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On the existence of weak solutions of anisotropic generally restrained plates

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Abstract

This paper presents investigations of free vibration of anisotropic plates of different geometrical shapes and generally restrained boundaries. The existence and uniqueness of weak solutions of boundary value problems and eigenvalue problems which correspond to the statical and dynamical behaviour of the mentioned plates is demonstrated. It is determined that when the plates have corner points formed by the intersection of edges free or elastically restrained against translation, the corresponding bilinear forms maintain the V – ellipticity property.

Also, an analytical formulation, based on the Ritz method and polynomial expressions as approximate functions for analysing the free vibrations of laminated plates with smooth and non-smooth boundary with non-classical edge supports is presented. Numerical results are presented for circular, elliptical and trapezoidal plates for different boundary conditions and material properties.

Keywords: Weak solutions; Vibration analysis; Elastically restrained edges; Composite material

1. Introduction

A typical method of solving boundary and eigenvalue problems for elliptic partial differential equations with variable coefficients is the variational method. It can be applied under comparatively weak conditions on diverse domains and elliptic differential operators of arbitrary order. On the other hand, since the variational method involves the minimization of functionals, which describe certain types of energy, it is more natural to look for a weak solution of the problem under consideration than to find its classical solution, which does not exist for many important engineering and mathematical physics problems. Weak solutions of boundary value or eigenvalue problems are useful because, unlike the classical solutions, they can be obtained for domains and data of the problems which are less smooth [1–11]. Hence, it is necessary to find sufficient conditions for the existence and uniqueness of the weak solutions of boundary and eigenvalue problems.

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Nomenclature

a, b, c, l	plate dimensions (Fig. 3)
$B(w, v)$	bilinear form
c_{ij}	unknown coefficients in the deflection function Eqs. (42) and (44)
$c_R(s)$	spring constant of the rotational restraint
$c_T(s)$	spring constant of the translational restraint
D	flexural rigidity of isotropic plate
D_{ij}	bending, twisting and coupling rigidities of anisotropic plate
D_0	reference rigidity of anisotropic plate $= E_L h^3/[12(1-\nu_{LT}\nu_{TL})]$
E_L, E_T	Young's moduli parallel and perpendicular to the fibres
G_{LT}	shear modulus of elasticity
h	plate thickness
$H^2(\Omega)$	Sobolev space of order two
n_1, n_2	components of the outward unit normal vector n to the boundary $\partial\Omega$
$p_i(x)$, $q_j(y)$	beam characteristic orthogonal polynomials
q	transversal load
R_i, T_i	nondimensional rotational and translational coefficients
s	arc length along the plate boundary
T_{max}	maximum kinetic energy
U_{max}	maximum strain energy due to plate bending
$U_{R,max}$	maximum strain energy stored in rotational springs
$U_{T,max}$	maximum strain energy stored in translational springs
u, w	deflection functions
x_1, x_2	cartesian coordinates
α, β	multi-index vectors
Ω, Ω^*	two-dimensional plate domains in Cartesian coordinates
$\hat{\Omega}$	plate domain in right triangular coordinates
$\partial\Omega$	plate boundary
ρ	mass density of the plate material
ν, ν_{LT}, ν_{TL}	Poisson's ratios
θ_1, θ_2	side angles of the trapezoidal plate (Fig. 3b)
ω	circular frequency of plate vibration

Composite structures, especially laminated composite plates, have been widely used in many engineering advantages of high strength and stiffness and light weight. Laminated composite plates allow the controllability of the structural properties, through changing the fibre orientation angles, the number of plies and selecting proper composite materials. With the wide use of composite plate structures in modern industries, mechanical analysis of plates of complex geometry becomes a relevant study. It is important to understand the free vibration and the flexural behaviour of these structural elements but the solutions to these plate problems are strongly dependent on the geometrical shapes, boundary conditions and material properties. It is widely recognised that closed form solutions are possible only for a few specific cases [12,13].

The determination of classical solutions (exact and/or approximate) which correspond to the statical and dynamical behaviour of anisotropic plates of different shapes and configurations, has been studied and is well documented. The bending of anisotropic plates subjected to different normal loads and boundary conditions has been extensively studied [14–16]. On the other hand, the vibration of anisotropic plates with different boundary conditions has received considerable attention from several investigators. Bert presented complete reviews on dynamics of composite and sandwich panels [17–19]. These compilations show that the results correspond mainly to rectangular shapes and classical boundary conditions. Nevertheless, studies on general quadrilateral plates, or polygonal plates with unequal side lengths are rather limited. Liew and his co-workers studied the behaviour of different plates using Ritz method with sets of two-dimensional plate functions, which express the entire plate domain into two implicitly related variables (see for instance [20–23]). Nallim et al. [24,25] analysed the statical and dynamical behaviour of thin fibre reinforced composite laminates plates of arbitrary quadrilateral geometry with different classical boundary conditions. Grossi and Lebedev [26] analysed the static and dynamic behaviour of anisotropic plates with corner points.

The problem of elastic edge restraints has received considerable attention, mainly in the case of isotropic and orthotropic plates. Nevertheless, analytical studies on the dynamical behaviour of composite laminated plates with edges elastically restrained are rather limited. Laura and Grossi [27] used the Ritz method with polynomial functions for the free vibration analysis of anisotropic rectangular plates of uniform thickness having all edges elastically restrained against rotation. Nallim et al. [28] presented a study of free vibration of anisotropic triangular plates with edges elastically restrained against rotation and translation. Liew et al. [29] analysed the vibration of laminated plates with edges elastically restrained. Ashour [30] analysed the buckling and vibration of cross-ply laminated plates with edges elastically restrained. Ashour [31] studied the vibration of angle-ply laminated rectangular plates with edges elastically restrained. Setoodeh and Karami [32] presented a solution for the vibration and buckling of composite laminates with edges elastically restrained.

The determination of existence and uniqueness of weak solutions for plates with elastically restrained boundaries is rather limited.

Balasundaram and Bhattacharyya [33] derived sufficient conditions for the existence and uniqueness of the solution of a general boundary value problem which includes as particular cases, the boundary value problems of the bending of elastic isotropic, orthotropic and anisotropic plates with variable thickness.

Bhattacharyya and Nataraj [34] used the weak variational formulation in the error estimates for the mixed finite element solution of four order elliptic problems with variable coefficients. Chudinovich and Constanda [35] determined the existence and uniqueness of weak solutions in the bending of plates with transverse shear deformation and with elastic boundary conditions. Grossi [36] determined the existence and uniqueness of the weak solution of boundary value problems and eigenvalue problems, which correspond, respectively, to the statical and dynamical behaviour of rectangular anisotropic plates with edges elastically restrained against rotation.

The present paper deals with the determination of sufficient conditions for the existence and uniqueness of the weak solutions of boundary value problems and eigenvalue problems, which correspond, respectively, to the statical and the free vibration analyse of anisotropic plates of different shapes and generally restrained boundaries.

In practice, the boundary conditions may not always correspond to the classical edge conditions: clamped, simply supported and free. The plate edges may experience partial resistance to rotation and translation, which may be modelled as a rotational and translational stiffness, respectively, along the edges. Also, in the study of elastic plates with edge beams, where it is difficult to find out suitable shape functions which satisfy the boundary conditions, it is possible to replace the real mechanical system with a plate supported with edges elastically restrained. For these reasons, in this paper boundaries elastically restrained against rotation and translation are considered. This includes the classical edge conditions, as simply particular cases.

This paper also presents results of the application of the Rayleigh–Ritz method used in conjunction with different sets of approximating functions, to investigate the vibrational characteristics of circular, elliptical and trapezoidal anisotropic plates having elastically restrained boundaries.

2. The weak solution

2.1. Smooth boundary

Let us consider a tapered anisotropic plate, of arbitrary shape and elastically supported along the edge by translational and rotational restraints, that in the equilibrium position covers a two-dimensional domain Ω, with smooth boundary $\partial\Omega$, as it is shown in Fig. 1. Suppose that the rotational restraint is characterized by the spring constant $c_{\mathrm{R}}(s)$, and the translational restraint by the spring constant $c_{\mathrm{T}}(s)$, where s is the arc length along the boundary $\partial\Omega$.

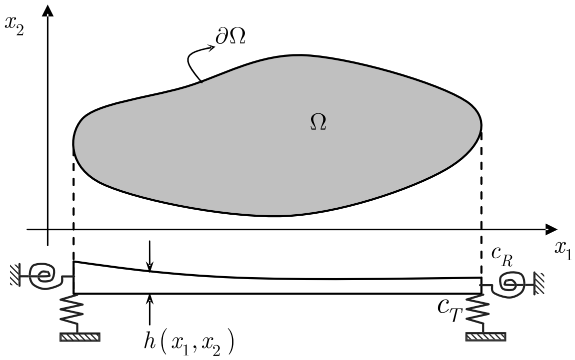


Fig. 1. Anisotropic plate with elastically restrained edges.

In order to obtain the weak formulation which corresponds to the mechanical system under study, we consider the domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial\Omega$ such that $\overline{\Omega} = \Omega \cup \partial\Omega$. We also consider the operator $A : D_A \to \mathbb{R}, \quad D_A \subset C^{(4)}(\overline{\Omega})$, given by

$$Au(x,t) = \sum_{|\alpha|,|\beta| \leqslant 2} (-1)^{|\alpha|} D^{\alpha}(a_{\alpha\beta}(x) D^{\beta} u(x,t)) = \sum_{|\alpha|,|\beta| \leqslant 2} (-1)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}} \left(a_{\alpha\beta}(x) \frac{\partial^{|\beta|} u(x,t)}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}} \right), \tag{1}$$

where $x = (x_1, x_2)$, t denotes the time, $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2)$ are multi-index vectors whose co-ordinates are non-negative integers and $|\alpha|$, $|\beta|$ are the sums $|\alpha| = \alpha_1 + \alpha_2$, $|\beta| = \beta_1 + \beta_2$. Let $a_{\alpha\beta}(x) \in C^{(|\alpha|)}(\Omega)$, $u(\cdot, t) \in C^{(4)}(\Omega)$. The summation in (1) is carried over all the vectors α and β for which $|\alpha|$, $|\beta| \leqslant 2$.

The equations which govern the statical and dynamical behaviour of arbitrary shape – isotropic, orthotropic and anisotropic – plates with complicating effects, are associated with operator A, as particular cases, [14,36].

The statical behaviour of the anisotropic plate when a load $q = q(x)$ is applied, is governed by the corresponding boundary conditions and the equation

$$Aw(x) = q(x), \tag{2}$$

with the operator A given by (1) and the coefficients $a_{\alpha\beta}(x)$ as defined in Appendix, i.e.:

$$\begin{aligned} Aw(x) = {} & \frac{\partial^2}{\partial x_1^2}\left(D_{11}(x) \frac{\partial^2 w(x)}{\partial x_1^2} + D_{12}(x) \frac{\partial^2 w(x)}{\partial x_2^2} + 2D_{16}(x) \frac{\partial^2 w(x)}{\partial x_1 \partial x_2} \right) \\ & + \frac{\partial^2}{\partial x_2^2}\left(D_{12}(x) \frac{\partial^2 w(x)}{\partial x_1^2} + D_{22}(x) \frac{\partial^2 w(x)}{\partial x_2^2} + 2D_{26}(x) \frac{\partial^2 w(x)}{\partial x_1 \partial x_2} \right) \\ & + \frac{\partial^2}{\partial x_1 \partial x_2}\left(2D_{16}(x) \frac{\partial^2 w(x)}{\partial x_1^2} + 2D_{26}(x) \frac{\partial^2 w(x)}{\partial x_2^2} + 4D_{66}(x) \frac{\partial^2 w(x)}{\partial x_1 \partial x_2} \right), \end{aligned} \tag{3}$$

$\forall x \in \Omega$, where w denotes the deflection of the mid-surface of the plate and the coefficients $D_{ij}(x)$, are the rigidities of the anisotropic material [16], which in terms of a x_1, x_2, x_3 co-ordinate system, are given by

$$D_{ij}(x) = D_{ij}(x_1, x_2) = \int_{-h(x)/2}^{h(x)/2} \overline{Q}_{ij} x_3^2 \, dx_3, \tag{4}$$

where the $\overline{Q}_{ij}$ are the transformed reduced stiffnesses.

The mathematical model allows the consideration of a composite plate. Let us consider a symmetric laminate of uniform thickness h, made up of a number of layers each consisting of unidirectional fibre reinforced composite material. The coefficients D_{ij}, are given by

$$D_{ij} = \sum_{k=1}^{N_1} \int_{-h_k/2}^{h_{k+1}/2} \overline{Q}_{ij} x_3^2 \, dx_3, \tag{5}$$

where h_k and h_{k+1} are the distances from the mid-plane to the top and bottom surface of the kth layer, and N_l is the total number of laminated layers.

Free transverse vibrations of the described plates are governed by the following partial differential equation

$$A(u(x,t)) = -\rho h(x)\frac{\partial^2 u(x,t)}{\partial t^2}, \tag{6}$$

where ρ denotes the density of the plate material and $h(x)$ the plate thickness.

Now let us introduce, as coordinate variables, the arc length s of the boundary $\partial\Omega$ and the distance n measured from the boundary and along the exterior normal to $\partial\Omega$. Let us consider that $\partial\Omega$ is an smooth curve represented in the parametric form by the C^1 function $\gamma : [0,l] \to \mathbb{R}^2$; $\gamma = (\gamma_1(s), \gamma_2(s)), s \in [0,l]$, where $l = l(\partial\Omega)$ is the length of the boundary $\partial\Omega$. If $\varphi = \varphi(s)$ denotes the angle made by the tangent to $\partial\Omega$ with the positive x_1 we have the following transformation equations [37]:

$$x_1(n,s) = \gamma_1(s) + n\sin\varphi(s), \quad x_2(n,s) = \gamma_2(s) - n\cos\varphi(s), \tag{7}$$

and the well known relations

$$\left.\frac{\partial u(x,t)}{\partial x_1}\right|_{\partial\Omega} = \frac{\partial u(\gamma_1(s),\gamma_2(s),t)}{\partial n}n_1 - \frac{\partial u(\gamma_1(s),\gamma_2(s),t)}{\partial s}n_2, \tag{8}$$

$$\left.\frac{\partial u(x,t)}{\partial x_2}\right|_{\partial\Omega} = \frac{\partial u(\gamma_1(s),\gamma_2(s),t)}{\partial n}n_2 + \frac{\partial u(\gamma_1(s),\gamma_2(s),t)}{\partial s}n_1. \tag{9}$$

The application of the calculus of variations allows us to obtain the boundary conditions which correspond to a vibrating anisotropic plate of arbitrary shape and smooth boundary elastically restrained against rotation and translation. These are given by (see [37])

$$c_R(s)\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = \left(M_1(u)n_1^2(s) + M_2(u)n_2^2(s) + 2H_{12}(u)n_1(s)n_2(s)\right)\big|_{\partial\Omega}, \tag{10}$$

$$\begin{aligned} c_T(s)u|_{\partial\Omega} = &-\left(\frac{\partial M_1(u)}{\partial x_1} + \frac{\partial H_{12}(u)}{\partial x_2}\right)n_1(s) - \left(\frac{\partial M_2(u)}{\partial x_2} + \frac{\partial H_{12}(u)}{\partial x_1}\right)n_2(s) \\ &- \frac{\partial}{\partial s}\left((M_2(u) - M_1(u))n_1(s)n_2(s) + H_{12}(u)(n_1^2(s) - n_2^2(s))\right)\big|_{\partial\Omega}, \end{aligned} \tag{11}$$

where

$$u = u(x,t), \quad M_1(u) = -\left(D_{11}\frac{\partial^2 u}{\partial x_1^2} + D_{12}\frac{\partial^2 u}{\partial x_2^2} + 2D_{16}\frac{\partial^2 u}{\partial x_1 \partial x_2}\right),$$

$$M_2(u) = -\left(D_{22}\frac{\partial^2 u}{\partial x_2^2} + D_{12}\frac{\partial^2 u}{\partial x_1^2} + 2D_{26}\frac{\partial^2 u}{\partial x_1 \partial x_2}\right),$$

$$H_{12}(u) = -\left(D_{16}\frac{\partial^2 u}{\partial x_1^2} + D_{26}\frac{\partial^2 u}{\partial x_2^2} + 2D_{66}\frac{\partial^2 u}{\partial x_1 \partial x_2}\right).$$

In Eqs. (10) and (11) the coefficients $c_R(s)$ and $c_T(s)$ denote the rotational and translational stiffnesses per unit length along the boundary. It is well known that if a differential operator is of order $2m$, boundary conditions containing derivatives of orders at most $m-1$ are called stable conditions and those containing derivatives of orders higher than $m-1$ are called unstable conditions. Consequently, when the operator in Eq. (1) is of order four and $0 \leqslant c_R, c_T < \infty$, Eqs. (10) and (11) correspond to unstable boundary conditions. When c_R, $c_T \to \infty$, the resulting conditions are stable.

Let $H^2(\Omega)$ be the Sobolev space $H^2(\Omega) = \{u \in L^2(\Omega); D^\alpha u \in L^2(\Omega), \forall\alpha, 0 \leqslant |\alpha| \leqslant 2\}$, equipped with the norm

$$\|u\|_{H^2(\Omega)} = \left(\sum_{|\alpha|\leqslant 2}\int_\Omega (D^\alpha u)^2\,dx\right)^{\frac{1}{2}}.$$

The stable and unstable boundary conditions are of different nature so in order to clearly distinguish them, it is useful to introduce the space V of elements of the Sobolev space $H^2(\Omega)$, which satisfy the corresponding stable homogeneous boundary conditions.

Consider the boundary value problem given by Eq. (2) and the boundary conditions (10) and (11) when the variable t is deleted. Then $u(x,t)$ is replaced by $w(x)$. Now this boundary value problem is transformed into one that leads to the concept of weak solution. If we let c_R, $c_T \to \infty$, in Eqs. (10) and (11), these conditions are reduced to $w(x)|_{\partial\Omega}= \frac{\partial w(x)}{\partial n}\Big|_{\partial\Omega} = 0$. Consequently, since a weak solution of Eq. (2), is a function from the Sobolev space $H^2(\Omega)$, the space V is given by

$$V = \left\{ v; v \in H^2(\Omega), v|_{\partial\Omega} = \frac{\partial v}{\partial n}\Big|_{\partial\Omega} = 0 \text{ in the sense of traces} \right\}. \tag{12}$$

When the coefficients c_R and c_T take finite values, there are no stable boundary conditions and the space V can be taken as $V = \{v;\ v \in H^2(\Omega)\}$.

First we assume that $q(x) \in C(\overline{\Omega})$, and that $w \in C^{(4)}(\overline{\Omega})$ is the classical solution of the problem (2) and (10, 11). If we take an arbitrary function $v \in V$, and multiply Eq. (2) by this function and integrate the result over the domain Ω we get

$$\int_\Omega A(u(x))v(x)\mathrm{d}x = \int_\Omega q(x)v(x)\mathrm{d}x. \tag{13}$$

Now it is necessary to use the Green formula

$$\int_\Omega w\frac{\partial v}{\partial x_i}\,\mathrm{d}x = \int_{\partial\Omega} wvn_i\,\mathrm{d}s - \int_\Omega v\frac{\partial w}{\partial x_i}\mathrm{d}x, \quad i = 1,2, \ \ \forall w, v \in H^{(1)}(\Omega),$$

where n_i denotes the components of the normal exterior to the boundary of Ω. If we apply this formula to the left hand side of Eq. (13) we obtain

$$\begin{aligned} B(w,v) = {} & -\int_\Omega \left(M_1(w)\frac{\partial^2 v}{\partial x_1^2} + M_2(w)\frac{\partial^2 v}{\partial x_2^2} + 2H_{12}(w)\frac{\partial^2 v}{\partial x_1 \partial x_2}\right)\mathrm{d}x + \int_{\partial\Omega} (M_1(w)n_1 + H_{12}(w)n_2)\frac{\partial v}{\partial x_1}\mathrm{d}s \\ & + \int_{\partial\Omega} (M_2(w)n_2 + H_{12}(w)n_1)\frac{\partial v}{\partial x_2}\mathrm{d}s - \int_{\partial\Omega}\left(\frac{\partial M_1(w)}{\partial x_1} + \frac{\partial H_{12}(w)}{\partial x_2}\right)vn_1\,\mathrm{d}s \\ & - \int_{\partial\Omega}\left(\frac{\partial M_2(w)}{\partial x_2} + \frac{\partial H_{12}(w)}{\partial x_1}\right)vn_2\,\mathrm{d}s. \end{aligned} \tag{14}$$

Since $v \in H^2(\Omega)$, the derivatives $\partial v/\partial x_i \in H^1(\Omega)$, $i = 1,2$ have traces, then the derivatives $\partial v(s)/\partial n$, $\partial v(s)/\partial s$ can be defined. Consequently, if we replace Eqs. (8) and (9) with $u = v$ in Eq. (14) it follows

$$\begin{aligned} B(w,v) = {} & -\int_\Omega \left(M_1(w)\frac{\partial^2 v}{\partial x_1^2} + M_2(w)\frac{\partial^2 v}{\partial x_2^2} + 2H_{12}(w)\frac{\partial^2 v}{\partial x_1 \partial x_2}\right)\mathrm{d}x \\ & + \int_{\partial\Omega} \left[M_1(w)n_1^2 + M_2(w)n_2^2 + 2H_{12}(w)n_1n_2\right]\frac{\partial v}{\partial n}\mathrm{d}s \\ & + \int_{\partial\Omega} \left[(M_2(w) - M_1(w))n_1n_2 + H_{12}(w)(n_1^2 - n_2^2)\right]\frac{\partial v}{\partial s}\mathrm{d}s \\ & + \int_{\partial\Omega}\left(-\left(\frac{\partial M_1(w)}{\partial x_1} + \frac{\partial H_{12}(w)}{\partial x_2}\right)n_1 - \left(\frac{\partial M_2(w)}{\partial x_2} + \frac{\partial H_{12}(w)}{\partial x_1}\right)n_2\right)v\,\mathrm{d}s. \end{aligned} \tag{15}$$

On the other hand, if we denote $P = (M_2 - M_1)n_1n_2 + H_{12}(n_1^2 - n_2^2)$, we have [37]

$$\int_{\partial\Omega} P\frac{\partial v}{\partial s}\mathrm{d}s = -\int_{\partial\Omega}\frac{\partial P}{\partial s}v\,\mathrm{d}s, \tag{16}$$

and replacing Eq. (16) in Eq. (15) it leads to

$$B(w,v) = -\int_\Omega \left(M_1(w)\frac{\partial^2 v}{\partial x_1^2} + M_2(w)\frac{\partial^2 v}{\partial x_2^2} + 2H_{12}(w)\frac{\partial^2 v}{\partial x_1 \partial x_2} \right) \mathrm{d}x$$
$$+ \int_{\partial\Omega} \left[M_1(w)n_1^2 + M_2(w)n_2^2 + 2H_{12}(w)n_1 n_2 \right] \frac{\partial v}{\partial n} \mathrm{d}s$$
$$+ \int_{\partial\Omega} \left[-\left(\frac{\partial M_1(w)}{\partial x_1} + \frac{\partial H_{12}(w)}{\partial x_2} \right) n_1 - \left(\frac{\partial M_2(w)}{\partial x_2} + \frac{\partial H_{12}(w)}{\partial x_1} \right) n_2 \right] v \, \mathrm{d}s$$
$$- \int_{\partial\Omega} \frac{\partial}{\partial s} \left[(M_2(w) - M_1(w)) n_1 n_2 + H_{12}(w) \left(n_1^2 - n_2^2 \right) \right] v \, \mathrm{d}s. \tag{17}$$

Finally, from Eqs. (10) and (11) we obtain

$$B(w,v) = -\int_\Omega \left(M_1(w)\frac{\partial^2 v}{\partial x_1^2} + M_2(w)\frac{\partial^2 v}{\partial x_2^2} + 2H_{12}(w)\frac{\partial^2 v}{\partial x_1 \partial x_2} \right) \mathrm{d}x + \int_{\partial\Omega} c_\mathrm{R}(s) \frac{\partial w}{\partial n}\frac{\partial v}{\partial n} \mathrm{d}s + \int_{\partial\Omega} c_\mathrm{T}(s) w v \, \mathrm{d}s. \tag{18}$$

The double integral in (18) constitutes the bilinear form $A(w,v)$ associated with the differential operator A defined in (3) and the curvilinear integrals constitute the boundary bilinear form $a(w,v)$. The equality (13) now assumes the form

$$B(w,v) = \int_\Omega q v \, \mathrm{d}x = (q,v)_{L^2(\Omega)}.$$

Now it is possible to weaken the assumptions. Let $q(x) \in L^2(\Omega)$, $D_{ij}(x)$, $c_\mathrm{R}(s)$, $c_\mathrm{T}(s)$ bounded measurable functions in Ω, and $h \in C(\overline{\Omega})$. A function $w \in H^2(\Omega)$ is called a weak solution of the boundary value problem (2), (10) and (11) if

$$\text{(i) } w \in H^2(\Omega), \tag{19}$$
$$\text{(ii) } B(w,v) = (q,v)_{L^2(\Omega)}, \quad \forall v \in V. \tag{20}$$

2.2. Non-smooth boundary

Now let us assume that the boundary $\partial\Omega$ consists of a finite number of smooth curves and therefore has at most a finite number of corner points. To be definite let us suppose that the four points P_i, $i = 1,\ldots,4$, divide the boundary in the disjoint parts $\partial\Omega_i$, $i = 1,\ldots,4$, represented parametrically by the functions $\gamma^{(i)}$, $i = 1,\ldots,4$ respectively, as it is shown in Fig. 2. In this case Eq. (16) is not valid. The functions $n_1(s)$, $n_2(s)$ are not continuous, and we would get additional terms in the corner points:

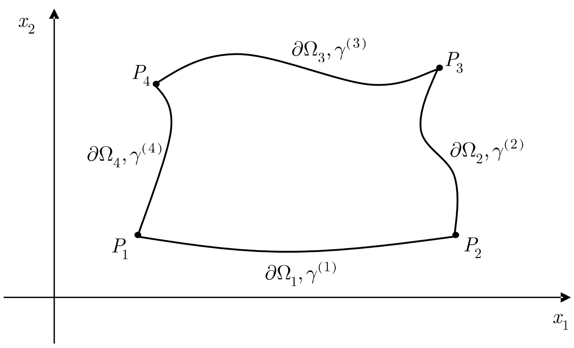


Fig. 2. Anisotropic plate with corner points.

$$\int_{\partial\Omega} P\frac{\partial v}{\partial s}\mathrm{d}s = \sum_{i=1}^{4}\int_{\partial\Omega_i} P\frac{\partial v}{\partial s}\mathrm{d}s$$
$$= -\int_{\partial\Omega}\frac{\partial P}{\partial s}v\,\mathrm{d}s + (Pv)(\gamma_1^{(1)}(s),\gamma_2^{(1)}(s))\Big|_{l_0}^{l_1} + (Pv)(\gamma_1^{(2)}(s),\gamma_2^{(2)}(s))\Big|_{l_1}^{l_2} + (Pv)(\gamma_1^{(3)}(s),\gamma_2^{(3)}(s))\Big|_{l_2}^{l_3}$$
$$+ (Pv)(\gamma_1^{(4)}(s),\gamma_2^{(4)}(s))\Big|_{l_3}^{l_4}, \tag{21}$$

with $l_0 = 0$, $l_i = l(\partial\Omega_1 \cup \ldots \cup \partial\ \Omega_i)$, $i = 1,\ldots,4$. Replacing Eq. (21) in Eq. (15) and taking into account the boundary conditions (10) and (11) we get

$$B(w,v) = -\int_{\Omega}\left(M_1(w)\frac{\partial^2 v}{\partial x_1^2} + M_2(w)\frac{\partial^2 v}{\partial x_2^2} + 2H_{12}(w)\frac{\partial^2 v}{\partial x_1\partial x_2}\right)\mathrm{d}x + \int_{\partial\Omega} c_{\mathrm{R}}(s)\frac{\partial w}{\partial n}\frac{\partial v}{\partial n}\mathrm{d}s$$
$$+ \int_{\partial\Omega} c_{\mathrm{T}}(s)wv\,\mathrm{d}s + \sum_{i=1}^{4}(Pv)(\gamma_1^{(i)}(s),\gamma_2^{(i)}(s))\Big|_{l_{i-1}}^{l_i}. \tag{22}$$

3. The continuity and V – ellipticity of the bilinear form *B*

As stated above, from Eq. (18) we have $B(w,v) = A(w,v) + a(w,v)$, where

$$A(w,v) = -\int_{\Omega}\left(M_1(w)\frac{\partial^2 v}{\partial x_1^2} + M_2(w)\frac{\partial^2 v}{\partial x_2^2} + 2H_{12}(w)\frac{\partial^2 v}{\partial x_1\partial x_2}\right)\mathrm{d}x, \tag{23}$$

$$a(w,v) = \int_{\partial\Omega} c_{\mathrm{R}}(s)\frac{\partial w}{\partial n}\frac{\partial v}{\partial n}\mathrm{d}s + \int_{\partial\Omega} c_{\mathrm{T}}(s)wv\,\mathrm{d}s. \tag{24}$$

If we use the notation introduced in Eq. (1) and define the coefficients

$$K_{ij}(x) = D_{ij}(x),\quad i,j = 1,2,\qquad K_{21} = K_{12},\quad K_{i3}(x) = 2D_{i6}(x),\quad i = 1,2,$$
$$K_{3j}(x) = 2D_{j6}(x),\quad j = 1,2,\qquad K_{33}(x) = 4D_{66}(x),$$

and the multi-index vectors $\alpha_1 = (2,0)$, $\alpha_2 = (0,2)$, $\alpha_3 = (1,1)$, Eq. (23) is reduced to

$$A(w,v) = \int_{\Omega}\left[\sum_{i=1}^{3}\left(\sum_{j=1}^{3}K_{ij}(x)D^{\alpha_j}w(x)\right)D^{\alpha_i}v(x)\right]\mathrm{d}x.$$

Consequently we have

$$|A(w,v)| \leqslant \int_{\Omega}\left[\sum_{i=1}^{3}\left(\sum_{j=1}^{3}|K_{ij}(x)||D^{\alpha_j}w(x)|\right)|D^{\alpha_i}v(x)|\right]\mathrm{d}x$$
$$\leqslant K\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{\Omega}|D^{\alpha_j}w(x)||D^{\alpha_i}v(x)|\mathrm{d}x$$
$$\leqslant K\sum_{i=1}^{3}\sum_{j=1}^{3}\left[\left(\int_{\Omega}|D^{\alpha_j}w(x)|^2\mathrm{d}x\right)^{1/2}\left(\int_{\Omega}|D^{\alpha_i}v(x)|^2\mathrm{d}x\right)^{1/2}\right]$$
$$\leqslant K\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\|w\|_{H^2(\Omega)}\|v\|_{H^2(\Omega)}\right),$$

where $K = \max_{1\leqslant i,j\leqslant 3}(\|K_{ij}\|_{L^\infty(\Omega)})$. From this inequality there follows the existence of a constant $C_1 > 0$, such that

$$|A(w,v)| \leqslant C_1\|w\|_{H^2(\Omega)}\|v\|_{H^2(\Omega)},\quad \forall w,v \in H^2(\Omega). \tag{25}$$

From (24) we have

$$|a(w,v)| \leqslant \int_{\partial\Omega} \left| c_{\mathrm{R}}(s) \frac{\partial w}{\partial n} \frac{\partial v}{\partial n} \right| \mathrm{d}s + \int_{\partial\Omega} |c_{\mathrm{T}}(s) w v| \mathrm{d}s \leqslant c_{R_0} \int_{\partial\Omega} \left| \frac{\partial w}{\partial n} \frac{\partial v}{\partial n} \right| \mathrm{d}s + c_{T_0} \int_{\partial\Omega} |wv| \mathrm{d}s, \tag{26}$$

where $c_{R_0} = \|c_{\mathrm{R}}\|_{L^\infty(\partial\Omega)}$ and $c_{T_0} = \|c_{\mathrm{T}}\|_{L^\infty(\partial\Omega)}$.

Since $w, v \in H^2(\Omega)$, then $\frac{\partial w}{\partial x_i}, \frac{\partial v}{\partial x_i} \in H^1(\Omega), i = 1, 2$, and consequently these functions have traces which belong to $L^2(\partial\Omega)$. Moreover, from the trace theorem [38,39] there exist a constant $C_2 > 0$ such that

$$\left\| \frac{\partial w}{\partial n} \right\|_{L^2(\partial\Omega)} \leqslant C_2 \|w\|_{H^2(\Omega)},$$

$$\left\| \frac{\partial v}{\partial n} \right\|_{L^2(\partial\Omega)} \leqslant C_2 \|v\|_{H^2(\Omega)}.$$

Then we have

$$\int_{\partial\Omega} \left| \frac{\partial w}{\partial n} \right| \left| \frac{\partial v}{\partial n} \right| \mathrm{d}s \leqslant \left(\int_{\partial\Omega} \left| \frac{\partial w}{\partial n} \right|^2 \mathrm{d}s \right)^{1/2} \left(\int_{\partial\Omega} \left| \frac{\partial v}{\partial n} \right|^2 \mathrm{d}s \right)^{1/2} \leqslant C_2^2 \|w\|_{H^2(\Omega)} \|v\|_{H^2(\Omega)}.$$

Besides, there exists a constant C_3, such that

$$\|w\|_{L^2(\partial\Omega)} \leqslant C_3 \|w\|_{H^1(\Omega)} \leqslant C_3 \|w\|_{H^2(\Omega)},$$

$$\|v\|_{L^2(\partial\Omega)} \leqslant C_3 \|v\|_{H^1(\Omega)} \leqslant C_3 \|v\|_{H^2(\Omega)}.$$

In consequence, we have

$$\int_{\partial\Omega} |w||v| \mathrm{d}s \leqslant \left(\int_{\partial\Omega} |w|^2 \mathrm{d}s \right)^{1/2} \left(\int_{\partial\Omega} |v|^2 \mathrm{d}s \right)^{1/2} \leqslant C_3^2 \|w\|_{H^2(\Omega)} \|v\|_{H^2(\Omega)}.$$

From the replacement of these estimates in Eq. (26) there follows

$$|a(w,v)| \leqslant C_4 \|w\|_{H^2(G)} \|v\|_{H^2(G)}, \forall v, w \in H^2(\Omega), \tag{27}$$

where $C_4 = \max\{c_{R_0} C_2^2, c_{T_0} C_3^2\}$.

From Eqs. (25) and (27) we have

$$|B(w,v)| \leqslant C_5 \|w\|_{H^2(\Omega)} \|v\|_{H^2(\Omega)}, \quad \forall v, w \in H^2(\Omega), \tag{28}$$

where $C_5 = C_1 + C_4$.

The inequality (28) implies that $B(w,v)$ is continuous on the product space $H^2(\Omega) \times H^2(\Omega)$.

Now it is necessary to prove that the bilinear form $B(w,v)$ is V – elliptic, in order to demonstrate that the problem under consideration has exactly one weak solution w [38,39]. If we replace $w = v$, in Eqs. (23) and (24) we obtain

$$\begin{aligned} B(v,v) &= A(v,v) + a(v,v) \\ &= \int_\Omega \left(D_{11} \left(\frac{\partial^2 v}{\partial x_1^2} \right)^2 + 2D_{12} \frac{\partial^2 v}{\partial x_1^2} \frac{\partial^2 v}{\partial x_2^2} + D_{22} \left(\frac{\partial^2 v}{\partial x_2^2} \right)^2 + 4D_{16} \frac{\partial^2 v}{\partial x_1^2} \frac{\partial^2 v}{\partial x_1 \partial x_2} \right. \\ &\qquad \left. + 4D_{26} \frac{\partial^2 v}{\partial x_2^2} \frac{\partial^2 v}{\partial x_1 \partial x_2} + 4D_{66} \left(\frac{\partial^2 v}{\partial x_1 \partial x_2} \right)^2 \right) \mathrm{d}x + \int_{\partial\Omega} c_{\mathrm{R}}(s) \left(\frac{\partial v}{\partial n} \right)^2 \mathrm{d}s + \int_{\partial\Omega} c_{\mathrm{T}}(s) v^2 \mathrm{d}s. \end{aligned} \tag{29}$$

It is known from the theory of elasticity that the quadratic form which represents twice the potential energy density of an elastic body is positive definite, i.e., there exists a constant $C_6 > 0$ so that

$$2W(\mathbf{u}) = \sum_{i,k,l,m=1}^{3} c_{iklm} \varepsilon_{ik}(\mathbf{u}) \varepsilon_{lm}(\mathbf{u}) \geqslant C_6 \left(\sum_{i,k=1}^{3} \varepsilon_{ik}^2(\mathbf{u}) \right), \tag{30}$$

where c_{iklm} are the stiffness matrix coefficients, ε_{ik} the strains and $\mathbf{u}$ is the displacement vector in terms of a x_1, x_2, x_3 co-ordinate system. Under the assumptions of the considered anisotropic plate theory, the rigidities are given by (4). Then, the integration in the inequality (30) with respect to x_1, x_2 and x_3, leads to

$$\int_\Omega \left[\int_{-h(x)/2}^{h(x)/2}\left(\overline{Q}_{11}\left(\frac{\partial^2 w(x)}{\partial x_1^2}\right)^2 + 2\overline{Q}_{12}\frac{\partial^2 w(x)}{\partial x_1^2}\frac{\partial^2 w(x)}{\partial x_2^2} + 4\overline{Q}_{16}\frac{\partial^2 w(x)}{\partial x_1^2}\frac{\partial^2 w(x)}{\partial x_1 \partial x_2} + \overline{Q}_{22}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 + 4\overline{Q}_{26}\frac{\partial^2 w(x)}{\partial x_2^2}\frac{\partial^2 w(x)}{\partial x_1 \partial x_2} + 4\overline{Q}_{66}\left(\frac{\partial^2 w(x)}{\partial x_1 \partial x_2}\right)^2\right)x_3^2 \mathrm{d}x_3\right]\mathrm{d}x \geqslant C_6 \int_\Omega \frac{h^3(x)}{12}\left[\left(\frac{\partial^2 w(x)}{\partial x_1^2}\right)^2 + 2\left(\frac{\partial^2 w(x)}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2 w(x)}{\partial x_2^2}\right)^2\right]\mathrm{d}x.$$

Then

$$\int_\Omega \left[\int_{-h(x)/2}^{h(x)/2}\left(\overline{Q}_{11}\left(\frac{\partial^2 w(x)}{\partial x_1^2}\right)^2 + 2\overline{Q}_{12}\frac{\partial^2 w(x)}{\partial x_1^2}\frac{\partial^2 w(x)}{\partial x_2^2} + 4\overline{Q}_{16}\frac{\partial^2 w(x)}{\partial x_1^2}\frac{\partial^2 w(x)}{\partial x_1 \partial x_2} + \overline{Q}_{22}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 + 4\overline{Q}_{26}\frac{\partial^2 w(x)}{\partial x_2^2}\frac{\partial^2 w(x)}{\partial x_1 \partial x_2} + 4\overline{Q}_{66}\left(\frac{\partial^2 w(x)}{\partial x_1 \partial x_2}\right)^2\right)x_3^2 \mathrm{d}x_3\right]\mathrm{d}x \geqslant C_7 \int_\Omega \left[\left(\frac{\partial^2 w(x)}{\partial x_1^2}\right)^2 + 2\left(\frac{\partial^2 w(x)}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2 w(x)}{\partial x_2^2}\right)^2\right]\mathrm{d}x, \tag{31}$$

where $C_7 = \frac{C_6}{12}\min_{x\in\overline{\Omega}}\{h(x)^3\}$. When $c_\mathrm{T}(s) \geqslant c_\mathrm{M} > 0$, with c_M constant, from Eqs. (29) and (31) with $w = v$ we have

$$B(v,v) \geqslant C_7 \int_\Omega \left(\left(\frac{\partial^2 v}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 v}{\partial x_1 \partial x_2}\right)^2 + \left(\frac{\partial^2 v}{\partial x_2^2}\right)^2\right)\mathrm{d}x + \int_{\partial\Omega} c_\mathrm{T}(s) v^2 \mathrm{d}s \geqslant C_8\left[\sum_{|\alpha|=2}\int_\Omega (D^\alpha v)^2 \mathrm{d}x + \int_{\partial\Omega} v^2 \mathrm{d}s\right], \tag{32}$$

where $C_8 = \min\{C_7, c_\mathrm{M}\}$.

By applying Friedrichs inequality in the case of a domain $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary $\partial\Omega$ [38,39]

$$\|u\|^2_{H^2(\Omega)} \leqslant C_9\left(\sum_{|\alpha|=2}\int_\Omega (D^\alpha u)^2 \mathrm{d}x + \int_{\partial\Omega} u^2(s)\mathrm{d}s\right), \quad C_9 > 0, \ \forall u \in H^2(\Omega),$$

we obtain

$$B(v,v) \geqslant \frac{C_8}{C_9}\|v\|^2_{H^2(\Omega)}, \quad \forall v \in V. \tag{33}$$

The inequality (33) implies that $B(w,v)$ is $V-$elliptic. Since it has been demonstrated that $B(w,v)$ is continuous on the product space $H^2(\Omega) \times H^2(\Omega)$ and it is $V-$elliptic, the boundary value problem under consideration has exactly one weak solution. In the case of symmetrically laminated plates the coefficients D_{ij} are given by Eq. (5) and the demonstration of the continuity and $V-$ellipticity of $B(w,v)$ is totally analogue.

In the case of non-smooth boundary, Eq. (22) of the bilinear form includes the terms $\sum_{i=1}^4 (Pv)(\gamma_1^{(i)}(s), \gamma_2^{(i)}(s))\Big|_{l_{i-1}}^{l_i}$, which do not allow to use the above demonstration to prove the $V-$ellipticity of the $B(w,v)$ form. Nevertheless, the application of the techniques of the variational calculus leads to the conclusion that when the boundary is simply supported (with or without rotational restraints) the function w equals zero along the boundary. In consequence v also equals zero and

$$\sum_{i=1}^4 (Pv)\left(\gamma_1^{(i)}(s), \gamma_2^{(i)}(s)\right)\Big|_{l_{i-1}}^{l_i} = 0. \tag{34}$$

When the boundary is free or elastically restrained against translation in the neighbourhood of the corner points it is $v \neq 0$ and additional boundary conditions exist [37]. These are given by

$$\begin{aligned}
&P(\gamma_1^{(1)}(l_1), \gamma_2^{(1)}(l_1)) - P(\gamma_1^{(2)}(l_1), \gamma_2^{(2)}(l_1)) = 0,\\
&P(\gamma_1^{(2)}(l_2), \gamma_2^{(2)}(l_2)) - P(\gamma_1^{(3)}(l_2), \gamma_2^{(3)}(l_2)) = 0,\\
&P(\gamma_1^{(3)}(l_3), \gamma_2^{(3)}(l_3)) - P(\gamma_1^{(4)}(l_3), \gamma_2^{(4)}(l_3)) = 0,\\
&P(\gamma_1^{(1)}(l_0), \gamma_2^{(1)}(l_0)) - P(\gamma_1^{(4)}(l_4), \gamma_2^{(4)}(l_4)) = 0.
\end{aligned} \tag{35}$$

The use of Eq. (35) leads again to Eq. (34).

As a consequence of Eq. (34) the proof of the V – ellipticity of $B(w, v)$ is the same as in the case of smooth boundary.

4. The boundary and eigenvalue problem

Free transverse vibrations of the anisotropic plate described above are governed by the corresponding boundary conditions and Eq. (6). In the case of normal modes of vibrations we take $u(x,t) = w(x)\cos\omega t$, consequently Eq. (6) is reduced to

$$A(w(x)) - \rho h(x)\omega^2 w(x) = 0, \tag{36}$$

where ω is the radian natural frequency.

Let us consider the eigenvalue problem given by Eq. (36) and the boundary conditions (10) and (11) with $u(x,t)$ replaced by $w(x)$. We rewrite it as the problem of finding a number λ and a function w such that

(i) $w \in H^2(\Omega), w \neq 0,$
(ii) $B(w, v) - \lambda(w, v) = 0, \forall v \in V,$

where V is the space of functions which satisfy the corresponding homogeneous stable boundary conditions, $\lambda = \rho h_0 \omega^2$, $(w, v) = \int_\Omega f(x)w(x)v(x)\mathrm{d}x$ and the thickness of the plate is given by $h(x) = h_0 f(x)$, where V is the space of elements which satisfy the corresponding stable homogeneous boundary conditions, previously defined. Since the bilinear form $B(w,v)$ is symmetric (as it can be seen from Eqs. (23) and (24)), continuous and V – elliptic, it has a countable set of eigenvalues which are given by [38,39]

$$\lambda_1 = \min\left\{\frac{B(v,v)}{(v,v)}, v \in V, v \neq 0\right\},$$
$$\lambda_n = \min\left\{\frac{B(v,v)}{(v,v)} v \in V, v \neq 0, (v, v_1) = 0, \ldots, (v, v_{n-1}) = 0\right\}, \quad n = 2, 3, \ldots$$

Let us introduce a new inner product $((\cdot,\cdot))$ in space V, given by $((w,v)) = B(w,v)$, $\forall w,v \in V$. If the sequence $\{v_i(x)\}$ is a base in the space $(V,((\cdot,\cdot)))$, the Ritz method leads to the equation

$$\begin{vmatrix} ((v_1, v_1)) - \lambda(v_1, v_1) & \ldots & ((v_1, v_n)) - \lambda(v_1, v_n) \\ ((v_n, v_1)) - \lambda(v_n, v_1) & \ldots & ((v_n, v_n)) - \lambda(v_n, v_n) \end{vmatrix} = 0. \tag{37}$$

Approximate eigenvalues can be obtained from this equation using adequate functions $v_i(x)$.

5. Analytical approximate solution

5.1. Strain and kinetic energies

The maximum strain energy of the anisotropic plate described above, can be expressed in rectangular coordinates as

$$U_{\max} = \frac{1}{2}\int_\Omega \left[D_{11}\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} + D_{22}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 + 4D_{16}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_1 \partial x_2} + 4D_{26}\frac{\partial^2 w}{\partial x_2^2}\frac{\partial^2 w}{\partial x_1 \partial x_2} + 4D_{66}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2\right]\mathrm{d}x, \tag{38}$$

where the integration is carried out over the entire plate domain Ω and the coefficients D_{ij} are given by Eq. (4) and/or Eq. (5).

The maximum kinetic energy for free transverse vibrations of the plate is given by

$$T_{\max} = \frac{\rho\omega^2}{2}\int_{\Omega} hw^2\,\mathrm{d}x. \tag{39}$$

Finally, the maximum potential energy due to the rotational and translational restraints on the boundary are respectively given by

$$U_{R,\max} = \frac{1}{2}\int_{\partial\Omega} c_{\mathrm{R}}\left(\frac{\partial w}{\partial n}\right)^2 \mathrm{d}s = \frac{1}{2}\int_{\partial\Omega} c_{\mathrm{R}}\left(\frac{\partial w}{\partial x_1}n_1 + \frac{\partial w}{\partial x_2}n_2\right)^2 \mathrm{d}s, \tag{40}$$

$$U_{T,\max} = \frac{1}{2}\int_{\partial\Omega} c_{\mathrm{T}}w^2\,\mathrm{d}s. \tag{41}$$

5.2. Smooth boundary

Let us consider an elliptical anisotropic plate as shown in Fig. 3a. In the case of composite material, the fibre angle is indicated by β measured from the x_1 axis to the fibre orientation. The considered laminate is of uniform thickness h and, in general, it is made up of a number of layers each consisting of unidirectional fibre reinforced composite material, and all laminae having equal thicknesses.

Let us introduce the following co-ordinates $x = x_1/a$, $y = x_2/b$, where a and b are the semi-major and minor axes of the ellipse (Fig. 3a). After this co-ordinate transformation the two dimensional plate domain Ω is transformed in

$$\Omega^* = \left\{(x,y), -1 \leqslant x \leqslant 1, -\sqrt{1-x^2} \leqslant y \leqslant \sqrt{1-x^2}\right\}.$$

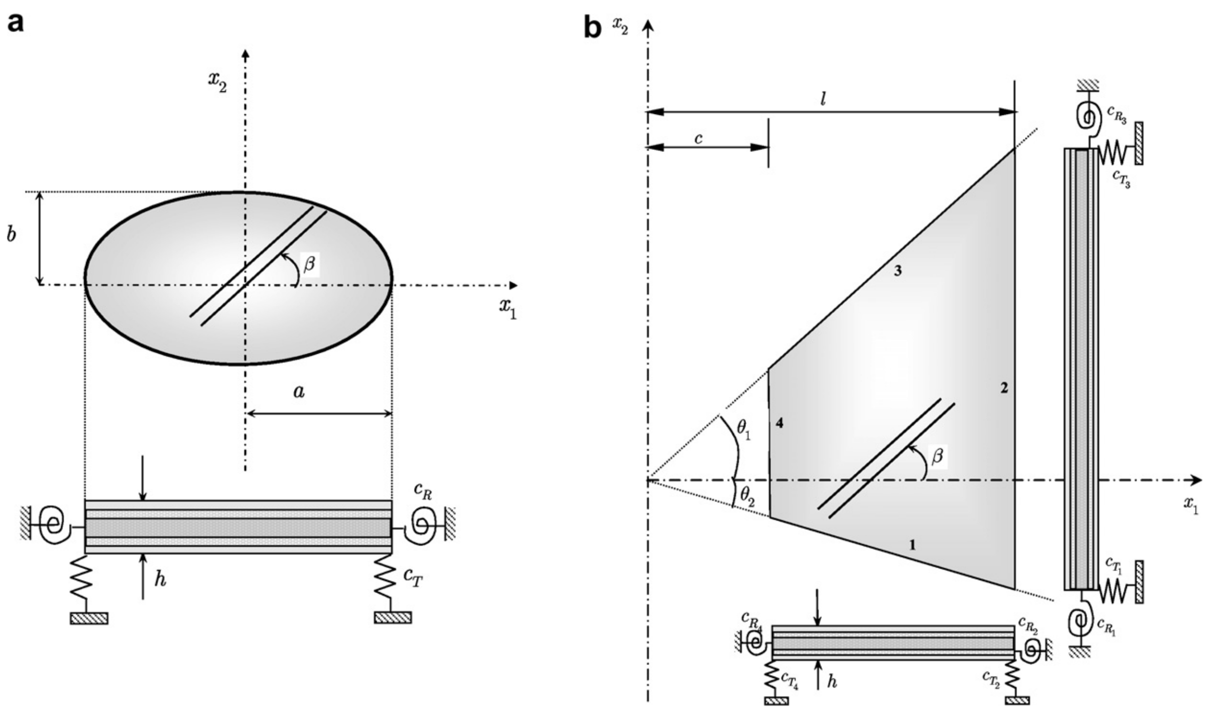


Fig. 3. General description of the plate geometries (a) elliptical plate (b) trapezoidal plate.

The plate deflection is represented by the following approximate function

$$w(x,y) = \sum_{i=1}^{M}\sum_{j=1}^{N} c_{ij}x^{i-1}y^{j-1}(x^2+y^2-1)^p, \quad (42)$$

Table 1
Convergence of frequency parameters $\omega a^2\sqrt{\rho h/D_0}$, for symmetrically laminated graphite-epoxy elliptical plates with aspect ratio $a/b = 2$ and stacking sequence $(-\beta, \beta, \beta, -\beta)$

β		Mode sequence number					
		1	2	3	4	5	6
Clamped							
30°	Present (4 × 4)	11.6408	17.9079	28.6163	29.0343	39.2058	43.9370
	Present (5 × 5)	11.6379	17.9022	27.4212	29.0330	37.0721	41.9376
	Present (6 × 6)	11.6379	17.8889	27.3915	28.9727	37.0478	39.7933
	Present (7 × 7)	11.6379	17.8888	27.3371	28.9726	36.9160	39.6787
	Present (8 × 8)	11.6379	17.8886	27.3366	28.9711	36.9153	39.4991
	Present (9 × 9)	11.6379	17.8885	27.3349	28.9711	36.9111	39.4955
	Present (10 × 10)	11.6379	17.8884	27.3348	28.9710	36.9110	39.4869
60°	Present (4 × 4)	19.9234	24.8471	35.8877	47.2497	53.0701	62.3723
	Present (5 × 5)	19.8474	24.8390	31.1138	46.8896	52.7422	60.7105
	Present (6 × 6)	19.8473	24.5531	31.0905	38.8169	52.7105	60.5350
	Present (7 × 7)	19.8435	24.5523	30.38099	38.7610	48.1139	52.6762
	Present (8 × 8)	19.8435	24.5330	30.3781	37.3171	48.0017	52.6754
	Present (9 × 9)	19.8434	24.5330	30.3141	37.3090	45.4093	52.6729
	Present (10 × 10)	19.8434	24.5324	30.3139	37.1388	45.3900	52.6710
Simply supported							
30°	Present (4 × 4)	5.6325	10.9589	19.3809	23.2373	34.3465	45.3799
	Present (5 × 5)	5.6289	10.9344	19.3749	19.5843	26.9199	35.4697
	Present (6 × 6)	5.6289	10.9056	19.1467	19.4887	26.7986	31.1930
	Present (7 × 7)	5.6288	10.9052	19.1463	19.3198	26.3396	30.8435
	Present (8 × 8)	5.6288	10.9047	19.1414	19.3171	26.3364	30.2541
	Present (9 × 9)	5.6288	10.9047	19.1414	19.3128	26.3230	30.2370
	Present (10 × 10)	5.6288	10.9047	19.1413	19.3128	26.3229	30.2114
60°	Present (4 × 4)	9.5079	13.9353	26.6385	35.2172	41.3175	45.9420
	Present (5 × 5)	9.4355	13.8964	19.8031	34.8089	39.9247	43.9391
	Present (6 × 6)	9.4354	13.6009	19.7100	27.3221	34.6310	42.2869
	Present (7 × 7)	9.4335	13.6004	18.8879	27.1380	34.5677	36.6845
	Present (8 × 8)	9.4334	13.5860	18.8846	25.3029	34.5628	36.4022
	Present (9 × 9)	9.4334	13.5859	18.8230	25.2908	32.9025	34.5592
	Present (10 × 10)	9.4334	13.5857	18.8226	25.0970	32.8717	34.5575
Free							
30°	Present (4 × 4)	2.4484	5.3011	8.7558	13.5754	13.9454	24.5528
	Present (5 × 5)	2.3696	5.1668	8.5328	11.2504	11.9576	19.2085
	Present (6 × 6)	2.3695	5.1498	7.7936	11.2128	11.8892	17.5239
	Present (7 × 7)	2.3661	5.1359	7.7804	11.0296	11.7478	16.0857
	Present (8 × 8)	2.3661	5.1359	7.7212	11.0294	11.7422	15.9459
	Present (9 × 9)	2.3661	5.1359	7.7211	11.0230	11.7413	15.7152
	Present (10 × 10)	2.3661	5.1359	7.7209	11.0230	11.7413	15.7127
60°	Present (4 × 4)	1.2903	4.1599	5.3008	13.4718	24.1362	25.1998
	Present (5 × 5)	1.2044	4.1576	5.2822	9.4773	10.7532	20.6047
	Present (6 × 6)	1.2044	3.5005	5.2538	9.4734	10.7038	17.1869
	Present (7 × 7)	1.2015	3.5003	5.2461	7.1511	10.4751	17.0922
	Present (8 × 8)	1.2015	3.4609	5.2459	7.1506	10.4448	12.3206
	Present (9 × 9)	1.2015	3.4609	5.2454	6.9443	10.4384	12.3199
	Present (10 × 10)	1.2015	3.4607	5.2454	6.9442	10.4267	11.6581

where c_{ij} are unknown coefficients and the parameter p depends on the boundary conditions; $p = 2$ is adopted when it is rigidly clamped, $p = 1$ when the plate is simply supported and $p = 0$ when it is free or elastically restrained along the boundary.

The corresponding boundary conditions are given by Eqs. (10) and (11). In this study the spring coefficients $c_R(s)$ and $c_T(s)$ have been considered constant along the boundary, and the following non dimensional parameters have been defined $R = c_R a/D_{11}$ and $T = c_T a^3/D_{11}$.

The Ritz method is used to generate the following non dimensional frequency coefficients $\omega a^2\sqrt{\rho h/D_{11}}$ and $\omega a^2\sqrt{\rho h/D_0}$ where D_0 is the reference flexural rigidity $D_0 = E_L h^3/12(1 - \nu_{LT}\nu_{TL})$. The subscript L and T represent the directions parallel with and perpendicular to the fibre direction.

5.3. Non-smooth boundary

Let us consider a composite plate with a general trapezoidal planform as shown in Fig. 3b. The angles of the plate sides θ_1 and θ_2 are measured from the x_1 axis to sides **3** and **1** respectively and are defined negative when measured clock-wise.

Let us introduce non-orthogonal right triangular co-ordinates x, y. They are related to the x_1, x_2 co-ordinates by

$$x = \frac{x_1}{l}, \quad y = \frac{x_2}{x_1}\cot\theta_1. \tag{43}$$

After this co-ordinate transformation the two-dimensional plate domain Ω is transformed in $\hat{\Omega} = \{(x, y), c/l \leqslant x \leqslant 1, \tan\theta_2 \cot\theta_1 \leqslant y \leqslant 1\}$ (see Fig. 3b). The plate deflection is represented by a set of beam characteristic orthogonal polynomials $p_i(x)$ and $q_j(y)$ as

Table 2
Frequency parameters $\omega a^2\sqrt{\rho h/D_{11}}$, for circular plates of generalized anisotropy with edges elastically restrained against rotation and translation

Material properties	$R = c_R a/D_{11}$	$T = c_T a^3/D_{11}$		Mode sequence number 1	2	3	4
$D_{22}/D_{11} = 1/2$	0	0	Present	3.1925	4.8126	8.1380	9.2047
$D_{12}/D_{11} = 3/10$		10		3.0585	5.2729	5.9155	7.6486
$D_{66}/D_{11} = 1/2$		100		4.2034	8.3781	12.3652	13.8972
$D_{16}/D_{11} = 1/3$		1000		4.3978	9.0524	14.5745	15.7685
$D_{26}/D_{11} = 1/3$		10000		4.4187	9.1279	14.8294	15.9881
		∞		4.4562	9.3034	15.4392	16.4853
			Bambill et al. [42]	4.4802	–	–	–
	10	∞		7.4263	12.9062	18.8495	20.5157
	100			8.2738	13.8357	20.5600	21.5190
	1000			8.4004	13.9645	20.8714	21.6852
	10000			8.4224	13.9784	20.9408	21.7432
	∞			9.5694	15.3527	23.5055	23.5190
			Bambill et al. [42]	9.6242	–	–	–
$D_{22}/D_{11} = 1/4$	0	0	Present	3.3147	5.1520	6.2605	9.4616
$D_{12}/D_{11} = 1/3$		10		2.8657	4.3388	5.8933	6.2788
$D_{66}/D_{11} = 1/2$		100		3.9399	7.0160	11.1083	11.8202
$D_{16}/D_{11} = 1/5$		1000		4.1343	7.7675	12.7208	13.5073
$D_{26}/D_{11} = 1/3$		10000		4.1559	7.8645	12.9906	13.7210
		∞		4.2344	8.7664	14.1852	14.6733
			Bambill et al. [42]	4.2759	–	–	–
	10	∞		7.0787	11.9949	17.6236	18.0361
	100			7.8327	12.8896	18.5891	19.4631
	1000			7.9409	13.0213	18.7475	19.7206
	10000			7.9580	13.0400	18.7846	19.7906
	∞			9.0877	14.3947	20.6129	22.3705
			Bambill et al. [42]	9.1466	–	–	–

Table 3
Convergence of frequency parameters $\omega l^2/h\sqrt{\rho/E_L}$, for symmetrically laminated E-glass-epoxy trapezoidal plates, with $\theta_1 = 36.87°$, $\theta_2 = 0°$, $c/l = 0.25$ and stacking sequence $(-\beta, \beta, \beta, -\beta)$

β		Mode sequence number					
		1	2	3	4	5	6
CCCC							
30°	4 × 4	30.4161	49.7905	65.2323	80.2623	97.2895	119.008
	5 × 5	30.3851	49.1289	64.0410	76.2810	90.9063	114.779
	6 × 6	30.3762	49.0148	63.6996	75.7062	88.2658	108.291
	7 × 7	30.3760	48.9845	63.5738	75.3793	87.6232	107.478
	8 × 8	30.3759	48.9838	63.5673	75.3684	87.5624	107.399
	9 × 9	30.3759	48.9811	63.5198	75.3175	87.4093	107.001
	10 × 10	30.3759	48.9748	63.5198	75.3069	87.3340	106.709
60°	4 × 4	31.6269	50.7935	68.8166	78.7007	100.278	118.689
	5 × 5	31.5819	49.6738	67.3430	76.5984	95.0772	112.321
	6 × 6	31.5720	49.6118	66.4925	75.2268	91.7392	105.703
	7 × 7	31.5707	49.5554	66.3866	75.0494	90.4129	103.744
	8 × 8	31.5707	49.5554	66.3866	75.0495	90.4129	103.744
	9 × 9	31.5706	49.5489	66.3697	75.0487	89.4123	102.060
	10 × 10	31.5705	49.5461	66.3632	75.0433	88.5436	100.342
SSSS							
30°	4 × 4	16.1454	31.9497	45.6298	70.2315	89.5856	108.955
	5 × 5	16.0608	31.0643	43.3937	54.4097	69.8263	88.9880
	6 × 6	16.0434	30.6321	42.8095	53.6047	65.5206	83.0780
	7 × 7	16.0406	30.5619	42.5142	52.3852	63.8513	81.8971
	8 × 8	16.0377	30.5467	42.4541	52.2622	63.0467	79.6361
	9 × 9	16.0345	30.5316	42.4325	52.2006	62.8474	79.4403
	10 × 10	16.0345	30.5316	42.4325	52.2006	62.8474	79.4403
	Lim et al. [43]	16.0416	30.569	42.469	52.252	62.889	79.310
60°	4 × 4	16.5852	32.0139	47.9376	67.6119	85.8515	11.5839
	5 × 5	16.5391	30.6275	45.2772	53.1895	71.8791	94.7399
	6 × 6	16.5167	30.3514	44.4224	52.0466	66.7259	79.8927
	7 × 7	16.5155	30.2527	43.9883	51.1379	65.3315	78.8920
	8 × 8	16.5145	30.2462	43.8822	50.9905	64.1001	76.1087
	9 × 9	16.5138	30.2393	43.8631	50.9387	63.8823	75.8000
	10 × 10	16.5138	30.2393	43.8631	50.9387	63.8823	75.8000
	Lim et al. [43]	16.5145	30.249	43.876	50.960	63.844	75.621
FFFF							
30°	4 × 4	7.1164	11.3566	17.4943	21.6055	38.2086	44.2072
	5 × 5	6.5805	10.3906	16.6420	19.6421	32.8092	37.3277
	6 × 6	6.5494	10.2870	16.0097	17.7729	28.1202	31.1221
	7 × 7	6.5269	10.2354	15.8768	17.4707	26.8433	30.6100
	8 × 8	6.5259	10.2260	15.8214	17.3485	26.3093	30.1226
	9 × 9	6.5259	10.2256	15.820	17.3351	26.1262	30.0904
	10 × 10	6.5259	10.2256	15.8188	17.3347	26.1141	30.0824
60°	4 × 4	7.0732	10.5800	17.6982	22.7236	36.8067	42.7418
	5 × 5	6.4403	9.8293	16.8402	20.5344	31.7597	35.3489
	6 × 6	6.4036	9.7556	15.7883	18.4410	26.6067	31.0657
	7 × 7	6.3744	9.7184	15.6445	18.1190	26.1304	28.7599
	8 × 8	6.3737	9.7117	15.5376	17.9686	25.7466	28.1767
	9 × 9	6.3736	9.7115	15.5357	17.9501	25.6790	27.8622
	10 × 10	6.3736	9.7115	15.5347	17.9494	25.6710	27.8487

$$w(x,y) = \sum_{i=1}^{M} \sum_{j=1}^{N} c_{ij} p_i(x) q_j(y), \tag{44}$$

where c_{ij} are the unknown coefficients. The procedure used is the construction of the orthogonal polynomials as has been developed by Bhat [40,41]. The Ritz method is used to generate the following non-dimensional frequency coefficients $\omega l^2/h\sqrt{\rho/E_L}$, and $\omega l^2\sqrt{\rho h/D_0}(1-c_l)2\tan\theta$.

In this case, the corresponding boundary conditions are given by Eqs. (10) and (11), with the non-dimensional spring coefficients given by $R_i = c_{R_i} l/D_0$ and $T_i = c_{T_i} l^3/D_0,\ i = 1,2,3,4$.

6. Verification and numerical applications

6.1. Circular and elliptical plates

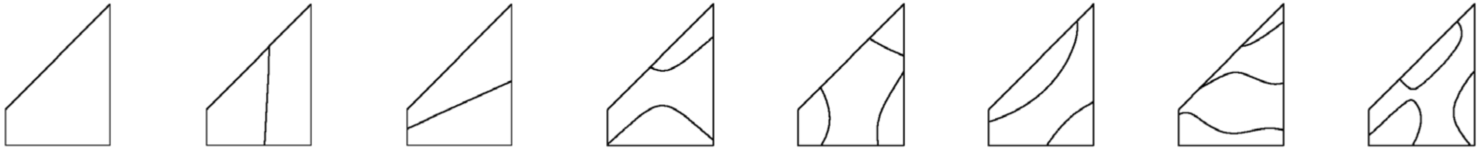
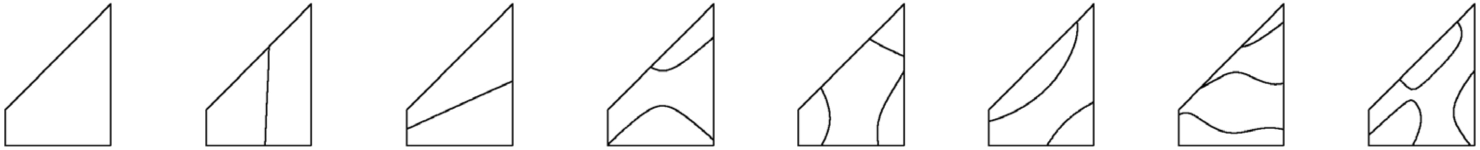
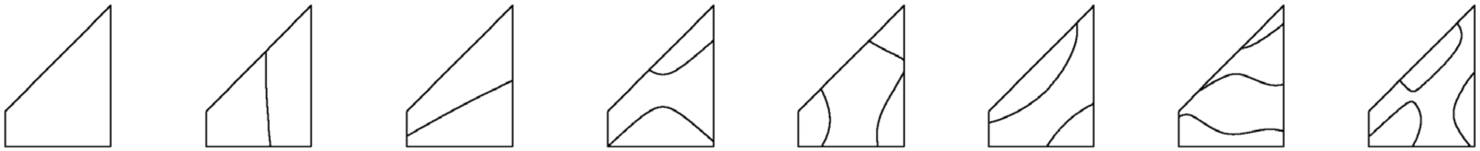
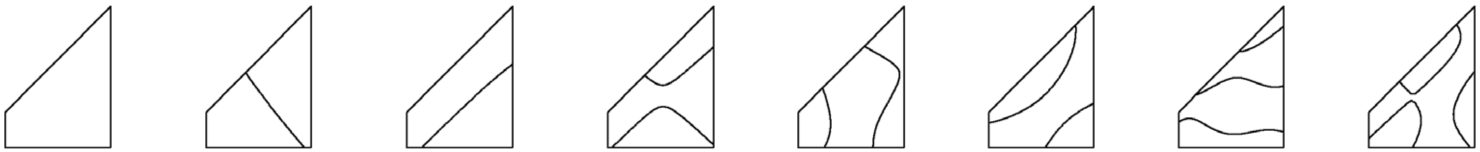
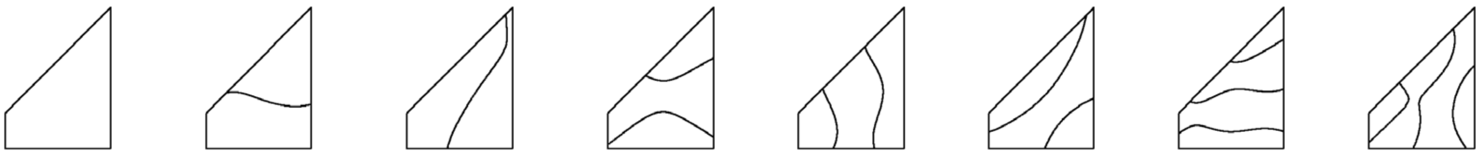
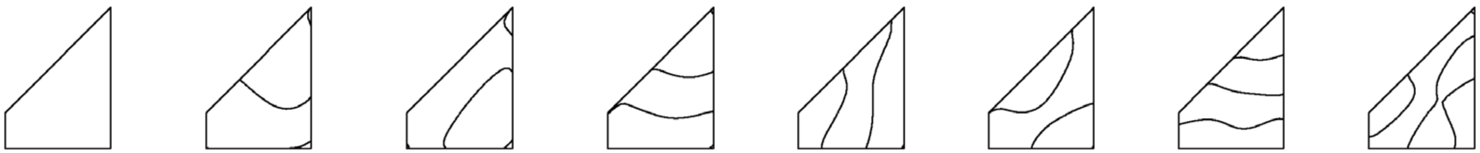
Results of a convergence study of eigenvalues $\omega a^2\sqrt{\rho h/D_0}$, for elliptical composite plates are presented in Table 1. Four-ply graphite-epoxy laminates ($E_L/E_T = 40$, $G_{LT}/E_T = 0.5$, $\nu_{LT} = 0.25$), with stacking

Table 4
Frequency parameters $\omega l^2\sqrt{\rho h/D_0}(1-c_l)2\tan\theta$, for symmetrically laminated graphite-epoxy trapezoidal plates, with $\theta_1=\theta_2=\theta = 20.556°$, $c/l = 0.25$, stacking sequence $(-\beta,\beta,\beta,-\beta)$

R_2	$T_1=T_3=T_4$	β		Mode sequence number							
				1	2	3	4	5	6	7	8
0	∞ (S-S-S-S)	15°	Present	14.963	28.333	41.915	48.387	60.151	71.794	83.353	94.895
0.01	10000			14.389	28.201	41.076	49.191	59.878	76.470	83.993	97.535
0.1	1000			13.908	26.917	37.524	44.855	53.462	63.995	68.353	79.589
1	100			11.692	18.968	23.381	27.678	32.749	38.454	45.296	53.554
10	10			6.3840	10.839	18.079	22.545	27.489	35.560	43.095	53.107
100	1			4.4740	9.7582	18.923	24.139	29.096	38.124	46.170	57.140
1000	0.1			4.2170	9.6827	19.129	24.484	29.462	38.651	46.839	58.019
10000	0.01			4.1902	9.6758	19.152	24.522	29.502	38.709	46.913	58.116
∞	0 (F-C-F-F)			4.1872	9.6748	19.153	24.524	29.505	38.623	46.854	57.849
∞	0 (F-C-F-F)		Liew and Lim [23]	4.1872	9.6743	19.151	24.516	29.499	38.566	46.385	56.755
0	∞ (S-S-S-S)	30°	Present	17.789	35.854	47.185	59.702	76.151	89.364	90.616	112.70
0.01	10000			17.342	35.188	45.830	59.508	74.761	86.785	92.070	115.26
0.1	1000			16.499	32.322	40.844	51.535	62.241	69.889	72.591	85.599
1	100			12.512	20.114	23.891	29.629	36.330	43.311	47.123	56.987
10	10			6.2432	11.803	16.820	24.687	32.210	41.211	46.428	55.921
100	1			3.6382	10.937	16.813	25.629	33.558	43.330	49.249	58.648
1000	0.1			3.2185	10.879	16.886	25.824	33.840	43.747	49.799	59.210
10000	0.01			3.1731	10.873	16.894	25.845	33.871	43.793	49.860	59.271
∞	0 (F-C-F-F)			3.1677	10.871	16.892	25.840	33.868	43.732	49.775	59.082
∞	0 (F-C-F-F)		Liew and Lim [23]	3.1672	10.870	16.878	25.829	33.862	43.694	49.692	58.586
0	∞ (S-S-S-S)	45°	Present	20.5866	38.9951	55.697	64.683	83.250	97.647	108.352	120.266
0.01	10000		Present	20.2942	37.9614	54.151	62.587	80.481	95.844	103.855	117.885
0.1	1000		Present	18.8377	33.0843	44.783	51.147	61.613	72.120	74.801	83.313
1	100		Present	12.5204	19.3269	23.560	28.729	33.903	43.666	47.688	54.096
10	10		Present	5.8569	10.4467	14.571	22.881	29.383	41.782	45.229	53.344
100	1		Present	2.7454	8.9430	13.397	23.045	29.811	42.876	46.265	55.005
1000	0.1		Present	2.0942	8.7991	13.291	23.097	29.900	43.054	46.459	55.282
10000	0.01		Present	2.0164	8.7851	13.281	23.102	29.910	43.074	46.480	55.311
∞	0 (F-C-F-F)		Present	2.0060	8.7812	13.275	23.099	29.904	42.983	46.329	54.785
∞	0 (F-C-F-F)		Liew and Lim [23]	2.0039	8.7734	13.250	23.087	29.873	42.950	46.274	54.594

Edge **2** elastically restrained against rotation ($R_1 = R_3 = R_4 = 0$), edges **1**, **3** and **4** elastically restrained against translation ($T_2 = \infty$).

Table 5
Frequency parameters $\omega l^2\sqrt{\rho h/D_0}$ and nodal patterns, for symmetrically laminated right graphite-epoxy trapezoidal plates, with $\theta_1 = 45°$, $\theta_2 = 0°$, $c/l = 0.25$, stacking sequence $(45°, -45°, -45°, 45°)$

$T_1 = T_2 = T_3 = T_4$	Mode sequence number							
	1	2	3	4	5	6	7	8
0.01	0.251878	0.349767	0.370891	6.96855	18.50615	22.2894	31.5337	38.9943
								
0.1	0.795649	1.10550	1.17162	7.09288	18.55363	22.32470	31.56243	39.01926
								
1	2.49031	3.47835	3.67092	8.24218	19.02233	22.67522	31.85023	39.26820
								
10	7.27497	10.57559	11.15468	15.55778	23.15863	25.91945	34.72386	41.67411
								
100	16.06498	28.27007	29.97741	39.84647	43.27316	45.58573	56.35957	60.10431
								
1000	22.64374	47.90213	51.59762	79.11992	85.54646	87.49987	111.9065	121.9051
								

Edges **1**–**4** elastically restrained against translation.

sequence $(-\beta, \beta, \beta, -\beta)$ and aspect ratio $a/b = 2$ are considered. The rate of convergence of eigenvalues is shown for clamped, simply supported and free boundaries. The convergence of the mentioned eigenvalues is studied by increasing the numbers M, N in Eq. (42). It can be seen that M, $N = 10$, is adequate to reach a stable convergence, specially in the case of the lower frequencies. Therefore it was decided to use M, $N = 8$ to generate further results since there is no drastic change.

Table 2 depicts values of the non-dimensional frequency coefficient $\omega a^2\sqrt{\rho h/D_{11}}$ for circular plates of generalized anisotropy. The results are presented for two different material properties and growing values of the restraint coefficients $R = c_R\ a/D_{11}$ and $T = c_T a^3/D_{11}$. The comparison of results for simply supported and clamped plates with those of Bambill et al. [42] shows that the present values are lower, in consequence more accurate, since the Ritz method gives upper bounds for eigenvalues.

6.2. Trapezoidal plates

Results of a convergence study of eigenvalues $\omega l^2/h\sqrt{\rho/E_L}$ for trapezoidal plates are presented in Table 3. Four-ply E-glass-epoxy laminates ($E_L = 60.7$ GPa, $E_T = 24.8$ GPa, $G_{LT} = 12$ GPa, $\nu_{LT} = 0.23$), with stacking sequence $(-\beta, \beta, \beta, -\beta)$.

When treating with classical boundary conditions, the symbolism CSFF, for example, identifies a plate with edge **1** clamped, edge **2** simply supported and edges **3** and **4** free, (see Fig. 3b).

The rate of convergence of eigenvalues is shown for clamped, simply supported and free boundaries. The convergence of the mentioned eigenvalues is studied by increasing the numbers M, N in Eq. (44). In this case it is also adequate to use M, $N = 8$ to generate the results with sufficient accuracy from an engineering viewpoint. The results for simply supported laminates are in good agreement with those of Lim et al. [43].

Table 4 depicts values of the non-dimensional frequency coefficient $\omega l^2\sqrt{\frac{\rho h}{D_0}}(1 - c_l)2\tan\theta$, for a trapezoidal plate. Four-ply graphite-epoxy laminates ($E_L/E_T = 40$, $G_{LT}/E_T = 0.5$, $\nu_{LT} = 0.25$), with stacking sequence $(-\beta, \beta, \beta, -\beta)$ are considered. The results are presented for different values of β and the restraint coefficients $R_2 = c_{R_2} l/D_0$ and $T_i = c_{T_i} l^3/D_0, i = 1, 3, 4$. The results for cantilever plates are compared with those of Liew and Lim [23] and very good agreement is obtained. Finally, Table 5 depicts the first eight non dimensional frequency parameters $\omega l^2\sqrt{\rho h/D_0}$ and the corresponding nodal patterns, for symmetrically laminated $(45°, -45°, -45°, 45°)$ graphite-epoxy trapezoidal plates, with $\theta_1 = 45°$, $\theta_2 = 0°$, $c/l = 0.25$.

7. Concluding remarks

The existence and uniqueness of the weak solutions of boundary value problems and eigenvalue problems, which correspond to the anisotropic plates analysed has been demonstrated. Two classes of boundaries have been considered:

(i) smooth boundaries of arbitrary shape,
(ii) piecewise smooth boundaries having a finite number of corner points.

The use of the weak solution theory enables a substantial generalisation of assumptions concerning the smoothness of coefficients of the differential operator (1) and of the functions which appear respectively in Eqs. (2) and (6).

It has been determined that when the plates have corner points formed by the intersection of edges free or elastically restrained against translation, the corresponding bilinear form maintains the V – ellipticity property. This property is given by Eq. (33) and it guaranties that the weak solution is unique. In practice this inequality shows that for a system involving a V – elliptic bilinear form, it is possible to obtain a large displacement only by a great expenditure if energy.

It is also the purpose of the present paper to present some technical results for the natural frequencies of circular, elliptical and trapezoidal plates of generalized anisotropy or made of composite materials and resting on elastic supports. The Ritz method has been employed by using different polynomial expressions as trial functions which satisfy only the stable boundary conditions. As it was expected convergence of frequencies is monotonic, and successively upper bounds in the values of the frequency parameters are obtained as additional terms are taken in the corresponding approximation functions, in spite of the fact that the co-ordinate functions do not satisfy the unstable boundary conditions. Since the combinations of boundary conditions, along with specific values for the stiffness constants for the restraints are prohibitively large in number, results are presented for only a few cases.

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Appendix. Definition of coefficients $a_{\alpha\beta}$ in Eq. (1)

α	β	$a_{\alpha\beta}$
(2,0)	(2,0)	D_{11}
(2,0)	(0,2)	D_{12}
(2,0)	(1,1)	$2D_{16}$
(0,2)	(2,0)	D_{12}
(0,2)	(0,2)	D_{22}
(0,2)	(1,1)	$2D_{26}$
(1,1)	(2,0)	$2D_{16}$
(1,1)	(0,2)	$2D_{26}$
(1,1)	(1,1)	$4D_{66}$

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