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Natural frequencies of symmetrically laminated elliptical and circular plates

Liz G. Nallim [a,b,*], Ricardo O. Grossi [a,b]

[a] CONICET, Av. Rivadavia 1917, C1033AAJ, Buenos Aires, Argentina
[b] Instituto de Ingeniería Civil y Medio Ambiente de Salta, Universidad Nacional de Salta, Avda. Bolivia 5150, 4400 Salta, Argentina

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ABSTRACT

Elliptical and circular fibre reinforced composite plates are important structural elements in modern engineering structures. Vibration analysis of these elements are of interest to structural designers. The present paper deals with the free transverse vibration analysis of symmetrically laminated solid and annular elliptic and circular plates with several complicating effects.

The approach developed is based on the Rayleigh–Ritz method where the deflection of the plate is approximated by a general shape function of polynomial type.

The analysis includes several complicating effects, such as the presence of an internal hole, an internal ring support, several concentrated masses and the boundary elastically restrained against rotation and translation.

Several examples are solved and some results which correspond to particular cases are compared with existing values in the literature. New results are also presented for cross-ply and angle-ply elliptical and circular laminates with different boundary conditions.

The algorithm developed can be applied to a wide range of elastic restraint conditions, to any aspect ratio and to higher modes. The effect of the restraint parameters along the boundary on the natural frequencies for plates with these complicating effects is considered.

1. Introduction

Anisotropic plates, especially laminated composite plates, have been widely used in diverse industries, e.g. aerospace, automobile, etc. The determination of the transverse vibrational characteristics of these plates is of great importance. Laminated plates have the advantage of controllability of their structural properties by changing the fibre orientation and the number of plies. Also, it is important to consider elastic restraints along the boundaries since, for instance, ideal clamped edges are practically impossible to achieve. Besides, the elastic restraints along the boundaries have a significant influence in the dynamic behaviour of the plates.

The study of vibration problems of elliptical and circular plates has been investigated extensively. It is not possible to give a detailed account because of the very large number of papers that have been published; nevertheless some important references will be cited. Early studies have been reviewed by Leissa. The monograph [1] contains a detailed review of the investigations on the vibration of plates of various shapes, including circular and elliptical planforms. In subsequent articles [2–4], Leissa provides information on plate vibration including different complicating effects. Although there are many papers on the vibration of elliptical and circular plates, they refer mainly to isotropic and polar orthotropic material. More recent works include different types of thickness variations and non-uniform boundary conditions. Rajalingham et al. [5], studied the vibration of clamped isotropic elliptic plates using the exact vibration modes of circular plates as shape functions. Young and Dickinson [6] studied plates with curved edges and internal cut-outs. The Ritz method was used to obtain an eigenvalue equation for the free vibration of a class of plates, which involved curved boundaries defined by polynomial expressions. Chakraverty and Petyt [7] analysed the natural frequencies of non-homogeneous elliptic and circular plates using two dimensional boundary characteristic orthogonal polynomials in the Rayleigh Ritz method. Lim et al. [8] treated the problem of free vibration for doubly connected composite plates with super-elliptical boundaries. The frequency response was analysed in a globally continuous plate domain by the p-Ritz method. Nallim et al. [9] analysed the fundamental frequency of transverse vibration of a circular plate of rectangular orthotropy carrying a central mass using the Ritz method with simple polynomial expressions. Chakraverty et al. [10] analysed the vibrations of isotropic annular elliptic plates using two

* Corresponding author at: Instituto de Ingeniería Civil y Medio Ambiente de Salta, Universidad Nacional de Salta, Avda. Bolivia 5150, 4400 Salta, Argentina. Tel./fax: +54 387 4255351.
E-mail addresses: lnallim@unsa.edu.ar, lgnallim@yahoo.com.ar (L.G. Nallim), grossiro@unsa.edu.ar (R.O. Grossi).

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Nomenclature

a,b	semi-major and semi-minor elliptical plate axes
$\bar{A}$	domain occupied by the plate in $\bar{x},\bar{y}$ coordinates
A	domain occupied by the plate in x,y coordinates
c_{ij}	unknown coefficients in the deflection function
$c_R(\bar{s}), c_T(\bar{s})$	spring constant of the rotational and translational restraint
D	flexural rigidity (isotropic plate)
D_{ij}	bending, twisting and coupling rigidities (laminated plate)
D_0	reference rigidity of laminated plates$= E_1h^3[12(1-\nu_{12}\nu_{21})$
E_1,E_2	Young's moduli parallel and perpendicular to the fibres
G_{12}	shear modulus of elasticity
h	plate thickness
$n_{\bar{x}}, n_{\bar{y}}$	components of the outward unit normal vector $\bar{n}$ to the boundary $\partial\bar{A}$
$p_{ij}(x,y)$	polynomials
R,T	nondimensional rotational and translational coefficients
$\bar{s}$	arc length along the boundary of the plate
$T_{\max}$	maximum kinetic energy
$U_{\max}$	maximum strain energy due to plate bending
$U_{R,\max}$	maximum strain energy stored in rotational springs
$U_{T,\max}$	maximum strain energy stored in translational springs
w	deflection function
$\bar{x},\bar{y},\bar{z}$	Cartesian coordinates
x,y,z	non-dimensional coordinates
ρ	mass density of the plate material
ν,ν_{12},ν_{21}	Poisson's ratios
$\partial\bar{A}$	plate boundary
ω	circular natural frequency
Ω	non-dimensional frequency parameter

dimensional boundary characteristic orthogonal polynomials in the Rayleigh–Ritz method. Bayer et al. [11] investigated the effect of parabolic variation of thickness on the frequency parameters of isotropic clamped elliptical plates. Two different approximated methods, the moment method and the Rayleigh–Ritz method were used to solve the problem. Kim [12] studied the free vibration problem of elliptic and circular plates with rectilinear orthotropic material. This problem was studied by using the Rayleigh–Ritz method with products of simple polynomials as the admissible functions. Hassan studied the free transverse vibrations of isotropic elliptical and circular plates with half of the boundary simply supported and the rest free [13], and with half of the boundary clamped and the rest free [14]. In both papers the computations have been carried out by using the Rayleigh–Ritz method. In all the above papers only the classical boundary conditions were considered.

Kukla and Szewczyk [15] presented the solution to the problem of the free axisymmetric vibration of annular isotropic plates with elastic concentric supports. The exact solution was obtained by applying the Green's function method.

Analysis of the literature shows that the study of vibration problems of elliptical and circular plates with complicating effects has attracted the attention of many researchers. As far as the problem of free vibrations of elliptical or circular composite material plates with concentrated masses, internal cutouts, internal ring supports and generally restrained boundaries is concerned, no information is available in the literature.

In the present paper the Ritz method is used to develop a general algorithm for the dynamical analysis of symmetrically laminated composite elliptical and circular plates with several complicating effects. The deflection of the plate is approximated by a general polynomial shape function. To demonstrate the validity and efficiency of the algorithm, several numerical examples are presented and some particular cases are compared with results presented by other authors.

The algorithm developed can be applied to a wide range of elastic restraint conditions, different symmetric laminates, elliptic and circular geometries, an internal hole, an internal ring support and concentrated masses. The effect of the fibre orientation on the natural frequencies of plates with these complicating effects is considered. Several sets of vibration mode shapes are included, to provide a better understanding of the dynamical behaviour of these plates.

New results are also presented for cross ply and angle ply laminates with different boundary conditions. Cases of circular plates not treated previously in the literature are also included.

2. Analysis

2.1. Solid plate

Consider a thin, symmetrically laminated elliptical composite plate elastically supported along its boundary by translational and rotational springs as shown in Fig. 1a. It is supposed that the rotational restraint is characterized by the spring constant $c_R(\bar{s})$, and the translational restraint by the spring constant $c_T(\bar{s})$, where $\bar{s}$ is the arc length along the boundary $\partial\bar{A}$. The laminate is of uniform thickness h and is made up of a number of layers of unidirectional fibre reinforced composite material (Fig. 1b). The fibre orientation is indicated by the angle β, measured from the $\bar{x}$ axis to the fibre direction.

The present study is based on the classical laminated plate theory (CLPT) [16,17], where it is assumed that the Kirchhoff hypothesis holds. Consequently the displacements in the $\bar{x},\bar{y},\bar{z}$ directions, respectively denoted by $\bar{u},\bar{v},\bar{w}$, are given by

$$\bar{u}(\bar{x},\bar{y},\bar{z},t) = -\bar{z}\frac{\partial W(\bar{x},\bar{y},t)}{\partial\bar{x}},$$
$$\bar{v}(\bar{x},\bar{y},\bar{z},t) = -\bar{z}\frac{\partial W(\bar{x},\bar{y},t)}{\partial\bar{y}}, \quad \bar{w}(\bar{x},\bar{y},\bar{z},t) = W(\bar{x},\bar{y},t), \qquad (1)$$

where $W(\bar{x},\bar{y},t)$ denotes the mid-plane deflection.

2.2. Annular plate with an internal ring support

Consider the plate described in Section 2.1 with the addition of the following characteristics:

(a) The plate has a central cut-out described by the equation $\bar{x}^2/a_2^2+\bar{y}^2/b_2^2-1=0$, see also Fig. 1c.The inner boundary is free.

(b) The plate is supported by an internal ring described by the equation $\bar{x}^2/a_1^2+\bar{y}^2/b_1^2-1=0$, see also Fig. 1c.

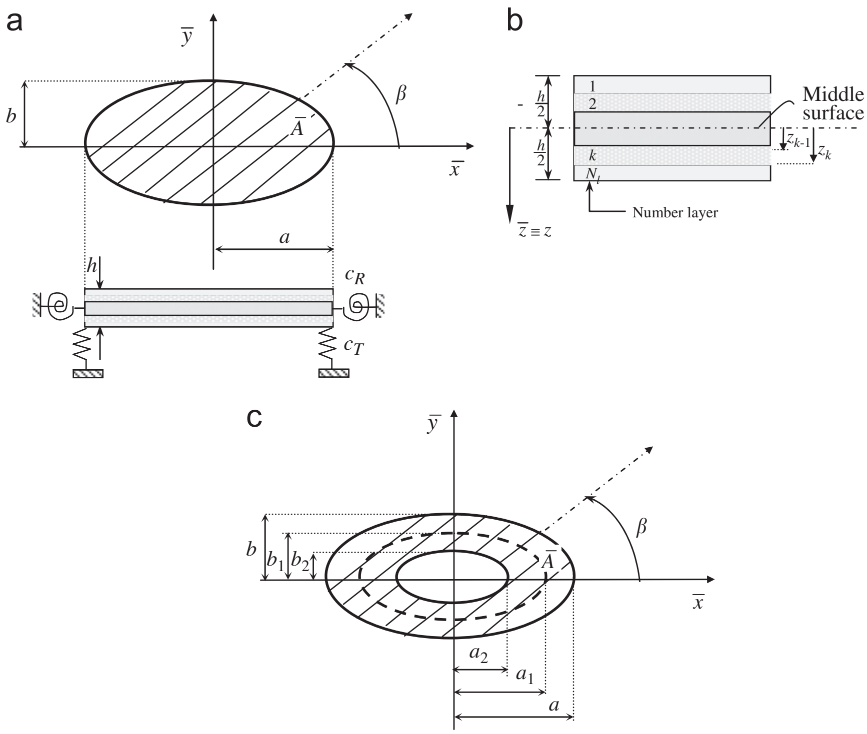

Fig. 1. Mechanical system under study: (a) solid elliptical plate; (b) geometry of an *N*-layered symmetric laminate; and (c) annular elliptical plate.

2.3. The plate and concentrated mass energies

In the case of normal modes of vibration it is assumed that the transverse displacement of any point of the plate is a sinusoidal function of time, such that $W(\bar{x},\bar{y},t) = w(\bar{x},\bar{y})\sin\omega t$. The maximum strain energy of the mechanical system under study when describing small amplitude simple harmonic motion is given by

$$\bar{U}_{\max} = \frac{1}{2}\iint_{\bar{A}}\left[D_{11}\left(\frac{\partial^2 w}{\partial\bar{x}^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial\bar{x}^2}\frac{\partial^2 w}{\partial\bar{y}^2} + D_{22}\left(\frac{\partial^2 w}{\partial\bar{y}^2}\right)^2 + 4D_{16}\left(\frac{\partial^2 w}{\partial\bar{x}^2}\right)\left(\frac{\partial^2 w}{\partial\bar{x}\partial\bar{y}}\right) + 4D_{26}\left(\frac{\partial^2 w}{\partial\bar{y}^2}\right)\left(\frac{\partial^2 w}{\partial\bar{x}\partial\bar{y}}\right) + 4D_{66}\left(\frac{\partial^2 w}{\partial\bar{x}\partial\bar{y}}\right)^2\right]\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}, \tag{2}$$

where D_{ij} are the laminate stiffness coefficients, which are obtained by integrating the material properties of each layer of the composite plate [16,17], and $\bar{A}$ denotes the plate domain in the $\bar{x},\bar{y}$ system for both cases described in Sections 2.1 and 2.2.

For simplicity and generality it is convenient to introduce the following change of variables:

$$x = \frac{\bar{x}}{a}, \quad y = \frac{\bar{y}}{b}, \tag{3}$$

where a and b are the semi-major and semi-minor axes of the ellipse, as is shown in Fig. 1a. This change of variables applied to Eq. (2) leads to

$$U_{\max} = \frac{1}{2}\iint_{A}\left[D_{11}\frac{b}{a^3}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{2}{ab}D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + D_{22}\frac{a}{b^3}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 + \frac{4}{a^2}D_{16}\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial x\partial y}\right) + \frac{4}{b^2}D_{26}\left(\frac{\partial^2 w}{\partial y^2}\right)\left(\frac{\partial^2 w}{\partial x\partial y}\right) + \frac{4}{ab}D_{66}\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y, \tag{4}$$

where A is a circular domain for the plate described in Section 2.1 and an annular circular domain for the plate described in Section 2.2.

On the other hand, the maximum kinetic energy of the plate in the $\bar{x},\bar{y}$ system is given by

$$\bar{T}_{\max} = \frac{1}{2}h\rho\omega^2\iint_{\bar{A}} w^2\,\mathrm{d}\bar{x}\,\mathrm{d}\bar{y}, \tag{5}$$

where ω is the circular natural frequency, ρ is the mass density per unit volume, and h is the laminate thickness. Several internal concentrated masses m_i located at the points $(\bar{x}_{m_i},\bar{y}_{m_i})$, can be handled straightforwardly by including their contribution to the kinetic energy,

$$T_{m_c} = \frac{1}{2}\sum_{i=1}^{N_m} m_i w^2(\bar{x}_{m_i},\bar{y}_{m_i}), \tag{6}$$

where N_m is the number of concentrated masses.

The change of variables (3) leads to the following expression for the total maximum kinetic energy

$$T_{\max} = \frac{1}{2}h\rho\omega^2 a\,b\iint_{A} w^2\,\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\sum_{i=1}^{N_m} m_i w^2(x_{m_i},y_{m_i}). \tag{7}$$

2.4. The potential energies due to the elastic restraints

The maximum potential energy due to the rotational restraint on the boundary, in the $\bar{x},\bar{y}$ system, is given by

$$\bar{U}_{R,\max} = \frac{1}{2}\oint_{\partial\bar{A}} c_R(\bar{s})\left(\frac{\partial w}{\partial\bar{n}}\right)^2\mathrm{d}\bar{s} = \frac{1}{2}\oint_{\partial\bar{A}} c_R(\bar{s})\left(\frac{\partial w}{\partial\bar{x}}n_{\bar{x}} + \frac{\partial w}{\partial\bar{y}}n_{\bar{y}}\right)^2\mathrm{d}\bar{s}, \tag{8}$$

where $\partial\bar{A}$ is the boundary of the plate domain $\bar{A}$ and $n_{\bar{x}}$ and $n_{\bar{y}}$ are the components of the outward normal vector to this boundary

$$\bar{n} = (n_{\bar{x}}, n_{\bar{y}}) = \left(\frac{\bar{x}b^2}{\sqrt{b^4\bar{x}^2 + a^4\bar{y}^2}}, \frac{\bar{y}a^2}{\sqrt{b^4\bar{x}^2 + a^4\bar{y}^2}}\right). \quad (9)$$

For a function $u : \mathbb{R}^2 \to \mathbb{R}$ defined in the image of a smooth curve Γ represented parametrically by the application $r : [c, d] \to \mathbb{R}^2$, $r(t) = (r_1(t), r_2(t))$, it is well known that the line integral of the function u along the curve Γ is given by

$$\oint_{\Gamma} u(x, y)\mathrm{d}s = \int_c^d u[r(t)]||r'(t)||\mathrm{d}t, \quad (10)$$

where t now denotes a parameter, $r'(t) = \mathrm{d}r(t)/\mathrm{d}t$ and $||r'(t)||$ is given by

$$||r'(t)|| = \sqrt{(r_1'(t))^2 + (r_2'(t))^2}.$$

In the case of the boundary $\partial\bar{A}$ of the elliptical plate, the parametric equations are given by

$$\bar{x} = a\cos t,\ \ \bar{y} = b\sin t, \quad \text{with} \quad t \in [0, 2\pi]. \quad (11)$$

In consequence, the application of Eqs. (3), (10) and (11) in the expression (8) leads to

$$U_{R,\max} = \frac{1}{2}\int_0^{2\pi} c_R(t)\left[\frac{1}{a^2}\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\cos^2 t}{\cos^2 t + (a/b)^2\sin^2 t}\right) + \frac{2}{ab}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\left(\frac{\sin t\cos t}{(b/a)\cos^2 t + (a/b)\sin^2 t}\right) + \frac{1}{b^2}\left(\frac{\partial w}{\partial y}\right)^2 \times \left(\frac{\sin^2 t}{(b/a)^2\cos^2 t + \sin^2 t}\right)\right] a\sqrt{1 - k^2\cos^2 t}\,\mathrm{d}t, \quad (12)$$

where $k = (1/a)\sqrt{a^2 - b^2}$.

The maximum potential energy due to the translational restraint on the boundary is given by

$$\bar{U}_{T,\max} = \frac{1}{2}\oint_{\partial\bar{A}} c_T(\bar{s})w^2(\bar{s})\mathrm{d}\bar{s}. \quad (13)$$

Taking into account Eqs. (3), (10) and (11) the expression (13) reduces to

$$U_{T,\max} = \frac{1}{2}\int_0^{2\pi} c_T(s)w^2 a\sqrt{1 - k^2\cos^2 t}\,\mathrm{d}t. \quad (14)$$

Table 1
Frequency parameters $\omega a^2\sqrt{\rho h/D}$, of isotropic elliptical solid and annular plates $\nu = 1/3$ with different aspect ratios and boundary conditions

a/b	$a_2/a = b_2/b$	$N \times M$	C–F			SS–F		
			Ω_1	Ω_2	Ω_3	Ω_1	Ω_2	Ω_3
1	0	6 × 6	10.216	21.260	21.260	4.9838	13.940	13.940
		7 × 7	10.216	21.260	21.260	4.9838	13.940	13.940
		8 × 8	10.216	21.260	21.260	4.9838	13.940	13.940
		9 × 9	10.216	21.260	21.260	4.9838	13.940	13.940
		10 × 10	10.216	21.260	21.260	4.9838	13.940	13.940
		11 × 11	10.216	21.260	21.260	4.9838	13.940	13.940
		12 × 12	10.216	21.260	21.260	4.9838	13.940	13.940
		Ref. [10]	10.22	21.26	21.26	4.984	13.94	13.94
	0.4	6 × 6	13.618	21.567	21.567	5.1113	13.966	13.966
		7 × 7	13.548	21.440	21.440	4.8969	13.896	13.896
		8 × 8	13.548	20.676	20.676	4.8969	13.463	13.463
		9 × 9	13.516	20.572	20.572	4.7937	13.375	13.375
		10 × 10	13.516	19.971	19.971	4.7937	12.825	12.825
		11 × 11	13.505	19.917	19.917	4.7579	12.752	12.752
		12 × 12	13.505	19.604	19.604	4.7579	12.320	12.320
		Ref. [10]	13.50	19.46	19.48	4.748	12.06	12.10
2.0	0	6 × 6	27.378	39.499	56.320	13.271	23.726	39.435
		7 × 7	27.377	39.499	55.985	13.271	23.726	38.454
		8 × 8	27.377	39.499	55.985	13.271	23.723	38.454
		9 × 9	27.377	39.497	55.976	13.271	23.723	38.425
		10 × 10	27.377	39.497	55.976	13.271	23.723	38.425
		11 × 11	27.377	39.497	55.976	13.271	23.723	38.425
		12 × 12	27.377	39.497	55.976	13.271	23.723	38.425
		Ref. [10]	27.38	39.50	55.98	13.27	23.72	38.43
	0.4	6 × 6	36.893	42.437	64.778	13.435	24.163	43.230
		7 × 7	36.668	42.184	64.210	12.753	24.008	41.906
		8 × 8	36.668	41.792	62.951	12.753	23.607	41.906
		9 × 9	36.531	41.595	62.482	12.422	23.447	41.612
		10 × 10	36.531	41.344	57.952	12.422	23.038	38.268
		11 × 11	36.428	41.211	57.774	12.301	22.918	37.854
		12 × 12	36.428	41.122	55.177	12.301	22.663	34.063
		Ref. [10]	36.35	41.04	54.03	12.26	22.53	31.85

3. The boundary and compatibility conditions

The boundary conditions which correspond to the elastic restraints [18], are given by

$$c_R(\bar{s})\frac{\partial w(\alpha_1(\bar{s}),\alpha_2(\bar{s}),t)}{\partial \bar{n}} = M_n(\alpha_1(\bar{s}),\alpha_2(\bar{s}),t), \tag{15}$$

$$c_T(\bar{s})w(\alpha_1(\bar{s}),\alpha_2(\bar{s}),t) = -Q_n - \frac{\partial M_{ns}(\alpha_1(\bar{s}),\alpha_2(\bar{s}),t)}{\partial \bar{s}}, \tag{16}$$

where M_n is the bending moment, M_{ns} is the twisting moment and Q_n is the transverse shear force.

The presence of a ring support introduces constraints on the displacement, slope and curvature. In consequence, the corresponding conditions in the $\bar{x},\bar{y}$ system are given by

$$w(\bar{x},\bar{y})|_{\Gamma_1} = 0, \tag{17}$$

$$\left.\frac{\partial w(\bar{x},\bar{y})}{\partial \bar{n}}\right|_{\Gamma_1^{(-)}} = \left.\frac{\partial w(\bar{x},\bar{y})}{\partial \bar{n}}\right|_{\Gamma_1^{(+)}}, \tag{18}$$

$$M_{1n}|_{\Gamma_1} = M_{2n}|_{\Gamma_1}, \tag{19}$$

where $(\partial w(\bar{x},\bar{y})/\partial \bar{n})|_{\Gamma_1^{(-)}}$ and $(\partial w(\bar{x},\bar{y})/\partial \bar{n})|_{\Gamma_1^{(+)}}$ denote the corresponding lateral derivatives.

The curve Γ_1 is given by the equation $\bar{x}^2/a_1^2 + \bar{y}^2/b_1^2 - 1 = 0$ (see Fig. 1c) and M_{1n}, M_{2n} are the corresponding bending moments.

4. The eigenvalue problem

It is well known that when using the Ritz method with a complete set of trial functions it is possible to ignore the natural boundary conditions. It is sufficient that they satisfy the geometrical ones since, as the number of co-ordinate functions approaches infinity, the natural boundary conditions will be exactly satisfied [19]. This is a transcendental characteristic of the Ritz method, especially when dealing with problems for which such satisfaction is very difficult to achieve. In the present problem this property is crucial since the natural boundary conditions (15) and (16), which correspond to the elastically restrained boundary, and the condition (19) which corresponds to the internal ring support, are extremely difficult to satisfy. In consequence, the assumed shape function is given by

$$w(x,y) = \sum_{i=1}^{M}\sum_{j=1}^{N} c_{ij}p_{ij}(x,y),$$
$$p_{ij}(x,y) = x^{i-1}y^{j-1}(x^2+y^2-r_p^2)^{b_c}, \tag{20}$$

where c_{ij} are unknown coefficients, the parameter b_c depends on the boundary conditions; $b_c = 1$ is adopted when the plate is simply supported, $b_c = 2$ when it is rigidly clamped and $b_c = 0$ when it is free or elastically restrained along the boundary. On the other hand, the parameter $r_p = (a_1/a) = (b_1/b)$, $0 \leqslant r_p < 1$, with $b_c = 1$ allows the support along an elliptical ring to be considered. Also a point support can be considered by setting $r_p = 0$ and $b_c = 1$.

The Ritz procedure requires the minimisation of the energy functional $F = U_{max} + U_{R,max} + U_{T,max} - T_{max}$, with respect to each of

Table 2
Frequency parameters $\omega a^2\sqrt{\rho h/D}$, of isotropic elliptical solid and annular plates $\nu = 0.3$ with different aspect ratios and boundary conditions

a/b	$a_2/a = b_2/b$	$N \times M$	R–F, $R = 100$				T–F, $T = 100$			
			Ω_1	Ω_2	Ω_3	Ω_4	Ω_1	Ω_2	Ω_3	Ω_4
1	0	6 × 6	10.020	20.867	20.867	34.265	4.7320	12.107	12.107	19.101
		7 × 7	10.019	20.866	20.866	34.265	4.7287	12.102	12.102	19.101
		8 × 8	10.019	20.858	20.858	34.226	4.7287	12.080	12.080	19.071
		9 × 9	10.019	20.858	20.858	34.226	4.7286	12.080	12.080	19.071
		10 × 10	10.019	20.858	20.858	34.226	4.7286	12.080	12.080	19.071
		11 × 11	10.019	20.858	20.858	34.226	4.7286	12.080	12.080	19.071
		12 × 12	10.019	20.858	20.858	34.226	4.7286	12.080	12.080	19.071
		Ref. [20]	10.019	–	–	–	4.729	–	–	–
	0.4	6 × 6	13.429	21.879	21.879	33.668	5.1527	12.378	12.378	18.559
		7 × 7	13.336	21.775	21.775	33.668	4.9094	12.345	12.345	18.559
		8 × 8	13.336	21.177	21.177	33.196	4.9094	12.152	12.152	18.411
		9 × 9	13.272	21.059	21.059	33.196	4.7340	12.112	12.112	18.411
		10 × 10	13.272	20.356	20.356	32.674	4.7340	11.856	11.856	18.333
		11 × 11	13.244	20.262	20.262	32.674	4.6498	11.804	11.804	18.333
		12 × 12	13.244	19.712	19.712	32.049	4.6498	11.468	11.468	18.240
2	0	6 × 6	26.433	38.338	59.030	67.527	10.561	16.658	24.004	25.124
		7 × 7	26.419	38.338	54.690	67.438	10.549	16.658	23.654	23.996
		8 × 8	26.419	38.264	54.690	67.421	10.549	16.622	23.654	23.996
		9 × 9	26.419	38.264	54.411	67.419	10.549	16.622	23.606	23.995
		10 × 10	26.419	38.263	54.411	67.419	10.549	16.622	23.606	23.995
		11 × 11	26.419	38.263	54.403	67.418	10.549	16.622	23.605	23.995
		12 × 12	26.419	38.263	54.403	67.418	10.549	16.622	23.605	23.995
		Ref. [20]	25.606	–	–	–	11.004	–	–	–
	0.4	6 × 6	35.617	41.684	66.606	69.860	11.617	17.101	24.361	25.697
		7 × 7	35.259	41.023	62.619	65.999	11.144	17.078	24.153	24.333
		8 × 8	35.051	40.792	62.123	65.484	11.144	16.939	24.153	24.220
		9 × 9	35.051	40.446	60.692	62.123	10.775	16.899	24.029	24.189
		10 × 10	35.051	40.446	60.692	62.123	10.775	16.769	23.916	24.029
		11 × 11	34.930	40.270	60.257	61.725	10.592	16.726	23.860	23.981
		12 × 12	34.930	40.049	55.997	61.725	10.592	16.586	23.320	23.981

Table 3
Frequency parameters $\omega a^2\sqrt{\rho h/D}$, of isotropic solid elliptical plates with different boundary conditions, $\nu = 0.3$

$R = c_R a/D$	a/b		Mode	$T = c_T a^3/D$				
				1	10	100	1000	∞
0	1.2	Ref. [20]	1	1.447	3.850	5.739	6.052	6.089
		Present	1	1.4522	3.8352	5.7101	6.0291	6.066
			2	2.0408	6.1474	12.954	15.007	15.251
			3	2.1386	6.5308	15.014	18.281	18.682
			4	6.5332	9.6028	20.580	27.689	28.617
	1.4	Ref. [20]	1	1.511	4.215	6.950	7.500	7.565
		Present	1	1.5261	4.1801	6.8429	7.4087	7.4773
			2	2.0957	6.3424	13.874	16.557	16.897
			3	2.2813	7.0439	17.712	23.520	24.319
			4	6.8331	9.9339	21.465	29.570	30.699
	2.0	Ref. [20]	1	1.648	4.976	11.004	13.458	13.798
		Present	1	1.7333	5.0186	10.549	12.875	13.213
			2	2.2674	6.9231	16.622	22.602	23.641
			3	2.6717	8.3648	23.606	35.852	38.326
			4	7.1433	10.506	23.995	42.030	46.150
1	1.2	Ref. [20]	1	1.458	4.084	6.753	7.279	7.340
		Present	1	1.4628	4.0792	6.7353	7.2710	7.3347
			2	2.5620	6.1865	13.450	16.112	16.439
			3	3.0811	6.6957	15.327	19.371	19.897
			4	7.4636	10.080	20.626	28.635	29.761
	1.4	Ref. [20]	1	1.517	4.384	7.955	8.813	8.916
		Present	1	1.5352	4.3787	7.8854	8.7785	8.8897
			2	2.5906	6.3807	14.360	17.768	18.218
			3	3.7077	7.4073	17.858	24.678	25.692
			4	7.8036	10.442	21.506	30.584	31.948
	2.0	Ref. [20]	1	1.650	5.029	11.736	14.921	15.382
		Present	1	1.7399	5.1375	11.441	14.594	15.082
			2	2.6925	6.9577	17.013	24.088	25.401
			3	5.9950	9.7097	23.637	37.052	39.972
			4	8.1151	11.009	24.023	43.162	48.046
100	1.2	Ref. [20]	1	1.472	4.486	10.069	12.003	12.226
		Present	1	1.4783	4.4937	10.048	12.040	12.283
			2	3.5061	6.2827	15.364	22.244	23.202
			3	4.7452	7.1253	16.543	26.118	27.718
			4	10.055	11.611	20.824	35.461	38.383
	1.4	Ref. [20]	1	1.528	4.703	11.425	14.491	14.866
		Present	1	1.5488	4.7382	11.345	14.629	15.086
			2	3.5287	6.4782	16.235	24.753	26.123
			3	6.2616	8.4154	18.439	32.649	35.851
			4	10.548	12.101	21.685	38.053	41.642
	2.0	Ref. [20]	1	1.655	5.176	14.643	23.906	25.606
		Present	1	1.7509	5.3993	14.499	24.155	26.419
			2	3.6116	7.0564	18.583	33.786	38.264
			3	11.030	12.773	23.780	46.269	54.411
			4	12.144	13.762	24.143	52.240	67.419
∞	1.2	Ref. [20]	1	1.473	4.497	10.225	12.258	12.490
		Present	1	1.4787	4.5051	10.202	12.299	12.554
			2	3.5344	6.2862	15.461	22.677	23.682
			3	4.7964	7.1413	16.605	26.644	28.338
			4	10.159	11.678	20.836	36.068	39.150
	1.4	Ref. [20]	1	1.529	4.712	11.592	14.829	15.223
		Present	1	1.5491	4.7485	11.506	15.417	15.466
			2	3.5577	6.4818	16.330	26.595	26.719
			3	6.3444	8.4549	18.469	36.447	36.774
			4	10.660	12.174	21.696	42.271	42.531
	2.0	Ref. [20]	1	1.655	5.182	14.794	24.638	26.467
		Present	1	1.7512	5.4079	14.640	24.896	27.377

Table 3 (*continued*)

$R = c_R a/D$	a/b	Mode	$T = c_T a^3/D$				
			1	10	100	1000	∞
		2	3.6428	7.0604	18.665	34.613	39.497
		3	11.155	12.855	23.789	47.143	55.976
		4	12.377	13.939	24.150	53.144	69.858

Table 4

Frequency parameters $\omega a^2\sqrt{\rho h/D}$, of isotropic solid elliptical plates with two boundary conditions, $\nu = 0.3$

a/b		r_m				
		0.0	0.1	0.2	0.3	0.7
Simply supported						
1	Ref. [20]	4.936	4.231	2.752	3.399	2.588
	Ref. [21]	4.89	4.21	3.75	3.42	2.63
	Present	4.9351	4.2335	3.7548	3.4051	2.5974
1.25	Ref. [20]	6.428	5.522	4.902	4.444	3.389
	Ref. [21]	6.33	5.45	4.86	4.42	3.40
	Present	6.3935	5.4777	4.8536	4.3982	3.3496
1.5	Ref. [20]	8.420	7.263	6.459	5.866	4.485
	Ref. [21]	8.21	7.05	6.28	5.71	4.39
	Present	8.2822	7.0739	6.2526	5.6558	4.2911
2.0	Ref. [20]	13.798	12.002	10.730	9.776	7.519
	Ref. [21]	13.11	11.22	9.95	9.03	6.91
	Present	13.213	11.174	9.8031	8.8206	6.6213
Clamped						
1	Ref. [20]	10.220	8.116	6.874	6.055	4.383
	Ref. [21]	10.11	8.08	6.90	6.12	4.49
	Present	10.216	8.1246	6.8970	6.0858	4.4190
1.25	Ref. [20]	13.129	10.458	8.879	7.829	5.677
	Ref. [21]	13.04	10.41	8.89	7.87	5.77
	Present	13.229	10.491	8.8892	7.8345	5.6772
1.5	Ref. [20]	16.770	13.458	11.466	10.131	7.369
	Ref. [21]	16.84	13.38	11.40	10.09	7.37
	Present	17.129	13.492	11.381	10.004	7.2170
2.0	Ref. [20]	26.467	21.538	18.472	16.388	11.998
	Ref. [21]	26.82	21.07	17.84	15.74	11.44
	Present	27.377	21.131	17.613	15.377	10.973

the c_{ij} coefficients

$$\frac{\partial}{\partial c_{ij}}(F) = 0, \quad i = 1, \ldots, M, \quad j = 1, \ldots, N. \tag{21}$$

For a non-trivial solution, the determinant of the coefficient matrix should be equal to zero:

$$\sum_i \sum_j [K_{ijkh} - \Omega^2 M_{ijkh}] c_{ij} = 0, \tag{22}$$

where Ω is the non-dimensional frequency coefficient $\Omega = \sqrt{\rho h/D_0}\,\omega a^2$ and D_0 is the reference flexural rigidity which is given by $D_0 = E_1 h^3/(12(1 - \nu_{12}\nu_{21}))$.

The other parameters in Eq. (22) are:

$$K_{ijkh} = \frac{D_{11}}{D_0} P^{(1)}_{ijkh} + \frac{D_{22}}{D_0}\left(\frac{a}{b}\right)^4 P^{(2)}_{ijkh} + \frac{D_{12}}{D_0}\left(\frac{a}{b}\right)^2 P^{(3)}_{ijkh} + 4\frac{D_{66}}{D_0}\left(\frac{a}{b}\right)^2 P^{(4)}_{ijkh} + 2\frac{D_{16}}{D_0}\left(\frac{a}{b}\right) P^{(5)}_{ijkh} + 2\frac{D_{26}}{D_0}\left(\frac{a}{b}\right)^3 P^{(6)}_{ijkh}$$
$$+ R\left[\left(\frac{a}{b}\right) R^{(1)}_{ijkh} + \left(\frac{a}{b}\right)^2 R^{(2)}_{ijkh} + \left(\frac{a}{b}\right)^3 R^{(3)}_{ijkh}\right] + T\left(\frac{a}{b}\right) T^{(1)}_{ijkh}$$

$$M_{ijkh} = \iint_A p_{ij} p_{kh}\,dx\,dy + \sum_{q=1}^{N_m} r_{m_q} \pi p_{ij}(x_{m_q}, y_{m_q}) p_{kh}(x_{m_q}, y_{m_q}),$$

$$R = \frac{c_R a}{D_0}, \quad T = \frac{c_T a^3}{D_0}, \quad r_{m_q} = \frac{m_q}{m_p} = \frac{m_q}{\pi a b \rho h},$$

$$P^{(1)}_{ijkh} = \iint_A \frac{\partial^2 p_{ij}}{\partial x^2}\frac{\partial^2 p_{kh}}{\partial x^2}\,dx\,dy, \quad P^{(2)}_{ijkh} = \iint_A \frac{\partial^2 p_{ij}}{\partial y^2}\frac{\partial^2 p_{kh}}{\partial y^2}\,dx\,dy,$$

$$P^{(3)}_{ijkh} = \iint_A \left(\frac{\partial^2 p_{ij}}{\partial x^2}\frac{\partial^2 p_{kh}}{\partial y^2} + \frac{\partial^2 p_{ij}}{\partial y^2}\frac{\partial^2 p_{kh}}{\partial x^2}\right) dx\,dy,$$

$$P^{(4)}_{ijkh} = \iint_A \frac{\partial^2 p_{ij}}{\partial x \partial y}\frac{\partial^2 p_{kh}}{\partial x \partial y}\,dx\,dy,$$

$$P^{(5)}_{ijkh} = \iint_A \left(\frac{\partial^2 p_{ij}}{\partial x^2}\frac{\partial^2 p_{kh}}{\partial x \partial y} + \frac{\partial^2 p_{ij}}{\partial x \partial y}\frac{\partial^2 p_{kh}}{\partial x^2}\right) dx\,dy,$$

$$P^{(6)}_{ijkh} = \iint_A \left(\frac{\partial^2 p_{ij}}{\partial y^2}\frac{\partial^2 p_{kh}}{\partial x \partial y} + \frac{\partial^2 p_{ij}}{\partial x \partial y}\frac{\partial^2 p_{kh}}{\partial y^2}\right) dx\,dy,$$

$$R^{(1)}_{ijkh} = \int_0^{2\pi} \left.\frac{\partial p_{ij}}{\partial x}\frac{\partial p_{kh}}{\partial x}\right|_{\substack{x=\cos t\\ y=\sin t}} \frac{\cos^2 t}{\cos^2 t + (a/b)^2 \sin^2 t} \sqrt{1 - k^2\cos^2 t}\,dt,$$

$$R^{(2)}_{ijkh} = \int_0^{2\pi} \left.\frac{\partial p_{ij}}{\partial x}\frac{\partial p_{kh}}{\partial y}\right|_{\substack{x=\cos t\\ y=\sin t}} \frac{\cos t\ \sin t}{(b/a)^2\cos^2 t + (a/b)^2 \sin^2 t} \sqrt{1 - k^2\cos^2 t}\,dt,$$

$$R^{(3)}_{ijkh} = \int_0^{2\pi} \left.\frac{\partial p_{ij}}{\partial y}\frac{\partial p_{kh}}{\partial y}\right|_{\substack{x=\cos t\\ y=\sin t}} \frac{\sin^2 t}{(b/a)^2\cos^2 t + \sin^2 t} \sqrt{1 - k^2\cos^2 t}\,dt,$$

$$T^{(1)}_{ijkh} = \int_0^{2\pi} p_{ij} p_{kh}\big|_{\substack{x=\cos t\\ y=\sin t}} \sqrt{1 - k^2\cos^2 t}\,dt.$$

5. Verification and numerical results

5.1. Generalities

A computer code, based on the variational algorithm developed in this paper, was implemented for the analysis of elliptic and circular plates having different material properties and boundary conditions. In order to check the accuracy of the algorithm, the frequency parameters were computed for a number of plate problems for which comparison values were available in the literature. Additionally, a great number of problems were solved and since the number of cases was extremely large, results were selected for the most significant cases. The analytical expressions obtained allow the adoption of different values for the following parameters:

- Number of layers, stacking sequences and angle of fibre orientation.

- Elastic properties of each material layer.
- Aspect ratio a/b.
- Rotational and translational restraint coefficients.
- Position of an internal ring or a point support.

Let us introduce the terminology to be used for describing the boundary conditions at both the inner and outer edges. In the case of annular plates, the designation C–F, identifies a plate with the outer edge clamped and the inner edge free and SS–F corresponds to a simply supported outer edge. When the external edge is elastically restrained against rotation or translation, the designations R or T are used.

5.2. Convergence and results comparison for isotropic plates

Results of a convergence study of eigenvalues $\omega a^2\sqrt{\rho h/D}$, where $D = Eh^3/12(1-\nu^2)$, of isotropic elliptical solid and annular plates with $\nu = 1/3$ and different aspect ratios (a/b, a_2/a, b_2/b) and boundary conditions are presented in Table 1. The rate of convergence of the first three eigenvalues is shown for C–F and SS–F cases. It is well known that the Ritz method, in the case of eigenvalues, gives upper bounds. The convergence of the mentioned eigenvalues is studied by gradually increasing the number of polynomials in the approximating function. It can be seen that for solid plates $M = 6$, $N = 6$, is sufficient for converged results. In the case of annular plates $M = 12$, $N = 12$, is sufficient to obtain satisfactory convergence for the first eigenvalues. However, for annular elliptical plates values of M, N greater than 12 could be convenient to give converged result for the second and third natural frequencies. This is due to presence of the free inner boundary. The comparison with results of Chakraverty et al. [10] shows a very close agreement.

Table 2 depicts values of the first four frequency parameters $\omega a^2\sqrt{\rho h/D}$, of isotropic elliptical solid and annular plates with $\nu = 0.3$, different aspect ratios and R–F and T–F boundary conditions. The comparison with results of Achong [20] shows a very close agreement in the case of circular plates. For elliptical plates Achong considered rotational springs acting on rotations of w with respect to the variable r. For this reason the values depicted in Table 3 for the frequency parameters $\omega a^2\sqrt{\rho h/D}$ of isotropic elliptical solid plates present a discrepancy when $R = c_R a/D \gg 1$ and $a/b > 1.5$. As it was expected, when the boundary is only restrained against translation, the agreement is good.

In Table 4, for comparison with Refs. [20,21], values of the fundamental frequency parameter $\omega a^2\sqrt{\rho h/D}$, for solid elliptical and circular plates with a concentrated mass and two different boundary conditions, are included.

5.3. Convergence and results comparison for composite plates

In this and the following section two kinds of composite materials are used:

(i) Graphite/epoxy: $E_1 = 138$ GPa, $E_2 = 8.96$ GPa, $G_{12} = 7.1$ GPa, $\nu_{12} = 0.3$,
(ii) E-glass/epoxy: $E_1 = 60.7$ GPa, $E_2 = 24.8$ GPa, $G_{12} = 12.0$ GPa, $\nu_{12} = 0.23$.

Table 5
Frequency parameters $\omega a^2\sqrt{\rho h/D_0}$, for $[(30^\circ, -30^\circ)_2]_{sym}$ Graphite/epoxy laminated solid and annular elliptical plates with different aspect ratios and boundary conditions

a/b	$a_2/a = b_2/b$	$N \times M$	C–F				SS–F			
			Ω_1	Ω_2	Ω_3	Ω_4	Ω_1	Ω_2	Ω_3	Ω_4
1	0	6 × 6	6.6455	10.256	15.605	16.553	3.1866	6.2345	10.929	11.085
		7 × 7	6.6455	10.256	15.566	16.553	3.1865	6.2345	10.929	10.952
		8 × 8	6.6455	10.256	15.566	16.552	3.1865	6.2341	10.927	10.951
		9 × 9	6.6455	10.256	15.565	16.552	3.1865	6.2341	10.927	10.947
		10 × 10	6.6455	10.256	15.565	16.552	3.1865	6.2341	10.927	10.947
		11 × 11	6.6455	10.256	15.565	16.552	3.1865	6.2341	10.927	10.947
		12 × 12	6.6455	10.256	15.565	16.552	3.1865	6.2341	10.927	10.947
	0.4	6 × 6	8.9810	10.942	16.368	17.568	3.2857	6.3478	10.823	11.972
		7 × 7	8.9491	10.907	16.136	17.484	3.1568	6.3237	10.708	11.808
		8 × 8	8.9408	10.755	15.238	17.404	3.1536	6.2096	10.220	11.772
		9 × 9	8.9264	10.732	15.043	17.331	3.0928	6.1843	10.061	11.727
		10 × 10	8.9187	10.620	14.286	17.229	3.0909	6.0591	9.3590	11.682
		11 × 11	8.9119	10.610	14.183	17.137	3.0718	6.0429	9.2146	11.634
		12 × 12	8.9055	10.555	13.792	17.053	3.0701	5.9577	8.5977	11.582
2	0	6 × 6	13.159	20.782	31.420	32.527	6.3117	12.770	21.485	22.347
		7 × 7	13.159	20.781	31.329	32.527	6.3116	12.770	21.485	22.063
		8 × 8	13.159	20.781	31.328	32.527	6.3116	12.769	21.482	22.061
		9 × 9	13.159	20.781	31.326	32.527	6.3116	12.769	21.482	22.054
		10 × 10	13.159	20.781	31.326	32.527	6.3116	12.769	21.482	22.054
		11 × 11	13.159	20.781	31.326	32.527	6.3116	12.769	21.482	22.054
		12 × 12	13.159	20.781	31.326	32.527	6.3116	12.769	21.482	22.054
	0.4	6 × 6	17.778	22.121	32.194	35.369	6.5147	13.005	21.289	24.225
		7 × 7	17.713	22.040	31.743	35.201	6.2617	12.952	21.064	23.906
		8 × 8	17.699	21.722	30.025	35.036	6.2558	12.720	20.138	23.840
		9 × 9	17.670	21.667	29.656	34.889	6.1370	12.663	19.829	23.758
		10 × 10	17.657	21.435	28.206	34.678	6.1342	12.408	18.495	23.668
		11 × 11	17.643	21.410	28.023	34.488	6.0967	12.371	18.221	23.574
		12 × 12	17.632	21.298	27.267	34.316	6.0945	12.196	17.050	23.469

Results of a convergence study of eigenvalues $\omega a^2\sqrt{\rho h/D_0}$ of elliptical solid and annular laminated plates are presented in Tables 5 and 6. Four-ply graphite/epoxy laminates with stacking sequence $[(30^\circ,-30^\circ)_2]_{sym}$ are considered.

The rate of convergence of eigenvalues is shown for C–F and SS–F in Table 5 and for R–F and T–F in Table 6. It has been demonstrated that $N=12$, $M=12$, is sufficient, from an engineering viewpoint, to obtain satisfactory convergence for the first few eigenvalues.

In Table 7, for comparison purpose with Ref. [8], values of the first eight frequency parameters $\omega a^2\sqrt{\rho h/D_0}$ of E-glass/epoxy $[(-\beta,\beta,-\beta,\beta)_2]_{sym}$ annular circular plates and two boundary conditions, are included. Good agreement can be observed.

Table 6

Frequency parameters $\omega a^2\sqrt{\rho h/D_0}$, for $[(30^\circ,-30^\circ)_2]_{sym}$ Graphite/epoxy laminated solid and annular elliptical plates with different aspect ratios and boundary conditions

a/b	$a_2/a=b_2/b$	$N\times M$	R–F, $R=100$				T–F, $T=100$			
			Ω_1	Ω_2	Ω_3	Ω_4	Ω_1	Ω_2	Ω_3	Ω_4
1	0	6 × 6	6.5582	10.180	16.285	16.330	3.1035	6.0126	9.9476	12.109
		7 × 7	6.5565	10.178	15.522	16.326	3.1003	6.0048	9.9369	10.394
		8 × 8	6.5565	10.169	15.510	16.304	3.1003	5.9817	9.8830	10.360
		9 × 9	6.5565	10.169	15.472	16.304	3.1003	5.9816	9.8829	10.260
		10 × 10	6.5565	10.169	15.472	16.304	3.1003	5.9814	9.8820	10.259
		11 × 11	6.5565	10.169	15.471	16.304	3.1003	5.9814	9.8820	10.257
		12 × 12	6.5565	10.169	15.471	16.304	3.1003	5.9814	9.8820	10.257
	0.4	6 × 6	8.8885	11.022	16.964	18.069	3.3637	6.2150	10.166	12.367
		7 × 7	8.8398	10.987	16.774	17.460	3.2080	6.1924	10.109	11.047
		8 × 8	8.8277	10.834	16.093	17.387	3.2005	6.0842	9.8365	10.963
		9 × 9	8.7955	10.800	15.866	17.304	3.0772	6.0627	9.7520	10.835
		10 × 10	8.7878	10.647	14.980	17.223	3.0740	5.9588	9.3897	10.807
		11 × 11	8.7735	10.625	14.789	17.150	3.0155	5.9357	9.2680	10.771
		12 × 12	8.7664	10.513	14.041	17.048	3.0137	5.8207	8.7154	10.737
2	0	6 × 6	13.023	20.605	32.222	32.749	5.9528	11.461	17.461	20.479
		7 × 7	13.019	20.600	31.151	32.216	5.9490	11.454	17.453	18.340
		8 × 8	13.019	20.580	31.126	32.181	5.9490	11.421	17.412	18.317
		9 × 9	13.019	20.580	31.041	32.181	5.9489	11.421	17.411	18.216
		10 × 10	13.019	20.580	31.040	32.180	5.9489	11.421	17.411	18.215
		11 × 11	13.019	20.580	31.039	32.180	5.9489	11.421	17.411	18.214
		12 × 12	13.019	20.580	31.039	32.180	5.9489	11.421	17.411	18.214
	0.4	6 × 6	17.633	22.263	33.477	36.385	6.4601	11.815	17.789	21.004
		7 × 7	17.535	22.180	33.110	35.135	6.1775	11.778	17.731	19.219
		8 × 8	17.514	21.866	31.812	35.002	6.1659	11.615	17.412	19.168
		9 × 9	17.449	21.784	31.373	34.836	5.9392	11.575	17.307	19.022
		10 × 10	17.436	21.471	29.672	34.678	5.9340	11.405	16.821	19.000
		11 × 11	17.406	21.415	29.312	34.530	5.8259	11.361	16.660	18.950
		12 × 12	17.395	21.188	27.873	34.328	5.8233	11.168	15.872	18.9214

Table 7

Frequency parameters $\omega a^2\sqrt{\rho h/D_0}$, for $[(-\beta,\beta,-\beta,\beta)_2]_{sym}$ E-glass/epoxy laminated circular annular plates with two boundary conditions

$a_2/a=b_2/b$	β	$N\times M$	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_7	Ω_8
0.3	15°		SS–F							
		Present	3.7825	9.5390	11.890	18.324	19.192	29.055	29.618	30.809
		Ref. [8]	3.7520	9.2565	11.276	18.145	19.057	28.640	29.410	30.953
			C–F							
		Present	9.3484	14.955	17.775	25.360	26.347	37.104	37.818	42.551
		Ref. [8]	9.4193	14.539	17.096	24.993	26.070	36.688	37.638	42.573
	30°		SS–F							
		Present	3.7842	9.9929	11.567	18.866	19.497	30.015	30.484	30.583
		Ref. [8]	3.7520	9.6648	11.017	18.653	19.354	29.725	30.398	30.773
			C–F							
		Present	9.3484	15.490	17.379	26.013	26.639	38.317	38.880	42.344
		Ref. [8]	9.4168	15.018	16.755	25.563	26.340	37.988	38.673	42.725
	45°		SS–F							
		Present	3.7844	10.316	11.302	19.218	19.581	30.271	30.684	31.048
		Ref. [8]	3.7473	9.9575	10.785	18.959	19.452	30.508	30.645	30.845
			C–F							
		Present	9.3476	15.879	17.055	26.469	26.675	39.110	39.424	42.168
		Ref. [8]	9.4073	15.410	16.397	25.888	26.413	38.853	39.138	42.818

Table 8
Frequency parameters $\omega a^2\sqrt{\rho h/D_0}$, for $[(\beta,-\beta)_2]_{sym}$ E-glass/epoxy laminated solid elliptical plates with different aspect ratios and boundary conditions

	R											
	$a/b=1.5$						$a/b=2$					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
$\beta=30^\circ$												
$T=\infty$												
0	6.1079	14.255	19.980	25.780	31.984	40.847	9.4905	18.463	30.897	2.870	47.073	47.159
1	7.9143	15.921	21.835	27.382	33.713	42.430	11.875	20.634	32.906	35.402	48.995	49.505
10	11.349	20.183	27.080	32.388	39.449	48.183	17.154	26.631	39.513	43.665	56.307	58.103
100	12.684	22.299	29.898	35.401	43.190	52.311	19.521	29.878	43.768	48.900	61.941	64.402
∞	12.879	22.633	30.355	35.908	43.849	52.750	19.885	30.410	44.509	49.807	62.354	65.562
$T=100$												
0	5.7688	12.379	16.176	19.825	22.343	27.408	8.4113	14.682	21.770	21.901	27.288	29.330
1	7.2253	13.217	16.654	20.050	22.407	27.423	9.9360	15.476	21.974	21.984	27.334	29.356
10	9.5911	14.884	17.550	20.536	22.539	27.468	12.421	17.028	22.144	22.422	27.455	29.434
100	10.384	15.527	17.865	20.735	22.591	27.492	13.224	17.612	22.201	22.608	27.517	29.479
∞	10.495	15.621	17.910	20.764	22.598	27.496	13.334	17.696	22.209	22.635	27.526	29.488
$T=10$												
0	3.9705	6.2855	7.1389	9.3112	10.620	14.550	4.7606	6.7907	8.2851	9.8351	12.896	16.543
1	4.3402	6.2935	7.4056	9.7234	11.266	16.468	5.0040	6.7999	9.2771	10.249	14.249	17.557
10	4.6872	6.3075	7.7816	10.434	12.401	18.327	5.2637	6.8169	10.712	11.003	16.719	19.620
100	4.7638	6.3122	7.8911	10.677	12.801	19.026	5.3266	6.8230	11.130	11.279	17.603	20.458
∞	4.7735	6.3129	7.9058	10.711	12.858	19.128	5.3346	6.8238	11.186	11.318	17.728	20.583
$T=1$												
0	1.5449	2.1188	2.3454	5.8101	7.4244	12.225	1.7212	2.2630	2.6691	6.1241	9.6615	14.405
1	1.5637	2.5211	3.7620	6.8496	8.6377	14.691	1.7345	2.6198	5.5917	7.1589	11.688	15.719
10	1.5777	2.9841	5.1083	8.3531	10.527	17.017	1.7459	3.0714	8.3808	8.7314	15.044	18.308
100	1.5805	3.1020	5.4316	8.8120	11.142	17.893	1.7483	3.1952	9.0781	9.2425	16.171	19.329
∞	1.5808	3.1172	5.4731	8.8743	11.227	18.019	1.7486	3.2116	9.1684	9.3133	16.329	19.480
$\beta=45^\circ$												
$T=\infty$												
0	6.5008	13.996	21.896	24.602	33.255	38.457	10.336	18.796	30.265	36.315	45.043	50.229
1	8.2190	15.675	23.602	26.240	34.922	40.088	12.551	20.898	32.267	38.621	47.003	52.420
10	11.837	19.991	29.112	31.230	40.835	45.820	18.166	27.074	38.964	47.235	54.411	61.265
100	13.422	22.202	32.596	34.188	45.073	49.770	21.131	30.810	43.557	53.799	60.270	68.791
∞	13.665	22.560	33.203	34.689	45.856	50.096	21.620	31.463	44.396	55.045	60.623	70.293
$T=100$												
0	6.0632	12.076	16.980	19.126	22.591	26.553	8.9096	14.527	21.078	22.557	27.579	28.313
1	7.4144	12.940	17.310	19.420	22.631	26.561	10.239	15.287	21.333	22.586	27.648	28.322
10	9.7788	14.645	17.997	20.034	22.720	26.582	12.625	16.813	21.883	22.651	27.831	28.349
100	10.624	15.298	18.263	20.279	22.757	26.593	13.458	17.401	22.110	22.680	27.925	28.366
∞	10.744	15.392	18.302	20.314	22.763	26.593	13.575	17.487	22.143	22.684	27.939	28.370
$T=10$												
0	3.9983	6.2228	7.1805	9.0138	10.647	14.647	4.7676	6.7271	8.3085	9.4774	12.984	15.259
1	4.3402	6.2260	7.5047	9.3878	11.342	15.626	4.9934	6.7316	9.4006	9.8461	14.416	16.307
10	4.6799	6.2316	8.0163	9.9866	12.579	17.351	5.2486	6.7402	10.472	11.186	17.161	18.232
100	4.7566	6.2335	8.1779	10.177	13.026	17.957	5.3118	6.7432	10.686	11.767	18.208	18.940
∞	4.7663	6.2337	8.1999	10.203	13.090	18.042	5.3200	6.7437	10.716	11.847	18.360	19.042
$T=1$												
0	1.5423	2.1164	2.3467	5.3709	7.4266	12.401	1.7167	2.2606	2.6699	5.5655	9.7498	12.932
1	1.5625	2.4902	3.8406	6.4329	8.7163	13.738	1.7325	2.5914	5.7084	6.6234	11.870	14.346
10	1.5771	2.8828	5.4217	7.8404	10.732	16.022	1.7450	2.9764	8.0943	8.9474	15.527	16.852
100	1.5800	2.9773	5.8337	8.2382	11.405	16.802	1.7476	3.0771	8.5345	9.8477	16.836	17.744
∞	1.5803	2.9894	5.8878	8.2911	11.499	16.911	1.7479	3.0903	8.5943	9.9679	17.023	17.872
$\beta=60^\circ$												
$T=\infty$												
0	6.9160	13.699	23.456	23.838	34.243	36.294	11.263	18.960	29.436	40.112	43.025	52.900
1	8.5323	15.371	25.129	25.408	35.821	37.982	13.290	20.975	31.423	42.195	45.019	54.929
10	12.311	19.740	30.101	31.117	41.866	43.677	19.195	27.326	38.226	51.027	52.551	63.945
100	14.199	22.108	33.054	35.337	46.690	47.484	22.903	31.669	43.258	58.604	59.306	72.894
∞	14.506	22.506	33.563	36.128	47.631	47.725	23.567	32.481	44.227	58.972	60.966	74.793
$T=100$												
0	6.3558	11.754	17.652	18.445	22.649	25.626	9.3932	14.316	20.423	23.077	27.466	27.608
1	7.5954	12.6473	17.8692	18.8186	22.6748	25.6455	10.530	15.062	20.742	23.084	27.479	27.696
10	9.9404	14.396	18.371	19.572	22.734	25.828	12.801	16.599	21.411	23.104	27.512	27.933
100	10.845	15.065	18.588	19.859	22.759	25.844	13.676	17.207	21.680	23.114	27.529	28.056
∞	10.978	15.162	18.621	19.901	22.763	25.845	13.803	17.296	21.719	23.116	27.533	28.075

Table 8 (*continued*)

	R											
	$a/b = 1.5$						$a/b = 2$					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
$T = 10$												
0	4.0091	6.1517	7.2103	8.8342	10.446	13.986	4.7615	6.6581	8.3242	9.2679	12.661	14.468
1	4.3333	6.1549	7.5806	9.1630	11.185	14.977	4.9782	6.6620	9.4906	9.5865	14.190	15.504
10	4.6690	6.1594	8.2205	9.6445	12.541	16.540	5.2328	6.6684	10.088	11.603	17.225	17.227
100	4.7467	6.1608	8.4382	9.7900	13.042	17.048	5.2976	6.6706	10.252	12.369	17.813	18.440
∞	4.7565	6.1610	8.4686	9.8097	13.114	17.120	5.3060	6.6709	10.274	12.478	17.896	18.620
$T = 1$												
0	1.5379	2.1137	2.3476	5.1255	7.1146	11.670	1.7113	2.2578	2.6703	5.2600	9.2879	12.026
1	1.5609	2.4567	3.9014	6.1906	8.4914	13.064	1.7304	2.5610	5.7932	6.3130	11.572	13.482
10	1.5764	2.7890	5.6918	7.4532	10.689	15.194	1.7440	2.8874	7.6369	9.4371	15.597	15.796
100	1.5794	2.8668	6.1939	7.7891	11.435	15.868	1.7468	2.9713	8.0110	10.562	16.560	17.099
∞	1.5797	2.8767	6.2611	7.8334	11.541	15.961	1.7471	2.9824	8.0615	10.718	16.668	17.318

Table 9

Frequency parameters $\omega a^2\sqrt{\rho h/D_0}$, for two cross-ply (1) $[(0^{\circ},90^{\circ})_2]_{\mathrm{sym}}$, (2) $[(90,0)_2]_{\mathrm{sym}}$, E-glass/epoxy laminated solid elliptical plates with different aspect ratios and boundary conditions

	R											
	$a/b = 1.5$						$a/b = 2$					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
(1)												
$T = \infty$												
0	5.9816	14.414	19.277	27.328	30.374	40.456	9.3177	18.020	31.253	32.263	45.022	49.084
1	7.8252	16.055	21.214	28.789	32.231	42.438	11.732	20.274	33.266	34.811	47.479	50.914
10	11.212	20.357	26.364	33.790	38.003	49.208	16.941	26.311	39.940	42.984	56.091	58.183
100	12.454	22.469	28.913	37.047	41.354	53.505	19.201	29.365	44.067	48.046	61.977	63.833
∞	12.630	22.797	29.311	37.609	41.907	54.235	19.544	29.847	44.759	48.914	63.020	64.299
$T = 100$												
0	5.6705	12.597	15.822	20.556	21.939	27.186	8.2812	14.720	21.683	22.358	26.857	30.401
1	7.1654	13.414	16.367	20.716	22.028	27.205	9.8517	15.568	21.778	22.534	26.886	30.453
10	9.5463	15.075	17.355	21.085	22.205	27.246	12.396	17.194	21.958	22.921	26.953	30.618
100	10.325	15.721	17.693	21.245	22.271	27.262	13.215	17.796	22.022	23.080	26.981	30.720
∞	10.432	15.815	17.740	21.269	22.281	27.264	13.327	17.883	22.031	23.103	26.986	30.738
$T = 10$												
0	3.9705	6.3252	7.1234	9.8930	10.079	14.260	4.7689	6.8334	8.2752	10.395	12.069	18.502
1	4.3455	6.3400	7.3648	10.267	10.807	16.165	5.0144	6.8481	9.2223	10.784	13.574	19.279
10	4.6964	6.3676	7.6657	11.035	11.968	19.045	5.2782	6.8761	10.498	11.601	16.020	21.237
100	4.7737	6.3775	7.7427	11.335	12.328	19.887	5.3420	6.8864	10.846	11.932	16.785	22.175
∞	4.7834	6.3788	7.7526	11.379	12.377	19.999	5.3502	6.8879	10.891	11.981	16.888	22.322
$T = 1$												
0	1.5474	2.1201	2.3450	6.5281	6.8219	11.851	1.7247	2.2647	2.6688	7.0966	8.4324	16.696
1	1.5646	2.5410	3.7333	7.6182	7.9895	14.499	1.7359	2.6390	5.5410	7.9231	10.832	17.649
10	1.5782	3.0633	4.9505	9.0675	10.014	18.135	1.7459	3.0714	8.3808	8.7314	15.044	18.308
100	1.5809	3.2053	5.0215	9.5836	10.590	19.145	1.7490	3.2856	8.7305	10.022	15.273	21.122
∞	1.5813	3.2240	5.2484	9.6566	10.667	19.278	1.7493	3.3048	8.8080	10.103	15.407	21.294
(2)												
$T = \infty$												
0	7.1727	13.505	23.085	24.979	34.399	36.000	11.871	18.900	28.866	42.144	42.643	54.036
1	8.7270	15.183	24.787	26.470	35.931	37.708	13.772	20.869	30.871	44.190	44.588	55.959
10	12.596	19.632	29.831	32.262	42.049	43.454	19.828	27.363	37.841	51.906	53.504	65.051
100	14.684	22.098	32.789	36.957	47.207	47.229	24.086	32.142	43.179	58.272	62.793	75.030
∞	15.035	22.518	33.298	37.880	47.881	48.274	24.893	33.072	44.233	58.585	64.902	76.258
$T = 100$												
0	6.5243	11.631	17.961	18.395	22.528	25.873	9.6744	14.223	20.359	23.300	27.394	27.550
1	7.7015	12.548	18.127	18.785	22.547	25.886	10.702	14.982	20.699	23.303	27.481	27.562
10	10.044	14.354	18.538	19.571	22.592	25.914	12.919	16.585	21.417	23.309	27.592	27.716
100	10.993	15.042	18.729	19.869	22.612	25.927	13.839	17.230	21.702	23.313	27.606	27.838
∞	11.135	15.141	18.759	19.912	22.615	25.928	13.975	17.325	21.743	23.314	27.609	27.858
$T = 10$												
0	4.0229	6.1492	7.2183	8.9478	10.055	14.388	4.7687	6.6609	8.3260	9.3674	12.015	14.857
1	4.3362	6.1505	7.6021	9.2747	10.843	15.327	4.9789	6.6619	9.5068	9.6884	13.643	15.866

Table 9 (*continued*)

	R											
	$a/b=1.5$						$a/b=2$					
	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6	Ω_1	Ω_2	Ω_3	Ω_4	Ω_5	Ω_6
10	4.6715	6.1521	8.2962	9.7245	12.289	16.763	5.2357	6.6635	10.166	11.767	16.859	17.472
100	4.7499	6.1526	8.5441	9.8508	12.817	17.220	5.3019	6.6639	10.309	12.654	17.984	18.162
∞	4.7598	6.1526	8.5793	9.8674	12.893	17.283	5.3105	6.6401	10.328	12.784	18.054	18.358
$T=1$												
0	1.5385	2.1137	2.3479	5.3651	6.4773	12.231	1.7120	2.2581	2.6704	5.4731	8.3454	12.571
1	1.5609	2.4552	3.9183	6.3990	8.0024	13.515	1.7304	2.5608	5.8071	6.5010	10.872	13.949
10	1.5765	2.7732	5.7943	7.5747	10.393	15.454	1.7441	2.8732	7.7535	9.6297	15.193	16.084
100	1.5794	2.8440	6.3445	7.8733	11.187	16.063	1.7469	2.9491	8.0865	10.902	16.751	16.810
∞	1.5798	2.8529	6.4192	7.9120	11.298	16.146	1.7472	2.9589	8.1302	11.084	16.843	17.048

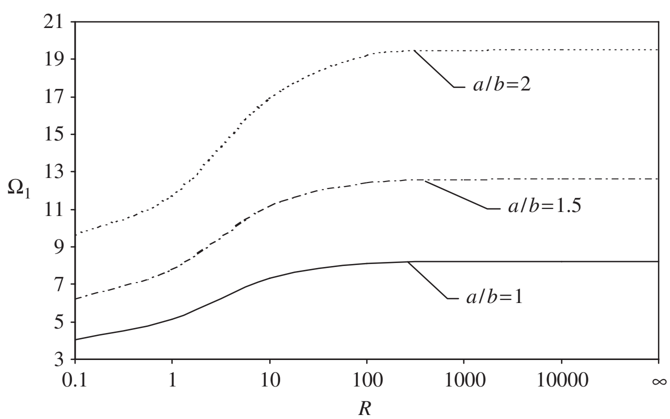

Fig. 2. Variation of the fundamental frequency coefficient Ω_1 with the rotational restraint parameter R ($T=\infty$), of circular and elliptical cross-ply $[(0^\circ, 90^\circ)_2]_{sym}$ E-glass/epoxy laminates.

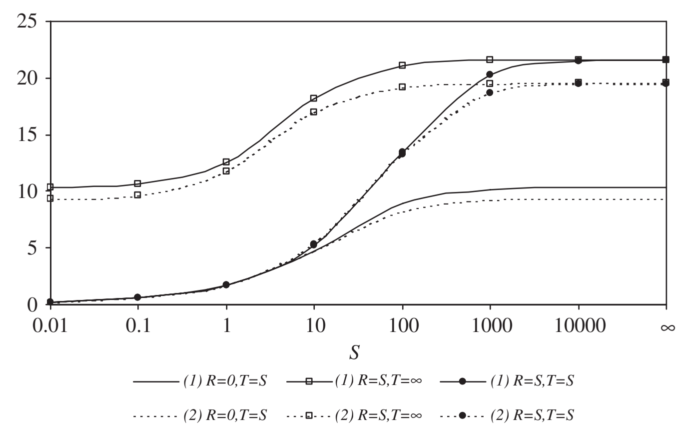

Fig. 4. Variation of the fundamental frequency coefficient Ω_1 with the translational and the rotational restraint parameters T and R for: (1) angle-ply $[(45^\circ, -45^\circ)_2]_{sym}$ and (2) cross-ply $[(0^\circ, 90^\circ)_2]_{sym}$ E-glass/epoxy composite laminates with $a/b=2$.

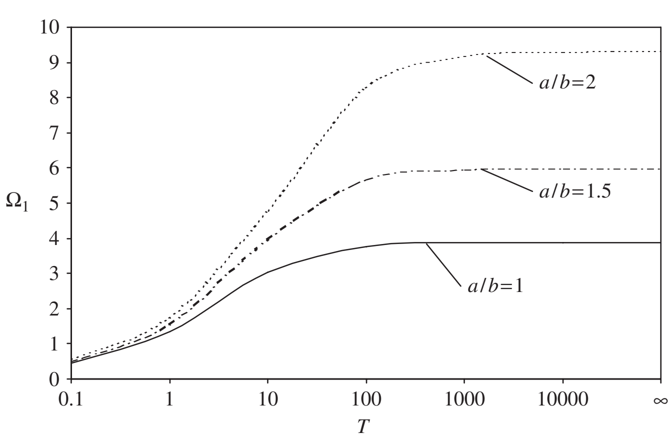

Fig. 3. Variation of the fundamental frequency coefficient Ω_1 with the translational restraint parameter T ($R=0$), of circular and elliptical cross-ply $[(0^\circ, 90^\circ)_2]_{sym}$ E-glass/epoxy laminates.

5.4. New results

In this section new results are presented. Table 8 shows values of the frequency parameters $\omega a^2\sqrt{\rho h/D_0}$, for angle-ply $[(\beta, -\beta)_2]_{sym}$ E-glass/epoxy elliptical laminates with different aspect ratios and boundary conditions. Table 9 depicts values of the frequency parameters $\omega a^2\sqrt{\rho h/D_0}$, for two cross-ply $[(0,90)_2]_{sym}, [(90,0)_2]_{sym}$ E-glass/epoxy laminated elliptical plates, different aspect ratios and boundary conditions.

In Figs. 2–4 the fundamental frequency coefficient $\Omega_1 = \omega_1 a^2\sqrt{\rho h/D_0}$ of laminated elliptical plates is plotted against the restraint parameters R and T. Fig. 2 shows the variation of Ω_1 for various values of the rotational restraint R, while Fig. 3 shows the variation of Ω_1 for various values of the translational restraint T. In both figures the effect due to three different aspect ratio is included. It can be observed that a major increase of frequency occurs when the elastic restraint values are in the interval (1–100). Fig. 4 shows the variation of the fundamental frequency coefficient Ω_1 for various values of the rotational and the translational restraint parameters: (a) $R=0$, $T=S$, (b) $R=S$, $T=\infty$ and (c) $R=T=S$. In this case, Ω_1 is plotted against S for (1) angle-ply $[(45^\circ, -45^\circ)_2]_{sym}$ and (2) cross-ply $[(0^\circ, 90^\circ)_2]_{sym}$ E-glass/epoxy composite laminates with $a/b=2$. The obtained curves illustrate the intervals of variation of the restraint parameters for which the frequency coefficient Ω_1 is sensitive to the values of R or T. The frequencies which correspond to the angle-ply case are higher in all cases, specially when the restraint parameters vary in the interval $(100, \infty)$.

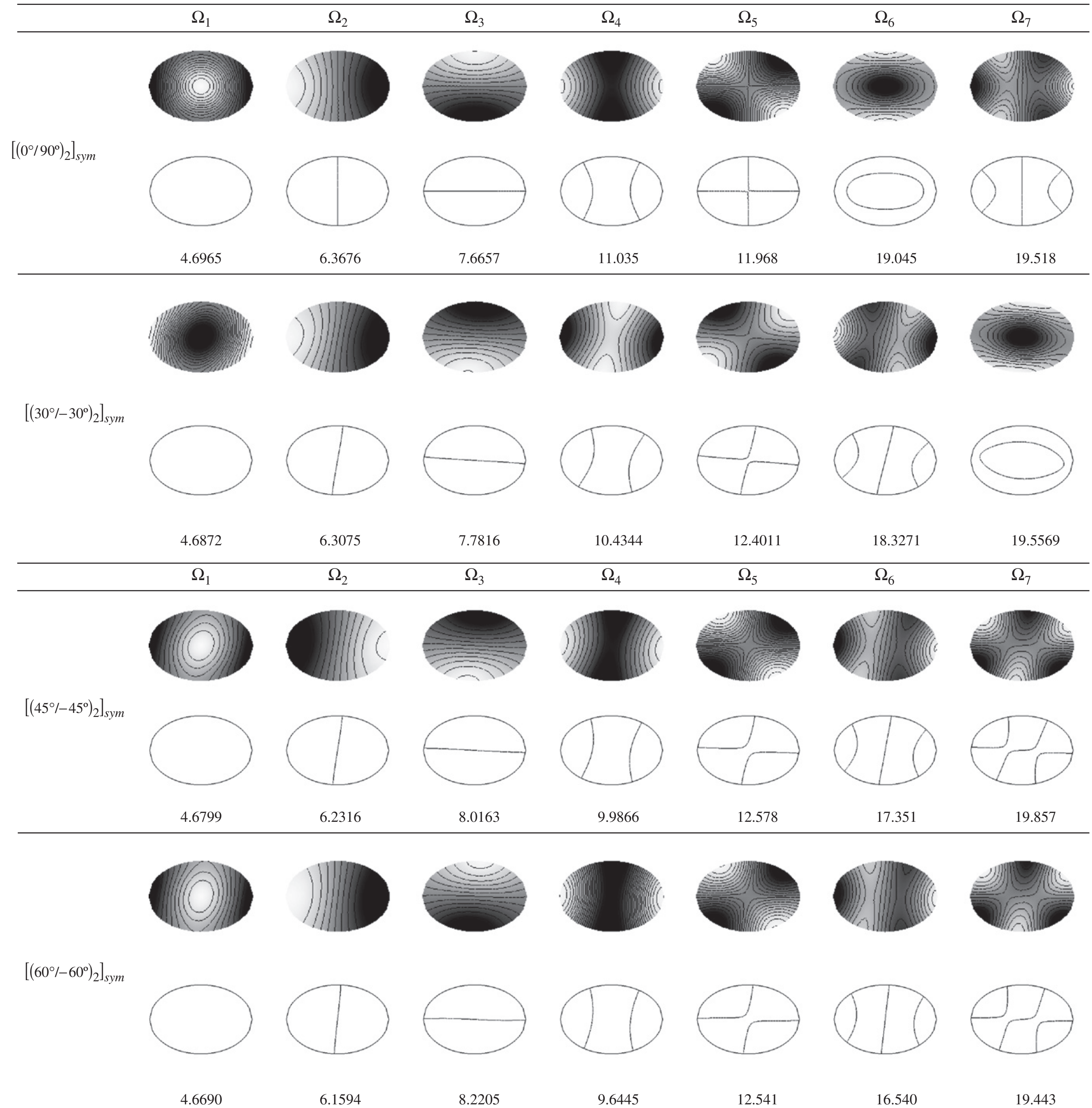

Fig. 5. First seven natural free vibration frequencies and mode shapes for elliptical ($a/b = 1.5$) cross-ply and angle-ply E-glass epoxy laminates, with $T = 10$, $R = 10$.

Also, in this section, the first seven natural free vibration frequencies, mode shapes and nodal patterns of different solid and annular elliptical angle-ply and cross-ply E-glass/epoxy laminated plates with different boundary conditions are shown. Fig. 5 corresponds to elliptical plates ($a/b = 1.5$) with elastically restrained edges for $T = 10$ and $R = 10$. Figs. 6 and 7 present the results which correspond to SS–F and C–F annular elliptical laminates with $a_2/a = b_2/b = 0.25$ and $a/b = 1.5$. Finally, Fig. 8 shows the results corresponding to elliptical laminates ($a/b = 1.5$) with a concentric ring support ($a_1/a = b_1/b = 0.5$).

6. Conclusions

A simple, computationally efficient and accurate approximate approach has been developed for the determination of natural frequencies and mode shapes of free vibration of symmetrically laminated cross-ply and angle-ply elliptical solid and annular plates. The approach is based on the Rayleigh–Ritz method with polynomial expressions as approximate functions. The obtained algorithm is very general and also attractive regarding its versatility in handling any boundary conditions, including edges

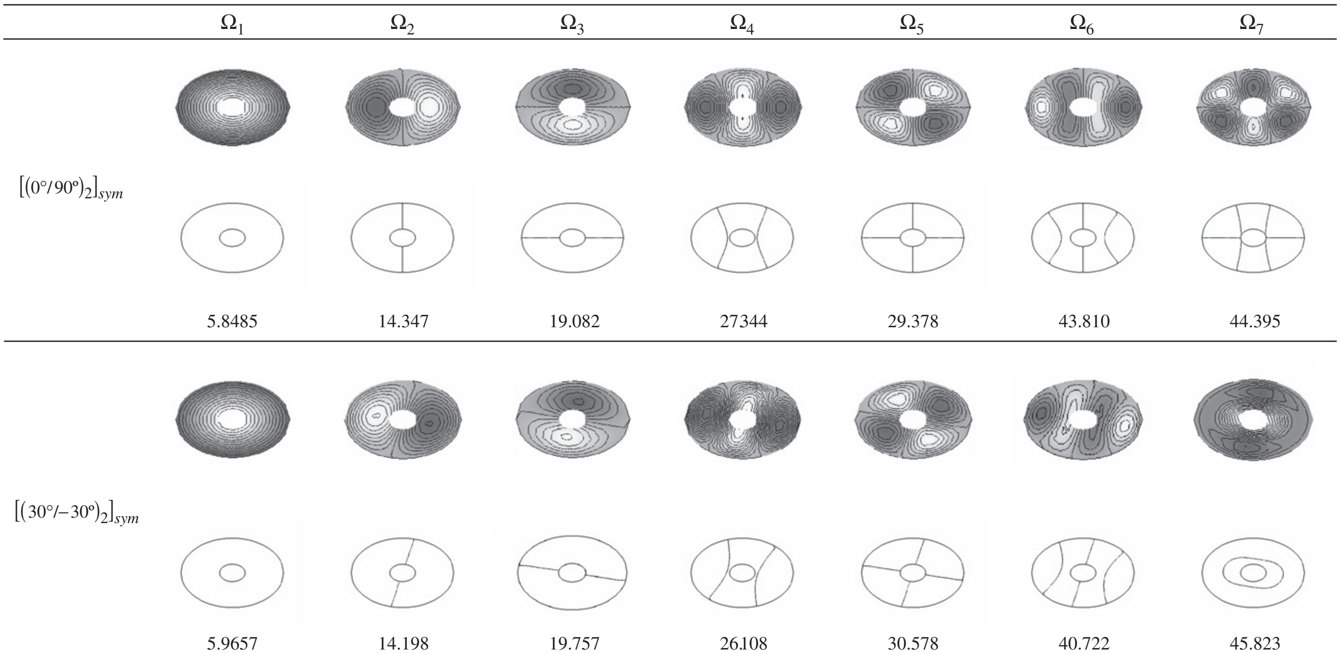

Fig. 6. First seven natural free vibration frequencies and mode shapes for SS–F annular elliptical cross-ply and angle-ply E-glass epoxy laminates with $a_2/a = b_2/b = 0.25$, $a/b = 1.5$.

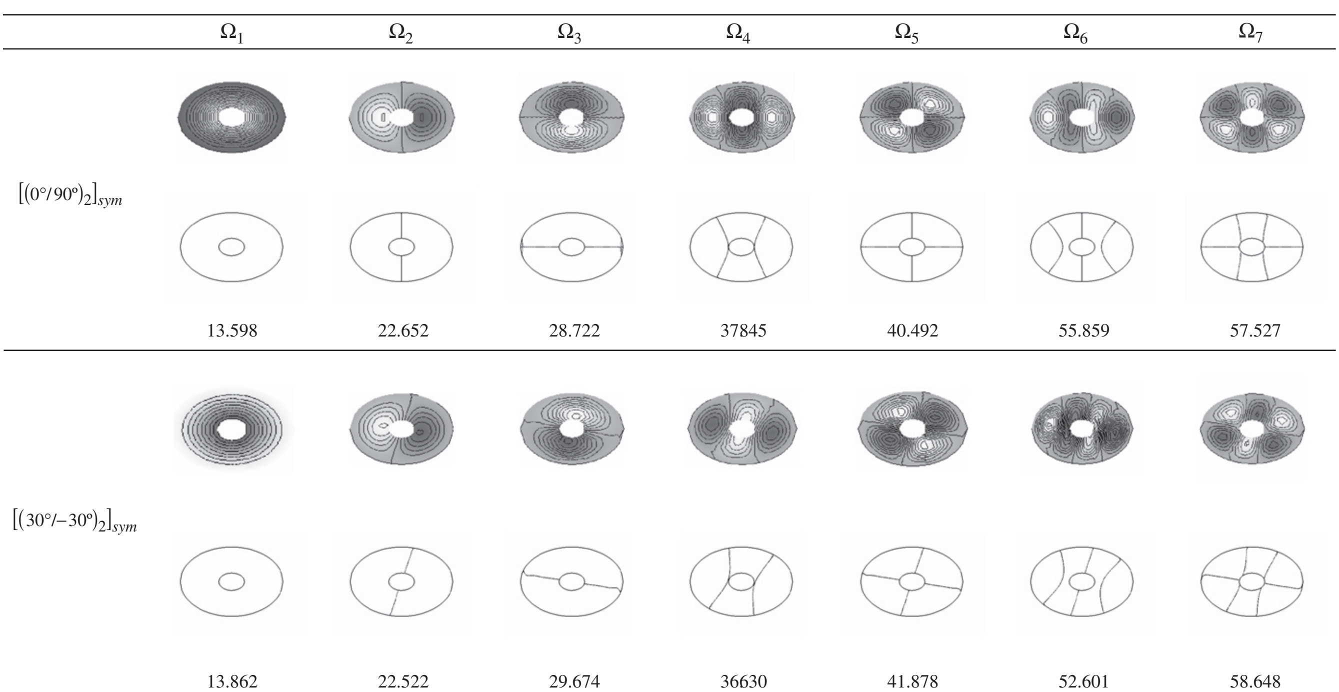

Fig. 7. First seven natural free vibration frequencies and mode shapes for C–F annular elliptical cross-ply and angle-ply E-glass epoxy laminates with $a_2/a = b_2/b = 0.25$, $a/b = 1.5$.

elastically restrained against rotation and against translation. Besides, it takes into account a great variety of anisotropic characteristics, geometric planforms (including annular plates), a concentric ring support and the presence of concentrated masses. Circular isotropic plates and classical boundary conditions can be easily generated as particular cases.

Close agreement with results presented by previous investigators is demonstrated for several examples. New results are presented for several symmetrically laminated composite circular and elliptical plates with elastically restrained edges. These results may provide useful information for structural designers and engineers and the method may be easily modified to apply to static deflection problems.

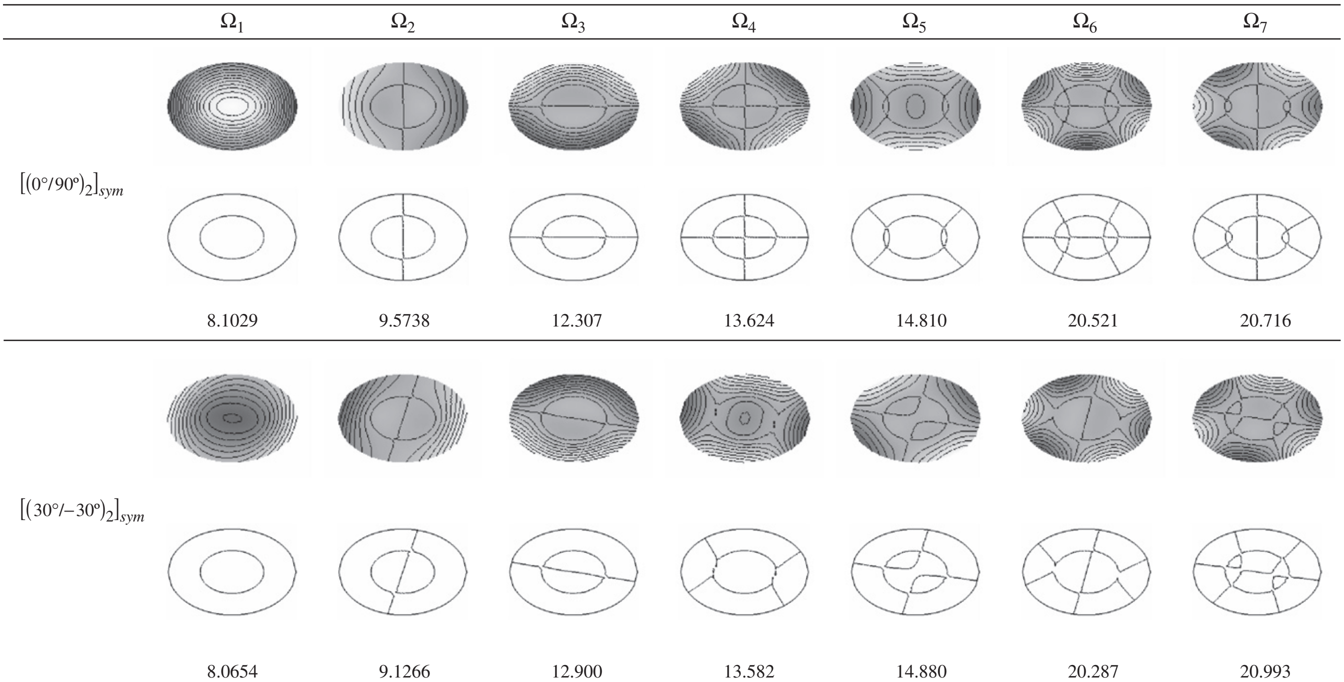

Fig. 8. First seven natural free vibration frequencies and mode shapes for solid elliptical cross-ply and angle-ply E-glass epoxy laminates $a/b = 1.5$, with a concentric ring support ($a_1/a = b_1/b = 0.5$).

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References

[1] Leissa AW. Vibration of plates, NASA SP-160. Washington DC: Office of Technology Utilization, NASA; 1969.
[2] Leissa AW. Recent research in plate vibrations, 1973–1979: classical theory. The Shock and Vibration Digest 1977;9(10):13–24.
[3] Leissa AW. Plate vibration research, 1976–1980: classical theory. The Shock and Vibration Digest 1981;13(9):11–22.
[4] Leissa AW. Recent studies in plate vibrations, 1981–1985, part I: classical theory. The Shock and Vibration Digest 1987;19(2):11–8.
[5] Rajalingham C, Bhat RB, Xistris GD. Vibration of clamped elliptic plates using exact circular plate modes as shape functions in Rayleigh–Ritz method. International Journal of Mechanical Science 1994;36(3):231–46.
[6] Young PG, Dickinson SM. Further studies on the vibration of plates with curved edges, including complicating effects. Journal of Sound and Vibration 1994;177(1):93–109.
[7] Chakraverty S, Petyt M. Natural frequencies for free vibration of non homogeneous elliptic and circular plates using two-dimensional orthogonal polynomials. Applied Mathematical Modelling 1997;21(7):399–417.
[8] Lim CW, Kitipornchai S, Liew KM. A free vibration analysis of doubly connected super-elliptical laminated composite plates. Composites Science and Technology 1998;58(3–4):435–45.
[9] Nallim LG, Grossi R, Laura PA. Transverse vibrations of circular plates of rectangular orthotropy carrying a central concentrated mass. Journal of Sound and Vibration 1998;216(2):337–41.
[10] Chakraverty S, Bhat RB, Stiharu I. Free vibration of annular elliptic plates using boundary characteristic orthogonal polynomials as shape functions in the Rayleigh–Ritz method. Journal of Sound and Vibration 2001;241(3): 524–39.
[11] Bayer I, Guven U, Altay G. A parametric study on vibrating clamped elliptical plates with variable thickness. Journal of Sound and Vibration 2002;254(1): 179–88.
[12] Kim SC. Natural frequencies of orthotropic, elliptical and circular plates. Journal of Sound and Vibration 2003;259(3):733–45.
[13] Hassan SM. Numerical solution for frequencies and mode shapes of elliptical plate half of whose boundary is simply supported and the rest free. International Journal of Mechanical Science 2004;46(12):1747–61.
[14] Hassan SM. Free transverse vibrations of elliptical plates of variable thickness with half of the boundary clamped and the rest free. International Journal of Mechanical Science 2004;46(12):1861–82.
[15] Kukla S, Szewczyk M. Frequency analysis of annular plates with elastic concentric supports by Green's function method. Journal of Sound and Vibration 2007;300(1–2):387–93.
[16] Reddy JN. Mechanics of Laminated Anisotropic Plates: Theory and Analysis. Boca Raton, FL: CRC Press; 1997.
[17] Whitney JM. Structural Analysis of Laminated Anisotropic Plates. Pennsylvania, USA: Technomic Pub Co; 1987.
[18] Grossi R, Nallim LG. Boundary and eigenvalue problems for generally restrained anisotropic plates. Journal of Multi-body Dynamics 2003;217(3): 241–51.
[19] Rektorys K. Variational methods in mathematics, science and engineering. Dordrecht: D. Reidel; 1980.
[20] Achong A. Vibrational analysis of circular and elliptical plates carrying point and ring masses and with edges elastically restrained. Journal of Sound and Vibration 1995;185(1):157–68.
[21] Sonemblum M, Gil E, Laura P A, Filipich P, Bergman A, Sanzi H C. A note on vibrations of elliptical plates carrying concentric, concentrated masses. Applied Acoustics 1989;28(1):1–7.