH}^*$ and leave the rest of the details to the reader.

Assume that a proof P uses the gmix rule only in its last step, which must be of the form ($*$). Let $s(P)$ and $g(P)$ denote the size and grade of this proof, respectively. The following is an outline of all the cases that should be dealt with:

(A) $s(P) = 3$ (its least possible value).

In this case, both upper sequents $\Gamma \Rightarrow \alpha$ and $\Delta \Rightarrow \beta$ must be initial sequents and it is easy to show that the lower sequent is also an initial sequent.

(B) $s(P) > 3$ and $g(P) = 1$.

In this case, α is either $\top$, $\bot$ or a propositional variable. We distinguish the following cases:

(1) $\Gamma \Rightarrow \alpha$ results from an application of a left-introduction rule or contraction.
(2) $\Delta \Rightarrow \beta$ results from an application of a right-introduction rule or contraction.
(3) $\Delta \Rightarrow \beta$ results from an application of a left-introduction rule in which α is not the main formula.

None of these cases directly involve the formula α. Thus we can "push up" the use of the gmix rule and use the induction hypothesis on the size of the proof to eliminate it.

(C) $s(P) > 3$ and $g(P) > 1$.

We subdivide this case into three subcases.

(1) $\Gamma \Rightarrow \alpha$ is an initial sequent.
In this case $\Delta \Rightarrow \beta$ cannot be an initial sequent, so we have two possibilities:
  (i) α is not the main formula in $\Delta \Rightarrow \beta$.
  As in case (B), the application of the gmix rule can be "pushed up" and eliminated by the induction hypothesis on the size $s(P)$.
  (ii) α is the main formula in $\Delta \Rightarrow \beta$.
  Since $\Gamma \Rightarrow \alpha$ is an initial sequent, there are three possibilities:
  - α appears in Γ: depending on which rule introduces α each case may be easily dealt with separately.
  - $\bot$ appears in Γ: in this case, $\Gamma, \Delta_\alpha \Rightarrow \beta$ is also an initial sequent.
  - $\alpha = \top$: this case is not possible, since α is the main formula in $\Delta \Rightarrow \beta$.

(2) $\Gamma \Rightarrow \alpha$ is not an initial sequent and α is not the main formula in $\Gamma \Rightarrow \alpha$.
As in case (B), the application of the gmix rule can be "pushed up" and eliminated by the induction hypothesis on the size $s(P)$.

(3) $\Gamma \Rightarrow \alpha$ is not an initial sequent and α is the main formula in $\Gamma \Rightarrow \alpha$. We distinguish the possibilities for $\Delta \Rightarrow \beta$:
   (i) $\Delta \Rightarrow \beta$ is an initial sequent.
   If $\beta = \alpha$, then $\Gamma, \Delta_\alpha \Rightarrow \beta$ follows from $\Gamma \Rightarrow \alpha$ by the weakening rule. In the other cases, $\Gamma, \Delta_\alpha \Rightarrow \beta$ is an initial sequent.
   (ii) $\Delta \Rightarrow \beta$ is not an initial sequent, but α is not the main formula in $\Delta \Rightarrow \beta$.
   (iii) $\Delta \Rightarrow \beta$ is not an initial sequent and α is the main formula in $\Delta \Rightarrow \beta$.

We now give the details of some selected cases:

Example of case (C-3-ii): Suppose $\Delta \Rightarrow \beta$ is of the form $\Delta, \delta_1 \to \delta_2 \Rightarrow \beta_1 \to \beta_2$ and is obtained by an application of the rule $(\to\Rightarrow\to)$. Note that $\alpha \in \Delta$, $\alpha \neq \delta_1 \to \delta_2$. Thus the following sequents have gmix-free proofs:

$$\Delta, \delta_1 \Rightarrow \beta_1 \qquad \Delta, \beta_1 \Rightarrow \delta_1 \qquad \Delta, \delta_2 \Rightarrow \beta_2 \qquad \Delta, \beta_2 \Rightarrow \delta_2$$

so that proof P ends like this:

$$\dfrac{\Gamma \Rightarrow \alpha \qquad \dfrac{\Delta, \delta_1 \Rightarrow \beta_1 \qquad \Delta, \beta_1 \Rightarrow \delta_1 \qquad \Delta, \delta_2 \Rightarrow \beta_2 \qquad \Delta, \beta_2 \Rightarrow \delta_2}{\Delta, \delta_1 \to \delta_2 \Rightarrow \beta_1 \to \beta_2}\,(\to\Rightarrow\to)}{\Gamma, \Delta_\alpha, \delta_1 \to \delta_2 \Rightarrow \beta_1 \to \beta_2}\,(gmix)$$

Using the proofs of $\Gamma \Rightarrow \alpha$ and $\Delta, \delta_1 \Rightarrow \beta_1$, we build a proof:

$$\dfrac{\Gamma \Rightarrow \alpha \qquad \Delta, \delta_1 \Rightarrow \beta_1}{\Gamma, \Delta_\alpha, \delta_1 \Rightarrow \beta_1}\,(gmix)$$

whose size is strictly less that $s(P)$. By the induction hypothesis, the sequent $\Gamma, \Delta_\alpha, \delta_1 \Rightarrow \beta_1$ has a gmix-free proof. Analogously, the following sequents have gmix-free proofs as well:

$$\Gamma, \Delta_\alpha, \beta_1 \Rightarrow \delta_1 \qquad \Gamma, \Delta_\alpha, \delta_2 \Rightarrow \beta_2 \qquad \Gamma, \Delta_\alpha, \beta_2 \Rightarrow \delta_2.$$

Combining these proofs, we get a gmix-free proof:

$$\dfrac{\Gamma, \Delta_\alpha, \delta_1 \Rightarrow \beta_1 \qquad \Gamma, \Delta_\alpha, \beta_1 \Rightarrow \delta_1 \qquad \Gamma, \Delta_\alpha, \delta_2 \Rightarrow \beta_2 \qquad \Gamma, \Delta_\alpha, \beta_2 \Rightarrow \delta_2}{\Gamma, \Delta_\alpha, \delta_1 \to \delta_2 \Rightarrow \beta_1 \to \beta_2}\,(\to\Rightarrow\to)$$

Example of case (C-3-iii): Assume α is the main formula both in $\Gamma \Rightarrow \alpha$ and $\Delta \Rightarrow \beta$. Moreover, assume that $\Gamma \Rightarrow \alpha$ derives from an application of rule $(\Rightarrow\to)$ and $\Delta \Rightarrow \beta$ from an application of rule $(\to\Rightarrow\to)$. Thus $\alpha = \alpha_1 \to \alpha_2$ and $\beta = \beta_1 \to \beta_2$. We write Δ_k to indicate that the formula α occurs k times in the multiset Δ. Then the structure of the proof P is:

$$\dfrac{\Gamma \Rightarrow \alpha_1 \to \alpha_2 \qquad \Delta_k \Rightarrow \beta_1 \to \beta_2}{\Gamma, \Delta_{k-r} \Rightarrow \beta_1 \to \beta_2}\,(gmix)$$

where $k \geq r \geq 1$ and the upper sequents have proofs of the form:

$$\dfrac{\Delta_{k-1}, \alpha_1 \Rightarrow \beta_1 \qquad \Delta_{k-1}, \beta_1 \Rightarrow \alpha_1 \qquad \Delta_{k-1}, \alpha_2 \Rightarrow \beta_2 \qquad \Delta_{k-1}, \beta_2 \Rightarrow \alpha_2}{\Delta_k \Rightarrow \beta_1 \to \beta_2}\ (\to\Rightarrow\to)$$

$$\dfrac{\Gamma, \alpha_1 \Rightarrow \alpha_2 \qquad \Gamma, \alpha_2 \Rightarrow \alpha_1}{\Gamma \Rightarrow \alpha_1 \to \alpha_2}\ (\Rightarrow\to)$$

In these proofs, the upper sequents all have gmix-free proofs.

We further distinguish two cases:

(a) $k \neq r$ or $k = r = 1$.

Consider the following proof of $\Gamma, \Delta_{k-r}, \beta_1 \Rightarrow \beta_2$:

$$\dfrac{\dfrac{\dfrac{\Delta_{k-1}, \beta_1 \Rightarrow \alpha_1 \qquad \Gamma, \alpha_1 \Rightarrow \alpha_2}{\Delta_{k-1}, \beta_1, \Gamma \Rightarrow \alpha_2}\ (gmix) \qquad \Delta_{k-1}, \alpha_2 \Rightarrow \beta_2}{\Delta_{k-1}, \beta_1, \Gamma, \Delta_{k-1} \Rightarrow \beta_2}\ (gmix)}{\Gamma, \Delta_{k-r}, \beta_1 \Rightarrow \beta_2}\ (c)$$

In the proof of $\Delta_{k-1}, \beta_1, \Gamma \Rightarrow \alpha_2$, the gmix formula is α_1, so its grade is less that $g(P)$. Thus this proof may be replaced with a gmix-free proof. Then $\Delta_{k-1}, \beta_1, \Gamma, \Delta_{k-1} \Rightarrow \beta_2$ has a proof which uses the gmix rule only in its last step. In this proof α_2 is the gmix formula, so its grade is strictly less than $g(P)$. Then the sequent $\Delta_{k-1}, \beta_1, \Gamma, \Delta_{k-1} \Rightarrow \beta_2$ has a gmix-free proof, and so does $\Gamma, \Delta_{k-r}, \beta_1 \Rightarrow \beta_2$.

In a similar way, $\Gamma, \Delta_{k-r}, \beta_2 \Rightarrow \beta_1$ also has a gmix-free proof. We can provide an adequate proof for $\Gamma, \Delta_{k-r} \Rightarrow \beta_1 \to \beta_2$ as in the previous case.

(b) $k = r > 1$.

We build the following proof of $\Gamma, \Delta_{k-r}, \beta_1 \Rightarrow \beta_2$:

$$\dfrac{\dfrac{\dfrac{\Gamma \Rightarrow \alpha_1 \to \alpha_2 \qquad \Delta_{k-1}, \beta_1 \Rightarrow \alpha_1}{\Gamma, \Delta_{k-r}, \beta_1 \Rightarrow \alpha_1}\ (gmix) \qquad \dfrac{\Gamma, \alpha_1 \Rightarrow \alpha_2 \qquad \dfrac{\Gamma \Rightarrow \alpha_1 \to \alpha_2 \qquad \Delta_{k-1}, \alpha_2 \Rightarrow \beta_2}{\Gamma, \Delta_{k-r}, \alpha_2 \Rightarrow \beta_2}\ (gmix)}{\Gamma, \alpha_1, \Gamma, \Delta_{k-r} \Rightarrow \beta_2}\ (gmix)}{\Gamma, \Delta_{k-r}, \beta_1, \Gamma, \Gamma, \Delta_{k-r} \Rightarrow \beta_2}\ (gmix)}{\Gamma, \Delta_{k-r}, \beta_1 \Rightarrow \beta_2}\ (c)\ .$$

Since the sizes of the proofs for $\Gamma, \Delta_{k-r}, \beta_1 \Rightarrow \alpha_1$ and $\Gamma, \Delta_{k-r}, \alpha_2 \Rightarrow \beta_2$ are strictly less than $s(P)$, the induction hypothesis allows us to replace them with gmix-free proofs. Now the proof for $\Gamma, \alpha_1, \Gamma, \Delta_{k-r} \Rightarrow \beta_2$ has grade less than $g(P)$ because its gmix formula is α_2. Then, regardless of the size of this proof, we may apply the induction hypothesis and provide a gmix-free proof of this same sequent. We get now to the proof of $\Gamma, \Delta_{k-r}, \beta_1, \Gamma, \Gamma, \Delta_{k-r} \Rightarrow \beta_2$, which, again, has grade strictly less than $g(P)$. By the induction hypothesis, we conclude that there is a gmix-free proof of $\Gamma, \Delta_{k-r}, \beta_1 \Rightarrow \beta_2$.

In a similar way the sequent $\Gamma, \Delta_{k-r}, \beta_2 \Rightarrow \beta_1$ also may be shown to have a gmix-free proof. We finally put both gmix-free proofs together to get:

$$\dfrac{\Gamma, \Delta_{k-r}, \beta_1 \Rightarrow \beta_2 \qquad \Gamma, \Delta_{k-r}, \beta_2 \Rightarrow \beta_1}{\Gamma, \Delta_{k-r} \Rightarrow \beta_1 \to \beta_2}\,(\Rightarrow\to)$$

Another example of case (C-3-iii): Suppose $\Gamma \Rightarrow \alpha$ derives from an application of rule $(\to\Rightarrow\to)$ and $\Delta \Rightarrow \beta$ from an application of rule $(\to\Rightarrow)$, in both cases α being the main formula. Thus $\alpha = \alpha_1 \to \alpha_2$ and the structure of the proof P is:

$$\dfrac{\Gamma, \gamma_1 \to \gamma_2 \Rightarrow \alpha_1 \to \alpha_2 \qquad \Delta_k \Rightarrow \beta}{\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \beta}\,(gmix)$$

with $k \geq r \geq 1$ and the following derivations for the upper sequents:

$$\dfrac{\Gamma, \gamma_1 \Rightarrow \alpha_1 \qquad \Gamma, \alpha_1 \Rightarrow \gamma_1 \qquad \Gamma, \gamma_2 \Rightarrow \alpha_2 \qquad \Gamma, \alpha_2 \Rightarrow \gamma_2}{\Gamma, \gamma_1 \to \gamma_2 \Rightarrow \alpha_1 \to \alpha_2}\,(\to\Rightarrow\to)$$

$$\dfrac{\Delta_{k-1} \Rightarrow \alpha_1 \qquad \Delta_{k-1}, \alpha_2 \Rightarrow \beta}{\Delta_k \Rightarrow \beta}\,(\to\Rightarrow)$$

In these last two derivations each upper subsequent has a gmix-free proof. We must distinguish two possible cases for r.

(a) $k \geq r > 1$.

Consider the following proof:

$$\dfrac{\dfrac{(6) \qquad (7)}{\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r}, \Gamma, \gamma_1 \to \gamma_2, \Rightarrow \beta}\,(\to\Rightarrow)}{\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \beta}\,(c)$$

where

$$\dfrac{\dfrac{\Gamma, \gamma_1 \to \gamma_2 \Rightarrow \alpha_1 \to \alpha_2 \qquad \Delta_{k-1} \Rightarrow \alpha_1}{\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \alpha_1}\,(gmix) \qquad \Gamma, \alpha_1 \Rightarrow \gamma_1}{\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r}, \Gamma \Rightarrow \gamma_1}\,(gmix) \tag{6}$$

$$\dfrac{\Gamma, \gamma_2 \Rightarrow \alpha_2 \qquad \dfrac{\Gamma, \gamma_1 \to \gamma_2 \Rightarrow \alpha_1 \to \alpha_2 \qquad \Delta_{k-1}, \alpha_2 \Rightarrow \beta}{\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r}, \alpha_2 \Rightarrow \beta}\,(gmix)}{\Gamma, \gamma_2, \Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \beta}\,(gmix) \tag{7}$$

The two topmost applications of (gmix) have size strictly less than $s(P)$. Hence, $\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \alpha_1$ and $\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r}, \alpha_2 \Rightarrow \beta$ have gmix-free proofs. Moreover, the (gmix) applications at the bottom of (6) and

(7) have grade less that $g(P)$. Consequently, $\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \beta$ has a gmix-free proof.

(b) $k \geq r = 1$.

We build the following proof:

$$\dfrac{\dfrac{\Delta_{k-1} \Rightarrow \alpha_1 \qquad \Gamma, \alpha_1 \Rightarrow \gamma_1}{\Gamma, \Delta_{k-1} \Rightarrow \gamma_1}\,(gmix) \qquad \dfrac{\Gamma, \gamma_2 \Rightarrow \alpha_2 \qquad \Delta_{k-1}, \alpha_2 \Rightarrow \beta}{\Gamma, \gamma_2, \Delta_{k-1} \Rightarrow \beta}\,(gmix)}{\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-1} \Rightarrow \beta}\,(\to\Rightarrow)$$

Here both applications of (gmix) have grade less than $g(P)$. Consequently, $\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \beta$ has a gmix-free proof.

Yet another example of case (C-3-iii)*:* Now consider the case in which both $\Gamma \Rightarrow \alpha$ and $\Delta \Rightarrow \beta$ come from an application of the rule $(\to\Rightarrow\to)$. Then $\alpha = \alpha_1 \to \alpha_2$, $\beta = \beta_1 \to \beta_2$, and we have a proof like this:

$$\dfrac{(8) \qquad (9)}{\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \beta_1 \to \beta_2}\,(gmix)$$

where $k \geq r \geq 1$ and

$$\dfrac{\Gamma, \gamma_1 \Rightarrow \alpha_1 \qquad \Gamma, \alpha_1 \Rightarrow \gamma_1 \qquad \Gamma, \gamma_2 \Rightarrow \alpha_2 \qquad \Gamma, \alpha_2 \Rightarrow \gamma_2}{\Gamma, \gamma_1 \to \gamma_2 \Rightarrow \alpha_1 \to \alpha_2}\,(\to\Rightarrow\to) \tag{8}$$

$$\dfrac{\Delta_{k-1}, \alpha_1 \Rightarrow \beta_1 \qquad \Delta_{k-1}, \beta_1 \Rightarrow \alpha_1 \qquad \Delta_{k-1}, \alpha_2 \Rightarrow \beta_2 \qquad \Delta_{k-1}, \beta_2 \Rightarrow \alpha_2}{\Delta_k \Rightarrow \beta_1 \to \beta_2}\,(\to\Rightarrow\to) \tag{9}$$

all of whose upper sequents have gmix-free proofs.

In this case we must also distinguish two cases for r.

(a) $r > 1$.

Since the size of the following proof

$$\dfrac{\Gamma, \gamma_1 \to \gamma_2 \Rightarrow \alpha_1 \to \alpha_2 \qquad \Delta_{k-1}, \alpha_1 \Rightarrow \beta_1}{\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r}, \alpha_1 \Rightarrow \beta_1}\,(gmix).$$

is less than $s(P)$, there is a gmix-free proof of the lower sequent. Then we can build a proof

$$\dfrac{\Gamma, \gamma_1 \Rightarrow \alpha_1 \qquad \Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r}, \alpha_1 \Rightarrow \beta_1}{\Gamma, \gamma_1, \Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \beta_1}\,(gmix).$$

whose grade is less than $g(P)$. Consequently, there is a gmix-free proof for

$$\Gamma, \gamma_1, \Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \beta_1 \tag{10}$$

In a similar way, there are gmix-free proofs for

$$\Gamma, \beta_1, \Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \gamma_1 \tag{11}$$

$$\Gamma, \gamma_2, \Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \beta_2 \tag{12}$$

$$\Gamma, \beta_2, \Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \gamma_2 \tag{13}$$

We now use these sequents to build a gmix-free proof of $\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \beta_1 \to \beta_2$:

$$\dfrac{\dfrac{(10) \qquad (11) \qquad (12) \qquad (13)}{\Gamma, \Gamma, \gamma_1 \to \gamma_2, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \beta_1 \to \beta_2}\,(\to\Rightarrow\to)}{\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-r} \Rightarrow \beta_1 \to \beta_2}\,(c)$$

(b) $r = 1$.

The grade of the following proof:

$$\dfrac{\Gamma, \gamma_1 \Rightarrow \alpha_1 \qquad \Delta_{k-1}, \alpha_1 \Rightarrow \beta_1}{\Gamma, \gamma_1, \Delta_{k-1} \Rightarrow \beta_1}\,(gmix).$$

is less than $g(P)$, so the sequent $\Gamma, \gamma_1, \Delta_{k-1} \Rightarrow \beta_1$ has a gmix-free proof. The same can be said of

$$\Gamma, \beta_1, \Delta_{k-1} \Rightarrow \gamma_1, \quad \Gamma, \gamma_2, \Delta_{k-1} \Rightarrow \beta_2, \quad \text{and} \quad \Gamma, \beta_2, \Delta_{k-1} \Rightarrow \gamma_2.$$

Then we can produce a gmix-free proof for $\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-1} \Rightarrow \beta_1 \to \beta_2$:

$$\dfrac{\Gamma, \gamma_1, \Delta_{k-1} \Rightarrow \beta_1 \qquad \Gamma, \beta_1, \Delta_{k-1} \Rightarrow \gamma_1 \qquad \Gamma, \gamma_2, \Delta_{k-1} \Rightarrow \beta_2 \qquad \Gamma, \beta_2, \Delta_{k-1} \Rightarrow \gamma_2}{\Gamma, \gamma_1 \to \gamma_2, \Delta_{k-1} \Rightarrow \beta_1 \to \beta_2}\,(\to\Rightarrow\to)$$

■

THEOREM 3.4. (Cut elimination theorem) *If a sequent s is provable in **GsH**, then it is provable without using the cut rule.*

PROOF. If Q is a proof of a sequent s in ***GsH***, we may consider each instance of the cut rule in Q as an instance of the gmix rule, thus turning Q into a proof of s in ***GsH****. By the previous lemma, there is a proof Q' of s in ***GsH**** without any instances of the gmix rule. This proof Q' is clearly a proof of s in ***GsH*** without the cut rule. ■

A direct consequence of the cut elimination theorem is the following important property of ***GsH***.

COROLLARY 3.5. ***GsH** has the subformula property, i.e., any provable sequent s in **GsH** has a proof in which every formula appearing in it is a subformula of some formula in s.*

PROOF. Looking at the inference rules of $\boldsymbol{GsH}$, it is clear that any cut-free proof of s has the subformula property. ■

Following [10, p. 245], we say that a logic **L** has the *disjunction property* when for any formulas α and β, if $\alpha \vee \beta$ is provable in **L** then either α or β is provable in it.

COROLLARY 3.6. *The logic* **GsH** *has the disjunction property.*

PROOF. Let $\alpha \vee \beta$ be a provable formula in **GsH** and let Q be a cut-free proof of the sequent $\Rightarrow \alpha \vee \beta$. Looking at the inference rules of $\boldsymbol{GsH}$ it is evident that the last inference rule in Q must be an instance of ($\Rightarrow\vee$). Hence, either $\Rightarrow \alpha$ or $\Rightarrow \beta$ is provable. ■

A logic **L** is *decidable* if there exists an algorithm that determines whether a formula α is provable or not. The decidability of intuitionistic logic may be derived from the corresponding cut-elimination theorem (see [10, 11]). To prove the decidability of **GsH** we will follow a similar procedure as the one used for the intuitionistic logic.

We say that a proof Q of a sequent s has *redundancies*, if the same sequent appears twice in a branch of Q. A sequent $\Gamma \Rightarrow \alpha$ is *2-reduced* if every formula in Γ occurs at most twice, and is *1-reduced* if every formula of its antecedent occurs only once.

If we apply the contraction rule repeatedly to a sequent $\Gamma \Rightarrow \alpha$ we can obtain a 1-reduced sequent which we denote by $\Gamma^* \Rightarrow \alpha$. We say that $\Gamma^* \Rightarrow \alpha$ is the *contraction* of $\Gamma \Rightarrow \alpha$.

We will need the following lemma whose proof is similar to that of [10, Lemma 4.10, p. 220].

LEMMA 3.7. *Let* $\Gamma \Rightarrow \alpha$ *be a provable sequent in* $\boldsymbol{GsH}$*. There exists a cut-free proof of* $\Gamma^* \Rightarrow \alpha$ *in which every sequent is 2-reduced.*

PROOF. We use induction on the length of the cut-free proof of $\Gamma \Rightarrow \alpha$. If $\Gamma \Rightarrow \alpha$ is an initial sequent, so is $\Gamma^* \Rightarrow \alpha$. Now assume the cut-free proof of $\Gamma \Rightarrow \alpha$ ends with an instance of some rule (r) which introduces a new connective or with the weakening rule. There are three possible tree-situations:

$$\frac{\Delta_1 \Rightarrow \beta_1}{\Gamma \Rightarrow \alpha}\ (r)$$

$$\frac{\Delta_1 \Rightarrow \beta_1 \qquad \Delta_2 \Rightarrow \beta_2}{\Gamma \Rightarrow \alpha}\ (r)$$

$$\frac{\Delta_1 \Rightarrow \beta_1 \qquad \Delta_2 \Rightarrow \beta_2 \qquad \Delta_3 \Rightarrow \beta_3 \qquad \Delta_4 \Rightarrow \beta_4}{\Gamma \Rightarrow \alpha}\ (r)$$

By the induction hypothesis, there is a cut-free proof of each $\Delta_i^* \Rightarrow \beta_i$ that contains only 2-reduced sequents. Applying the same rule (r) to the sequents $\Delta_i^* \Rightarrow \beta_i$ we get $\Gamma' \Rightarrow \alpha$, where Γ' is a multiset of formulas with the same root set as Γ. Since each $\Delta_i^* \Rightarrow \beta_i$ is 1-reduced, $\Gamma' \Rightarrow \alpha$ is 2-reduced (the main formula of rule (r) may occur in Δ_i^*). From $\Gamma' \Rightarrow \alpha$, by an application of the contraction rule, we get a cut-free proof of $\Gamma^* \Rightarrow \alpha$ which contains only 2-reduced sequents.

If the last rule in the cut-free proof of $\Gamma \Rightarrow \alpha$ is the contraction rule, let $\Gamma_0 \Rightarrow \alpha$ be the upper sequent of this instance of the rule. By the induction hypothesis, there is a cut-free proof of $\Gamma_0^* \Rightarrow \alpha$ that contains just 2-reduced sequents. But $\Gamma_0^* \Rightarrow \alpha$ clearly coincides with $\Gamma^* \Rightarrow \alpha$. ■

Theorem 3.8. *The logic* **GsH** *is decidable.*

Proof. Consider a sequent $\Gamma \Rightarrow \alpha$. Using the contraction and weakening rules it is clear that the sequent $\Gamma \Rightarrow \alpha$ is provable if and only if $\Gamma^* \Rightarrow \alpha$ is provable. Thus, we may assume that $\Gamma \Rightarrow \alpha$ is 1-reduced.

If $\Gamma \Rightarrow \alpha$ is provable, by Lemma 3.7, we may assume that it has a cut-free proof Q in which every sequent is 2-reduced. Furthermore, we may assume that Q has no redundancies. Hence, we seek a cut-free proof Q of $\Gamma \Rightarrow \alpha$ all of whose sequents are 2-reduced and which has no redundancies. Since Q is cut-free, it has the subformula property. Thus every formula that appears in Q is a subformula of a formula in its end sequent $\Gamma \Rightarrow \alpha$. This implies that the number of formulas that may appear in Q is finite. Moreover, since every sequent in Q is 2-reduced, the number of sequents that may appear in Q is finite. Finally, since Q has no redundancies, the number of possible trees that may be built with these sequents is finite. Thus, there is an effective procedure for seeking a possible proof of $\Gamma \Rightarrow \alpha$: build all possible trees with 2-reduced sequents that use subformulas of $\Gamma \Rightarrow \alpha$. ■

4. Completeness

Let **Fm** be the algebra of formulas over the language of semi-Heyting algebras on some countable set of variables Var. We denote by $Hom(\mathbf{Fm}, \mathbf{A})$ the set of homomorphisms of **Fm** to a semi-Heyting algebra **A**. If $\mathcal{K}$ is a class of semi-Heyting algebras, we consider the 1-assertional logic of $\mathcal{K}$ given by

$\Sigma \models_{\mathcal{K}} \alpha$ if and only if for each $\mathbf{A} \in \mathcal{K}$ and each $f \in Hom(\mathbf{Fm}, \mathbf{A})$,

$$f(\alpha) = 1 \text{ holds whenever } f(\sigma) = 1 \text{ holds for all } \sigma \in \Sigma,$$

where $\Sigma \cup \{\alpha\}$ is a set of formulas.

We will prove that the logic **GsH** is sound and complete with respect to the class of semi-Heyting algebras, that is, the consequence relations $\vdash_{\mathbf{GsH}}$ and $\models_{\mathcal{SH}}$ coincide.

We need some elementary properties of semi-Heyting algebras that will allow us to prove the main result of this section.

LEMMA 4.1. *Let* $\mathbf{A} = \langle A; \wedge, \vee, \rightarrow, \bot, \top\rangle \in \mathcal{SH}$ *and* $a, b, c, d, e \in A$.

(*a*) *If* $c \leq a$ *and* $c \wedge b \leq d$, *then* $c \wedge (a \rightarrow b) \leq d$.

(*b*) *If* $c \wedge a \leq b$ *and* $c \wedge b \leq a$, *then* $c \leq a \rightarrow b$.

(*c*) *If* $a \wedge b \leq c$, $a \wedge c \leq b$, $a \wedge d \leq e$ *and* $a \wedge e \leq d$, *then* $a \wedge (b \rightarrow d) \leq c \rightarrow e$.

PROOF.

(*a*) Indeed $c \wedge (a \rightarrow b) = c \wedge ((c \wedge a) \rightarrow (c \wedge b)) = c \wedge (c \rightarrow (c \wedge b)) = c \wedge b \leq d$.

(*b*) Note that $c = c \wedge \top = c \wedge ((c \wedge a \wedge b) \rightarrow (c \wedge b \wedge a)) = c \wedge ((c \wedge a) \rightarrow (c \wedge b)) = c \wedge (a \rightarrow b) \leq a \rightarrow b$.

(*c*) This may be proved as follows: $a \wedge (b \rightarrow d) \wedge (c \rightarrow e) = a \wedge ((a \wedge b) \rightarrow (a \wedge d)) \wedge ((a \wedge c) \rightarrow (a \wedge e)) = a \wedge ((a \wedge b \wedge c) \rightarrow (a \wedge d \wedge e)) \wedge ((a \wedge c \wedge b) \rightarrow (a \wedge e \wedge d)) = a \wedge ((a \wedge b \wedge c) \rightarrow (a \wedge d \wedge e)) = a \wedge ((a \wedge b) \rightarrow (a \wedge d)) = a \wedge (b \rightarrow d)$.

■

Consider a sequent $\Gamma \Rightarrow \alpha$ and let $\mathbf{A} \in \mathcal{SH}$ and $f \in Hom(\mathbf{Fm}, \mathbf{A})$. We say that the sequent $\Gamma \Rightarrow \alpha$ is *valid in* $\mathbf{A}$ *under the interpretation* f, symbolically

$$\mathbf{A}, f \models \Gamma \Rightarrow \alpha,$$

if $f(\Gamma^{\wedge}) \leq f(\alpha)$ where $\Gamma^{\wedge} = \begin{cases} \bigwedge_{\gamma \in \Gamma} \gamma & \text{si } \Gamma \neq \emptyset, \\ \top & \text{si } \Gamma = \emptyset. \end{cases}$

THEOREM 4.2. (Soundness) *Let* $\Sigma \cup \{\alpha\}$ *be a set of formulas. If* $\Sigma \vdash_{\mathbf{GsH}} \alpha$, *then* $\Sigma \models_{\mathcal{SH}} \alpha$.

PROOF. Let $\mathbf{A} \in \mathcal{SH}$, $f \in Hom(\mathbf{Fm}, \mathbf{A})$ and assume that $f(\sigma) = \top$ for every $\sigma \in \Sigma$. Since $\Sigma \vdash_{\mathbf{GsH}} \alpha$, there is a proof Q of the sequent $\Rightarrow \alpha$ whose leaves are either initial sequents or sequents of the form $\Rightarrow \sigma$ for some $\sigma \in \Sigma$. We claim that $\mathbf{A}, f \models s$ for every sequent s in the proof Q. Indeed, this is clear for the initial sequents and for those of the form $\Rightarrow \sigma$, $\sigma \in \Sigma$, since $f(\sigma) = \top$ for very $\sigma \in \Sigma$. Moreover, every rule of the sequent calculus $\boldsymbol{GsH}$ preserves the validity of sequents in $\mathbf{A}$ under the interpretation f. This is a straightforward verification for most of the rules. For rules $(\to\Rightarrow)$, $(\Rightarrow\to)$ and $(\to\Rightarrow\to)$, the necessary algebraic properties of semi-Heyting algebras are the ones proven in Lemma 4.1. We have thus proved that $\mathbf{A}, f \models\Rightarrow \alpha$, so $f(\alpha) = \top$. ■

We would like to show now that the converse of Theorem 4.2 also holds. Fix $\Sigma \subseteq Fm$ and define the following relation on Fm:

$$\alpha \equiv_\Sigma \beta \text{ if and only if } \Sigma \vdash_{\mathbf{GsH}} \alpha \to \beta \text{ and } \Sigma \vdash_{\mathbf{GsH}} \beta \to \alpha.$$

The properties proved in the following lemma will allow us to prove that $\equiv_\Sigma$ is an equivalence relation on Fm.

LEMMA 4.3. *For $\Sigma \cup \{\alpha, \beta\} \subseteq Fm$, we have that*

(*a*) $\alpha \equiv_\Sigma \top$ *if and only if* $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \Rightarrow \alpha$.

(*b*) $\alpha \equiv_\Sigma \beta$ *if and only if* $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \alpha \Rightarrow \beta$ *and* $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \beta \Rightarrow \alpha$.

PROOF. To prove (*a*), suppose $\alpha \equiv_\Sigma \top$. In particular, $\Sigma \vdash_{\mathbf{GsH}} \top \to \alpha$ and there is a proof of $\Rightarrow \top \to \alpha$ from $\{\Rightarrow \sigma : \sigma \in \Sigma\}$. Hence the following is a proof of $\Rightarrow \alpha$ from $\{\Rightarrow \sigma : \sigma \in \Sigma\}$:

$$\dfrac{\dfrac{\dfrac{\Rightarrow \sigma : \sigma \in \Sigma}{\vdots}}{\Rightarrow \top \to \alpha} \qquad \dfrac{\Rightarrow \top \qquad \alpha \Rightarrow \alpha}{\top \to \alpha \Rightarrow \alpha}\,(\to\Rightarrow)}{\Rightarrow \alpha}\,(\text{cut}).$$

Now assume that $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \Rightarrow \alpha$. Then we have the following proofs

$$\dfrac{\alpha \Rightarrow \top \qquad \dfrac{\dfrac{\dfrac{\Rightarrow \sigma : \sigma \in \Sigma}{\vdots}}{\Rightarrow \alpha}}{\top \Rightarrow \alpha}\,(\text{w})}{\Rightarrow \alpha \to \top}\,(\Rightarrow\to) \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow \sigma : \sigma \in \Sigma}{\vdots}}{\Rightarrow \alpha}}{\top \Rightarrow \alpha}\,(\text{w}) \qquad \alpha \Rightarrow \top}{\Rightarrow \top \to \alpha}\,(\Rightarrow\to)$$

We now turn to (*b*). Suppose $\alpha \equiv_\Sigma \beta$. Then $\Sigma \vdash_{\mathbf{GsH}} \alpha \to \beta$ and $\Sigma \vdash_{\mathbf{GsH}} \beta \to \alpha$. We can build the following proof of $\alpha \Rightarrow \beta$:

$$\dfrac{\dfrac{\dfrac{\Rightarrow \sigma : \sigma \in \Sigma}{\vdots}}{\Rightarrow \alpha \to \beta} \qquad \dfrac{\alpha \Rightarrow \alpha \qquad \beta \Rightarrow \beta}{\alpha, \alpha \to \beta \Rightarrow \beta}\,(\to\Rightarrow)}{\alpha \Rightarrow \beta}\,(\text{cut})\,.$$

Analogously there is a proof of $\beta \Rightarrow \alpha$ from $\{\Rightarrow \sigma : \sigma \in \Sigma\}$. Conversely, assume that $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \alpha \Rightarrow \beta$ and $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \beta \Rightarrow \alpha$. Applying the rule $(\Rightarrow\to)$ we can show that $\Sigma \vdash_{\mathbf{GsH}} \alpha \to \beta$ and $\Sigma \vdash_{\mathbf{GsH}} \beta \to \alpha$. ■

LEMMA 4.4. *For every $\Sigma \subseteq Fm$, the relation $\equiv_\Sigma$ is an congruence relation on* **Fm**.

PROOF. Let $\alpha, \beta, \gamma \in Fm$. Since $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \alpha \Rightarrow \alpha$, by Lemma 4.3, we get that $\alpha \equiv_\Sigma \alpha$. By definition, $\equiv_\Sigma$ is symmetric. Now assume that $\alpha \equiv_\Sigma \beta$ and $\beta \equiv_\Sigma \gamma$. By Lemma 4.3 we deduce that $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \alpha \Rightarrow \beta$, $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \beta \Rightarrow \alpha$, $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \beta \Rightarrow \gamma$ and $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \gamma \Rightarrow \beta$. Using the cut rule, we get $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \alpha \Rightarrow \gamma$ and $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \gamma \Rightarrow \alpha$. Again, by Lemma 4.3, we finally get $\alpha \equiv_\Sigma \gamma$.

Since the rules for $\vee$ and $\wedge$ are the same as in the sequent calculus for intuitionistic logic, it is clear that $\equiv_\Sigma$ is preserved by both these operations. Now let $\alpha_1, \beta_1, \alpha_2, \beta_2 \in Fm$ such that $\alpha_1 \equiv_\Sigma \beta_1$ and $\alpha_2 \equiv_\Sigma \beta_2$. By Lemma 4.3, it follows that $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \alpha_1 \Rightarrow \beta_1$, $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \beta_1 \Rightarrow \alpha_1$, $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \alpha_2 \Rightarrow \beta_2$ and $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \beta_2 \Rightarrow \alpha_2$. Using rule $(\to\Rightarrow\to)$ we deduce that $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \alpha_1 \to \alpha_2 \Rightarrow \beta_1 \to \beta_2$ and $\{\Rightarrow \sigma : \sigma \in \Sigma\} \vdash^{seq}_{\boldsymbol{GsH}} \beta_1 \to \beta_2 \Rightarrow \alpha_1 \to \alpha_2$. By an new application of Lemma 4.3 we get that $\alpha_1 \to \alpha_2 \equiv_\Sigma \beta_1 \to \beta_2$. ■

LEMMA 4.5. *Given $\Sigma \subseteq Fm$, $\mathbf{Fm}/{\equiv_\Sigma}$ is a semi-Heyting algebra.*

PROOF. Since the rules for $\vee$, $\wedge$, $\top$ and $\bot$ are the same as in the sequent calculus for intuitionistic logic, it is clear that $\mathbf{Fm}/{\equiv_\Sigma}$ is a bounded distributive lattice. Moreover, Lemma 2.1 guarantees that the defining equations for semi-Heyting algebras are also satisfied. ■

In the following theorem, we denote by $[\![\alpha]\!]_\Sigma$ the equivalence class of α in $Fm/{\equiv_\Sigma}$.

THEOREM 4.6. (Completeness) *Let $\Sigma \cup \{\alpha\} \subseteq Fm$. Then $\Sigma \vdash_{\mathbf{GsH}} \alpha$ if and only if $\Sigma \models_{\mathcal{SH}} \alpha$.*

Proof. Half of this result was proved in Theorem 4.2. Now assume that $\Sigma \models_{\mathcal{SH}} \alpha$. We consider the semi-Heyting algebra $\langle Fm/{\equiv_\Sigma}, \vee_\Sigma, \wedge_\Sigma, \to_\Sigma, \top_\Sigma, \bot_\Sigma \rangle$ given in Lemma 4.5. Now observe that for every $\sigma \in \Sigma$:

$$\dfrac{\dfrac{\Rightarrow \sigma}{\top \Rightarrow \sigma}\,(\mathrm{w}) \qquad \sigma \Rightarrow \top}{\Rightarrow \top \to \sigma}\,(\Rightarrow\to) \qquad\qquad \dfrac{\sigma \Rightarrow \top \qquad \dfrac{\Rightarrow \sigma}{\top \Rightarrow \sigma}\,(\mathrm{w})}{\Rightarrow \sigma \to \top}\,(\Rightarrow\to)$$

Thus $\Sigma \vdash_{\mathbf{GsH}} \top \to \sigma$ and $\Sigma \vdash_{\mathbf{GsH}} \sigma \to \top$. Then $[\![\sigma]\!]_\Sigma = [\![\top]\!]_\Sigma$ for every $\sigma \in \Sigma$. Therefore, by the hypothesis, $[\![\alpha]\!]_\Sigma = [\![\top]\!]_\Sigma$ and, consequently, $\Sigma \vdash_{\mathbf{GsH}} \top \to \alpha$. Hence we may build the following proof for $\Rightarrow \alpha$ from $\{\Rightarrow \sigma : \sigma \in \Sigma\}$:

$$\dfrac{\dfrac{\dfrac{\Rightarrow \sigma : \sigma \in \Sigma}{\vdots}}{\Rightarrow \top \to \alpha} \qquad \dfrac{\Rightarrow \top \qquad \alpha \Rightarrow \top}{\top \to \alpha \Rightarrow \alpha}\,(\to\Rightarrow)}{\Rightarrow \alpha}\,(\mathrm{cut}).$$

■

Joining this result with [7, Theorem 4.2] we get the following summary theorem.

Theorem 4.7. *Given $\Sigma \cup \{\alpha\} \subseteq Fm$, the following conditions are equivalent:*

(i) $\Sigma \vdash_{\mathbf{HsH}} \alpha$

(ii) $\Sigma \vdash_{\mathbf{GsH}} \alpha$

(iii) $\Sigma \models_{\mathcal{SH}} \alpha$.

The Completeness Theorem may be reformulated as stating an algebraization relation in the sense of W. J. Blok and D. Pigozzi [5]. Indeed the theorem implies that

$$\Sigma \vdash_{\mathbf{GsH}} \alpha \text{ if and only if } \{\sigma \approx \top : \sigma \in \Sigma\} \models_{\mathcal{SH}} \alpha \approx \top$$

(here the equational consequence relation of $\mathcal{SH}$ is defined in the usual way). Moreover, since

$$\alpha \approx \beta \;=\!\mid\models_{\mathcal{SH}} (\alpha \to \beta) \wedge (\beta \to \alpha) \approx \top,$$

we conclude that the variety of semi-Heyting algebras is the equivalent variety semantics of the logic **GsH** (and **HsH**).

By the general theory of algebraizability (see [5, Corollary 2.9]), we also get that given $\{\epsilon_i, \delta_i : i \in I\} \cup \{\alpha, \beta\} \subseteq Fm$, the following two conditions hold:

(*a*) $\{\epsilon_i \approx \delta_i : i \in I\} \models_{\mathcal{SH}} \alpha \approx \beta$ iff $\{(\epsilon_i \to \delta_i) \wedge (\delta_i \to \epsilon_i) : i \in I\} \vdash_{\mathbf{GsH}} (\alpha \to \beta) \wedge (\beta \to \alpha)$.

(*b*) $\alpha \dashv\vdash_{\mathbf{GsH}} (\alpha \to \top) \wedge (\top \to \alpha)$.

As a direct consequence of the algebraization relation and the decidability of the logic **GsH**, we obtain that the variety of semi-Heyting algebras has a *decidable equational theory*, that is, there is an algorithm that decides whether an equation holds in $\mathcal{SH}$ or not.

Corollary 4.8. *The equational theory of the variety $\mathcal{SH}$ is decidable.*

Another immediate consequence from the algebraization relation is the fact that the Lindenbaum–Tarski algebras of **GsH** coincide with the free semi-Heyting algebras. Moreover, an easy consequence of the disjunction property of **GsH** yields the indecomposability of free algebras in $\mathcal{SH}$.

References

[1] Abad, M., J. M. Cornejo, and J. P. Díaz Varela, The variety generated by semi-Heyting chains, *Soft Computing* 15(4):721–728, 2010.

[2] Abad, M., J. M. Cornejo, and J. P. Díaz Varela, The variety of semi-Heyting algebras satisfying the equation $(0 \to 1)^* \vee (0 \to 1)^{**} \approx 1$, *Reports on Mathematical Logic* 46:75–90, 2011.

[3] Abad, M., J. M. Cornejo, and J. P. Díaz Varela, Free-decomposability in varieties of semi-Heyting algebras, *Mathematical Logic Quarterly* 58(3):168–176, 2012.

[4] Abad, M., J. M. Cornejo, and J. P. Díaz Varela, Semi-Heyting algebras term-equivalent to Gödel algebras, *Order* 30(2):625–642, 2013.

[5] Blok, W. J., and D. Pigozzi, Algebraizable logics, *Memoirs of the American Mathematical Society* 77(396):vi+78 pp., 1989.

[6] Cornejo, J. M., Semi-intuitionistic logic, *Studia Logica* 98(1–2):9–25, 2011.

[7] Cornejo, J. M., and I. D. Viglizzo, On some semi-intuitionistic logics, *Studia Logica* 103(2):303–344, 2015.

[8] Cornejo, J. M., The semi Heyting-Brouwer logic, *Studia Logica* 103(4):853–875, 2015.

[9] Font, J. M., R. Jansana, and D. Pigozzi, A survey of abstract algebraic logic, Abstract algebraic logic, Part II (Barcelona, 1997), *Studia Logica* 74(1–2):13–97, 2003.

[10] Galatos, N., P. Jipsen, T. Kowalski, and H. Ono, *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*, Studies in Logic and the Foundations of Mathematics 151, Elsevier B.V., Amsterdam, xxii+509 pp., 2007.

[11] Gentzen, G., Untersuchungen über das logische Schließen I, *Mathematische Zeitschrift* 39(1):176–210, 1935.

[12] Pelletier, F. J., and A. Urquhart, Synonymous logics, *Journal of Philosophical Logic* 32(3):259–285, 2003.

[13] Sankappanavar, H. P., Semi-Heyting algebras: an abstraction from Heyting algebras, *Proceedings of the 9th "Dr. Antonio A. R. Monteiro" Congress*, Universidad Nacional del Sur, Bahía Blanca, 2008, pp. 33–66.

[14] Sankappanavar, H. P., Expansions of semi-Heyting algebras I: discriminator varieties, *Studia Logica* 98(1–2):27–81, 2011.

D. Castaño, J. M. Cornejo
Departamento de Matemática
Universidad Nacional del Sur
Alem 1253, Bahía Blanca, Argentina
e-mail: diego.castano@uns.edu.ar

J. M. Cornejo
e-mail: jmcornejo@uns.edu.ar

D. Castaño, J. M. Cornejo
INMABB - UNS - CONICET
Alem 1253, Bahía Blanca, Argentina