Impedance function calculation for the study of waves in locally periodic arrangements

2015 Eur. J. Phys. 36 055034

(http://iopscience.iop.org/0143-0807/36/5/055034)

IOP Publishing European Journal of Physics

Eur. J. Phys. **36** (2015) 055034 (16pp) doi:10.1088/0143-0807/36/5/055034

Impedance function calculation for the study of waves in locally periodic arrangements

B J Gómez[1,2], C E Repetto[1,2], C R Stia[1,2] and R Welti[2]

[1] Instituto de Física Rosario (CONICET-UNR), Bv. 27 de Febrero 210 Bis, S2000EZP Rosario, Argentina
[2] Laboratorio de Vibraciones y Ondas, Departamento de Física, Escuela de Formación Básica, Facultad de Ciencias Exactas, Ingeniería y Agrimensura (UNR), Pellegrini 250, S2000BTP Rosario, Argentina

E-mail: weltireinaldo@arnet.com.ar

Received 11 February 2015, revised 28 May 2015
Accepted for publication 16 June 2015
Published 4 August 2015

Abstract

In this paper, we study theoretically the propagation of waves in locally periodic systems. To this end, we employ a simple method based on the impedance function which is similar to the one used in the transmission line theory. Results are presented for the particular cases of mass-loaded strings arrangements. These systems are easily reproducible in laboratory of introductory and/or advanced courses of physics, and contain all the main characteristics of locally periodic media. Transmission coefficient and normal frequencies are calculated for both mono and diatomic chains, and compared with previously computed and experimental data. Moreover, we show that the corresponding eigenfunctions can be obtained easily from the knowledge of the impedance function.

Keywords: impedance function, locally periodic media, transmission coefficient, normal modes

1. Introduction

Ordinary macroscopic crystals can be considered as strictly periodic. Hence, the quantum problem of a particle moving in a periodic potential leads to energy levels falling into allowed continuous bands, separated by forbidden energy gaps. Indeed, this band structure is a characteristic of solid state physics, where the Bloch theorem is valid [1, 2]. Band gap structures have been studied recently for the case of composite materials as for instance phononic [3, 4] and photonic crystals [5, 6]. These composites are built as laminar structures, alternating two or more materials with different physical properties. Only recently has the study of these locally periodic systems received a lot of attention, especially in the field of communication technology. One interesting property of these composites is the possibility to develop and fabricate electronic devices capable to transmit in higher frequency ranges.

Although the theoretical description of the locally periodic systems becomes more complicated, analytical solutions can be found for optical [7], quantum [8, 9] as well as mechanical systems [10]. In particular, Griffiths and Steinke [10] proposed a theoretical treatment of locally periodic waves using a non-relativistic quantum scattering model based on the transfer matrix calculation, that can be applied to different areas of physics. Moreover, these authors showed that the band structure becomes evident even for a small number of repeated structures.

The study of mechanical waves in loaded strings and sound waves in tubes is important for different reasons. On one hand, it is of great pedagogical value because it is possible to perform accurate measurements in easily modifiable and inexpensive systems. On the other hand, the understanding of these simple systems may provide a deeper insight into the behaviour of more complex ones characterized by parameters with local periodicity [10, 11]. It is well known that the discrete structure of a solid can be simulated by concentrated masses on a vibrating string [12, 13], giving a dispersion relation similar to the phononic one. This approach can be made provided that the force between masses is linear and the mass of the string is neglected. From the experimental side, the validity of these assumptions can be checked properly. For instance, the linear force behaviour can be verified from the dispersion relation of a bare string, i.e. a linear dispersion relation for a taut string ensures that the forces between masses are linear [13]. Also, it is necessary to use either large wavelengths or concentrated masses heavier enough than the mass of the string. However, even in this case, the system is continuous with infinite degrees of freedom. As we will see below, the corresponding dispersion relation presents features such as band-gaps patterns, and the analogy with the phononic dispersion could be made only for the first band.

In this paper, we study the wave propagation in locally periodic systems, focusing our attention in a homogeneous string loaded with a finite number of masses. After imposing adequate boundary conditions, both transmission coefficients and eigenfrequencies are computed for several arrangements by using a method based on the impedance function. It is noted that the impedance method was introduced in the framework of the transmission line theory, and also used in acoustics to analyse wave propagation in tubes [14] as well as in quantum mechanics to study the transmission through 1D finite lattices [15–17].

Finally, we also show that the impedance function $Z(x)$ allows to infer the behaviour of the wave functions of the system. Moreover, we display that the eigenfunctions can be obtained explicitly from the direct integration of the function $Z(x)$.

The present paper is organized as follows. In section 2 the impedance method is described for a homogeneous string as well as for a N-mass string system. Corresponding transmission coefficients and eigenfrequencies are obtained by imposing proper boundary conditions. In section 3, the impedance function is generalized for the acoustic and quantum mechanics analogues, showing that their wavefunctions can be obtained through the integration of the impedance function. Finally, in section 4, calculations for both mono and diatomic ten-mass strings are presented and compared with previous theoretical and experimental results. Also here, the explicit form of the wavefunctions are shown for the simple case of the string with two concentrated masses.

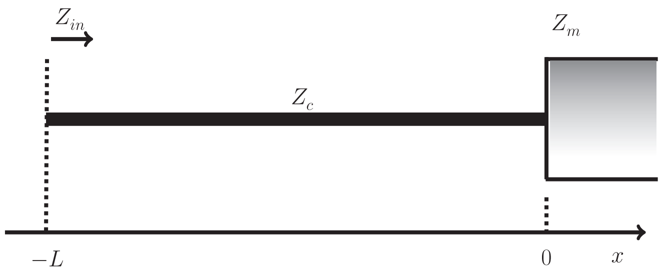

Figure 1. String of length L and characteristic impedance Z_c, connected at $x = 0$ to an element of mechanical impedance Z_m. The input impedance Z_{in} is calculated from the impedance function evaluated at $x = -L$.

2. Theoretical method

2.1. Homogeneous string

Firstly, we consider a uniform string of length L, linear mass density μ, and tension T, on which a wave propagates along the x axis. The mechanical impedance Z [18, 19] is defined as the ratio of the transverse force acting on an element x of the string and the transverse speed v of this point. If $\Psi(x, t)$ represents the transverse displacement of the string elements, we can write

$$Z = -T \cdot \frac{1}{v}\frac{\partial \Psi}{\partial x}. \tag{1}$$

For the particular case of an infinite string, the precedent equation defines the so-called characteristic impedance of the string: $Z_c = \pm\mu c$, where the sign + (−) stands for waves propagating in the positive (negative) direction of the x axis, and c represents the modulus of the propagation velocity. Thus, the characteristic impedance Z_c is a number that depends solely on the properties of the medium as, for instance, the tension and the linear mass density of the string.

Conversely, if there are waves propagating in both directions along the string, equation (1) defines the impedance Z of the string as a function of the x coordinate. In particular, the harmonics solutions to the wave equation Ψ can be written as:

$$\Psi(x, t) = \Phi(x)\mathrm{e}^{j\omega t} = \left(A\mathrm{e}^{-jkx} + B\mathrm{e}^{jkx}\right)\mathrm{e}^{j\omega t}, \tag{2}$$

where ω is the angular frequency, k is the wavenumber and A (B) is the amplitude of the incident (reflected) wave. Then, the impedance function, Z, at any value of the x coordinate is determined by

$$Z(x) = Z_c\frac{A\mathrm{e}^{-jkx} - B\mathrm{e}^{jkx}}{A\mathrm{e}^{-jkx} + B\mathrm{e}^{jkx}}. \tag{3}$$

If the string is connected at $x = 0$ to an impedance Z_m (see figure 1), it is verified that

$$Z(0) = Z_m. \tag{4}$$

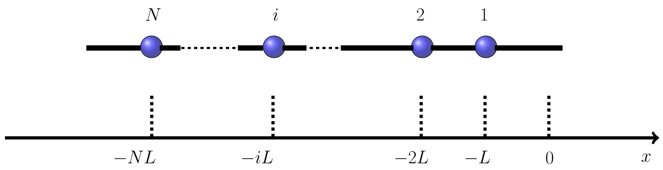

Figure 2. *N*-mass string system. The masses are equally distributed along the homogeneous string, being *L* the distance between masses.

Then, from equation (3), the relationship between amplitudes is obtained as follows:

$$\frac{B}{A} = \frac{Z_c - Z_m}{Z_c + Z_m}, \tag{5}$$

and the impedance function reduces to

$$Z(x) = Z_c \frac{Z_m - jZ_c \tan(kx)}{Z_c - jZ_m \tan(kx)}. \tag{6}$$

In analogy with transmission line theory, Z_m acts as a *load* impedance.

Finally, the input impedance Z_{in} at the left end of the string (see figure 1), is obtained as the impedance function at $x = -L$:

$$Z_{in} = Z(-L) = Z_c \frac{Z_m + jZ_c \tan(kL)}{Z_c + jZ_m \tan(kL)}. \tag{7}$$

2.2. String loaded with N concentrated masses

In the following, we consider a system composed by a string with *N* attached masses as sketched in figure 2. For the sake of simplicity, we assume that the masses are identical and located at the positions x_i ($i = 1, 2, \ldots, N$), separated by a distance *L*. However, the method may be applied for any mass arrangements, not only for regularly distributed along the string.

In the case that the parameters describing the systems are continuous or piecewise continuous (e.g. the linear density μ and the tension *T*), the impedance *Z* (equation (6)) results continuous as a function of the x-coordinate. However, as a point mass *M* has an infinite mass density, the function *Z* presents discontinuities at each x_i position. The value of this *jump* can be obtained as follows. Firstly, by applying Newton's second law for the concentrated mass *M* located at $x = x_i$, it is obtained that:

$$-T\frac{\partial \Psi_{i+1}(x_i, t)}{\partial x} + T\frac{\partial \Psi_i(x_i, t)}{\partial x} = M\frac{\mathrm{d}v_i(x_i, t)}{\mathrm{d}t} = jM\omega v_i, \tag{8}$$

where

$$v_i = \frac{\partial \Psi_i}{\partial t} = j\omega\Psi_i \tag{9}$$

is the transversal speed of the mass. Here, Ψ_{i+1} and Ψ_i are, respectively, the wavefunctions to the left and to the right of that point. Secondly, another equation is obtained from the proper continuity condition applied to the displacements at x_i:

$$\Psi_{i+1}(x_i, t) = \Psi_i(x_i, t). \tag{10}$$

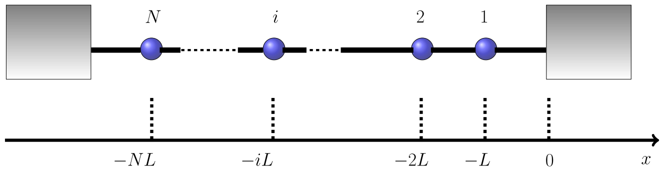

Figure 3. Idem as figure 2, but in this case the string is fixed at both ends.

Then, the ratio between (8) and (9) gives the value of the jump of Z

$$Z(x_i - \epsilon) - Z(x_i + \epsilon) = jM\omega \tag{11}$$

provided that $\epsilon > 0$ is small enough.

In general, a system can be thought as partitioned in different sections. The input impedance $Z_{\text{in}}^{(n)}$ at the beginning of the nth section may be expressed as a function of the impedance $Z_{\text{in}}^{(n-1)}$ that takes into account the whole information of the $n - 1$ preceding sections [14]:

$$Z_{\text{in}}^{(n)} = Z_n \frac{Z_{\text{in}}^{(n-1)} + jZ_n \tan(k_n l_n)}{Z_n + Z_{\text{in}}^{(n-1)} j \tan(k_n l_n)}, \tag{12}$$

where k_n is the wavenumber corresponding to the nth section of length l_n and characteristic impedance Z_n. In the particular case considered here, $l_n = L$, $Z_n = Z_c$ and $k_n = k$. It is worth mentioning that the impedance $Z_{\text{in}}^{(n-1)}$ represents the load impedance Z_{m} for the nth segment of the system in the same way as was shown for the homogeneous string case.

Finally, the impedance computation scheme allows one to obtain the transmission (and reflection) coefficient for the propagating waves through the system, as well as the frequencies related to the eigenmodes for finite arrangements.

2.3. Transmission coefficient and eigenmodes

With the aim to calculate the transmission coefficient, we consider a system connected at the ends to strings of infinite length (see figure 2). In this case, $Z(0) = Z_{\text{c}}$ is satisfied and the input impedance $Z_{\text{in}}^{(1)} = Z_{\text{c}}$. Similarly, at the left end of the whole arrangement, the boundary condition requires that $Z(-(N+1)L) = Z_{\text{c}}$. Therefore, an incident harmonic wave of frequency ω and amplitude A, traveling from the left, sees a structure whose input impedance Z_{in} is given by equation (12). Thus, there will be a reflected wave of amplitude B and a transmitted wave of amplitude C. Moreover, if I_{I}, I_{R} and I_{T} are the intensities of the incident, reflected and transmitted waves, respectively, it can be shown that [14]:

$$R = \frac{I_{\text{R}}}{I_{\text{I}}} = \left|\frac{B}{A}\right|^2 = \left(\frac{Z_{\text{c}} - Z_{\text{in}}}{Z_{\text{c}} + Z_{\text{in}}}\right)^2$$
$$T = \frac{I_{\text{T}}}{I_{\text{I}}} = \frac{Z_{\text{in}}}{Z_{\text{c}}}\left|\frac{C}{A}\right|^2 = \frac{4Z_{\text{c}}Z_{\text{in}}}{(Z_{\text{c}} + Z_{\text{in}})^2} = 1 - R, \tag{13}$$

where R and T are the intensity reflection and transmission coefficients, respectively. The input impedance Z_{in}, as a function of the frequency, can be calculated without difficulty by repeated application of equation (12), taking into account not only the impedance of the $N + 1$ pieces of the string, but also the N impedance jumps at the corresponding positions of the concentrated masses.

If now a system of finite length is considered, a similar procedure can be performed in order to obtain the eigenfrequencies of the normal modes. Figure 3 shows the system constructed from the string and the N concentrated masses, by fixing their ends at $x = 0$ and $x = -(N + 1)L$. Hence, the new boundary conditions on the impedance are:

$$|Z(x = 0)| \to \infty, \qquad |Z(x = -(N + 1)L)| \to \infty. \tag{14}$$

Applying equation (12) to the points x of the string between 0 and $-L$, the impedance is

$$Z_1(x) = Z_c \frac{Z(x = 0) - jZ_c \tan(kx)}{Z_c - jZ(x = 0)\tan(kx)} \tag{15}$$

that, by applying the boundary condition at $x = 0$ (equation (14)), reduces to

$$Z_1(x) = jZ_c \cot(kx). \tag{16}$$

Therefore, the input impedance Z_{in} at $x = -(N + 1)L$ can be obtained from equation (12) for each section of the system. As stated before, it has to be taken into account that the impedance is discontinuous at the positions where the masses are located. Finally, the frequency values at which the impedance goes to infinity are precisely the expected eigenfrequencies, as imposed by the second boundary condition given by equation (14).

3. Impedance and wavefunctions

In previous sections, we defined the (mechanical) impedance for the case of a taut string (see equation (1)). If the transversal displacements are given by equation (2), it is easy to show that the impedance becomes

$$Z_{mech}(x) = j\left(\frac{T}{\omega}\right)\frac{1}{\Phi(x)}\frac{d\Phi(x)}{dx}, \tag{17}$$

i.e., proportional to the logarithmic derivative of Φ with respect to the position.

This result can be extended to other classical or quantum equivalent systems. For instance, in the acoustic analogue, for a propagating wave through a fluid contained in a tube of transversal section S, the impedance can be written as follows

$$Z_{ac}(x) = \frac{p}{U} = -\mathcal{B}\frac{\frac{\partial\Psi}{\partial x}}{Sv}, \tag{18}$$

where p is the acoustic pressure, U the volume flux, and $\mathcal{B}$ the compressibility modulus. In the latter, Ψ represents the longitudinal displacements of the fluid. In the field of quantum mechanics, Khondker and co-workers [15] introduced the concept of generalized impedance, putting in evidence a correspondence between the quantum transport and the wave propagation in transmission lines. Hence, quantum impedance can be expressed as:

$$Z_q(x) = j\frac{1}{\Psi}\frac{\partial\Psi}{\partial x}. \tag{19}$$

By considering harmonic solutions (2), both $Z_{ac}(x)$ and $Z_q(x)$ can be reduced to similar expressions to equation (17).

As a consequence, the impedance for a general system may be re-written as follows:

$$Z(x) = \alpha\,\frac{d\,\ln(\Phi)}{dx}, \tag{20}$$

where α depends on the parameters characterizing the system under consideration.

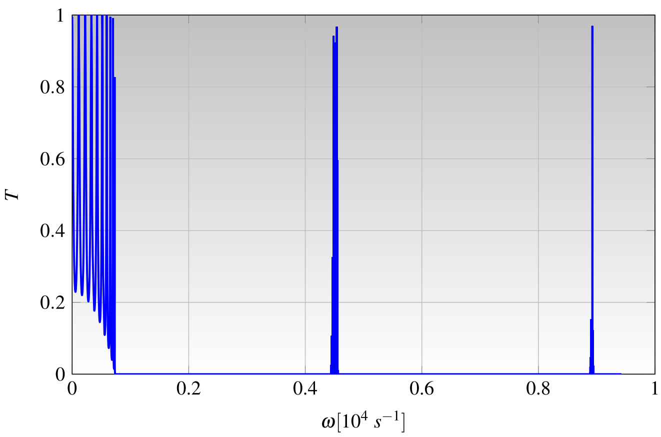

Figure 4. Transmission coefficient T as a function of the angular frequency ω for a ten-mass string. The linear mass density of the string is $\mu_L = 5.7573 \times 10^{-4}$ kg m^{-1} and the attached masses are $M = 1.86$ g. The distance between masses is $L = 0.226$ m.

Moreover, the solutions Φ can be obtained from the direct integration of equation (20)

$$\int \frac{\mathrm{d}\Phi}{\Phi} = \frac{1}{\alpha} \int Z(x)\mathrm{d}x, \tag{21}$$

that finally leads to

$$\Phi(x) = \exp\left(\frac{1}{\alpha} \int Z(x)\mathrm{d}x\right). \tag{22}$$

For the cases in which the integration given by equation (22) becomes a difficult task, the knowledge of the impedance function could be an useful tool to elucidate the behaviour of the wavefunction. For example, considering equation (17), the positions where Z diverges (tends to zero) are correlated to the positions of the nodes (antinodes) of the wavefunction.

4. Results

4.1. Ten-mass string

In figure 4, the transmission coefficient for the case of a linear string loaded with 10 masses is shown in terms of the angular frequency ω. In the calculations, the string mass density is $\mu_L = 5.7573 \times 10^{-4}$ kg m^{-1}, each concentrated mass is $M = 1.86$ g, and the distance between masses is $L = 0.226$ m, as in the experiments performed by Lüerßen and co-workers [13]. In the figure, transmission bands separated by gaps of frequencies are observed in a similar way to the band structure in solids [10, 20, 21]. The existence of more than one band is due to the fact that the string has a non-negligible mass and the N-mass string system behaves as a continuum with infinite degrees of freedom.

The pass bands have a transmission peak structure. Out of these bands, the transmission is zero and the system behaves as a filter. Indeed, it has been shown that this kind of band structures begins to be evident from $N = 6$ masses [10].

The lower limit of the nth band appears approximately at a frequency equal to $nc/2L$ that corresponds to one of the well-known eigenfrequencies for a finite string of length L, fixed at

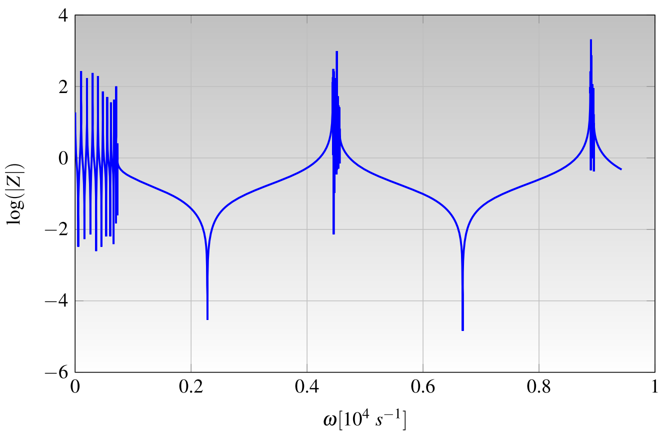

Figure 5. Eigenfrequencies for a ten-mass string system obtained from the peaks of $\log(|Z(x = -11L)|)$ (see text). The values of L, M, and μ are the same used in figure 4.

both ends. Also, it can be shown that $N - 1$ transmission peaks are present in each band. This result coincides with the one obtained previously for the quantum analogue of a N-delta functions potential with periodicity L [20].

To better understand the transmission band-gaps in locally periodic media, it is useful to study the normal modes of the N-mass string system fixed at both ends (in the quantum analogue, this is equivalent to determine the bound states corresponding to a periodic arrangement of delta functions in an infinite well potential). Figure 5 shows the logarithm of the impedance function $Z(x)$ at $x = -(N + 1)L$ as a function of the frequency ω, for the same parameters employed in figure 4. As discussed above, the frequency values for which this function goes to infinity correspond to the normal frequencies of the finite system. It is observed that the normal modes also appear grouped in well-defined frequency ranges or bands. In the first band, N normal modes are found, whereas in the higher ones, $N + 1$ modes are observed. The additional mode in the upper bands is the one of lower frequency and corresponds to solutions of a homogeneous string. This result agrees with the previous findings corresponding to the case of a string with two concentrated masses [22]. Then, the lowest N eigenmodes are associated with the concentrated masses motion as if the whole system would be thought as one of N degrees of freedom. In this case, the mass of the string influences only the eigenfrequencies values, but does not change the oscillating behaviour. Furthermore, the corresponding displacement distributions present discontinuous slope in the positions where the masses are located [22]. In contrast, the lowest modes in the upper allowed bands are those that present continuous slope in the positions of the masses. The point-masses are almost stationary for these modes and, as expected, their eigenvalues match the frequencies $nc/2L$ of a bare string of length L. In addition, all modes alternate between a symmetric and antisymmetric displacement distribution with respect to the midpoint of the system [22].

In figure 6, the lowest ten eigenfrequencies are shown as a function of kL and compared with previous measured data [13]. There is a notable good agreement between theory and experiments. From figure 6, it is also observed that data behaviour follows the trend of the well-known monoatomic chain dispersion relation given by:

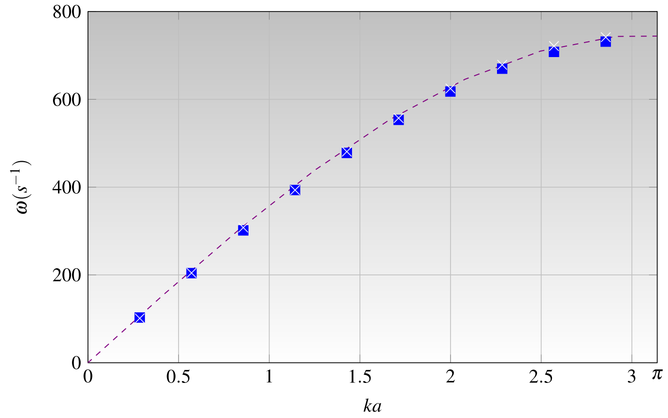

Figure 6. Eigenfrequencies of the lowest band versus the wave vectors for a monoatomic chain of ten masses. The values of L, M, and μ are the same used in figure 4. Present results: blue filled squares. Experimental results [13]: white crosses. Phonon dispersion relation (equation (23)): dashed line.

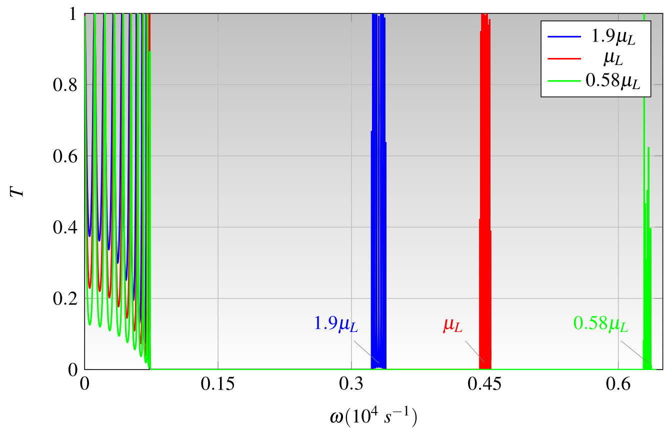

Figure 7. Idem figure 4, but for three different values of linear mass density. Blue lines: $\mu_L = 5.7573 \times 10^{-4}$ kg m^{-1}. Red lines: $1.9\mu_L$. Green lines: $0.58\mu_L$. Pass-bands are shifted toward higher frequencies as μ diminishes.

$$\omega(k) = \sqrt{\frac{4\,T}{mL}}\,\sin\left(\frac{kL}{2}\right). \tag{23}$$

This can be explained by taking into account that the mass of each string piece is about 14 times lesser than the loaded masses. The fact that the dispersion relation deviates from a linear function is due to the local periodicity imposed by the concentrated masses. Indeed, the discrepancies are more noticeable when the wavelength is comparable to the lattice parameter, i.e., to values of kL approaching to π. However, it is mentioned that the interaction between masses is still linear.

From a close inspection of figures 4 and 5, the frequency ranges at which the normal modes appear are practically the same to those of the transmission bands. As a consequence, the knowledge of the eigenvalues for the finite system allows one to infer the transmission

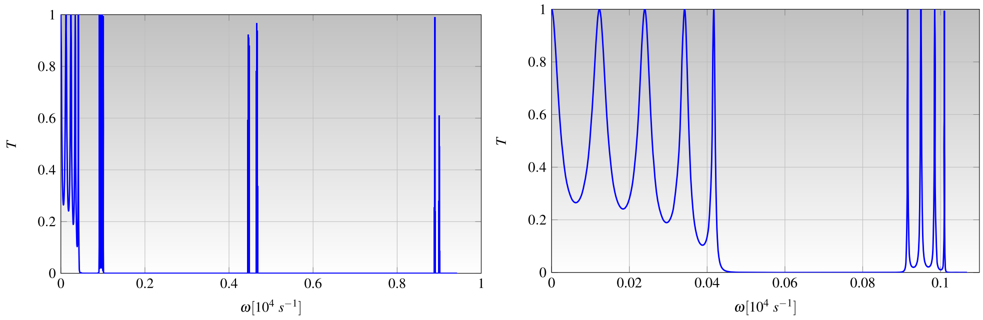

Figure 8. Transmission coefficient T versus the frequency ω for a diatomic chain with 10 masses. The linear mass density of the string is $\mu_L = 5.7573 \times 10^{-4}$ kg m^{-1} and the attached masses are $M = 2.53$g and $m = 0.553$ g. The distance between masses is $L = 0.226$ m. Right side: detail of the first transmission band.

behaviour through the same structure. Then, the lower limit of the transmission gap corresponds approximately to the highest eigenfrequency of the previous band, whereas the upper one is determined by a frequency rather close to the value $nc/2L$.

It is mentioned that the upper limit depends on the linear density μ through c, while the lower limit remains almost constant. This behaviour is shown in figure 7, where the transmission coefficient is computed for the following three different values of the mass density: $1.9\mu_L$, μ_L and $0.5\mu_L$, being $\mu_L = 5.7573 \times 10^{-4}$ kg m^{-1} the value used in the figures. As μ diminishes the gap end tends to larger values of frequency. In the limit $\mu \to 0$, the upper bands disappear which is consistent with the fact the system is considered as one of N degrees of freedom.

4.2. Diatomic string with ten concentrated masses

In this section a diatomic ten-mass string system is considered. In this case, the string of mass density $\mu_L = 5.7573 \times 10^{-4}$ kg m^{-1}, is loaded with five bead pairs of masses $M = 2.53$ g and $m = 0.553$ g. The distance between masses is $L = 22.6$ cm, being the lattice parameter $a = 2L$.

In figure 8, the transmission coefficient is shown as a function of the frequency ω. As in the case of the monoatomic chain, well-defined transmission bands are observed. However, the presence of different masses provokes that the transmission peaks in each band appear distributed in two groups. A detail of the first band is sketched at the right side of figure 8 for the sake of clarity.

This behaviour can also be understood from the analysis of the normal modes of the finite diatomic chain. Figure 9 represents the logarithm of the impedance function evaluated at the end of the finite system. Both acoustic and optical branches are clearly visible in each one of the bands. The lowest band is composed by five normal modes in both branches, whereas the upper bands are characterized by six normal modes in the acoustic branch and five modes in the optical one. This additional mode in the acoustic branch possesses the lowest frequency and is related to the modes of a homogeneous string of length L. Again, this mode is the one that defines the upper limit of the band gap.

Figure 10 shows the lowest band eigenvalues for the diatomic system obtained through the impedance method (blue squares) as a function of the the wave vectors ka. Experimental data of Lüerßen *et al* [13] is also included in the figure (crosses). Excellent agreement between present calculations and measurements is found for both acoustic and optical

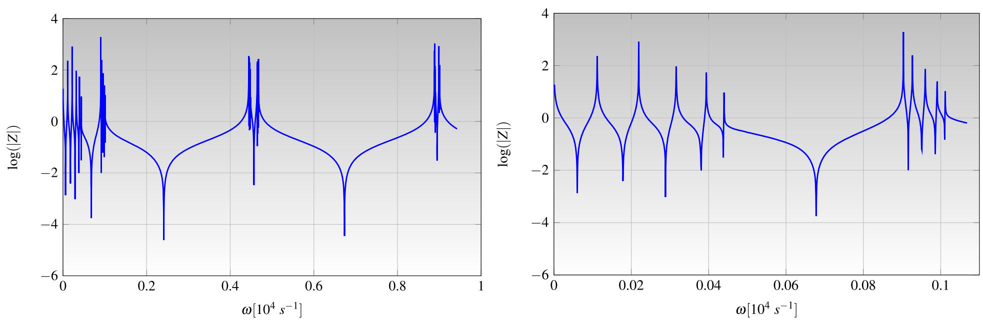

Figure 9. Logarithm of the impedance function Z evaluated at $x = -11L$ as a function of ω for the diatomic chain of figure 8. The peaks of the function correspond to the eigenfrequencies of the system (see text). Right: detail of the acoustic and optical branches in the lowest band.

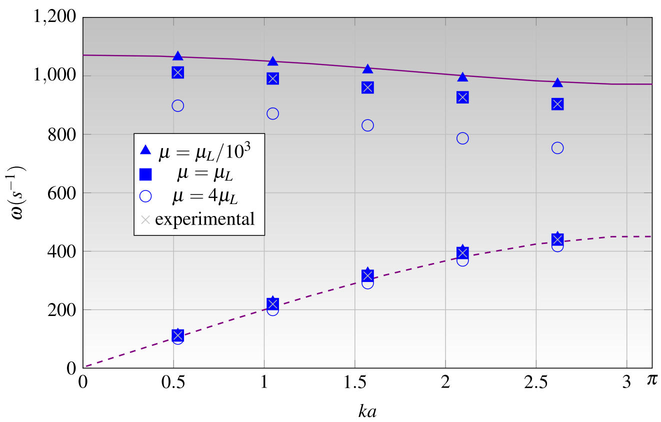

Figure 10. Eigenfrequencies of the lowest band as a function of the wave vectors for the 10 mass-string system considered in figure 9. Present results: full squares. Experimental results: crosses [13]). Acoustic and optical branches (equation (24)): dashed and full lines, respectively. Present results for $\mu_L/10^3$: triangles. Present results for $4\mu_L$: circles. Here, $\mu_L = 5.7573 \times 10^{-4}$ kg m^{-1} as in the experiments of [13].

branches. To consider the influence of the string segment mass on the motion of the system, the dispersion relation for both branches

$$\omega^2(k) = \frac{2T}{a}\left(\frac{1}{m} + \frac{1}{M}\right)\left[1 \pm \sqrt{1 - \frac{4mM}{(m+M)^2}\sin^2(ka)}\right] \tag{24}$$

is also depicted in figure 10 (dashed and full lines). No notable discrepancy is found between experimental data and the latter expression for the acoustic branch. On the contrary, it is observed in the optical branch that measurements are lower than the predictions given by equation (24) for a massless string. These differences may be explained by the relative weight of the string piece mass $\mu_L L$ with respect to the loaded masses, taking into account that the acoustic (optical) branch is associated to the heaviest (lightest) mass M (m) dynamics. In the case considered here, $\mu_L L = 0.13$ g, which is much smaller than the mass M, and measurements do not deviate appreciably from the dispersion relation in the acoustic branch in a similar way to the monotaomic case (see figure 6). Otherwise, the string mass piece differs from the lightest mass m by a factor of 4.25, which explains the observed behaviour in the optical branch. Of course, as the linear density μ decreases, this deviation also diminishes. For small enough values of μ, calculations fit well the dispersion relation for both branches as can be seen in figure 10 for a value of $\mu = \mu_L/1000$ (blue triangles). For the sake of completeness, results for a linear density value four times greater than the one used in the experiments (open circles) are also included. As expected, enhanced differences with the dispersion relation are found mostly in the optical branch.

4.3. Impedance function and wavefunctions. A two-mass string system

In this section, a two-mass string system fixed at both ends is considered as a simple illustration of the impedance method applied to the wave function calculations (see section 3). The choice of this simple system is made in order to compare easily our findings with its well-known eigenfunctions.

In this case, the impedance function for each partition of the system is obtained by using equations (15) and (16). It is found that:

$$Z_1(x) = jZ_c \cot(kx), \qquad 0 < x < -L, \tag{25}$$

$$Z_2(x) = Z_1(-L) + jM\omega = jZ_c C, \qquad x = -L, \tag{26}$$

$$Z_3(x) = jZ_c \frac{C - \tan(k(x + L))}{1 + C \tan(k(x + L))}, \qquad -2L < x < -L, \tag{27}$$

$$Z_4(x) = Z_3(-2L) + jM\omega = jZ_c D, \qquad x = -2L, \tag{28}$$

$$Z_5(x) = jZ_c \frac{D - \tan(k(x + 2L))}{1 + D \tan(k(x + 2L))}, \qquad -3L \leqslant x < -2L, \tag{29}$$

where

$$C = kL(M/\mu L) - \cot(kL) \tag{30}$$

$$D = \frac{C + \tan(kL)}{1 - C \tan(kL)} + kL(M/\mu L). \tag{31}$$

The impedance Z as a function of the position x can be useful to obtain information about the behaviour of the corresponding wavefunctions. To achieve this end, it is noted that expressions (25)–(29) also depend on the frequency. In figure 11, the logarithm of the end impedance $|Z_5|$, evaluated at $x = -3L$, is shown as a function of the frequency. In the computations, $\mu = 0.021\ \mathrm{kg\ m^{-1}}$, $M = \mu L$, and $L = 0.20$ m, are considered. The eigenfrequencies are obtained from the positions of the impedance maxima. These values are in agreement with the measured ones in [22].

To know the eigenmodes of the system, the impedance function $Z(x)$ must be evaluated at each of the normal frequencies. In figures 12 and 13, $|Z(x)|$ is represented as a function of x for the two lowest eigenfrequencies $f_1 = 3.76$ Hz and $f_2 = 7.13$ Hz, respectively. Figure 12 shows clearly that the impedance function assumes a very small value in the middle of the system. Therefore, from equation (17), the wavefunction presents an antinode at this position. At $x = -0.4$ and -0.2 where the masses are located, the impedance shows a discontinuity as expected from equation (11), where the jump value is proportional to the product $M\omega$. Moreover, it is noted that the masses will reach maximum amplitudes in this mode.

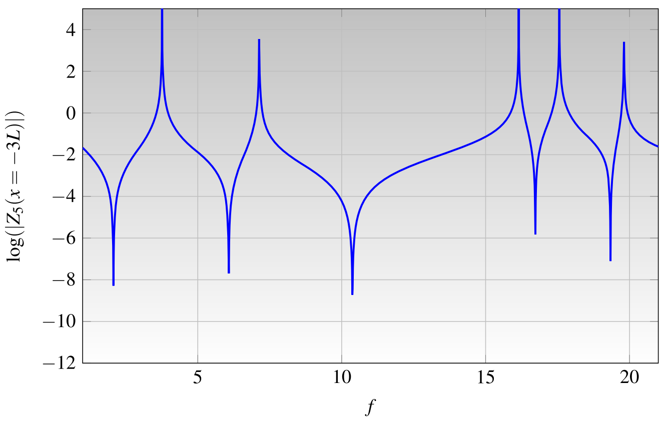

Figure 11. Logarithm of the function $Z_5(x)$ (see equation (29)) versus the frequency f for a two-mass string system fixed at both ends (see text). The eigenfrequencies of the system are those at which the peaks are present.

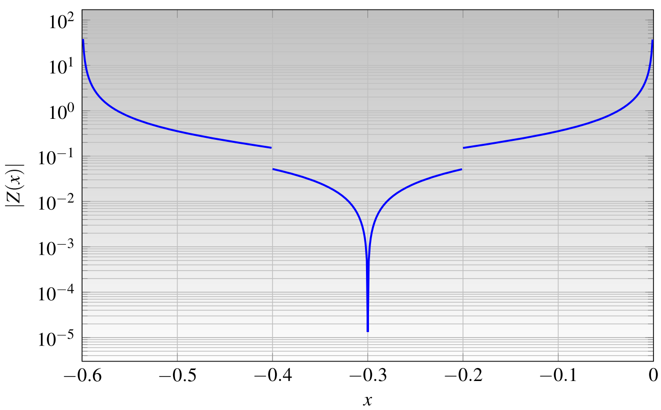

Figure 12. Impedance function $Z(x)$, for the lowest mode (f_1 = 3.76 Hz) of the two-mass string system considered in figure 11.

In contrast, for f_2 (see figure 13) the impedance presents a high peak in the middle point that implies that the corresponding wavefunction presents a node at this place. Also, it can be seen that the jump value is greater than the one discussed above for the first mode, because of the bigger value of the frequency in this case. Again, maximum displacements are reached at the positions where the masses are located.

In the present simple case considered here, it is possible to obtain easily the wavefunctions through the direct integration of the impedance function (see section 3). Applying equation (22), it is found that

$$\Phi_1(x) = A_1 \sin(kx) \quad -L < x < 0, \tag{32}$$

$$\Phi_3(x) = A_3(\cos[k(x+L)] + C \sin[k(x+L)]), \tag{33}$$

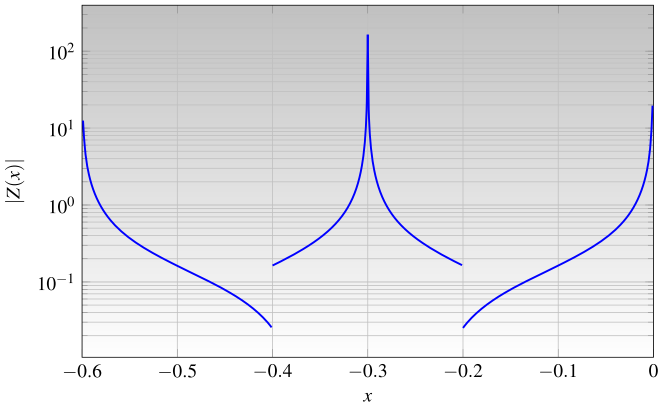

Figure 13. Idem as figure 12 but for the second mode ($f_2 = 7.13$ Hz).

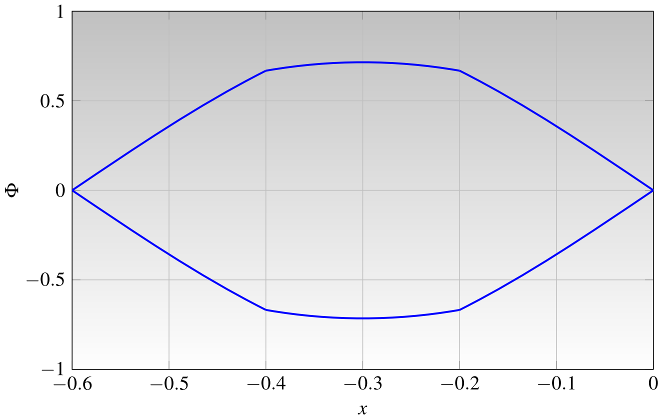

Figure 14. Wavefunction for the first mode of the system composed by two concentrated masses.

for $-2L < x < -L$, and

$$\Phi_5(x) = A_5(\cos[k(x + 2L)] + D\sin[k(x + 2L)]), \tag{34}$$

for $-3L < x < -2L$. Imposing the continuity of the wavefunction at $x = -L$ and $x = -2L$ we obtain:

$$\begin{aligned} A_3 &= -A_1 \sin(kL), \\ A_5 &= -A_1 \sin(kL)[\cos(kL) - C\sin(kL)]. \end{aligned} \tag{35}$$

Finally, the first two normal modes are plotted in figures 14 and 15, respectively, and correspond to those expected from the previous analysis. These eigenmodes present discontinuous slope where the masses are located. These findings are in agreement with the previous results shown in [22], where a comprehensive study, both theoretical and experimental, of this system was reported.

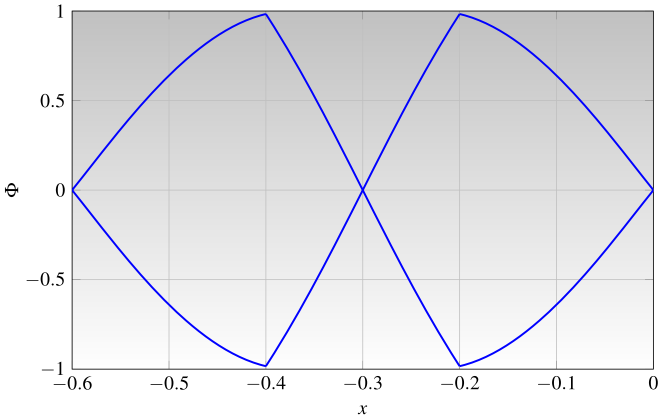

Figure 15. Wavefunction for the second mode of the system composed by two concentrated masses.

5. Conclusions

In this work, a simple method based on the impedance function was presented to study the propagation of waves through a locally periodic medium. Using a generalized impedance function, the method appears as a very useful tool since it can be easily applied to several physical systems that can be characterized by parameters to describe the discontinuous medium. Here, we used mass-chain systems as test cases considering that the corresponding mass density is discontinuous at the positions of the concentrated masses.

Computed transmission coefficients and normal eigenfrequencies for both mono and diatomic chains were in very good agreement with previous theoretical and experimental results [13].

Finally, it was shown that from the knowledge of the impedance function it is possible to infer the behaviour of the corresponding wavefunctions for finite systems. Moreover, it was shown that the wavefunctions can be obtained from the direct integration of the impedance. Indeed, for the particular case of a two-mass string studied here, our findings are in accord with previous results [22].

Acknowledgments

Authors acknowledge financial support from the Universidad Nacional de Rosario (PID ING408), and the Consejo Nacional de Investigaciones Científicas y Técnicas de la República Argentina.

References

[1] Bloch F 1928 *Z. Phys.* **52** 555–600
[2] Kronig R L and Penney W G 1931 *Proc. R. Soc.* A **130** 499–513
[3] Kushwaha M S, Halevi P, Dobrzynski L and Djafari-Rouhani B 1993 *Phys. Rev. Lett.* **71** 2022–5
[4] Ye Z and Hoskinson E 2000 *Appl. Phys. Lett.* **77** 4428–30
[5] Joannopoulos J D, Villeneuve P R and Fan S 1997 *Nature* **386** 143–9
[6] Scalora M, Bloemer M J, Pethel A S, Dowling J P, Bowden C M and Manka A S 1998 *J. Appl. Phys.* **83** 2377–83

[7] Abelès F 1950 *Ann. Phys. (Paris)* **5** 706–82
[8] Cvetič M and Pičman L 1981 *J. Phys. A: Math. Gen.* **14** 379–82
[9] Sprung D W L, Wu H and Martorell J J 1993 *Am. J. Phys.* **61** 1118–24
[10] Griffiths D J and Steinke C A 2001 *Am. J. Phys.* **69** 137–54
[11] Gómez B J, Repetto C E, Stia C R and Welti R 2009 *Eur. J. Phys.* **30** 1107–15
[12] Parmley S, Zobrist T, Clough T, Perez-Miller A, Makela M and Yu R 1995 *Am. J. Phys.* **63** 547–53
[13] Lüerßen D, Easwar N, Malhotra A, Hutchins L, Schulze K and Wilcox B 2004 *Am. J. Phys.* **72** 197–202
[14] Kinsler L E, Frey A R, Coppens A B and Sanders J V 2000 *Fundamentals of Acoustics* 4th edn (New York: Wiley)
[15] Khondker A N, Rezwan Khan M and Anwar A F M 1988 *J. Appl. Phys.* **63** 5191–3
[16] Chang C C and Kou C S 1999 *J. Phys. D: Appl. Phys.* **32** 139–46
[17] Gutiérrez-Medina B 2013 *Am. J. Phys.* **81** 104–11
[18] French A P 1982 *Vibrations and Waves* (New York: W W Norton and Company Inc.)
[19] Crawford F S 1967 *Waves* (*Berkeley Physics Course* vol 3) (New York: McGraw-Hill)
[20] Berman P R 2013 *Am. J. Phys.* **81** 190–201
[21] Vezzetti D J and Cahay M M 1986 *J. Phys. D: Appl. Phys.* **19** 53–55
[22] Gómez B J, Repetto C E, Stia C R and Welti R 2007 *Eur. J. Phys.* **28** 961–72