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Quantum chaotic resonances from short periodic orbits

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We present an approach to calculating the quantum resonances and resonance wave functions of chaotic scattering systems, based on the construction of states localized on classical periodic orbits and adapted to the dynamics. Typically only a few such states are necessary for constructing a resonance. Using only short orbits (with periods up to the Ehrenfest time), we obtain approximations to the longest-living states, avoiding computation of the background of short living states. This makes our approach considerably more efficient than previous ones. The number of long-lived states produced within our formulation is in agreement with the fractal Weyl law conjectured recently in this setting. We confirm the accuracy of the approximations using the open quantum baker map as an example.

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Quantum scattering in chaotic systems is a very active field in the area of quantum chaos, with current experimental realizations in ballistic quantum dots [1], microlasers [2] and microwave cavities [3], among others [4]. The fractal Weyl law [5], which relates the counting of resonances in the complex plane to the dimension of the trapped set of the corresponding classical dynamics, has attracted considerable attention [6–9] because resonances (or Gamow states) are central to the description of many aspects of wave scattering.

The decaying eigenstates associated with these quantum chaotic resonances are far from being fully understood. They have recently been shown [8,10] to display fractal structures in phase space when the resonance is long lived, that is, when the decay rate $\Gamma/\hbar$ remains finite as $\hbar \to 0$, and to be localized when the resonance is short lived, that is when $\hbar/\Gamma \to 0$. However, the semiclassical limit is much richer for scattering systems than for closed ones, for which the quantum ergodicity theorem [11] states that almost all states become uniform. Owing to the existence of different decay rates, nothing of this kind is available for scattering systems and we are still far from a complete description.

Our purpose here is to establish an approach to resonances and resonance wave functions based on short classical periodic orbits. The idea is to use the proliferation of periodic orbits in the phase space of chaotic systems to build an approximate basis of functions for the quantum Hilbert space. These functions are constructed in such a way as to contain dynamical information up to Ehrenfest time. This formulation has several virtues. First, the fractal Weyl law emerges very naturally from the theory and is seen to have a direct connection with periodic orbits. Second, we have an approximation to the quantum propagator that provides the long-lived states (which are usually dominant) without having to calculate short-lived states, therefore very significantly reducing the dimension of the matrices involved in the theory. Specifically, we achieve a power saving in the matrix dimension. Third, it turns out that usually only a few of our states are required to produce a quantum resonance, providing a way to quantitatively analyze scarring effects (anomalous localization of chaotic eigenstates around periodic orbits [12]). Finally, it opens a new and promising avenue for semiclassical approaches to resonance wave functions, which have so far been elusive.

A corresponding theory exists for closed systems in the form of *scar functions* [13,14], which have proved efficient in providing semiclassical approximations for quantum spectra and eigenstates of billiards [14] and quantum maps [15]. In the open systems considered here, the efficiency gain is considerably greater. An alternative periodic orbit approach to resonances already exists in terms of the semiclassical zeta function [16]. However, the orbits used are in general much longer than the ones considered here and the final result is an approximation to the spectral determinant that does not provide the wave functions.

For simplicity, we restrict ourselves to quantum maps, in which time evolution is discrete, the quantum Hilbert space has finite dimension $N=1/(2\pi\hbar)$ and the classical phase space is a torus. Open maps are defined by identifying a region of phase space—usually a strip of width M/N parallel to one of the axis—with a "hole" so that particles falling into that region are lost. This is a simplified but effective model for chaotic cavities with leads like the ones used in experiments with quantum dots. Quantum mechanically, the introduction of the hole corresponds to setting M rows (or columns) in the quantum propagator U to zero. Since it is no longer unitary, the new matrix $\tilde{U}$ has left and right eigenstates

$$\tilde{U}|\Psi_n^R\rangle = z_n|\Psi_n^R\rangle, \quad \langle\Psi_n^L|\tilde{U} = z_n\langle\Psi_n^L| \tag{1}$$

and we may choose the following normalization and orthogonality conditions:

$$\langle\Psi_n^R|\Psi_n^R\rangle = \langle\Psi_n^L|\Psi_n^L\rangle, \quad \langle\Psi_n^L|\Psi_m^R\rangle = \delta_{nm}. \tag{2}$$

The eigenvalues lie in the unit disk, $|z_n|^2 = e^{-\Gamma_n} \leq 1$, and the quantity $\Gamma_n \geq 0$ is interpreted as the decay rate. It was shown in [8] that in the semiclassical limit the long-lived left and right eigenstates localize on the stable and unstable mani-

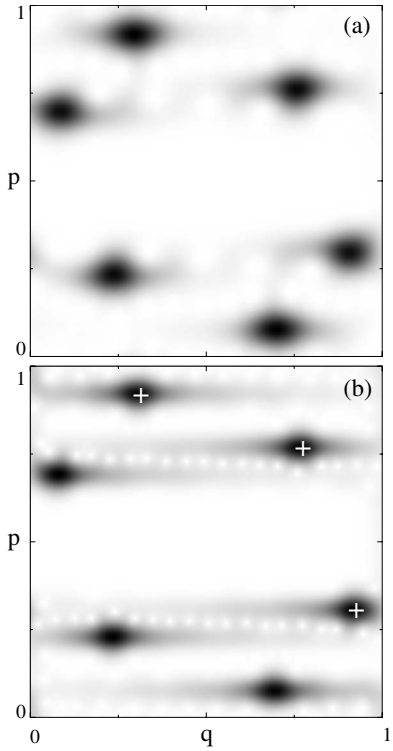

FIG. 1. Husimi representation of symmetrized right scar functions corresponding to a period 3 orbit of the triadic baker map, at (a) $N=81$ and (b) $N=243$. In panel (b) white crosses show the location of periodic points of the trajectory, and reflection by the diagonal produces a symmetric partner. Absolute value grows from white to black.

folds, respectively, of the classical trapped set (strange repeller).

We associate with every primitive periodic orbit γ of period L a total of L scar functions, as was done in [15] for closed maps. One starts by associating with each point of the orbit (q_j,p_j) a coherent state $|q_j,p_j\rangle$. In general, these states may have arbitrary deformations (squeezing) and this may be exploited [14]. For simplicity, we consider only circular states. One then builds a linear combination called "tube function," or periodic orbit mode,

$$|\phi_\gamma^k\rangle = \frac{1}{\sqrt{L}}\sum_{j=0}^{L-1} \exp\{-2\pi i(jA_\gamma^k - N\theta_j)\}|q_j,p_j\rangle. \tag{3}$$

Here k is any positive integer up to L, while $\theta_j=\Sigma_{l=0}^{j}S_l$ where S_l is the action acquired by the lth coherent state in one step of the map. The total action of the orbit is $\theta_L \equiv S_\gamma$. The quantity $A_\gamma^k=(NS_\gamma+k)/L$ is a Bohr-Sommerfeld-like eigenvalue,

$$U|\phi_\gamma^k\rangle \approx \frac{e^{2\pi i A_\gamma^k}}{\sqrt{\cosh \lambda}}|\phi_\gamma^k\rangle. \tag{4}$$

We denote by λ the Lyapounov exponent of the system.

The right and left scar functions associated with the periodic orbit are defined through the propagation, under the open map, of the tubes until around the Ehrenfest time $T_E=\frac{1}{\lambda}\ln N$. Namely,

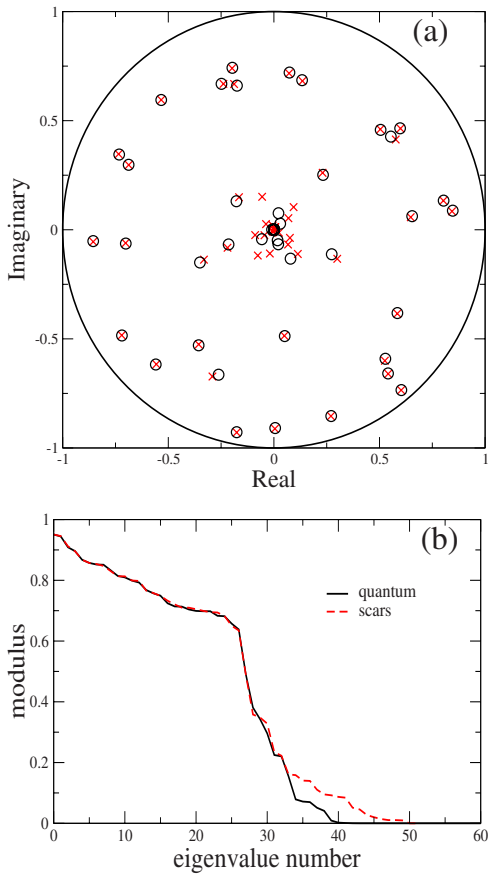

FIG. 2. (Color online) In (a) we show the spectrum of the open triadic baker map for $N=81$ (circles) and the spectrum of the scar matrix (crosses). Their moduli (ordered by decreasing value) are displayed in panel (b).

$$|\psi_{\gamma,k}^R\rangle = \frac{1}{\mathcal{N}_\gamma^R}\sum_{t=0}^{\tau} \tilde{U}^t e^{-2\pi i t A_\gamma^k}\cos\left(\frac{\pi t}{2\tau}\right)|\phi_\gamma^k\rangle, \tag{5}$$

and

$$\langle\psi_{\gamma,k}^L| = \frac{1}{\mathcal{N}_\gamma^L}\sum_{t=0}^{\tau} \langle\phi_\gamma^k|\tilde{U}^t e^{-2\pi i t A_\gamma^k}\cos\left(\frac{\pi t}{2\tau}\right). \tag{6}$$

The constants $\mathcal{N}^{R,L}$ are chosen such that $\langle\psi_{\gamma,k}^R|\psi_{\gamma,k}^R\rangle = \langle\psi_{\gamma,k}^L|\psi_{\gamma,k}^L\rangle$ and $\langle\psi_{\gamma,k}^L|\psi_{\gamma,k}^R\rangle=1$. The cosine is used to introduce a smooth cutoff, and the propagation time τ is taken of the order of T_E. In contrast to what is done for closed systems, we do not use negative powers of the matrix $\tilde{U}$; i.e., the tubes are propagated in only one direction in time. This implies that the phase-space support of right and left scar functions becomes localized on the unstable and stable manifolds, respectively, of the periodic orbit, in consonance with the properties of resonances. It is natural to order these resonant scar functions according to the modulus of $\langle\psi_{\gamma,k}^L|\tilde{U}|\psi_{\gamma,k}^R\rangle$ so that longest-living ones come first. Finally, we note that it is convenient to impose the symmetries of the map on these functions (if two orbits are related by symmetry, we build symmetric and antisymmetric scar functions).

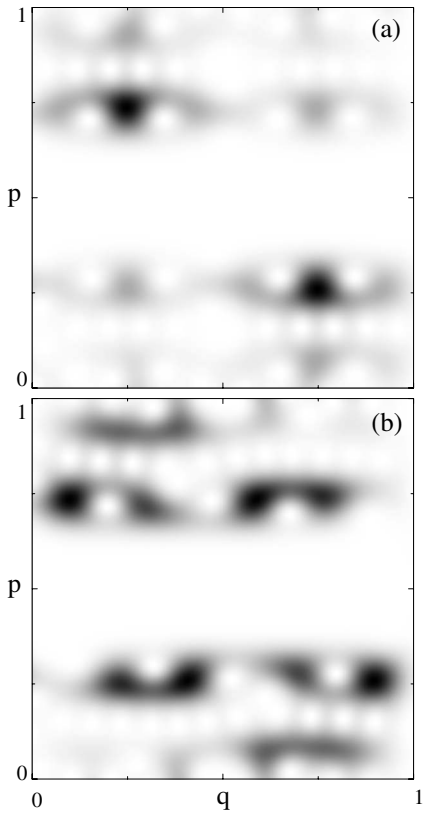

FIG. 3. Husimi representation of two exact right resonances, for the triadic baker map at $N=81$.

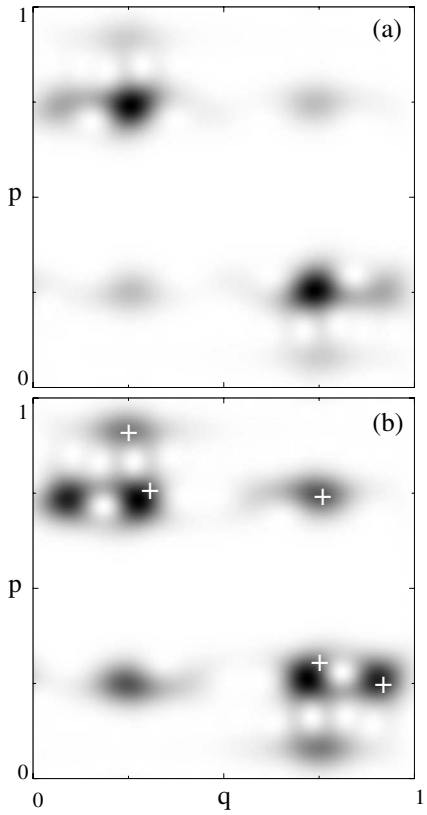

FIG. 4. Husimi representation of two symmetrized right scar functions corresponding to the same orbit of period 5, at $N=81$. White crosses show the periodic points of the orbit, and reflection through the diagonal produces a symmetric partner.

It is by now established [13–15] that it is only necessary to use short periodic orbits to obtain good approximations to quantum spectra and eigenstates. By 'short' we mean orbits with periods up to around the Ehrenfest time of the system. This is not unexpected, because T_E is the time when quantum interference effects become important. In the present case, since all periodic points are on the trapped set, the theory approximates only the long-lived states. Using the ordering mentioned above we construct the matrix $\langle \psi_n^L | \tilde{U} | \psi_m^R \rangle$, which we call the *scar matrix*, as an approximation to the "long-lived sector" of $\tilde{U}$. This matrix is by construction almost diagonal (in the sense that it equals a diagonal matrix plus a sparse one).

What is the dimension of the scar matrix? For chaotic systems the number of periodic points grows with the period L like e^{hL} where h is the topological entropy. Taking orbits with periods up to T_E we have e^{hT_E} periodic points and corresponding scar functions. However, for small openings h is related to the fractal (information) dimension of the trapped set by $d=2h/\lambda$ [17,18]. Since $e^{\lambda T_E}=N$ we conclude that the matrix dimension (and the number of long-lived states) scales with N as $N^{d/2}$, in agreement with the fractal Weyl law [5,7]. Note that the dimension of $\tilde{U}$ is N, so our approach leads to a power saving in the size of the matrices used. This therefore represents a considerable improvement in efficiency.

The above reasoning is in a sense complementary to the one presented in [6]. There the authors considered quantum states which escape from the system before the Ehrenfest time (short-lived states). As a consequence they were led to regions of phase space that are preimages of the hole. Conversely, we are attempting to construct the quantum states which do not escape from the system before the Ehrenfest time (long-lived states) and are thus lead to short periodic orbits on the repeller. Consistently, both approaches result in the fractal Weyl law.

As an example of the formalism, we use the triadic baker map, as in [8]. For this map the Lyapounov exponent is $\lambda = \ln 3$, and we choose the quantum dimension to be $N=3^k$ so that $T_E=k$. The trapped set is the Cartesian product Can $\times$ Can where Can is the usual middle-third Cantor set of dimension $\ln 2/\ln 3$. The fractal Weyl law therefore predicts that the number of long-lived states should grow like $N^{\ln 2/\ln 3}=2^k$ (this is actually the exact number if Walsh quantization [8] is used). Let us take for instance $k=4$ and build scar functions for orbits with period up to 5 (there are 51 periodic points in total). We illustrate this construction in Fig. 1, where we show the Husimi plots of a symmetrized right scar function corresponding to an orbit of period 3, at $N=81$ and $N=243$. It can clearly be seen that the probability extends along the unstable manifolds of this periodic orbit (q-axis direction). We have used $\tau=T_E$.

In Fig. 2 we present the exact quantum spectrum and the spectrum of the scar matrix (solution of a generalized eigenvalue problem since $\langle \psi_n^L | \psi_m^R \rangle \neq \delta_{nm}$), both for $N=81$. We see that the latter provides excellent approximations to the first

30 resonances: they are all reproduced accurately and moreover there are no spurious eigenvalues among them. We have verified that for $N=3^5$ and using orbits up to period 6 (matrix dimension 106) the first 55 resonances are reproduced accurately and without spurious eigenvalues. We have also verified that the method works well for other baker maps.

Figure 3 shows the Husimi functions of the right resonances number 3 and 12 (ordered according to decreasing moduli of eigenvalues), again for $N=81$. They are both strongly localized around periodic orbits, i.e., they are scarred. The corresponding eigenstates of the scar matrix are indistinguishable from the exact ones, showing that this matrix is indeed a good approximation to the long-lived sector of $\widetilde{U}$.

In Fig. 4 we show two symmetrized scar functions built from the same periodic orbit, of period 5. Because some of the periodic points are very near the symmetry lines in phase space (the diagonals of the square), we see that interference between the orbit and its symmetric partner makes the functions look rather different. Note the similarity between Fig. 3(a) and the scar function of Fig. 4(a). A single element of our base captures almost all the structure of an exact quantum eigenstate. On the other hand, the resonance shown in Fig. 3(b) results essentially from the combination of the scar function in Fig. 1(a) and the one in Fig. 4(b).

We postpone a more detailed analysis to a future publication, but the convenience of our approach to the study of the phase-space morphology of quantum resonances is clear. Indeed, scar functions have proved extremely useful in the study of scarring effects, providing for example ways to quantify scarring and to understand the influence of homoclinic motion on scars [19]. We expect it will also permit a more systematic study of scarring effects in open systems [20], a subject still in its infancy. For instance, there is certainly an interesting interplay between scarring and the decay rate, so that more scarred states are expected to live longer. We believe our approach will shed some light on this issue.

To conclude, we have introduced a theory based on short periodic orbits for quantum chaotic scattering. It may be argued that the use of the scar matrix offers no real advantage since its construction makes use of the quantum propagator. However, previous studies of closed systems [14] suggest that semiclassical approaches can be successfully implemented within this framework, because the propagation times involved are not longer than T_E. We are currently working in this direction. Another direction to follow is to adapt the theory to dielectric boundary conditions in order to treat microlasers, an application where spectacular manifestations of scarring can be observed [2].

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[1] S. Oberholzer, E. V. Sukhorukov, and C. Schonenberger, Nature (London) **415**, 765 (2002); W. Lu *et al.*, *ibid.* **423**, 422 (2003); J. Bylander, T. Duty, and P. Delsing, *ibid.* **434**, 361 (2005); E. V. Sukhorukov *et al.*, Nat. Phys. **3**, 243 (2007).

[2] C. Gmachl *et al.*, Science **231**, 486 (1998); S.-Y. Lee *et al.*, Phys. Rev. Lett. **93**, 164102 (2004); M. Lebental, J. S. Lauret, R. Hierle, and J. Zyss, Appl. Phys. Lett. **88**, 031108 (2006); T. Tanaka, M. Hentschel, T. Fukushima, and T. Harayama, Phys. Rev. Lett. **98**, 033902 (2007).

[3] C. Dembowski *et al.*, Phys. Rev. Lett. **93**, 134102 (2004); R. Schafer, H-J. Stockmann, T. Gorin, and T. H. Seligman, *ibid.* **95**, 184102 (2005); B. Dietz *et al.*, *ibid.* **98**, 074103 (2007); A. Backer *et al.*, *ibid.* **100**, 174103 (2008).

[4] *Trends in Quantum Chaotic Scattering*, special issue of J. Phys. A **38** (49) (2005).

[5] K. K. Lin and M. Zworski, Chem. Phys. Lett. **355**, 201 (2002); W. T. Lu, S. Sridhar, and M. Zworski, Phys. Rev. Lett. **91**, 154101 (2003).

[6] H. Schomerus and J. Tworzydlo, Phys. Rev. Lett. **93**, 154102 (2004).

[7] S. Nonnenmacher and M. Zworski, J. Phys. A **38**, 10683 (2005); Commun. Math. Phys. **269**, 311 (2007).

[8] J. P. Keating, M. Novaes, S. D. Prado, and M. Sieber, Phys. Rev. Lett. **97**, 150406 (2006); S. Nonnenmacher and M. Rubin, Nonlinearity **20**, 1387 (2007); J. P. Keating, S. Nonnenmacher, M. Novaes, and M. Sieber, *ibid.* **21**, 2591 (2008).

[9] D. L. Shepelyansky, Phys. Rev. E **77**, 015202(R) (2008).

[10] G. Casati, G. Maspero, and D. L. Shepelyansky, Physica D **131**, 311 (1999).

[11] A. I. Shnirelman, Usp. Mat. Nauk. **29**, 181 (1974); M. V. Berry, J. Phys. A **10**, 2083 (1977).

[12] E. J. Heller, Phys. Rev. Lett. **53**, 1515 (1984); E. B. Bogomolny, Physica D **31**, 169 (1988); M. V. Berry, Proc. R. Soc. London, Ser. A **423**, 219 (1989).

[13] E. Vergini, J. Phys. A **33**, 4709 (2000); E. Vergini and G. G. Carlo, *ibid.* **34**, 4525 (2001).

[14] E. Vergini and G. G. Carlo, J. Phys. A **33**, 4717 (2000); G. G. Carlo, E. Vergini, and P. Lustemberg, *ibid.* **35**, 7965 (2002).

[15] E. Vergini, D. M. Schneider, and A. M. F. Rivas, J. Phys. A **41**, 405102 (2008); L. Ermann and M. Saraceno, Phys. Rev. E **78**, 036221 (2008).

[16] P. Gaspard and S. A. Rice, J. Chem. Phys. **90**, 2242 (1989); P. Cvitanovic and B. Eckhardt, Phys. Rev. Lett. **63**, 823 (1989); Nonlinearity **6**, 277 (1993); P. Gaspard, D. Alonso, T. Okuda, and K. Nakamura, Phys. Rev. E **50**, 2591 (1994); A. Wirzba, Phys. Rep. **309**, 1 (1999).

[17] H. Kantz and P. Grassberger, Physica D **17**, 75 (1985).

[18] *Thermodynamics of Chaotic Systems*, edited by C. Beck and F. Schlogl (Cambridge University Press, Cambridge, 1993).

[19] D. A. Wisniacki, E. Vergini, R. M. Benito, and F. Borondo, Phys. Rev. Lett. **94**, 054101 (2005); Phys. Rev. Lett. **97**, 094101 (2006).

[20] D. Wisniacki and G. G. Carlo, Phys. Rev. E **77**, 045201(R) (2008).