Stationary arrays of vortices in Bose–Einstein condensates confined by a toroidal trap

2009 J. Phys. B: At. Mol. Opt. Phys. 42 145301

(http://iopscience.iop.org/0953-4075/42/14/145301)

J. Phys. B: At. Mol. Opt. Phys. **42** (2009) 145301 (7pp) doi:10.1088/0953-4075/42/14/145301

Stationary arrays of vortices in Bose–Einstein condensates confined by a toroidal trap

P Capuzzi[1,2] **and D M Jezek**[1,2]

[1] Departamento de Física, Facultad de Ciencias Exactas y Naturales, Universidad de Buenos Aires, RA-1428 Buenos Aires, Argentina
[2] Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina

E-mail: capuzzi@df.uba.ar and djezek@df.uba.ar

Received 19 February 2009, in final form 14 May 2009
Published 17 June 2009
Online at stacks.iop.org/JPhysB/42/145301

Abstract
We numerically study metastable arrays of vortices in three-dimensional Bose–Einstein condensates by solving the Gross–Pitaevskii equation with initial imprinted vorticity. We consider condensates confined by a harmonic plus Gaussian potential such as that used in a recent experiment. We analyse the energy barrier that prevents the vortices from leaving the trap and the spatial distribution of vortices for different trap parameters and winding numbers. For configurations forming rings of vortices we interpret the results in terms of the velocity fields produced by the vortices themselves and the spatial inhomogeneity of the condensate. For low enough densities, we found stationary configurations of multiply quantized vortices.

1. Introduction

Since the discovery of quantized vortices in helium 3 [1] and 4 [2], it has been known that the presence of quantized vortices is an unquestionable proof of the superfluidity of both ultracold liquids and gases [3]. In the context of Bose–Einstein condensation of atomic gases, following the investigation of single vortices in harmonic potentials, research has greatly diversified to include several new phenomena such as, e.g., the interplay of an additional rotating optical lattice to an existing vortex lattice [4], the study of topological spin-textures and novel strongly correlated phases in multi-component spinor gases [5], and the effects of long-range interactions [6]. Nonetheless, many aspects on the dynamics and stationary configurations of vortices in inhomogeneous superfluids still deserve to be investigated.

In a non-rotating parabolic trap, an off-axis vortex spirals away from the condensate in the presence of dissipation. This is due to the lack of an energy barrier between the vortex state and the ground state. The possibility of observing persistent flows is related to the degree of metastability of these systems. The larger the energy barrier, the longer the flow could survive [7]. A way to increase such an energy barrier is to alter the density profile of the ground state, which in turn can be achieved by varying the confinement. In particular, a density depression can be obtained by using a toroidal trap generated from the combination of a standard harmonic trap and a focused Gaussian beam. Density profile engineering thus appears as the simplest method to increase the vortex lifetime. When many vortices are involved, not only is the density relevant for the lifetime of the array but also the spatial distribution of the vortices. Furthermore, stationary arrays may serve to study more complex superflow patterns.

In this work, we obtain possible metastable configurations of vortices confined by toroidal traps by solving the full three-dimensional Gross–Pitaevskii equation with initial vorticity (GPEv), as we have described for a single vortex system [8]. Depending on the parameters of the confinement and the number of particles we have obtained either rings of singly quantized vortices or multiply quantized vortices. These configurations are obtained without imposing an external rotation as studied in previous works. In particular, the multiply quantized vortex has been treated in [9] where the full three-dimensional equation with an external rotation has been solved, while in a more recent work, approximated analytical solutions have been obtained in non-rotating traps [10].

By means of a constrained minimization procedure (CMP), we also explore the effects of the anisotropy of the confinement on the position dependence of the energy of a single vortex. We then address the issue of the structure of metastable arrays of vortices with the same circulation on pancake condensates. We compare these results with those given by the GPEv and with predictions based on the balance of the velocity fields developed for inhomogeneous superfluids. For a density profile with a deep enough local minimum, we found that multiply quantized vortices are energetically favoured instead of the rings of singly quantized vortices.

This paper is organized as follows. Section 2 introduces the standard model to describe vortices in Bose–Einstein condensates, section 3 discusses single vortex properties in anisotropic traps of current experiments, while sections 4.1 and 4.2 treat the cases of two and many vortices in pancake-shaped condensates, respectively. Finally, in section 5 the summary and concluding remarks are given.

2. The framework

At zero temperature an atomic Bose–Einstein condensate can be described by the Gross–Pitaevskii energy functional for its wavefunction ψ,

$$E[\psi] = \int \left(\frac{\hbar^2}{2m} |\nabla\psi|^2 + V_{\rm trap}|\psi|^2 + \frac{g}{2}|\psi|^4 \right) {\rm d}^3 r, \qquad (1)$$

where $V_{\rm trap}(\boldsymbol{r})$ is the external confining potential, m is the atom mass and $g = 4\pi a\hbar^2/m$ is the interaction strength with a being the s-wave scattering length. Minimization of the energy functional leads to the Gross–Pitaevskii equation (see, e.g., [11] and references therein), which reads

$$\left(-\frac{\hbar^2\nabla^2}{2m} + V_{\rm trap} + g|\psi|^2 \right) \psi = \mu\psi \qquad (2)$$

with μ being the chemical potential. If the initial state is vortex free and there is no imposed external rotation, the solution obtained is the ground state. Solutions of what we have called the GPEv are numerically found by initially imprinting a phase profile around the desired vorticity lines as described in [12]. In particular, a discrete set of N_v straight line vortices is imprinted using the wavefunction

$$\psi'(\boldsymbol{r}) = \left[\prod_{k=1}^{N_v} \frac{(x-x_k) + {\rm i}(y-y_k)}{\sqrt{(x-x_k)^2 + (y-y_k)^2}} \right] \psi_0(\boldsymbol{r}) \qquad (3)$$

as the initial guess for the minimization of the energy functional, where $\psi_0(\boldsymbol{r})$ is the ground-state wavefunction. We use a conjugate gradient method [13] to minimize the energy (1) as a function of ψ discretized in a spatial mesh of $256 \times 256 \times 128$ points in the x-, y- and z-directions, respectively. During the minimization, both the density profile and the velocity field are modified and as a consequence the vortex positions move either to infinity, corresponding to the ground state, or to a nearby metastable state if there exists an energy barrier between the initial state and the ground state. In other words, we obtain locally stable solutions of the Gross–Pitaevskii equation which include vortices.

In order to analyse the energy landscape, we have adopted a CMP which fixes the vortices' positions. We assume that the velocity fields corresponding to the vortex configurations are given by that of ψ' in (3) and minimize the energy as a function of the density profile only, while keeping the phases and thus the positions of the vortices fixed.

3. Vortices in anisotropic traps

In a recent experiment by Ryu *et al* [14], the persistence of flow around a toroidal condensate was observed for several seconds. The confinement was provided by a Gaussian potential added to the standard anisotropic harmonic oscillator potential as

$$V_{\rm trap}(\boldsymbol{r}) = \tfrac{1}{2}m\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right) + V_0\,{\rm e}^{-2(x^2+y^2)/w_0^2}, \qquad (4)$$

where w_0 is the width of the Gaussian beam and V_0 is its depth. This superflow can be produced by a singly quantized vortex oriented along the symmetry axis $(\hat{z})$ as obtained by solving the GPEv. The Gaussian beam created a hole in the density profile that pinned the vortex. In figure 1, we show two-dimensional views of the density profile obtained from the numerical solution of the GPEv for a condensate with the experimental parameters of [14]: $N = 2.5 \times 10^5$ particles, $V_0/h = 3.6$ kHz, $w_0 = 8\mu$ m and frequencies $\omega_x = 2\pi \times 51$ s^{-1}, $\omega_y = 2\pi \times 36$ s^{-1} and $\omega_z = 2\pi \times 25$ s^{-1}.

In a parabolic potential, a centred vortex is energetically unstable [15] and in the presence of dissipation it escapes from the trap causing the superflow to decay. In toroidal traps, on the other hand, vortices are known to be metastable (see, e.g., [16]), namely, they experience an energy barrier that may prevent them from leaving the trap at low temperature. It is therefore of fundamental importance to examine the energy landscape of the vortex in this case. To perform this analysis, we will assume that the vortex line is straight. Contrary to a three-dimensional condensate in a parabolic trap where vortices can bend [17], we have found straight single-vortex solutions of the GPEv for all the toroidal traps considered in this work. Therefore, we believe that this assumption is appropriate for describing these configurations.

It has been shown that for quasi-2D condensates the energy of a straight vortex, E_v, as a function of its position r_0 on the xy plane roughly follows the ground-state density profile $\rho_0(\boldsymbol{r})$ as (see, e.g., [8]),

$$E_v = \frac{1}{2}m \int {\rm d}^2 \boldsymbol{r}_\perp\,{\rm d}z\,\rho(\boldsymbol{r}) v^2(\boldsymbol{r}_\perp) \simeq \rho_0(r_0, z=0)F, \qquad (5)$$

thus indicating that the locations of the minima and maxima of the energy agree with those of the density profile. This approximation provides a direct mean to explore the energy landscape. One can immediately observe from (5) that as the energy barrier is proportional to the density, its shape can be controlled by the Gaussian beam through w_0 and V_0. However, on the one hand, the determination of an accurate non-phenomenological approximation for factor F is difficult, and on the other hand, equation (5) has been derived in pancake condensates where vortex lines can be assumed to be straight [17]. For these reasons, we have chosen to calculate the energy of the off-centred vortex directly using the CMP. We assume that the phase profile $\phi(\boldsymbol{r})$ of the condensate is fixed

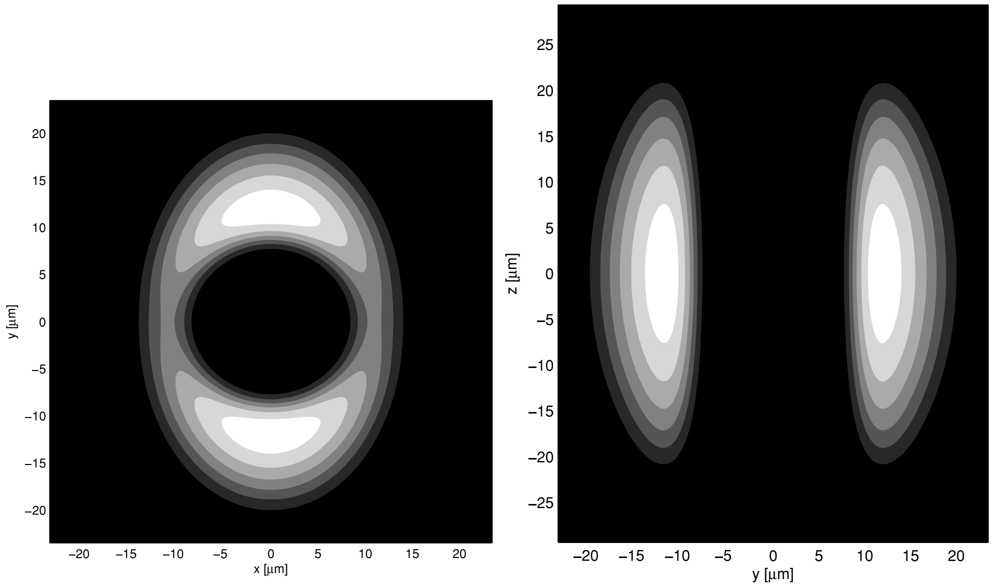

Figure 1. Two-dimensional density plots of a BEC of $N = 2.5 \times 10^5$ ^{23}Na atoms confined by the toroidal trap of [14]. The left panel depicts $\rho(x, y, z = 0)$ and the right panel $\rho(x = 0, y, z)$.

and behaves as that of an ideal vortex at a position $\boldsymbol{r}_0$, i.e., that it satisfies $\tan\phi(\boldsymbol{r}) = (y - y_0)/(x - x_0)$, and then we minimize the energy with respect to the density profile only (see, e.g., [11]). We note however that this assumption neglects the contribution of the background velocities that appear in inhomogeneous superfluids [12 ,18] and breaks down if the vortex is located close to the borders of the condensate where the effects of image vortices are likely to manifest [19–21]. The obtained energies are shown in figure 2 along the x- and y-directions. As qualitatively anticipated from (5) and the geometry of the confinement, the energy barrier experienced by a vortex displaced along the x-direction is smaller than when it is displaced along the y-direction. Experimentally, the trapping potential in a second set-up of Ryu *et al* [14] has been made closer to cylindrically symmetric to allow for the generation of higher circulation states. This finding has motivated the choice of the cylindrically symmetric trap for the study of vortex arrays to be considered hereafter.

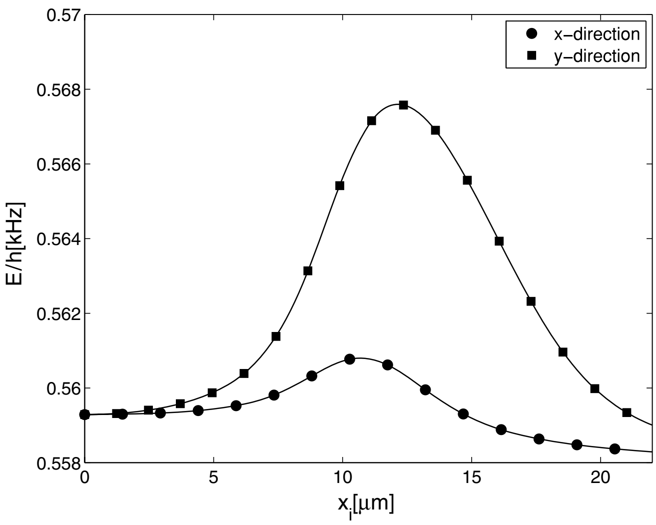

Figure 2. Energy of a vortex, E, as a function of its position along the x- (circles) and y-(squares) directions, for the same parameters as in figure 1.

4. Vortex arrays in pancake-shaped condensates

When several vortices are confined, the existence of stationary configurations is determined not only by the spatial inhomogeneity and geometry of the condensate but also by the presence of the other vortices. These configurations can be interpreted in terms of the balance of the velocity fields involved as introduced previously in [12] and in terms of the energy contributions by all the vortices as discussed here. We focus on cylindrically symmetric pancake-shaped condensates fixing $\omega_x = \omega_y = 2\pi \times 51$ s^{-1} and $\omega_z = 2\pi \times 400$ s^{-1}.

4.1. Two vortices

The simplest example of an array is a configuration with two singly quantized vortices with the same circulation. In stationary conditions, the positions of the vortices do not change and therefore the velocity field generated by a given vortex in the position of the other must be compensated by another velocity field, which we generically call background velocity, $\boldsymbol{v}_B$. This velocity originates from the local inhomogeneities and the shape of the density profile as discussed by several authors under different conditions [12, 18].

The velocity field of a straight vortex oriented along the $\hat{z}$-axis and crossing the xy plane at a position $\boldsymbol{r}_0$ is given by

$$\boldsymbol{v}_v(\boldsymbol{r}) = \frac{\hbar}{m} \frac{\hat{z} \times (\boldsymbol{r} - \boldsymbol{r}_0)}{|\boldsymbol{r} - \boldsymbol{r}_0|^2}, \tag{6}$$

where $\boldsymbol{r}$ lies in the xy plane. On the other hand, for the background velocity, $\boldsymbol{v}_B$, many approximate formulae are available [19, 22, 23] and have recently been tested [12]. The

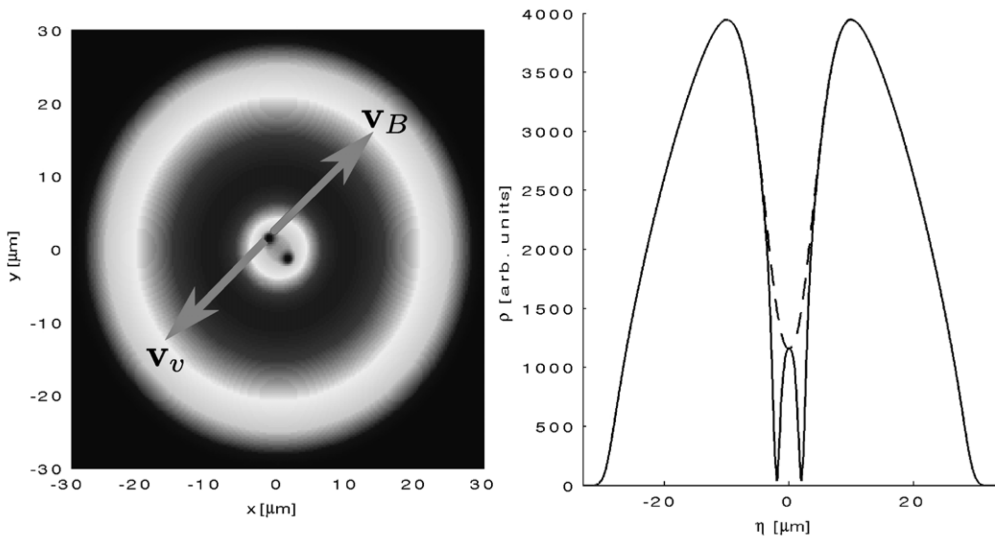

Figure 3. Density profile of an array of two vortices confined by the toroidal trap with $N = 10^6$, $V_0/h = 2$ kHz and $w_0 = 8$ μm. The left panel depicts a density plot of the array with arrows indicating the balancing contributions to the velocity field. The right panel shows the density profile of the array (solid line) together with the ground-state density (dashed line) along the direction in the xy plane joining the vortex cores, η.

crudest approximation, valid in the Thomas–Fermi (TF) limit, is given by [22, 24]

$$\boldsymbol{v}_B(\boldsymbol{r}) = -\frac{\hbar^2}{2m}\frac{\hat{z}\times\nabla\rho_0(\boldsymbol{r})}{\rho_0(\boldsymbol{r})}\ln\left(\frac{R_{\mathrm{TF}}}{\xi}\right), \tag{7}$$

where $\xi = [8\pi\rho_0(r_0)a]^{-1/2}$ is the healing length and R_{TF} is the TF radius of the cloud. Due to the cylindrical symmetry of the trapping potential, $\boldsymbol{v}_B$ only has an azimuthal component and thus it is only possible to find stationary configurations when the two points at which the vortices cross the xy plane are in the same line that the centre of the trap, $\boldsymbol{r} = 0$. Furthermore, since the vortices have the same circulation, they must be in opposite positions with respect to the same centre, namely, at $\boldsymbol{r}_0$ and $-\boldsymbol{r}_0$. Equating (6) and (7) at position $-\boldsymbol{r}_0$ determines the equilibrium radius r_{eq} and also shows that a larger value of r_{eq} requires a smaller derivative of the density profile. We want to note that in order to obtain stationary configurations using these velocity formulae the gradient of the density should point in the radially outward direction and thus the only possible radii should be smaller than the position of the density maximum. In this domain there are two solutions: one that corresponds to the desired local energy minimum and the other one, as we shall see, to an energy maximum.

Numerically, the locally stable vortex configuration is obtained exactly by solving the GPEv initiated with the phase imprinting technique. We found that the vortex lines are indeed straight and the locations found validate the interpretation in terms of the balance of velocities. However, a quantitative agreement requires the use of a more accurate formula for $\boldsymbol{v}_B$ (see [12]). This balance is schematically illustrated in figure 3 for $N = 10^6$ ^{23}Na atoms in the toroidal trap with $V_0/h = 2$ kHz and $w_0 = 8$ μm, corresponding to a shallower Gaussian depression within the range of experimental parameters.

Similarly to the single-vortex situation, the energy minima of two vortices placed at opposite positions correspond to a

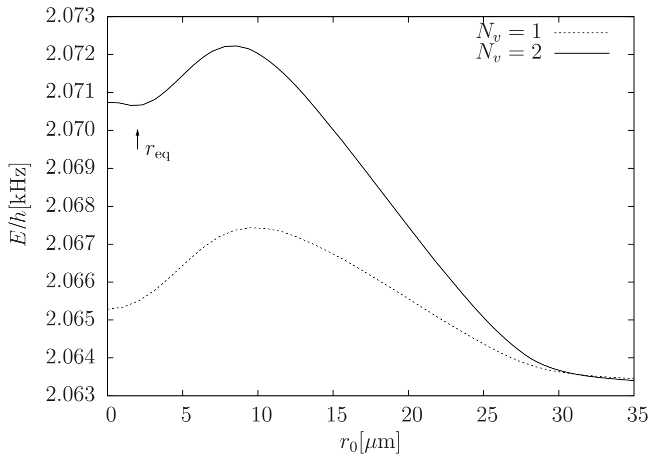

Figure 4. Energy of two ideal vortices in opposite positions, E, as a function of their common radius r_0 for the same parameters as in figure 3. The dashed line marks the result for a single vortex at a distance r_0.

stationary configuration of two vortices, which we found in good agreement with both the numerical solution of the GPEv and the balance of velocities. In figure 4, we plot the variational energy of the two vortices as a function of their positions. For large values of r_0, the vortices leave the condensate and the energy goes to the ground-state value, whereas for low values of r_0 the configuration displays an energy minimum and a barrier which corresponds to the energy maximum we have mentioned before.

It is worthwhile to stress that while the determination of the equilibrium configuration *via* the balance of velocities requires the calculation of the ground-state density profile only (cf equation (7)), its determination *via* the minimization of the energy of ideal vortices requires the calculation of several excited states, namely, the vortices' configurations. Moreover,

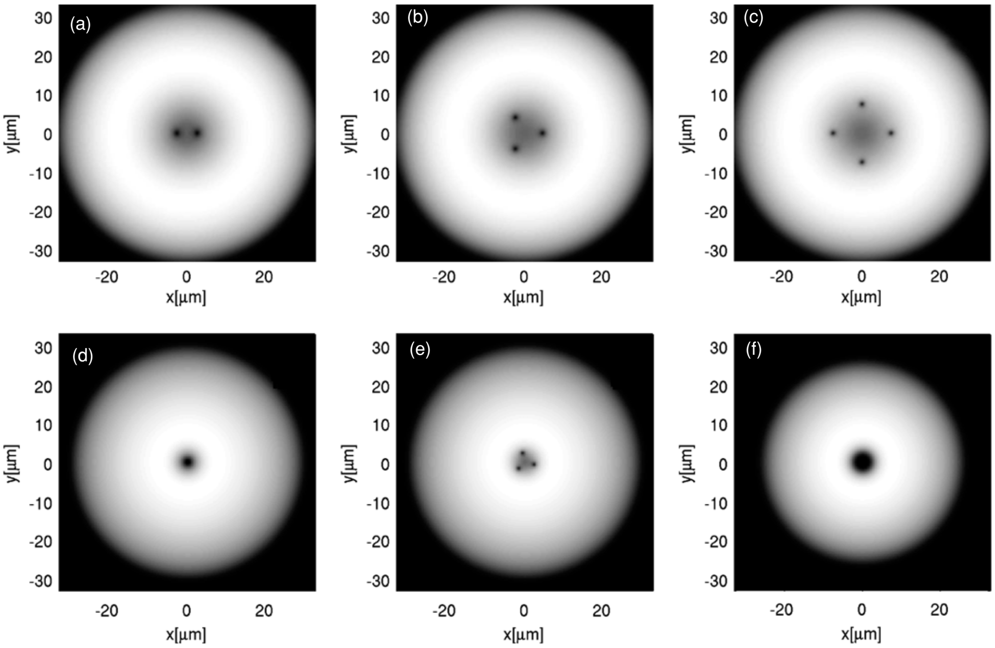

Figure 5. Two-dimensional view of the density profile at $z = 0$, $\rho(x, y, z = 0)$ for arrays with $N_v = 2$, 3 and 4 vortices for several values of the Gaussian beam and particle numbers. Panels (a)–(c) correspond to $V_0/h = 3$ kHz, $w_0 = 20\ \mu$m, and $N = 10^6$, (d)–(e) $V_0/h = 2.3$ kHz, $w_0 = 5.3\ \mu$m, and $N = 10^6$, and (f) $V_0/h = 2.3$ kHz, $w_0 = 5.3\ \mu$m, and $N = 5 \times 10^5$. Panels (d) and (f) correspond to multiply quantized vortices with winding numbers 2 and 4, respectively.

the same value of the ground-state density may be used for different arrays of vortices provided one knows the geometry of the array, as discussed in the following section.

4.2. Rings and multiply quantized vortices

Stationary configurations of rings of vortices can easily be predicted also for larger N_v. A ring of vortices is an array of regularly distributed vortices along a ring of radius r_0 in the xy plane. In a previous work [12], we have shown that the velocity field on a given vortex at position $\boldsymbol{r}_0$ due to the remaining $N_v - 1$ vortices in the ring gives

$$\boldsymbol{v}_v(\boldsymbol{r}_0) = \frac{\hbar}{m}\frac{(N_v - 1)}{2}\frac{\hat{z} \times \boldsymbol{r}_0}{|\boldsymbol{r}_0|^2}. \tag{8}$$

The balance equation in this case can be used to study how many vortices can be sustained and how the radii of their rings compare. Indeed, it is found that the radius increases with the increasing number of vortices and that the maximum number of vortices is related to the maximum of the logarithmic derivative of the density profile. We have fixed the trap frequencies and explored the parameter space (V_0, w_0) with realistic values in order to find different types of configurations. In particular, for a simply connected ground-state condensate we found arrays with up to four vortices. When imprinting a larger vorticity the remaining vortices escape from the condensate. Finally to obtain multiply quantized vortices, we also lower the number of particles. The density profiles of the metastable configurations at plane $z = 0$ obtained in this way are depicted in figure 5. For $N = 10^6$ (panels (a)–(e)), the results can be summarized as: (i) for a given set of parameters of the Gaussian beam, the radius of the ring increases with the number of vortices, (ii) for wider beams, i.e., larger w_0 (panels (a)–(c)) the system can sustain larger arrays and (iii) for lower densities at the trap centre, the radius of the ring decreases as well. Indeed, if the central density is below a certain threshold, one may have a multiply quantized vortex instead of a ring of several vortices. This type of vortices, also called giant vortices, have recently been obtained numerically in a 2D rotating two-species condensate [25]. Examples of the configurations we have obtained are shown in panels (d) and (f) for 2 and 4 winding numbers, respectively.

We have further evaluated the variational energy for the ring of vortices as a function of its radius r_0 with $N_v = 1, 2, 3,$ and 4. These curves are depicted in the top panel of figure 6. As in the case of two vortices, the energy minimum is shifted in a distance r_{eq} from the origin. While using a lower number of particles and thus a lower central density, the energy minimum remains at the origin and thus a multiple quantized vortex is more favourable. This is displayed in the bottom panel of figure 6. In the inset of the same figure, it may be seen that the presence of the multiply quantized vortex is manifested in the density by the broadening of the hole for increasing vorticity.

5. Summary and concluding remarks

We analysed metastable configurations of vortices in a non-rotating condensate of current experimental relevance. We evaluated the lowest energy barrier for the decay of single vortices in the experimental non-axisymmetric trap and considered cylindrically symmetric traps allowing the formation of configurations with larger vorticities. In the

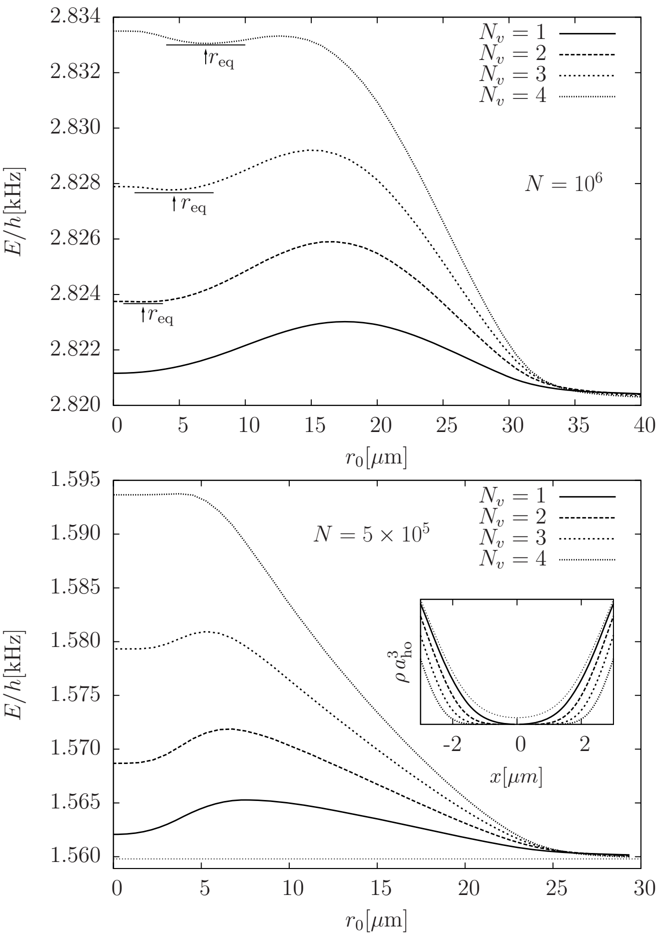

Figure 6. Energy, E, of a ring of N_v ideal vortices as a function of its radius, r_0, for several values of N_v. The top panel corresponds to the same parameters as panels (a)–(c) in figure 5, while the bottom panel corresponds to the parameters of panel (f) in figure 5. The vertical arrows mark the equilibrium radii of the rings. The inset in the bottom panel shows the density at $y = z = 0$ as obtained from the numerical solution of the GPEv for a multiply quantized vortex of charge N_v at the trap centre. The thin dotted lines correspond to ground-state results.

cylindrically symmetric trap the stationary solutions were either a ring of singly quantized vortices or a multiply quantized vortex localized at the trap center depending on the number of particles. Using the balance of the associated velocity fields, we predict the kind of stationary states we may obtain from the knowledge of the density profile of the ground state only. On the other hand, the knowledge of the energy landscape of the configuration obtained from a constrained minimization procedure allows to determine the metastable stationary configuration and additionally provides information on the energy barriers involved. Both predictions are in agreement with the full numerical solution of the Gross–Pitaevskii equation in three dimensions. At low but finite temperature, in the presence of a small normal cloud of uncondensed atoms, we expect these configurations to be rather robust as they are local minima of the energy. This is also supported by the experiments in [14] where the superflow survives even for systems with a condensate fraction as small as 20%. Furthermore, the obtained energy barriers could be used for calculating the vortex lifetimes by means of quantum tunnelling approaches [26].

Acknowledgments

This work was partially supported by grants PIP 5409/05 and 5138/05 from CONICET, PICT 31980/05 from ANPCyT and X607 from UBA, Argentina.

References

[1] Vollhardt D and Wölfe P 1990 *The Superfluid Phases of Helium 3* (Boca Raton, FL: CRC press)
[2] Donnelly R J 1991 *Quantized Vortices in Helium II* (Cambridge: Cambridge University Press)
[3] Zwierlein M W, Abo-Shaeer J R, Schirotzek A, Schunck C H and Ketterle W 2005 *Nature* **435** 1035
[4] Tung S, Schweikhard V and Cornell E A 2006 *Phys. Rev. Lett.* **97** 240402
Hafezi M, Rensen A S S, Demler E and Lukin M D 2007 *Phys. Rev.* A **76** 023613
[5] Yip S K 1999 *Phys. Rev. Lett.* **83** 4677
Mizushima T, Machida K and Kita T 2002 *Phys. Rev. Lett.* **89** 030401
Martikainen P, Collin A and Suominen K A 2002 *Phys. Rev.* A **66** 053604
Kita T, Mizushima T and Machida K 2002 *Phys. Rev.* A **66** 061601
Ho T L and Mueller E J 2002 *Phys. Rev. Lett.* **89** 050401
[6] Santos L, Shlyapnikov G V, Zoller P and Lewenstein M 2000 *Phys. Rev. Lett.* **85** 1791
Griesmaier A, Werner J, Hensler S, Stuhler J and Pfau T 2005 *Phys. Rev. Lett.* **94** 160401
Cooper N R, Rezayi E H and Simon S H 2005 *Phys. Rev. Lett.* **95** 200402
Zhang J and Zhai H 2005 *Phys. Rev. Lett.* **95** 200403
[7] Weiler C N, Neely T W, Scherer D R, Bradley A S, Davis M J and Anderson B P 2008 *Nature* **455** 948
[8] Jezek D M, Capuzzi P and Cataldo H M 2008 *J. Phys. B: At. Mol. Opt. Phys.* **41** 045304
[9] Simula T P, Virtanen S M M and Saloma M M 2002 *Phys. Rev.* A **65** 033614
[10] Cozzini M, Jackson B and Stringari S 2006 *Phys. Rev.* A **73** 013603
[11] Dalfovo F, Giorgini S, Pitaevskii L P and Stringari S 1999 *Rev. Mod. Phys.* **71** 463
[12] Jezek D M, Capuzzi P, Guilleumas M and Mayol R 2008 *Phys. Rev.* A **78** 053616
[13] Press W H, Teukolsky S A and Flannery B P 1992 *Numerical Recipes in Fortran 77: The Art of Scientific Computing* (Cambridge: Cambridge University Press)
[14] Ryu C, Andersen M F, Cladé P, Natarajan V, Helmerson K and Phillips W D 2007 *Phys. Rev. Lett.* **99** 260401
[15] Rokhsar D S 1997 *Phys. Rev. Lett.* **79** 2164
[16] Tempere J, Devreese J T and Abraham E R I 2001 *Phys. Rev.* A **64** 023603
[17] Modugno M, Pricoupenko L and Castin Y 2003 *Eur. Phys. J.* D **22**
García-Ripoll J J and Pérez-García V M 2001 *Phys. Rev.* A **63** 041603
García-Ripoll J J and Pérez-García V M 2001 *Phys. Rev.* A **64** 053611
[18] Mason P and Berloff N G 2008 *Phys. Rev.* A **77** 032107
[19] Lundh E and Ao P 2000 *Phys. Rev.* A **61** 063612

[20] Guilleumas M and Graham R 2001 *Phys. Rev.* A **64** 033607
[21] Kim J K and Fetter A L 2004 *Phys. Rev.* A **70** 043624
[22] Sheehy D E and Radzihovsky L 2004 *Phys. Rev.* A **70** 063620
[23] Nilsen H M, Baym G and Pethick C J 2006 *Proc. Natl Acad. Sci. USA* **103** 7978
[24] Svidzinsky A A and Fetter A L 2000 *Phys. Rev.* A **62** 063617
[25] Yang S J, Wu Q S, Zhang S N and Feng S 2008 *Phys. Rev.* A **77** 033621
[26] Kagan Yu and Leggett A J (ed) 1992 *Quantum Tunnelling in Condensed Media* (Amsterdam: North-Holland)