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Symmetric quadratic Hamiltonians with pseudo-Hermitian matrix representation

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HIGHLIGHTS

- Symmetric quadratic operators are useful models for many physical applications.
- Any such operator exhibits a pseudo-Hermitian matrix representation.
- Its eigenvalues are the natural frequencies of the Hamiltonian operator.
- The eigenvalues may be real or complex and describe a phase transition.

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ABSTRACT

We prove that any symmetric Hamiltonian that is a quadratic function of the coordinates and momenta has a pseudo-Hermitian adjoint or regular matrix representation. The eigenvalues of the latter matrix are the natural frequencies of the Hamiltonian operator. When all the eigenvalues of the matrix are real, then the spectrum of the symmetric Hamiltonian is real and the operator is Hermitian. As illustrative examples we choose the quadratic Hamiltonians that model a pair of coupled resonators with balanced gain and loss, the electromagnetic self-force on an oscillating charged particle and an active LRC circuit.

1. Introduction

In two recent papers Bender et al. [1] and Bender and Gianfreda[2] discussed two interesting physical problems: a pair of optical resonators with balanced gain and loss and the electromagnetic self-force on an oscillating charged particle, respectively. In both cases the authors resorted to

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Hamiltonians that are quadratic functions of the coordinates and momenta to describe the dynamics. They found that those quadratic Hamiltonians exhibit PT symmetry so that the quantum-mechanical counterparts show real spectra when PT symmetry is exact.

In the first case Bender et al. solved the Schrödinger equation in coordinate representation by writing each eigenfunction as the product of a Gaussian function times a polynomial function of the two coordinates and obtained suitable recurrence relations for the polynomials. In the second case Bender and Gianfreda [2] resorted to the approach proposed by Rossignoli and Kowalski [3] that consists in converting the quadratic Hamiltonian into a diagonal form by means of a canonical transformation of the creation and annihilation operators.

In two recent papers Fernández [4,5] proposed the application of a simple and straightforward algebraic method based on the construction of the adjoint or regular matrix representation of the Hamiltonian operator in a suitable basis set of operators [6,7]. The eigenvalues of such matrix representation are the natural frequencies of the Hamiltonian operator. Instead of invoking the PT symmetry of the problem the algebraic method takes advantage of the fact that those Hamiltonians are symmetric.

There are many other problems that can be modelled by quadratic Hamiltonians. For example, Schindler et al. [8] studied mutually coupled modes of a pair of active LRC circuits, one with amplification and another with an equivalent amount of attenuation, and found a remarkable agreement between theoretical results and experimental data. They argued that the gain and loss mechanism breaks Hermiticity while preserving PT symmetry. In a discussion of the bandwidth theorem Ramezani et al. [9] resorted to the same system of differential equations derived from Kirchhoff's laws.

The purpose of this paper is to apply the algebraic method to a general quadratic Hamiltonian in order to derive some general conclusion about its spectral properties. In Section 2 we outline the main ideas of the algebraic method. In Section 3 we apply the approach to a general quadratic Hamiltonian, derive the main result of this paper and illustrate the general results by means of two toy models. In Sections 4 and 5 we discuss the pair of resonators and the electromagnetic self-force mentioned above. In Section 6 we apply the algebraic method to the Hamiltonian associated to the differential equations for the active LRC circuit. Finally, in Section 7 we summarize the main results of the paper and draw conclusions.

2. The algebraic method

We begin the discussion of this section with some well known definitions that will facilitate the presentation of the algebraic method. Given a linear operator A its adjoint $A^\dagger$ satisfies

$$\langle f|\, A^\dagger \,|f\rangle = \langle f|\, A\, |f\rangle^* , \tag{1}$$

for any vector $|f\rangle$ in the Hilbert space where it is defined. If $A^\dagger = A$ we say that the operator A is symmetric. If $|\psi\rangle$ is an eigenvector of the symmetric operator H with eigenvalue E

$$H\,|\psi\rangle = E\,|\psi\rangle , \tag{2}$$

then $\langle f|\, H\, |f\rangle = \langle f|\, H\, |f\rangle^*$ leads to $(E - E^*)\,\langle\psi|\,\psi\rangle = 0$. Therefore, if $0 < \langle\psi|\,\psi\rangle < \infty$ then E is real.

The algebraic method enables us to solve the eigenvalue equation for a symmetric operator H when there exists a set of symmetric operators $S_N = \{O_1, O_2, \ldots, O_N\}$ that satisfy the commutation relations

$$[H, O_i] = \sum_{j=1}^{N} H_{ji} O_j. \tag{3}$$

Without loss of generality we assume that the operators in S_N are linearly independent; that is to say, the only solution to

$$\sum_{j=1}^{N} d_j O_j = 0, \tag{4}$$

is $d_i = 0, i = 1, 2, \ldots, N$. It follows from Eq. (3) and $[H, O_i]^\dagger = -[H, O_i]$ that

$$H_{ij}^* = -H_{ij}; \tag{5}$$

that is to say:

$$\mathbf{H}^{\dagger} = -\mathbf{H}^{t}. \tag{6}$$

Because of Eq. (3) it is possible to find an operator of the form

$$Z = \sum_{i=1}^{N} c_i O_i, \tag{7}$$

such that

$$[H, Z] = \lambda Z. \tag{8}$$

The operator Z is important for our purposes because

$$HZ\left|\psi\right\rangle = ZH\left|\psi\right\rangle + \lambda Z\left|\psi\right\rangle = (E+\lambda)Z\left|\psi\right\rangle, \tag{9}$$

that is to say, $Z\left|\psi\right\rangle$ is eigenvector of H with eigenvalue $E+\lambda$. Obviously, if $\left|\psi\right\rangle$ and $Z\left|\psi\right\rangle$ are normalizable, then both E and λ are real as argued above.

It follows from Eqs. (3), (7) and (8) that the coefficients c_i are solutions to the homogeneous linear system of equations

$$(\mathbf{H} - \lambda\mathbf{I})\mathbf{C} = 0, \tag{10}$$

where $\mathbf{H}$ is an $N \times N$ matrix with elements H_{ij}, $\mathbf{I}$ is the $N \times N$ identity matrix, and $\mathbf{C}$ is an $N \times 1$ column matrix with elements c_i. $\mathbf{H}$ is called the adjoint or regular matrix representation of the symmetric operator H in the operator basis set S_N [6,7]. Eq. (10) admits nontrivial solution if λ is a root of the secular determinant

$$\det(\mathbf{H} - \lambda\mathbf{I}) = 0. \tag{11}$$

If H is Hermitian, then all its eigenvalues are real and, consequently, all the roots λ_i, $i = 1, 2, \ldots, N$, of the characteristic polynomial (11) are real. These roots are obviously related to the natural frequencies of the quantum-mechanical system with Hamiltonian H. However, since the regular matrix representation of H is not normal: $\mathbf{H}\mathbf{H}^{\dagger} \neq \mathbf{H}^{\dagger}\mathbf{H}$ we cannot assure that it is always diagonalizable.

If λ is real then it follows from Eq. (8) and $[H, Z]^{\dagger} = -[H, Z^{\dagger}]$ that

$$[H, Z^{\dagger}] = -\lambda Z^{\dagger}, \tag{12}$$

where $Z^{\dagger}$, the adjoint of Z, is a linear combination like (7) with coefficients c_i^*. This equation tells us that if λ is a real root of the characteristic polynomial (11) then $-\lambda$ is also a root. In the language of quantum mechanics we often say that Z and $Z^{\dagger}$ are a pair of annihilation–creation or ladder operators because, in addition to (9), we also have

$$HZ^{\dagger}\left|\psi\right\rangle = (E-\lambda)Z^{\dagger}\left|\psi\right\rangle. \tag{13}$$

If N is odd then there is an operator Z_0 with eigenvalue $\lambda_0 = 0$ that commutes with H. If H is the Hamiltonian operator of a quantum-mechanical system then Z_0 is a constant of the motion. For concreteness, in what follows we assume that $N = 2K$ and $|\lambda_i| > 0$ for all $i = 1, 2, \ldots, K$. More precisely, we arrange the eigenvalues of $\mathbf{H}$ as follows:

$$-\lambda_K < -\lambda_{K-1} < \cdots < -\lambda_1 < 0 < \lambda_1 < \cdots < \lambda_K, \tag{14}$$

so that $-\lambda_i$ and λ_i are the eigenvalues of $\mathbf{H}$ associated to Z_i and $Z_i^{\dagger}$, respectively. Under these conditions any operator of the form

$$L = \sum_{i=1}^{K} l_i Z_i^{\dagger} Z_i, \tag{15}$$

commutes with H.

If at least one of the roots of the characteristic polynomial (11) is complex then we are sure that the spectrum of H is not real and that not all of its eigenvectors are normalizable.

Of particular interest for the present paper is the case where the basis operators satisfy

$$[O_i, O_j] = U_{ij}\hat{1}, \tag{16}$$

where $\hat{1}$ is the identity operator that we omit from now on. It follows from $[O_j, O_i] = -[O_i, O_j]$ and $[O_i, O_j]^\dagger = -[O_i, O_j]$ that

$$U_{ij} = -U_{ij}^* = -U_{ji}; \tag{17}$$

that is to say:

$$\mathbf{U}^\dagger = -\mathbf{U}^t = \mathbf{U}, \tag{18}$$

where $\mathbf{U}$ is the $N \times N$ matrix with elements U_{ij}. Under these conditions the well known Jacobi identity

$$[O_k, [H, O_i]] + [O_i, [O_k, H]] + [H, [O_i, O_k]] = 0, \tag{19}$$

reduces to

$$[O_k, [H, O_i]] = [O_i, [H, O_k]]. \tag{20}$$

Therefore, Eqs. (3), (16), (18), (20) and (6) lead to

$$\mathbf{H}^\dagger\mathbf{U} = \mathbf{U}\mathbf{H}. \tag{21}$$

Note that $\mathbf{H}$ and $\mathbf{H}^\dagger$ share eigenvalues:

$$\mathbf{H}^\dagger\mathbf{U}\mathbf{C} = \mathbf{U}\mathbf{H}\mathbf{C} = \lambda\mathbf{U}\mathbf{C}. \tag{22}$$

The matrix $\mathbf{U}$ is invertible because the operators in the set S_N are linearly independent. In fact, the commutator between O_k and the linear combination (4) yields

$$\sum_{j=1}^{N} U_{kj} d_j = 0, \quad k = 1, 2, \ldots, N, \tag{23}$$

so that the solution $d_j = 0$ for all j is unique if and only if $|\mathbf{U}| \neq 0$. Consequently, the regular matrix representation of a symmetric operator H in a basis set of symmetric operators that satisfy the commutation relations (16) is pseudo-Hermitian [10–12]:

$$\mathbf{H}^\dagger = \mathbf{U}\mathbf{H}\mathbf{U}^{-1}. \tag{24}$$

3. Quadratic Hamiltonians

The two quadratic Hamiltonians mentioned in the introduction [1,2] are particular cases of the general quadratic Hamiltonian

$$H = \sum_{i=1}^{2K} \sum_{j=1}^{2K} \gamma_{ij} O_i O_j, \tag{25}$$

where $\{O_1, O_2, \ldots, O_{2K}\} = \{x_1, x_2, \ldots, x_K, p_1, p_2, \ldots, p_k\}$, $[x_m, p_n] = i\delta_{mn}$, and $[x_m, x_n] = [p_m, p_n] = 0$. If $\boldsymbol{\gamma}^\dagger = \boldsymbol{\gamma}$, where $\boldsymbol{\gamma}$ is the matrix with elements γ_{mn}, then this quadratic Hamiltonian is symmetric. In this case the matrix $\mathbf{U}$ has the form

$$\mathbf{U} = i \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix} \tag{26}$$

where $\mathbf{0}$ and $\mathbf{I}$ are the $K \times K$ zero and identity matrices, respectively, so that $\mathbf{U}^\dagger = \mathbf{U}^{-1} = \mathbf{U}$.

We have thus arrived at the main result of the paper:

Theorem. *The regular or adjoint matrix representation* $\mathbf{H}$ *of a symmetric quadratic Hamiltonian like* (25) *is pseudo-Hermitian*

$$\mathbf{H}^\dagger = \mathbf{U}\mathbf{H}\mathbf{U}^{-1}, \tag{27}$$

where $\mathbf{U}$ *is given by Eq.* (26).

The matrices $\mathbf{H}$, $\boldsymbol{\gamma}$ and $\mathbf{U}$ are connected by

$$\mathbf{H} = (\boldsymbol{\gamma} + \boldsymbol{\gamma}^t)\mathbf{U}. \tag{28}$$

If $\mathbf{C}_i$ and $\mathbf{C}_j$ are two eigenvectors of $\mathbf{H}$ with eigenvalues λ_i and λ_j, respectively, then it is not difficult to prove that

$$\left(\lambda_j - \lambda_i^*\right) \mathbf{C}_i^\dagger \mathbf{U}\mathbf{C}_j = 0, \tag{29}$$

which is just the matrix version of the result proved some time ago by Mostafazadeh [10]. In particular, when $i = j$ we conclude that $\lambda_i = \lambda_i^*$ if $\mathbf{C}_i^\dagger \mathbf{U}\mathbf{C}_i \neq 0$ and that $\mathbf{C}_i^\dagger \mathbf{U}\mathbf{C}_i = 0$ if λ_i is a complex number. It is clear that the eigenvalues of $\mathbf{H}$ may be real or complex [10]. The occurrence of one or another will depend on the matrix elements that are given in terms of the parameters of the Hamiltonian operator. Therefore, all the symmetric quadratic Hamiltonians are bound to exhibit some regions in model-parameter space where the spectrum is real and other regions where it is complex. This result is independent of the existence of PT symmetry (or any other kind of antiunitary symmetry) in the problem. At the phase transition from real to complex eigenvalues at least one eigenvector of H is no longer normalizable, $\mathbf{C}_i^\dagger \mathbf{U}\mathbf{C}_i = 0$ and $\mathbf{H}$ ceases to be diagonalizable. A phase transition may also be interpreted as a broken Hermiticity.

In this paper we do not try to solve the eigenvalue equation completely by means of the algebraic method and simply obtain the eigenvalues of the adjoint matrix to determine whether the spectrum is real or not.

3.1. Simple examples

In this subsection we discuss two toy problems. The first one is given by the symmetric quadratic Hamiltonian

$$H = p^2 + \alpha x^2 + \frac{\beta}{2}\left(xp + px\right), \tag{30}$$

where α and β are positive real numbers. In this case the adjoint matrix representation reads

$$\mathbf{H} = i\begin{pmatrix} -\beta & 2\alpha \\ -2 & \beta \end{pmatrix}, \tag{31}$$

and its eigenvalues are $-\lambda_1$ and λ_1, where

$$\lambda_1 = \sqrt{4\alpha - \beta^2}. \tag{32}$$

It is clear that the spectrum of the symmetric operator (30) is real when $4\alpha - \beta^2 > 0$.

The ground-state eigenfunction is $\psi_0(x) = De^{-ax^2}$, where $a = \frac{\sqrt{4\alpha-\beta^2}}{4} + i\frac{\beta}{4}$, with eigenvalue $E_0 = \frac{\sqrt{4\alpha-\beta^2}}{2}$. This eigenfunction is square integrable when $4\alpha - \beta^2 > 0$. We clearly see the connection between the eigenvalues of $\mathbf{H}$ and the spectrum of H.

The results just obtained are not surprising because

$$H = \exp\left[-i\frac{\beta}{4}x^2\right] H_0 \exp\left[i\frac{\beta}{4}x^2\right],$$
$$H_0 = p^2 + \left(\alpha - \frac{\beta^2}{4}\right)x^2. \tag{33}$$

The second example

$$H = p_x^2 + p_y^2 + x^2 + y^2 + \beta xy, \tag{34}$$

is similar to the one chosen by Bender et al. [13] to illustrate a PT phase transition in a simple mechanical system. In this case the adjoint matrix representation reads

$$\mathbf{H} = i\begin{pmatrix} 0 & 0 & 2 & \beta \\ 0 & 0 & \beta & 2 \\ -2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}. \tag{35}$$

The characteristic polynomial has four roots $-\sqrt{\xi_2} < -\sqrt{\xi_1} < \sqrt{\xi_1} < \sqrt{\xi_2}$, where

$$\xi_1 = 2(2-\beta), \qquad \xi_2 = 2(2+\beta). \tag{36}$$

Therefore, the spectrum of the symmetric Hamiltonian (34) is real when $-2 < \beta < 2$.

In this case the ground state is $\psi_0(x, y) = De^{-a(x^2+y^2)-bxy}$, where

$$a = \frac{\sqrt{2-\sqrt{4-\beta^2}}\left(\sqrt{4-\beta^2}+2\right)}{4\beta}, \qquad b = \frac{\sqrt{2-\sqrt{4-\beta^2}}}{2}, \tag{37}$$

and the corresponding eigenvalue is $E_0 = 4a$. We appreciate that $\psi_0(x, y)$ is square integrable only if $-2 < \beta < 2$ as predicted by the algebraic method.

4. Coupled resonators with balanced gain and loss

From the classical equations of motion for the case of equal gain and loss between the two resonators Bender et al. [1] derived the following quadratic Hamiltonian

$$H = p_x p_y + \gamma(yp_y - xp_x) + \left(\omega^2 - \gamma^2\right)xy + \frac{\epsilon}{2}\left(x^2 + y^2\right), \tag{38}$$

where ω is the natural frequency of the oscillators, γ is related to the friction force and ϵ is a coupling strength. One can easily verify that this operator is symmetric.

The set of operators $\{O_1, O_2, O_3, O_4\} = \{x, y, p_x, p_y\}$ leads to the matrix representation [4]

$$\mathbf{H} = i\begin{pmatrix} \gamma & 0 & \epsilon & \omega^2-\gamma^2 \\ 0 & -\gamma & \omega^2-\gamma^2 & \epsilon \\ 0 & -1 & -\gamma & 0 \\ -1 & 0 & 0 & \gamma \end{pmatrix}, \tag{39}$$

with characteristic polynomial

$$\xi^2 + 2\xi\left(2\gamma^2 - \omega^2\right) - \epsilon^2 + \omega^4 = 0, \tag{40}$$

where $\xi = \lambda^2$. A necessary condition for the spectrum of the symmetric quadratic Hamiltonian (38) to be real is that the two roots of the polynomial (40) are real and positive. A more detailed discussion of this spectrum is given elsewhere [1,4].

5. Electromagnetic self-force

From the pair of classical equations of motion proposed by Englert [14], Bender and Gianfreda [2] derived the Hamiltonian function

$$H_c = \frac{p_x p_w - p_y p_z}{m\tau} + \frac{2p_z p_w}{m\tau^2} + \frac{wp_y + zp_x}{2} - \frac{mzw}{2} + kxy. \tag{41}$$

In this expression k is the restoring force of the oscillator, m the mass of the particle and τ is related to the classical radius of the charged particle. The quantum-mechanical version of this operator is PT-symmetric but its eigenvalues are not real because the PT symmetry is broken for all m, τ, k.

In order to illustrate how PT symmetry is broken the authors added two coupling terms and obtained the modified Hamiltonian operator

$$H = \frac{B(wp_z - zp_w)}{m\tau} + \frac{2p_z p_w}{m\tau^2} + \frac{p_x p_w - p_y p_z}{m\tau} - \frac{mzw}{2} + \frac{wp_y + zp_x}{2} + kxy + \frac{A(x^2+y^2)}{2}, \tag{42}$$

where every term is obviously symmetric. Following a recent communication [5] we choose the basis set of operators $\{O_1, O_2, \ldots, O_8\} = \{x, y, z, w, p_x, p_y, p_z, p_w\}$ and obtain the adjoint matrix representation

$$\mathbf{H} = i\begin{pmatrix} 0 & 0 & 0 & 0 & A & k & 0 & 0 \\ 0 & 0 & 0 & 0 & k & A & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{B}{m\tau} & 0 & 0 & 0 & -\frac{m}{2} \\ 0 & -\frac{1}{2} & -\frac{B}{m\tau} & 0 & 0 & 0 & -\frac{m}{2} & 0 \\ 0 & 0 & 0 & -\frac{1}{m\tau} & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{m\tau} & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{m\tau} & 0 & -\frac{2}{m\tau^2} & 0 & 0 & 0 & \frac{B}{m\tau} \\ -\frac{1}{m\tau} & 0 & -\frac{2}{m\tau^2} & 0 & 0 & 0 & -\frac{B}{m\tau} & 0 \end{pmatrix}. \tag{43}$$

The characteristic polynomial can be factorized as

$$\left(m^2\tau^2\xi - B^2 + m^2\right)\left[m^2\tau^2\xi^3 + \xi^2\left(m^2 - B^2\right) + \xi\left(2AB - 2km\right) - A^2 + k^2\right] = 0, \tag{44}$$

where $\xi = \lambda^2$. Obviously, one of the roots is

$$\xi = \frac{B^2 - m^2}{m^2\tau^2}, \tag{45}$$

and the remaining three ones are solutions to the cubic equation

$$m^2\tau^2\xi^3 + \left(m^2 - B^2\right)\xi^2 + 2\left(AB - km\right)\xi - A^2 + k^2 = 0. \tag{46}$$

It is clear that the spectrum of the Hamiltonian (42) will not be real unless the rhs of Eq. (45) as well as the three roots of Eq. (46) are positive numbers.

6. Active LRC circuits

From the first and second Kirchhoff's laws Schindler et al. [8] derived the following system of differential equations for the charges Q_1^c and Q_2^c in an LRC circuit

$$\begin{aligned} \frac{d^2Q_1^c}{d\tau^2} &= -\frac{1}{1-\mu^2}Q_1^c + \frac{\mu}{1-\mu^2}Q_2^c + \gamma\frac{dQ_1^c}{d\tau}, \\ \frac{d^2Q_2^c}{d\tau^2} &= \frac{\mu}{1-\mu^2}Q_1^c - \frac{1}{1-\mu^2}Q_2^c - \gamma\frac{dQ_2^c}{d\tau}, \end{aligned} \tag{47}$$

where τ is a dimensionless time and μ and γ are related to circuit features such as resistance, inductance and capacitance. This system of differential equations can be derived from the Hamiltonian function

$$H = p_x p_y + \frac{\gamma}{2}(xp_x - yp_y) + \left(\frac{1}{1-\mu^2} - \frac{\gamma^2}{4}\right)xy - \frac{\mu}{2\left(1-\mu^2\right)}\left(x^2+y^2\right), \tag{48}$$

where $x = Q_1^c, y = Q_2^c$, and p_x, p_y their conjugate momenta. The corresponding quantum-mechanical Hamiltonian operator is similar to the one in Eq. (38) and, therefore, symmetric. The resulting adjoint matrix

$$\mathbf{H} = i \begin{pmatrix} -\frac{\gamma}{2} & 0 & \frac{\mu}{\mu^2-1} & \frac{\gamma^2\left(\mu^2-1\right)+4}{4\left(1-\mu^2\right)} \\ 0 & \frac{\gamma}{2} & \frac{\gamma^2\left(\mu^2-1\right)+4}{4\left(1-\mu^2\right)} & \frac{\mu}{\mu^2-1} \\ 0 & -1 & \frac{\gamma}{2} & 0 \\ -1 & 0 & 0 & -\frac{\gamma}{2} \end{pmatrix}, \tag{49}$$

is pseudo-Hermitian as argued in Section 3. The characteristic polynomial is

$$\xi^2\left(\mu^2-1\right) + \xi\left[\gamma^2\left(\mu^2-1\right)+2\right] - 1 = 0, \tag{50}$$

where $\xi = \lambda^2$. Its two roots

$$\xi_1 = \frac{\sqrt{\gamma^4\left(\mu^2-1\right)^2 + 4\gamma^2\left(\mu^2-1\right) + 4\mu^2} + \gamma^2\left(1-\mu^2\right) - 2}{2\left(\mu^2-1\right)},$$
$$\xi_2 = \frac{\sqrt{\gamma^4\left(\mu^2-1\right)^2 + 4\gamma^2\left(\mu^2-1\right) + 4\mu^2} + \gamma^2\left(\mu^2-1\right) + 2}{2\left(1-\mu^2\right)}, \tag{51}$$

are the squares of the eigenfrequencies obtained by Schindler et al. [8] from Eq. (47).

The fact that the classical and quantal versions of the system have the same frequencies is due to the fact that $[Q_j, Q_k] = i\{Q_j, Q_k\}$, where $\{\ldots, \ldots\}$ is the classical Poisson bracket [15].

7. Conclusions

The main result of this paper is that the frequencies of any symmetric quadratic Hamiltonian like (25) are the eigenvalues of a nonnormal but pseudo-Hermitian matrix. Consequently, the eigenvalues of any Hamiltonian belonging to such family may be real or complex independently of the existence of a PT symmetry in the Hamiltonian.

The occurrence of real or complex eigenvalues depends on the matrix elements of the adjoint or regular matrix representation of the Hamiltonian that are functions of the Hamiltonian parameters. If the eigenvalues of the symmetric Hamiltonian are real its eigenvectors are normalizable and the operator is Hermitian. On the other hand, complex eigenvalues reveal that the norm of some eigenvectors are either zero or infinity. Looking for exact or broken PT symmetry is equivalent to finding whether the Hamiltonian is Hermitian or not.

Real and complex eigenvalues of the symmetric quadratic Hamiltonian correspond to real or complex eigenvalues of the adjoint matrix representation. Since the analysis of a matrix of finite dimension does not offer any difficulty we think that the algebraic method may be a suitable tool for the analysis of physical problems that can be modelled by symmetric quadratic Hamiltonians.

Note added in proof

We would like to mention another interesting application of a quadratic Hamiltonian given by two harmonic oscillators coupled by their angular momentum. It proved useful for the study of the dynamics of entanglement between two harmonic modes in stable and unstable regimes and also belongs to the class of quadratic operators discussed in this paper [16].

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