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Algebra Universalis

On implicator groupoids

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ABSTRACT. In a paper published in 2012, the second author extended the well-known fact that Boolean algebras can be defined using only implication and a constant, to De Morgan algebras—this result led him to introduce, and investigate (in the same paper), the variety $\mathcal{I}$ of algebras, there called implication zroupoids (**I**-zroupoids) and here called implicator groupoids ($\mathcal{I}$-groupoids), that generalize De Morgan algebras.

The present paper is a continuation of the paper mentioned above and is devoted to investigating the structure of the lattice of subvarieties of $\mathcal{I}$, and also to making further contributions to the theory of implicator groupoids. Several new subvarieties of $\mathcal{I}$ are introduced and their relationship with each other, and with the subvarieties of $\mathcal{I}$ which were already investigated in the paper mentioned above, are explored.

1. Introduction

Boolean algebras can be defined using only implication and a constant. In 2012, this result was extended to De Morgan algebras in [12] which led the second author of this paper to introduce, and investigate, the variety $\mathcal{I}$ of *implicator groupoids* (there called implication zroupoids).

Definition 1.1. An algebra $\mathbf{A} = \langle A, \rightarrow, 0 \rangle$, where $\rightarrow$ is binary, 0 is a constant, and x' is defined by $x' := x \rightarrow 0$, is called an *implicator groupoid* ($\mathcal{I}$-groupoid, for short) if $\mathbf{A}$ satisfies:

(I) $(x \rightarrow y) \rightarrow z \approx [(z' \rightarrow x) \rightarrow (y \rightarrow z)']'$,
($\mathrm{I_0}$) $0'' \approx 0$.

The term "implicator" has been extensively used in fuzzy logic and fuzzy set theory; see for example [9]. Throughout this paper, $\mathcal{I}$ denotes the variety of *implicator groupoids*.

The present paper is a continuation of [12] and is devoted to investigating the structure of the lattice of subvarieties of $\mathcal{I}$, and also to making further contributions to the theory of implicator groupoids. Several new subvarieties of $\mathcal{I}$ are introduced and their relationship with each other and with the subvarieties of $\mathcal{I}$, which were already investigated in [12], are explored.

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Definition 1.2. The varieties $\mathcal{MID}$, $\mathcal{JID}$, and $\mathcal{I}_{2,0}$ are defined relative to $\mathcal{I}$, respectively, by

(MID) $x \wedge x \approx x,$
(JID) $x \vee x \approx x,$
($\text{I}_{2,0}$) $x'' \approx x.$

The paper is organized as follows. In Section 2, we focus on the variety $\mathcal{I}_{2,0}$. We present several fundamental identities that hold in $\mathcal{I}_{2,0}$ and that play an essential role in the rest of the paper. We prove that $\mathcal{I}_{2,0} = \mathcal{MID} = \mathcal{JID}$. We also show that the implicator groupoids which are implication algebras in the sense of Abbott are precisely Boolean algebras.

In this paper, we use the characterizations of De Morgan algebras, Kleene algebras, and Boolean algebras obtained in [12] as definitions.

Definition 1.3. $\mathbf{A} \in \mathcal{I}$ is a *De Morgan algebra* (**DM**-algebra for short) if $\mathbf{A}$ satisfies the axiom

(DM) $(x \to y) \to x \approx x.$

A **DM**-algebra $\mathbf{A}$ is a *Kleene algebra* (**KL**-algebra for short) if $\mathbf{A}$ satisfies either of the equivalent axioms

(KL_1) $(x \to x) \to (y \to y)' \approx x \to x,$
(KL_2) $(y \to y) \to (x \to x) \approx x \to x.$

A **DM**-algebra $\mathbf{A}$ is a *Boolean algebra* (**BA**-algebra for short) if $\mathbf{A}$ satisfies the axiom

(BA) $x \to x \approx 0'.$

We denote by $\mathcal{DM}$, $\mathcal{KL}$, and $\mathcal{BA}$, respectively, the varieties of **DM**-algebras, **KL**-algebras, and **BA**-algebras.

Section 3 characterizes Boolean algebras and Kleene algebras as subvarieties of $\mathcal{I}_{2,0}$. In Section 4, we prove a Glivenko-like theorem for algebras in $\mathcal{I}$.

Throughout this paper, we use the following definitions:

$$(\text{M}) \quad x \wedge y := (x \to y')' \quad \text{and} \quad (\text{J}) \quad x \vee y := (x' \wedge y')'.$$

With each $\mathbf{A} \in \mathcal{I}$, we associate the following algebra: $\mathbf{A}^{\mathbf{mj}} := \langle A, \wedge, \vee, 0 \rangle$.

Definition 1.4. The varieties $\mathcal{SCP}$ and $\mathcal{MC}$ are defined relative to $\mathcal{I}$, respectively, by

(SCP) $x \to y \approx y' \to x',$
(MC) $x \wedge y \approx y \wedge x.$

Section 5 discusses the relationship between the varieties $\mathcal{SCP}$ and $\mathcal{MC}$. It is shown that $\mathcal{SCP} \subset \mathcal{MC}$, and, moreover, if we restrict them to $\mathcal{I}_{2,0}$, they coincide with each other.

Section 6 determines the algebras $\mathbf{A} \in \mathcal{I}$ such that the induced algebra $\mathbf{A}^{\mathbf{mj}} = \langle A, \wedge, \vee, 0 \rangle$ is a lattice. Surprisingly, it turns out that this is the case precisely when the absorption identity holds, which, in turn, is equivalent to

A being a De Morgan algebra. The section ends with another characterization of De Morgan algebras as a subvariety of $\mathcal{I}_{2,0}$. In Section 7, we give several interesting properties of $\mathcal{I}$ and of the variety $\mathcal{I}_{2,0} \cap \mathcal{MC}$.

Definition 1.5. The varieties $\mathcal{Z}, \mathcal{C}, \mathcal{A}, \mathcal{I}_{3,1}, \mathcal{I}_{1,0}$, and $\mathcal{ID}$ are defined relative to $\mathcal{I}$, respectively, by

$\quad$ (Z) $\quad x \to y \approx 0$,
$\quad$ (C) $\quad x \to y \approx y \to x$,
$\quad$ (CP) $\quad x \to y' \approx y \to x'$,
$\quad$ (A) $\quad (x \to y) \to z \approx x \to (y \to z)$,
$\quad$ (I$_{3,1}$) $\quad x''' \approx x'$,
$\quad$ (I$_{1,0}$) $\quad x' \approx x$,
$\quad$ (ID) $\quad x \to x \approx x$.

$\quad$ Section 8 investigates relationships among the varieties $\mathcal{Z}, \mathcal{C}, \mathcal{A}$, and $\mathcal{I}_{3,1}$. It is proved that $\mathcal{Z} \subset \mathcal{C} \subset \mathcal{A} \subset \mathcal{I}_{3,1}$. In Section 9, it is shown that $\mathcal{I}_{1,0} = \mathcal{ID} \cap \mathcal{A}$.

$\quad$ In Section 10, the three 2-element algebras $\mathbf{2}_z$, $\mathbf{2}_s$, and $\mathbf{2}_b$ of $\mathcal{I}$ are recalled and the varieties they generate, denoted $\mathcal{V}(\mathbf{2}_z)$, $\mathcal{V}(\mathbf{2}_s)$, and $\mathcal{V}(\mathbf{2}_b)$, respectively, are characterized. As a consequence, the variety of join semilattices with least element, denoted $\mathcal{SL}$, can be viewed as $\mathcal{C} \cap \mathcal{I}_{1,0} \subset \mathcal{I}$, from which it follows, in view of a well-known result (see [7]), that the congruence lattices of algebras in $\mathcal{I}$ do not satisfy any nontrivial lattice identities.

$\quad$ Section 11 proves that $\mathcal{MC} \cap \mathcal{MID} \cap \mathcal{A} = \mathcal{SL} \subset \mathcal{CP}$. In Section 12, we prove that $\mathcal{MC} \cap \mathcal{ID} = \mathcal{C} \cap \mathcal{I}_{1,0} = \mathcal{V}(2_s)$. The paper concludes with some remarks about results that are contained in [4], [5], and [6].

2. The variety $\mathcal{I}_{2,0}$

$\quad$ In this section, we focus on the variety $\mathcal{I}_{2,0}$, an important subvariety of $\mathcal{I}$. We present several fundamental identities that hold in $\mathcal{I}_{2,0}$ and play an essential role in the rest of this paper. We also give two new characterizations of $\mathcal{I}_{2,0}$ and show that the algebras in $\mathcal{I}$ that are implication algebras in the sense of Abbott are precisely Boolean algebras.

$\quad$ The following lemmas are frequently used in the sequel.

Lemma 2.1 ([12, Theorem 8.15]). *Let* $\mathbf{A} \in \mathcal{I}$. *Then the following are equivalent in* $\mathbf{A}$:

$\quad$ (a) $0' \to x \approx x$,
$\quad$ (b) $x'' \approx x$,
$\quad$ (c) $(x \to x')' \approx x$,
$\quad$ (d) $x' \to x \approx x$.

Lemma 2.2 ([12, Lemma 8.13 (e),(f)]). *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then* $\mathbf{A}$ *satisfies*

$\quad$ (a) $x' \to 0' \approx 0 \to x$,
$\quad$ (b) $0 \to x' \approx x \to 0'$.

Lemma 2.3 ([12, Theorem 8.15]). *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then* $\mathbf{A} \models x \wedge x \approx x$.

The following lemma is a restatement of [12, Theorem 5.1].

Lemma 2.4. *Let* $\mathbf{A} \in \mathcal{I}$. *Then* $\mathbf{A}$ *is a De Morgan algebra iff*
$\mathbf{A} \models (0 \to x) \to y \approx y$.

Lemma 2.5 ([12, Lemma 7.5b]). *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then*
$\mathbf{A} \models (x \to y'')' \approx (x \to y)'$.

Lemma 2.6 ([12, Theorem 7.6]). *Let* $\mathbf{A} \in \mathcal{I}$. *Then* $\mathbf{A} \models x''' \to y \approx x' \to y$.

The following lemma plays a crucial role in the rest of the paper.

Lemma 2.7. *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then* $\mathbf{A}$ *satisfies the following:*

(1) $(x \to 0') \to y \approx (x \to y') \to y$,

(2) $x \to (0 \to x)' \approx x'$,

(3) $(y \to x) \to y \approx (0 \to x) \to y$,

(4) $[(x \to 0') \to y]' \approx (0 \to x) \to y'$,

(5) $0 \to x \approx 0 \to (0 \to x)$,

(6) $[x' \to (0 \to y)]' \approx (0 \to x) \to (0 \to y)'$,

(7) $[x \to (y \to x)']' \approx (x \to y) \to x$,

(8) $0 \to [(0 \to x) \to (0 \to y')'] \approx 0 \to (x \to y)$,

(9) $0 \to \{(0 \to x) \to y'\} \approx x \to (0 \to y')$,

(10) $0 \to (0 \to x)' \approx 0 \to x'$,

(11) $0 \to (x \to y) \approx x \to (0 \to y)$,

(12) $[(0 \to x) \to y] \to x \approx [(y \to x) \to (0 \to x)']'$,

(13) $[\{(0 \to x) \to y\} \to x]' \approx (y \to x) \to (0 \to x)'$,

(14) $(0 \to x') \to (y \to x) \approx y \to x$,

(15) $x' \to (0 \to x) \approx 0 \to x$,

(16) $(y \to x)' \approx (0 \to x) \to (y \to x)'$,

(17) $(x \to y) \to (0 \to y)' \approx (x \to y)'$,

(18) $0 \to (x' \to y)' \approx x \to (0 \to y')$,

(19) $[(x \to y) \to x] \to [(y \to x) \to y] \approx x \to y$,

(20) $[x \to (y \to x')] \to x \approx x' \to (y \to x')'$,

(21) $x \to (y \to x') \approx y \to x'$,

(22) $0 \to (x \to y')' \approx 0 \to (x' \to y)$,

(23) $(x \to y) \to y' \approx y \to (x \to y)'$,

(24) $x \to [(y \to z') \to x]' \approx (x' \to y) \to \{(0 \to z) \to x'\}$,

(25) $[\{((x \to 0') \to y) \to z\} \to \{u \to ((0 \to x) \to y')\}']'$
$$\approx (z \to u) \to \{(0 \to x) \to y'\},$$

(26) $(z \to x) \to (y \to z) \approx (0 \to x) \to (y \to z)$,

(27) $(x' \to y) \to [(0 \to z) \to x'] \approx (0 \to y) \to [(0 \to z) \to x']$,

(28) $x \to [(y \to z') \to x]' \approx (0 \to y) \to [(0 \to z) \to x']$,

(29) $(x' \to y) \to (x \to y') \approx x \to y'$,

(30) $(x \to y')' \to (x' \to y)' \approx x \to y'$.

Proof. We only give proofs of items (24)–(30) here. We refer the interested reader to the appendix of the arxiv version, `arxiv:1509.03774v2[math.LO]` `9Jun2016`, of this paper which is available online at `http://www.arxiv.org`, where detailed proofs of all the items of this lemma are given.

(24): $x \to [(y \to z') \to x]' \overset{(7)}{\approx} [\{x \to (y \to z')\} \to x]'$

$\overset{(3)}{\approx} [\{0 \to (y \to z')\} \to x]' \overset{(11)}{\approx} [\{y \to (0 \to z')\} \to x]'$

$\overset{(I)}{\approx} (x' \to y) \to [(0 \to z') \to x]' \overset{2.2(a)}{\approx} (x' \to y) \to [(z \to 0') \to x]'$

$\overset{(4)}{\approx} (x' \to y) \to [(0 \to z) \to x'].$

(25): First we have $\{(0 \to x) \to y'\}' \to z \overset{2.2(a)}{\approx} [(x' \to 0') \to y']' \to z$

$\overset{(4)}{\approx} [(0 \to x') \to y] \to z \overset{2.2(a)}{\approx} [(x \to 0') \to y] \to z.$

Hence,

$$[(0 \to x) \to y']' \to z \approx [(x \to 0') \to y] \to z. \qquad (2.1)$$

Then, $(z \to u) \to [(0 \to x) \to y']$

$\overset{(I)}{\approx} [\{((0 \to x) \to y')' \to z\} \to \{u \to ((0 \to x) \to y')\}']'$

$\overset{eq(2.1)}{\approx} [\{((x \to 0') \to y) \to z\} \to \{u \to ((0 \to x) \to y')\}']'.$

(26): $(z \to x) \to (y \to z) \approx [\{(y \to z)' \to z\} \to \{x \to (y \to z)\}']'$

$\overset{(I)}{\approx} [\{(z' \to (y \to z)) \to (0 \to z)'\}' \to \{x \to (y \to z)\}']'$

$\overset{(21)}{\approx} [\{(y \to z) \to (0 \to z)'\}' \to \{x \to (y \to z)\}']'$

$\overset{(17)}{\approx} [(y \to z)'' \to \{x \to (y \to z)\}']'$

$\approx [\{(y \to z)' \to 0\} \to \{x \to (y \to z)\}']' \overset{(I)}{\approx} (0 \to x) \to (y \to z).$

(27): Observe that

$$[(z \to 0') \to x] \to x' \approx (0 \to z) \to x', \qquad (2.2)$$

since $[(z \to 0') \to x] \to x' \overset{(1)}{\approx} [(z \to 0') \to 0'] \to x'$

$\overset{(1)}{\approx} [(z \to 0) \to 0'] \to x' \approx [z' \to 0'] \to x' \overset{2.2(a)}{\approx} (0 \to z) \to x'.$

Hence, $(x' \to y) \to [(0 \to z) \to x']$

$\overset{(25)}{\approx} [\{((z \to 0') \to x) \to x'\} \to \{y \to ((0 \to z) \to x')\}']'$

$\overset{eq(2.2)}{\approx} [\{(0 \to z) \to x'\} \to \{y \to ((0 \to z) \to x')\}']'$

$\overset{(7)}{\approx} [\{(0 \to z) \to x'\} \to y] \to \{(0 \to z) \to x'\}$

$\overset{(3)}{\approx} (0 \to y) \to [(0 \to z) \to x'].$

(28): This is immediate from (24) and (27).

(29): $(x' \to y) \to (x \to y') \overset{(21)}{\approx} [y' \to (x' \to y)] \to (x \to y')$

$\overset{(26)}{\approx} [0 \to (x' \to y)] \to (x \to y') \overset{(11)}{\approx} [x' \to (0 \to y)] \to (x \to y')$

$\approx [x' \to (0 \to y'')] \to (x \to y') \overset{(18)}{\approx} [0 \to (x'' \to y')'] \to (x \to y')$

$\approx [0 \to (x \to y')'] \to (x \to y') \overset{(3)}{\approx} [(x \to y') \to (x \to y')'] \to (x \to y')$

$\approx [(x \to y')'' \to (x \to y')'] \to (x \to y') \overset{2.1(d)}{\approx} (x \to y')' \to (x \to y')$

$\overset{2.1(d)}{\approx} x \to y'.$

(30): $(x \to y')' \to (x' \to y)'$

$\overset{(I)}{\approx} [\{(x' \to y)'' \to (x \to y')\} \to \{0 \to (x' \to y)'\}']'$

$\approx [\{(x' \to y) \to (x \to y')\} \to \{0 \to (x' \to y)'\}']'$

$\overset{(18)}{\approx} [\{(x' \to y) \to (x \to y')\} \to \{x \to (0 \to y')\}']'$

$\overset{(11)}{\approx} [\{(x' \to y) \to (x \to y')\} \to \{0 \to (x \to y')\}']'$

$\overset{(17)}{\approx} [(x' \to y) \to (x \to y')]'' \approx (x' \to y) \to (x \to y') \overset{(29)}{\approx} x \to y'.$

This completes the proof. $\square$

Lemma 2.8. *Let* $\mathbf{A} \in \mathcal{I}$. *Then* $\mathbf{A}$ *satisfies*

(1) $[(x \to y) \to z]''' \approx [(x \to y) \to z]'$,
(2) $(x \to y) \to z \approx [(x \to y) \to z]''$,
(3) $(x \to y)' \approx (x'' \to y)'$,
(4) $x \wedge y \approx (x \wedge y)''$,
(5) $x \vee y \approx (x \vee y)''$,
(6) $x \wedge y \approx (x' \vee y')'$.

Proof. (1): $[(x \to y) \to z]''' \overset{(I)}{\approx} [(z' \to x) \to (y \to z)']''''$
$\overset{2.6}{\approx} [(z' \to x) \to (y \to z)']'' \overset{(I)}{\approx} [(x \to y) \to z]'.$

(2): $(x \to y) \to z \overset{(I)}{\approx} [(z' \to x) \to (y \to z)']'$
$\overset{(1)}{\approx} [(z' \to x) \to (y \to z)']''' \overset{(I)}{\approx} [(x \to y) \to z]''.$

(3): $(x \to y)' \approx (x \to y) \to 0 \overset{(I)}{\approx} [(0' \to x) \to (y \to 0)']'$

$\approx [\{(0 \to 0) \to x\} \to (y \to 0)']' \overset{(I)}{\approx} [\{(x' \to 0) \to (0 \to x)'\}' \to (y \to 0)']'$

$\overset{2.6}{\approx} [\{(x''' \to 0) \to (0 \to x)'\}' \to (y \to 0)']'$

$\overset{2.5}{\approx} [\{(x''' \to 0) \to (0 \to x'')'\}' \to (y \to 0)']' \overset{(I)}{\approx} [\{(0 \to 0) \to x''\} \to (y \to 0)']'$

$\approx [(0' \to x'') \to (y \to 0)']' \overset{(I)}{\approx} (x'' \to y) \to 0 \approx (x'' \to y)'.$

(4): $x \vee y \approx (x' \wedge y')' \approx (x' \rightarrow y'')'' \approx [(x \rightarrow 0) \rightarrow y'']''$
$\overset{(1)}{\approx} [(x \rightarrow 0) \rightarrow y'']'''' \approx (x' \rightarrow y'')'''' \approx (x' \wedge y')''' \approx (x \vee y)''$.

(5): $x \wedge y \overset{(M)}{\approx} (x \rightarrow y')' \overset{(3)}{\approx} (x'' \rightarrow y')' \approx [(x' \rightarrow 0) \rightarrow y']'$
$\overset{(1)}{\approx} [(x' \rightarrow 0) \rightarrow y']''' \approx (x'' \rightarrow y')''' \overset{(3)}{\approx} (x \rightarrow y')''' \overset{(M)}{\approx} (x \wedge y)''$.

(6): $x \wedge y \overset{(M)}{\approx} (x \rightarrow y')' \overset{(3)}{\approx} (x'' \rightarrow y')' \approx [(x' \rightarrow 0) \rightarrow y']'$
$\overset{(1)}{\approx} [(x' \rightarrow 0) \rightarrow y']''' \approx (x'' \rightarrow y')''' \overset{2.5}{\approx} (x'' \rightarrow y''')''' \overset{(M)}{\approx} (x'' \wedge y'')'' \approx (x' \vee y')'$.
This completes the proof.   □

The following theorem gives two new characterizations of $\mathcal{I}_{2,0}$.

Theorem 2.9. *Let* $\mathbf{A} \in \mathcal{I}$ *Then* $\mathcal{I}_{2,0} = \mathcal{MID} = \mathcal{JID}$.

Proof. From Lemma 2.3, we get $\mathcal{I}_{2,0} \subseteq \mathcal{MID}$. Using $x \wedge x \approx x$, we get

$$x \vee x \approx (x' \wedge x')' \overset{hyp}{\approx} x'' \overset{hyp}{\approx} (x \wedge x)'' \approx (x \rightarrow x')'''$$

$$\overset{2.8(3)}{\approx} (x'' \rightarrow x')''' \approx [(x' \rightarrow 0) \rightarrow x']''' \overset{2.8(1)}{\approx} [(x' \rightarrow 0) \rightarrow x']'$$

$$\approx (x'' \rightarrow x')' \overset{2.8(3)}{\approx} (x \rightarrow x')' \approx x \wedge x \overset{(MID)}{\approx} x,$$

proving $\mathcal{MID} \subseteq \mathcal{JID}$. Finally, from $x \vee x \approx x$ and Lemma 2.6, we have
$x'' \approx (x \vee x)'' \approx (x' \wedge x')''' \approx (x' \rightarrow x'')'''' \approx (x' \rightarrow x'')'' \approx (x' \wedge x')' \approx x \vee x \approx x$,
thus $\mathcal{JID} \subseteq \mathcal{I}_{2,0}$.   □

In the sequel we will use these names interchangeably.

We conclude this section by giving a characterization of those algebras of $\mathcal{I}$ that are implication algebras in the sense of Abbott [1]. An algebra $\mathbf{A} = \langle A, \rightarrow 0 \rangle$ is an *implication algebra in the sense of Abbott* if it satisfies

(1) $(x \rightarrow y) \rightarrow x \approx x$,
(2) $(x \rightarrow y) \rightarrow y \approx (y \rightarrow x) \rightarrow x$,
(3) $x \rightarrow (y \rightarrow z) \approx y \rightarrow (x \rightarrow z)$.

Theorem 2.10. $\mathbf{A} \in \mathcal{I}$ *is an Implication algebra in the sense of Abbott iff A is a Boolean algebra.*

Proof. Let $\mathbf{A} \in \mathcal{I}$. If $\mathbf{A}$ is a Boolean algebra, then clearly $\mathbf{A}$ is an Implication algebra in the sense of Abbott. For the converse, let $\mathcal{V}$ be the subvariety of $\mathcal{I}$ defined by (1), (2), and (3). Since $\mathcal{V}$ satisfies (1), clearly $\mathcal{V} \subseteq \mathcal{DM}$. To finish off the proof, it is sufficient to observe that the 3-element Kleene algebra and the 4-element De Morgan algebra fail to satisfy (2).   □

3. The varieties $\mathcal{BA}$ and $\mathcal{KL}$ as subvarieties of $\mathcal{MID}$

In this section, we describe the varieties of Boolean algebras and Kleene algebras as subvarieties of the variety $\mathcal{MID}$. In the next theorem, Boolean algebras are characterized relative to $\mathcal{MID}$.

Theorem 3.1. *Let* $\mathbf{A} \in \mathcal{MID}$. *Then the following are equivalent in* $\mathbf{A}$:

(a) $\mathbf{A} \models x \wedge x' \approx 0$,

(b) $\mathbf{A}$ *is a Boolean algebra.*

Proof. (a) $\Rightarrow$ (b): $0' \approx (x \wedge x')' = (x \rightarrow x'')'' \approx x \rightarrow x$, and hence,

$$(0 \rightarrow x) \rightarrow y \overset{2.2(b)}{\approx} (x' \rightarrow 0') \rightarrow y \approx \{x' \rightarrow (x \rightarrow x)\} \rightarrow y$$

$$\overset{hyp}{\approx} \{x' \rightarrow (x \rightarrow x'')\} \rightarrow y \overset{2.7(21)}{\approx} (x \rightarrow x) \rightarrow y \approx 0' \rightarrow y \overset{2.1(a)}{\approx} y.$$

So, it follows from Lemma 2.4 and (a) that (b) is true, while (b) $\Rightarrow$ (a) is straightforward. $\qquad\square$

Next, we present an axiomatization of Kleene algebras as a subvariety of $\mathcal{I}_{2,0}$. For this purpose, we need the following result.

Lemma 3.2. *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$ *and* $\mathbf{A}$ *satisfy* $(x \rightarrow x) \rightarrow (y \rightarrow y) \approx y \rightarrow y$. *Then* $\mathbf{A} \models [x \rightarrow (x' \rightarrow x')] \rightarrow 0' \approx 0 \rightarrow x$.

Proof. The following computations prove the lemma:

$$[x \rightarrow (x' \rightarrow x')] \rightarrow 0' \overset{(I)}{\approx} [(0'' \rightarrow x) \rightarrow \{(x' \rightarrow x') \rightarrow 0'\}']'$$

$$\approx [(0 \rightarrow x) \rightarrow \{(x' \rightarrow x') \rightarrow (0'' \rightarrow 0'')\}']' \overset{hyp}{\approx} [(0 \rightarrow x) \rightarrow (0'' \rightarrow 0'')']'$$

$$\approx [(0 \rightarrow x) \rightarrow 0'']' \approx (0 \rightarrow x)'' \approx 0 \rightarrow x. \qquad\square$$

Theorem 3.3. *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then the following are equivalent in* $\mathbf{A}$:

(1) $\mathbf{A} \models (x \rightarrow x) \rightarrow (y \rightarrow y) \approx y \rightarrow y$,

(2) $\mathbf{A}$ *is a Kleene algebra.*

Proof. (1) $\Rightarrow$ (2) : $(x \rightarrow y) \rightarrow x \overset{2.7(3)}{\approx} (0 \rightarrow y) \rightarrow x \overset{3.2}{\approx} 0' \rightarrow x \overset{2.7(3)}{\approx} x'' \overset{hyp}{\approx} x$. Hence, $\mathbf{A}$ is a De Morgan algebra and so, in view of (1), $\mathbf{A}$ is a Kleene algebra. Thus, (1) implies (2).

(2) $\Rightarrow$ (1): This follows directly from the definition of Kleene algebras. $\quad\square$

4. A Glivenko-like theorem for $\mathcal{I}$

Let $\mathbf{A} \in \mathcal{I}$. We let $A'' = \{a'' \mid a \in A\}$ and $\mathbf{A}'' := \langle A'', \rightarrow, 0 \rangle$.

Lemma 4.1. *Let* $\mathbf{A} = \langle A, \rightarrow, 0 \rangle \in \mathcal{I}$. *Then* $\mathbf{A}''$ *is a subalgebra of* $\mathbf{A}$.

Proof. Since $0 = 0''$ we obtain that $0 \in A''$. If $a, b \in A''$, $a = a_0''$ and $b = b_0''$ with $a_0, b_0 \in A$. By Lemma 2.8 (2), Lemma 2.8 (3) and 2.5, we have that

$$(a \rightarrow b)'' = (a_0'' \rightarrow b_0'')'' = ((a_0' \rightarrow 0) \rightarrow b_0'')'' \overset{2.8(2)}{=} (a_0' \rightarrow 0) \rightarrow b_0''$$

$$= a_0'' \rightarrow b_0'' = a \rightarrow b. \qquad\square$$

We now present a Glivenko-like result for algebras in $\mathcal{I}$. In view of Definition 1.3, the following Theorem is immediate from Lemma 4.1.

Theorem 4.2. *Let* $\mathbf{A} = \langle A, \rightarrow, 0 \rangle \in \mathcal{I}$; *then*

(a) $\langle A'', \rightarrow, 0 \rangle \in \mathcal{I}_{2,0}$,
(b) $\mathbf{A} \models (x'' \rightarrow y'') \rightarrow x'' \approx x''$ *iff* $\mathbf{A}''$ *is a De Morgan algebra,*
(c) $\mathbf{A} \models (y'' \rightarrow y'') \rightarrow (x'' \rightarrow x'') \approx x'' \rightarrow x''$ *iff* $\mathbf{A}''$ *is a Kleene algebra,*
(d) $\mathbf{A} \models x'' \rightarrow x'' \approx 0'$ *iff* $\mathbf{A}''$ *is a Boolean algebra.*

5. The varieties $\mathcal{SCP}$ and $\mathcal{MC}$

In this section, we describe the relationship between $\mathcal{SCP}$ and $\mathcal{MC}$.

Lemma 5.1. *Let* $\mathbf{A} \in \mathcal{SCP}$. *Then* $\mathbf{A}$ *satisfies*

$$0' \rightarrow [(x \rightarrow y) \rightarrow z] \approx (x \rightarrow y) \rightarrow z.$$

Proof. $(x \rightarrow y) \rightarrow z \overset{(I)}{\approx} [(x' \rightarrow y) \rightarrow (z \rightarrow x)']'$
$\overset{(SCP)}{\approx} 0' \rightarrow [(x' \rightarrow y) \rightarrow (z \rightarrow x)']' \overset{(I)}{\approx} 0' \rightarrow [(x \rightarrow y) \rightarrow z].$ $\square$

Theorem 5.2. $\mathcal{SCP} \subset \mathcal{MC}$.

Proof.

$$y \wedge x \approx (y \rightarrow x')' \overset{(SCP)}{\approx} (x'' \rightarrow y')' \approx [(x' \rightarrow 0'') \rightarrow y']' \overset{(SCP)}{\approx} [(0' \rightarrow x) \rightarrow y']'$$
$$\approx [(0' \rightarrow y) \rightarrow (0' \rightarrow x)']' \approx [(0' \rightarrow x)'' \rightarrow (0' \rightarrow y)']'$$
$$\approx [\{0' \rightarrow (0' \rightarrow x)\} \rightarrow (0' \rightarrow y)']' \overset{5.1}{\approx} [(0' \rightarrow x) \rightarrow (0' \rightarrow y)']'$$
$$\overset{(SCP)}{\approx} [(0' \rightarrow x) \rightarrow (y' \rightarrow 0'')']' \approx [(0' \rightarrow x) \rightarrow (y' \rightarrow 0)']'$$
$$\overset{(I)}{\approx} (x \rightarrow y')' \approx x \wedge y.$$

The example in Figure 1 shows (at 2,2) that the inclusion is proper. $\square$

$\rightarrow$:	0	1	2
0	0	0	0
1	0	0	0
2	0	0	1

$'$:	0	1	2
	0	0	0

$\wedge$:	0	1	2
0	0	0	0
1	0	0	0
2	0	0	0

FIGURE 1. The Example for Theorem 5.2

A much stronger version of Theorem 5.2 holds in $\mathcal{I}_{2,0}$. Indeed, $\mathcal{SCP}$ and $\mathcal{MC}$ will coincide in $\mathcal{I}_{2,0}$.

Theorem 5.3. $\mathcal{MC} \cap \mathcal{I}_{2,0} = \mathcal{SCP} \cap \mathcal{I}_{2,0}$.

Proof. Let $\mathbf{A} \in \mathcal{MC} \cap \mathcal{I}_{2,0}$. Observe that

$$x \rightarrow y \approx (x \rightarrow y'')'' \approx (x \wedge y')' \approx (y' \wedge x)' \approx (y' \rightarrow x')'' \approx y' \rightarrow x'.$$

Hence, $\mathcal{MC} \cap \mathcal{I}_{2,0} \subseteq \mathcal{SCP} \cap \mathcal{I}_{2,0}$. The other inclusion is proved in the preceding theorem. $\square$

6. The algebra $\mathbf{A^{mj}}$ as a lattice

In this section, we present necessary and sufficient conditions on algebras $\mathbf{A} \in \mathcal{I}$ under which the *derived* algebra $\mathbf{A^{mj}} = \langle A, \wedge, \vee, 0 \rangle$ is a lattice.

Lemma 6.1. *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *If* $\mathbf{A} \models x \wedge (x \vee y) \approx x$, *then*
$\mathbf{A} \models (x \rightarrow y) \rightarrow x \approx x$.

Proof. $(x \rightarrow y) \rightarrow x \overset{2.7(3)}{\approx} (0 \rightarrow y) \rightarrow x \approx (0 \rightarrow y)'' \rightarrow x$

$\overset{2.1(a)}{\approx} [0' \rightarrow (0 \rightarrow y)']' \rightarrow x \approx (0' \wedge (0 \rightarrow y)) \rightarrow x \approx (0' \wedge (0'' \rightarrow y'')'') \rightarrow x$

$\approx (0' \wedge (0'' \wedge y')') \rightarrow x \approx (0' \wedge (0' \vee y)) \rightarrow x \overset{hyp}{\approx} 0' \rightarrow x \overset{2.1(a)}{\approx} x,$

which completes the proof. $\qquad\square$

Lemma 6.2. *Let* $\mathbf{A} \in \mathcal{I}$. *If* $\mathbf{A} \models x \wedge (x \vee y) \approx x$, *then* $\mathbf{A} \models (x \rightarrow y) \rightarrow x \approx x$.

Proof. Let $a, b \in A$. Then using (2) and (3) of Lemma 2.8 and Lemma 2.5, $(a \rightarrow b) \rightarrow a = [(a \rightarrow b) \rightarrow a]'' = [(a'' \rightarrow b'') \rightarrow a'']''$. As $\mathbf{A}''$ is a subalgebra of $\mathbf{A}$ (see Lemma 4.1), we have $\mathbf{A}'' \models x \wedge (x \vee y) \approx x$. Hence, using $a'', b'' \in A''$, Lemma 6.1 implies $(a \rightarrow b) \rightarrow a = [a'']'' = a''$. So, $(a \rightarrow b) \rightarrow a = a'' = [a \wedge (a \vee b)]'' = a \wedge (a \vee b) = a$ by hypothesis and Lemma 2.8. $\qquad\square$

Theorem 6.3. *The following are equivalent in* $\mathbf{A} \in \mathcal{I}$:

(1) $\mathbf{A^{mj}}$ *is a lattice,*
(2) *Absorption law holds in* $\mathbf{A^{mj}}$,
(3) $(x \rightarrow y) \rightarrow x \approx x$ *holds in* $\mathbf{A}$,
(4) $\mathbf{A}$ *is a De Morgan algebra.*

Proof. $(1) \Rightarrow (2)$: This is trivial.
$(2) \Rightarrow (3)$: This follows from the preceding lemma.
$(3) \Rightarrow (4)$: This follows from Definition 2.2.
$(4) \Rightarrow (1)$: This was proved in [12]. $\qquad\square$

Here is another interesting (and surprising to us) characterization of De Morgan algebras. But first we need a lemma.

Lemma 6.4. *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$ *such that* $\mathbf{A} \models x \wedge 0 \approx 0$. *Then* $\mathbf{A} \models x \rightarrow 0' \approx 0'$.

Proof. Since $\mathbf{A} \in \mathcal{I}_{2,0}$, we have $x \rightarrow 0' \approx (x \rightarrow 0')'' \approx (x \wedge 0)'$, implying $x \rightarrow 0' \approx 0'$, since $\mathbf{A} \models x \wedge 0 \approx 0$. $\qquad\square$

Now we are ready to give the new characterization of De Morgan algebras.

Theorem 6.5. *Let* $\mathbf{A} \in \mathcal{I}$. *Then the following are equivalent:*

(1) (a) $\mathbf{A} \models x'' \approx x$,
　　(b) $\mathbf{A} \models x \wedge 0 \approx 0$,
(2) $\mathbf{A} \models (0 \rightarrow x) \rightarrow y \approx y$,
(3) $\mathbf{A}$ *is a De Morgan algebra.*

Proof. Suppose (1) holds. Then

$$(0 \to x) \to y \approx (0 \to x'') \to y \overset{2.2(2)}{\approx} (x' \to 0') \to y \approx 0' \to y \approx y.$$

Hence, (1) implies (2). The other implications follow from [12].　　□

7. Some properties of the varieties $\mathcal{I}$ and $\mathcal{I}_{2,0} \cap \mathcal{MC}$

In this section, we present more properties of algebras in $\mathcal{I}$ and also some properties of algebras in $\mathcal{I}_{2,0} \cap \mathcal{MC}$.

Theorem 7.1. *Let* $\mathbf{A} \in \mathcal{I}$.

(a) $\mathbf{A^{mj}}$ *satisfies*
　(1) $(x \vee y)' \approx x' \wedge y'$,
　(2) $(x \wedge y)' \approx x' \vee y'$.
(b) *The following are equivalent in* $\mathbf{A^{mj}}$:
　(1) $x \wedge y \approx y \wedge x$　(i.e., $\wedge$-commutative),
　(2) $x \vee y \approx y \vee x$　(i.e., $\vee$-commutative).
(c) *The following are equivalent in* $\mathbf{A^{mj}}$:
　(1) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$,
　(2) $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$.

Proof. (a): (1) $\Rightarrow$ (2): By Lemma 2.8 (1), we have that $(x \vee y)' \approx (x' \wedge y')'' \approx (x' \to y'')''' \approx [(x \to 0) \to y'']''' \approx [(x \to 0) \to y'']' \approx (x' \to y'')' \approx x' \wedge y'$.

(2) $\Rightarrow$ (1): From Lemma 2.5 and Lemma 2.8 (3), $x' \vee y' \approx (x'' \wedge y'')' \approx (x'' \to y''')'' \approx (x \to y''')'' \approx (x \to y')'' \approx (x \wedge y)'$.

(b): From (1), we have that $x \vee y \approx (x' \wedge y')' \approx (y' \wedge x')' \approx y \vee x$, implying (2). Suppose (2) holds. Then by Lemma 2.8 (6), $x \wedge y \approx (x' \vee y')' \approx (y' \vee x')' \approx y \wedge x$, yielding (1).

(c): (1) $\Rightarrow$ (2): $x \vee (y \wedge z) \overset{\text{def of } \vee}{\approx} [x' \wedge (y \wedge z)']' \overset{2.8(6)}{\approx} [x' \wedge (y' \vee z')'']'$

$\overset{2.8(5)}{\approx} [x' \wedge (y' \vee z')]' \overset{hyp}{\approx} [(x' \wedge y') \vee (x' \wedge z')]' \overset{2.8(4)}{\approx} [(x' \wedge y')'' \vee (x' \wedge z')'']'$

$\overset{\text{def of } \vee}{\approx} [(x \vee y)' \vee (x \vee z)']' \overset{2.8(6)}{\approx} (x \vee y) \wedge (x \vee z)$.

(2) $\Rightarrow$ (1): $x \wedge (y \vee z) \overset{2.8(6)}{\approx} [x' \vee (y \vee z)']' \overset{\text{def of } \vee}{\approx} [x' \vee (y' \wedge z')'']'$

$\overset{2.8(4)}{\approx} [x' \vee (y' \wedge z')]' \overset{hyp}{\approx} [(x' \vee y') \wedge (x' \vee z')]' \overset{2.8(5)}{\approx} [(x' \vee y')'' \wedge (x' \vee z')'']'$

$\overset{2.8(6)}{\approx} [(x \wedge y)' \wedge (x \wedge z)']' \overset{\text{def of } \vee}{\approx} (x \wedge y) \vee (x \wedge z)$.

This completes the proof.　　□

Theorem 7.2. *Let* $\mathbf{A} \in \mathcal{I}$. *Then the following hold in* $\mathbf{A^{mj}}$:

(a) $(x \wedge y) \wedge (x \vee y) \approx x \wedge y$,
(b) $(x \vee y) \vee (x \wedge y) \approx x \vee y$.

Proof. (a): $(x \wedge y) \wedge (x \vee y) \stackrel{\text{def of } \vee}{\approx} (x \wedge y) \stackrel{\text{def of } \wedge}{\approx} (x \rightarrow y')' \wedge (x' \rightarrow y'')''$

$$\stackrel{\text{def of } \wedge}{\approx} [(x \rightarrow y')' \rightarrow (x' \rightarrow y'')''']']' \stackrel{2.8(1)}{\approx} [(x \rightarrow y')' \rightarrow (x' \rightarrow y'')']'$$

$$\stackrel{2.8(3)}{\approx} [(x'' \rightarrow y')' \rightarrow (x''' \rightarrow y'')']' \stackrel{2.5}{\approx} [(x'' \rightarrow y''')' \rightarrow (x''' \rightarrow y'')']'.$$

Notice that by Lemma 2.6, $x'', y'' \in A''$. Hence, by Theorem 4.2 and Lemma 2.7 (30), we have that $[(x'' \rightarrow y''')' \rightarrow (x''' \rightarrow y'')']' \approx (x'' \rightarrow y''')'$. Then by Lemma 2.8 (3) and Lemma 2.5, we obtain $(x \wedge y) \wedge (x \vee y) \approx (x \rightarrow y')' \approx x \wedge y$.

(b): $(x \vee y) \vee (x \wedge y) \stackrel{\text{def of } \vee}{\approx} (x' \wedge y')' \vee (x \wedge y)$

$$\stackrel{\text{def of } \vee}{\approx} [(x' \wedge y')'' \wedge (x \wedge y)']' \stackrel{2.8(4)}{\approx} [(x' \wedge y') \wedge (x \wedge y)']'$$

$$\stackrel{2.8(6)}{\approx} [(x' \wedge y') \wedge (x' \vee y')'']']' \approx [(x' \wedge y') \wedge (x' \vee y')]' \stackrel{(a)}{\approx} [x' \wedge y']' \approx x \vee y,$$

completing the proof. $\qquad\square$

We should note here that the conclusions in the preceding theorem are known to hold in every Birkhoff system (see [8]).

Recall that $\mathcal{MC}$ denotes the subvariety of $\mathcal{I}$ defined by the identity $x \wedge y \approx y \wedge x$. Next, we consider the variety $\mathcal{I}_{2,0} \cap \mathcal{MC}$ and show that it has some interesting properties.

Theorem 7.3. *Let* $\mathbf{A} \in \mathcal{I}_{2,0} \cap \mathcal{MC}$. *Then* $\mathbf{A}^{\mathbf{mj}}$ *satisfies*

(a) $x \wedge x \approx x$,
(b) $x \vee x \approx x$,
(c) $x \vee y \approx y \vee x$,
(d) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$,
(e) $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$,
(f) $x \wedge (x \vee y) \approx x \vee (x \wedge y)$.

Proof. (a) and (b): These follow from Theorem 2.9.

(c): This follows from Theorem 7.1(2).

(d): $(x \wedge y) \vee (x \wedge z) \approx [(x \wedge y)' \wedge (x \wedge z)']'$

$$\stackrel{\text{def of } \vee}{\approx} [(x \wedge y)' \rightarrow (x \wedge z)'']'' \stackrel{2.8(4)}{\approx} [(x \wedge y)' \rightarrow (x \wedge z)]''$$

$$\stackrel{hyp}{\approx} [(x \wedge y)' \rightarrow (z \wedge x)]'' \stackrel{\text{def of } \wedge}{\approx} [(x \rightarrow y')'' \rightarrow (z \rightarrow x')']''$$

$$\stackrel{2.8(3)}{\approx} [(x \rightarrow y') \rightarrow (z \rightarrow x')']'' \stackrel{(I)}{\approx} [(y' \rightarrow z) \rightarrow x]'$$

$$\stackrel{\text{def of } \wedge}{\approx} (y' \rightarrow z) \wedge x \stackrel{hyp}{\approx} x \wedge (y' \rightarrow z)$$

$$\stackrel{\text{def of } \wedge}{\approx} [x \rightarrow (y' \rightarrow z)']' \stackrel{2.5}{\approx} [x \rightarrow (y' \rightarrow z'')']' \stackrel{\text{def of } \wedge}{\approx} x \wedge (y' \rightarrow z'')$$

$$\stackrel{(a)}{\approx} x \wedge (y' \rightarrow z'')'' \stackrel{\text{def of } \wedge}{\approx} x \wedge [y' \wedge z']' \stackrel{\text{def of } \vee}{\approx} x \wedge (y \vee z).$$

(e): This follows from (d) and Theorem 7.1(3).

(f): $x \vee (x \wedge y) \overset{\text{def of } \vee}{\approx} [x' \wedge (x \wedge y)']' \overset{\text{def of } \wedge}{\approx} [x' \to (x \wedge y)'']''$

$\overset{(a)}{\approx} [x' \to (x \wedge y)] \overset{hyp}{\approx} [x' \to (y \wedge x)] \overset{\text{def of } \wedge}{\approx} [x' \to (y \to x')']$

$\approx [(x \to 0) \to (y \to x')'] \overset{(a)}{\approx} [(x'' \to 0) \to (y \to x')']$

$\overset{(a)}{\approx} [(x'' \to 0) \to (y \to x')']'' \overset{(I)}{\approx} [(0 \to y) \to x']' \overset{2.2(a)}{\approx} [(y' \to 0') \to x']'$

$\overset{2.7(4)}{\approx} (0 \to y') \to x'' \overset{(a)}{\approx} (0 \to y') \to x \overset{2.7(3)}{\approx} (x \to y') \to x$

$\overset{2.7(7)}{\approx} [x \to (y' \to x)']' \overset{(a)}{\approx} [x \to (y' \to x'')']' \overset{\text{def of } \wedge}{\approx} [x \to (y' \wedge x')]'$

$\overset{hyp}{\approx} [x \to (x' \wedge y')]' \overset{2.8(4)}{\approx} [x \to (x' \wedge y')'']'$

$\overset{\text{def of } \vee}{\approx} [x \to (x \vee y)']' \approx x \wedge (x \vee y).$

This completes the proof. $\qquad \square$

We should mention here that the absorption identity $x \wedge (x \vee y) \approx x$ fails in $\mathcal{I}_{2,0} \cap \mathcal{MC}$; so, if $\mathbf{A} \in \mathcal{I}_{2,0} \cap \mathcal{MC}$, then $\mathbf{A}^{\mathbf{mj}}$ is not a lattice. Indeed, in Section 6, we have already shown that $\mathbf{A}^{\mathbf{mj}}$ is a lattice precisely when $\mathbf{A}$ is a De Morgan algebra.

Remark 7.4. We note here that $\mathcal{I}_{2,0}$ ($= \mathcal{MID}$) and $\mathcal{MC}$ are incomparable in the lattice of subvarieties of $\mathcal{I}$, as the examples in Figures 2 and 3 show.

$\to$:	0	1
0	0	0
1	0	0

$'$:	0	1
	0	0

$\wedge$:	0	1
0	0	0
1	0	0

FIGURE 2. $\mathcal{MC} \nsubseteq \mathcal{MID}$

$\to$:	0	1	2
0	0	1	2
1	1	1	2
2	2	1	2

$'$:	0	1	2
	0	1	2

$\wedge$:	0	1	2
0	0	1	2
1	1	1	2
2	2	1	2

FIGURE 3. $\mathcal{MID} \nsubseteq \mathcal{MC}$

8. The varieties $\mathcal{Z}$, $\mathcal{C}$, $\mathcal{A}$, and $\mathcal{I}_{3,1}$

In this section, our goal is to prove $\mathcal{Z} \subset \mathcal{C} \subset \mathcal{A} \subset \mathcal{I}_{3,1}$. For this purpose, we first need to establish the following lemma.

Lemma 8.1. *Let* $\mathbf{A} \in \mathcal{C}$. *Then* $\mathbf{A}$ *satisfies the identities*

(a) $0 \approx 0'$,

(b) $x' \approx (x \to x)'$,

(c) $(x \to y)' \approx (x' \to y)' \approx (x' \to y')' \approx (x \to y')'$,

(d) $x \to (x \to y) \approx y' \to x$.

Proof. (a): First we see that $0 = 0'' = 0' \to 0 = 0 \to 0'$ by (C). So,

$$0 = 0 \to 0' = 0'' \to 0' = (0' \to 0) \to 0' = [(0'' \to 0') \to (0 \to 0')']'$$
$$= [(0 \to 0') \to (0 \to 0')']' = [0 \to 0']' = 0'.$$

(b): $x' \approx x \to 0 \overset{(C)}{\approx} 0 \to x \overset{(a)}{\approx} (0 \to 0) \to x \overset{(I)}{\approx} [(x' \to 0) \to (0 \to x)']'$
$\overset{(C)}{\approx} [(x' \to 0) \to (x \to 0)']' \approx (x'' \to x'')' \overset{2.8(3)}{\approx} (x \to x'')' \overset{2.5}{\approx} (x \to x)'$.

(c): We have that

$$(x \to y)' \approx (x \to y) \to 0 \approx [(0' \to x) \to (y \to 0)']' \approx [(x \to 0') \to (y \to 0)']'$$
$$\approx [(x \to 0') \to y'']' \overset{(a)}{\approx} [(x \to 0) \to y'']' \approx (x' \to y'')' \overset{2.5}{\approx} (x' \to y)'.$$

Hence, it follows that $(x \to y)' \approx (x' \to y')' \approx (x \to y')'$.

(d): $x \to (x \to y) \overset{(C)}{\approx} (x \to y) \to x \overset{(I)}{\approx} [(x' \to x) \to (y \to x)']'$

$\overset{(2.8(3))}{\approx} [(x' \to x)'' \to (y \to x)']' \overset{(c)}{\approx} [(x \to x)'' \to (y \to x)']' \overset{(b)}{\approx} [x'' \to (y \to x)']'$
$\approx [(x' \to 0) \to (y \to x)']' \approx (0 \to y) \to x \approx (y \to 0) \to x \approx y' \to x.$ $\qquad\square$

Next, we describe the relationship among the varieties $\mathcal{Z}, \mathcal{C}, \mathcal{A}$, and $\mathcal{I}_{3,1}$.

Theorem 8.2. $\mathcal{Z} \subset \mathcal{C} \subset \mathcal{A} \subset \mathcal{I}_{3,1}$.

Proof. It is clear that $\mathcal{Z} \subset \mathcal{C}$. So, let $\mathbf{A} \in \mathcal{C}$. Notice that

$$(x \to y) \to z \approx [(z' \to x) \to (y \to z)']' \overset{2.8(3)}{\approx} [(z' \to x)'' \to (y \to z)']'$$

$\overset{8.1(c)}{\approx} [(z \to x)'' \to (y \to z)']' \overset{2.8(3)}{\approx} [(z \to x) \to (y \to z)']'$

$\overset{8.1(c)}{\approx} [(z \to x) \to (y \to z)]' \approx [(y \to z) \to (z \to x)]'$

$\overset{(I)}{\approx} [\{(z \to x)' \to y\} \to \{z \to (z \to x)\}']''$

$\overset{2.8(3)}{\approx} [\{(z \to x)' \to y\}'' \to \{z \to (z \to x)\}']''$

$\overset{8.1(c)}{\approx} [\{(z \to x) \to y\}'' \to \{z \to (z \to x)\}']''$

$\overset{2.8(3)}{\approx} [\{(z \to x) \to y\} \to \{z \to (z \to x)\}']''$

$\overset{2.8(3)}{\approx} [\{(z \to x) \to y\} \to \{z \to (z \to x)\}]'' \overset{8.1(d)}{\approx} [\{(z \to x) \to y\} \to (x' \to z)]''$

$\overset{2.5}{\approx} [\{(z \to x) \to y\} \to (x' \to z)'']'' \overset{8.1(c)}{\approx} [\{(z \to x) \to y\} \to (x \to z)'']''$

$$\overset{2.5}{\approx} [\{(z \to x) \to y\} \to (x \to z)]'' \overset{2.8(2)}{\approx} [(z \to x) \to y] \to (x \to z).$$

Also,

$$x \to (y \to z) \approx (y \to z) \to x \approx [(x' \to y) \to (z \to x)']'$$

$$\overset{2.8(3)}{\approx} [(x' \to y)'' \to (z \to x)']' \overset{8.1(c)}{\approx} [(x \to y)'' \to (z \to x)']'$$

$$\overset{2.8(3)}{\approx} [(x \to y) \to (z \to x)']' \overset{8.1(c)}{\approx} [(x \to y) \to (z \to x)]'$$

$$\overset{(I)}{\approx} [\{(z \to x)' \to x\} \to \{y \to (z \to x)\}']''$$

$$\overset{2.8(3)}{\approx} [\{(z \to x)' \to x\}'' \to \{y \to (z \to x)\}']''$$

$$\overset{8.1(c)}{\approx} [\{(z \to x) \to x\}'' \to \{y \to (z \to x)\}']''$$

$$\overset{2.8(3)}{\approx} [\{(z \to x) \to x\} \to \{y \to (z \to x)\}']''$$

$$\overset{8.1(c)}{\approx} [\{(z \to x) \to x\} \to \{y \to (z \to x)\}]''$$

$$\approx [\{x \to (x \to z)\} \to \{y \to (z \to x)\}]'' \overset{8.1(d)}{\approx} [(z' \to x) \to [y \to (z \to x)]]''$$

$$\overset{2.8(3)}{\approx} [(z' \to x)'' \to [y \to (z \to x)]]'' \overset{8.1(c)}{\approx} [(z \to x)'' \to [y \to (z \to x)]]''$$

$$\overset{2.8(3)}{\approx} [(z \to x) \to [y \to (z \to x)]]'' \approx [\{(z \to x) \to y\} \to (x \to z)]''$$

$$\approx [(z \to x) \to y] \to (x \to z).$$

Hence, $(x \to y) \to z \approx x \to (y \to z)$. For the second half, let $\mathbf{A} \in \mathcal{A}$.

$$x''' \approx [(x \to 0) \to 0]' \overset{(A)}{\approx} [x \to (0 \to 0]' \approx (x \to 0') \to 0 \overset{(A)}{\approx} x \to (0' \to 0)$$

$$\approx x \to 0'' \approx x \to 0 \approx x'.$$

Finally, Example 2 of Section 8 shows that the first two inclusions are strict, while the 2-element Boolean algebra illustrates the strictness of the third inclusion. $\qquad \square$

9. The varieties $\mathcal{I}_{1,0}$, $\mathcal{ID}$ and $\mathcal{A}$

We recall from [12] that $\mathcal{I}_{1,0}$ and $\mathcal{ID}$ denote the subvarieties of $\mathcal{I}$ defined, respectively, by $x' \approx x$ and $x \to x \approx x$. (The variety $\mathcal{ID}$ was actually called **IDMP** in [12].)

In this section, we wish to show that $\mathcal{I}_{1,0} = \mathcal{ID} \cap \mathcal{A}$. For this we need the following two lemmas.

Lemma 9.1. *Let* $\mathbf{A} \in \mathcal{I}_{1,0}$. *Then the following are true in* $\mathbf{A}$:
(a) $(y \to z) \to x \approx (x \to y) \to (z \to x)$,
(b) $(0 \to x) \to y \approx x \to y$,
(c) $x \to (y \to x) \approx (0 \to y) \to x$,
(d) $x \to (y \to x) \approx y \to x$.

Proof. Let $a, b \in A$.

(a): This is immediate from (I) and $a' \approx a$.

(b): Since $\mathbf{A} \in \mathcal{I}_{1,0}$, we get

$$a \to b = (a \to b)' = [(0' \to a) \to b'']' = (0 \to a) \to b.$$

(c): The proof is similar to that of (b).

(d): This follows from (b) and (c).   □

Lemma 9.2. *Let* $\mathbf{A} \in \mathcal{ID} \cap \mathcal{A}$. *Then the following are true in* $\mathbf{A}$:

(1) $y \to (z \to x) \approx x \to [0 \to \{y \to (z \to x')\}]$,

(2) $x \to (y \to x') \approx y \to x$,

(3) $x \to (0 \to x) \approx x'$,

(4) $(x \to y) \to [0 \to \{x \to (y \to (x \to y'))\}] \approx x \to y$,

(5) $(x \to y) \to [0 \to (x \to y)] \approx x \to y$,

(6) $x \to y' \approx x \to y$.

Proof. (1): $y \to (z \to x) \overset{(A)}{\approx} (y \to z) \to x \overset{(I)}{\approx} [(x' \to y) \to (z \to x)']'$

$\overset{(A)}{\approx} [\{x \to (0 \to y)\} \to (z \to x)']' \overset{(A)}{\approx} [\{x \to (0 \to y)\} \to (z \to x')]'$

$\overset{(A)}{\approx} [x \to \{(0 \to y) \to (z \to x')\}]' \overset{(A)}{\approx} [x \to \{0 \to (y \to (z \to x'))\}]'$

$\overset{(A)}{\approx} x \to [\{0 \to (y \to (z \to x'))\} \to 0] \overset{(A)}{\approx} x \to [0 \to \{(y \to (z \to x')) \to 0\}]$

$\overset{(A)}{\approx} x \to [0 \to \{y \to ((z \to x') \to 0)\}] \overset{(A)}{\approx} x \to [0 \to \{y \to (z \to (x' \to 0))\}]$

$\overset{(A)}{\approx} x \to [0 \to \{y \to (x \to (x \to (0 \to 0)))\}] \overset{(ID)}{\approx} x \to [0 \to \{y \to (z \to x')\}]$.

(2): $x \to (y \to x') \overset{(1)}{\approx} x \to [0 \to \{0 \to (y \to (x \to 0'))\}]$

$\overset{(ID)}{\approx} x \to [0 \to \{0 \to (y \to x')\}] \overset{(A)}{\approx} x \to [(0 \to 0) \to (y \to x')]$

$\overset{(ID)}{\approx} x \to [0 \to (y \to x')] \overset{(ID)}{\approx} x \to [0 \to \{y \to (x \to x)'\}]$

$\overset{(A)}{\approx} x \to [0 \to \{y \to (x \to x')\}] \overset{(1)}{\approx} y \to (x \to x) \overset{(ID)}{\approx} y \to x$.

(3): $x \to (0 \to x) \overset{(A)}{\approx} (x \to 0) \to x \overset{(I)}{\approx} [(x' \to x) \to (0 \to x)']'$

$\overset{(A)}{\approx} [\{x \to (0 \to x)\} \to (0 \to x)']' \overset{(A)}{\approx} [x \to \{(0 \to x) \to (0 \to x')\}]'$

$\overset{(A)}{\approx} [x \to (0 \to \{x \to (0 \to x')\})]' \overset{(1))}{\approx} [x \to (0 \to x)]' \overset{(A)}{\approx} x \to (0 \to x')$

$\overset{(A)}{\approx} x' \to x' \overset{(ID)}{\approx} x'$.

(4): $(x \to y) \to [0 \to \{x \to (y \to (x \to y'))\}]$

$\overset{(A)}{\approx} (x \to y) \to [0 \to \{x \to (y \to (x \to y)')\}] \overset{(1)}{\approx} x \to [y \to (x \to y)]$

$\overset{(A)}{\approx} (x \to y) \to (x \to y) \overset{(ID)}{\approx} x \to y$.

(5): $(x \to y) \to [0 \to (x \to y)] \overset{(ID)}{\approx} (x \to y) \to [0 \to \{(x \to x) \to y\}]$

$\qquad \overset{(A)}{\approx} (x \to y) \to [0 \to \{x \to (x \to y)\}]$

$\qquad \overset{(2)}{\approx} (x \to y) \to [0 \to \{x \to (y \to (x \to y'))\}] \overset{(4)}{\approx} x \to y.$

(6): $x \to y \overset{(5)}{\approx} (x \to y) \to [0 \to (x \to y)] \overset{(3)}{\approx} (x \to y)' \overset{(A)}{\approx} x \to y'.$ □

We are now ready to prove the theorem.

Theorem 9.3. $\mathcal{I}_{1,0} = \mathcal{ID} \cap \mathcal{A}.$

Proof. Let $\mathbf{A} \in \mathcal{I}_{1,0}$. Then using (b) and (d) of Lemma 9.1, we have $x \approx x' \approx 0 \to x \approx x \to (0 \to x) \approx x \to x$. Thus, $\mathcal{I}_{1,0} \subseteq \mathcal{ID}$. Observe that since $\mathcal{I}_{1,0} \subseteq \mathcal{I}_{2,0}$, we get

$(x \to y) \to z \overset{9.1(a)}{\approx} (z \to x) \to (y \to z) \overset{9.1(a)}{\approx} [(y \to z) \to z] \to [x \to (y \to z)]$

$\overset{hyp}{\approx} [(y \to z') \to z] \to [x \to (y \to z)] \overset{2.7(1)}{\approx} [(y \to 0') \to z] \to [a \to (y \to z)]$

$\overset{hyp}{\approx} [(y \to 0) \to z] \to [x \to (y \to z)] \overset{hyp}{\approx} (y \to z) \to [x \to (y \to z)]$

$\overset{hyp}{\approx} (y \to z) \to [x \to (y \to z)'] \overset{2.7(21)}{\approx} x \to (y \to z)' \overset{hyp}{\approx} x \to (y \to z).$

Hence $\mathcal{I}_{1,0} \subseteq \mathcal{ID} \cap \mathcal{A}$. From $x' \overset{(ID)}{\approx} (x \to x)' \overset{(A)}{\approx} x \to x' \overset{9.2(6)}{\approx} x \to x \approx x$, the other inclusion follows.   □

10. Axiomatization of the varieties generated by the 2-element algebras in $\mathcal{I}$

In this section, we axiomatize the varieties generated by 2-element algebras in $\mathcal{I}$. Recall from [12] that there are three 2-element algebras in $\mathcal{I}$, namely $\mathbf{2_z}$, $\mathbf{2_s}$, and $\mathbf{2_b}$, whose $\to$ operations are, respectively, as follows:

$\to$:	0	1
0	0	0
1	0	0

$\to$:	0	1
0	0	1
1	1	1

$\to$:	0	1
0	1	1
1	0	1

10.1. $\lor$-semilattices with the least element 0 as a subvariety of $\mathcal{I}$.
We first determine those algebras of $\mathcal{I}$ which are $\lor$-semilattices with the least element 0.

Let $\mathcal{SL}$ denote the subvariety of $\mathcal{I}$ defined by

$$x' \approx x, \qquad \text{and} \qquad (C) \quad x \to y \approx y \to x.$$

That is, $\mathcal{SL} := \mathcal{I}_{1,0} \cap \mathcal{C}.$

Lemma 10.1. $\mathcal{SL} \models x \to x \approx x.$

Proof. Let $\mathbf{A} \in \mathcal{SL}$. Then $x \to x \approx (x' \to x')' \overset{(C)}{\approx} [(x \to 0) \to (0 \to x)']'$ $\overset{(I)}{\approx} (0 \to 0) \to x \approx 0 \to x \approx x' \approx x$, proving the lemma. $\qquad \square$

Let $\mathcal{S}^0$ denote the variety of $\vee$-semilattices with the least element 0; that is, $\mathbf{A} = \langle A, \vee, 0 \rangle \in \mathcal{S}^0$ satisfies

(1) $(x \vee y) \vee z \approx x \vee (y \vee z)$,
(2) $x \vee x \approx x$,
(3) $x \vee y \approx y \vee x$,
(4) $x \vee 0 \approx x$.

The following lemma is easy.

Lemma 10.2. *Let $\mathbf{A} \in \mathcal{S}^0$. Then $\mathbf{A}$ satisfies $[(z' \vee x) \vee (y \vee z)']' \approx (x \vee y) \vee z$, where $x' := x \vee 0$.*

Theorem 10.3. *$\mathcal{SL}$ is term-equivalent to (in fact, is) $\mathcal{S}^0$. More precisely,*

(a) *Let $\mathbf{A} \in \mathcal{SL}$. Define $\mathbf{A}^\vee := \langle A, \vee, 0 \rangle$ be the algebra where $\vee := \to$. Then $\mathbf{A}^\vee \in \mathcal{S}^0$.*
(b) *Let $\mathbf{A} \in \mathcal{S}^0$. Define $\mathbf{A}^\to := \langle A, \to, 0 \rangle$ be the algebra where $\to := \vee$. Then $\mathbf{A}^\to \in \mathcal{SL}$.*
(c) *Let $\mathbf{A} \in \mathcal{SL}$. Then $\mathbf{A}^{\vee \to} = \mathbf{A}$.*
(d) *Let $\mathbf{A} \in \mathcal{S}^0$. Then $\mathbf{A}^{\to \vee} = \mathbf{A}$.*

Proof. (a): This follows from Lemma 10.1 and Theorem 9.3.
    (b): This follows from Lemma 10.2.
    (c) and (d): These proofs are left to the reader. $\qquad \square$

The following corollary gives an axiomatization for $\mathcal{V}(\mathbf{2_s})$.

Corollary 10.4. $\mathcal{V}(\mathbf{2_s}) = \mathcal{S}^0 = \mathcal{C} \cap \mathcal{I}_{1,0} = \mathcal{SL}$.

Corollary 10.5. *The class of congruence lattices $\mathbf{Con}\ \mathbf{A}$, where $\mathbf{A} \in \mathcal{I}$, does not satisfy any non-trivial lattice identity. In particular, the variety $\mathcal{I}$ is not congruence-modular.*

Proof. It is well known (see [7]) that the class of congruence lattices of semilattices does not satisfy any nontrivial lattice identity. $\qquad \square$

10.2. A characterization of the variety $\mathcal{V}(\mathbf{2_z})$. Recall from [12] $\mathcal{Z}$ denotes the subvariety of $\mathcal{I}$ defined by the identity $x \to y \approx 0$. Let $\mathbf{Eq}(\mathbf{A})$ denote the lattice of equivalence relations of $\mathbf{A}$. The following lemma is obvious.

Lemma 10.6. *Let $\mathbf{A} \in \mathcal{Z}$. Then*

(1) $\mathbf{Con}\ \mathbf{A} = \mathbf{Eq}(\mathbf{A})$,
(2) *$\mathbf{2_z}$ is the only nontrivial subdirectly irreducible algebra in $\mathcal{Z}$.*

Corollary 10.7. $\mathcal{Z} = \mathcal{V}(\mathbf{2_z})$.

We end this section by noting that the variety $\mathcal{V}(\mathbf{2_b})$, generated by the 2-element Boolean algebra $\mathbf{2_b}$, was axiomatized in [12].

11. The varieties $\mathcal{MC}$, $\mathcal{MID}$, $\mathcal{A}$ and $\mathcal{C}$

Our goal in this section is to prove that $\mathcal{MC} \cap \mathcal{MID} \cap \mathcal{A} = \mathcal{SL} \subset \mathcal{CP}$.

Lemma 11.1. *Let* $\mathbf{A} \in \mathcal{MC} \cap \mathcal{MID} \cap \mathcal{A}$. *Then* $\mathbf{A}$ *satisfies*

(a) $x \to (x \to 0') \approx x$,

(b) $x \vee y \approx x \to [0 \to \{y \to (0 \to (0 \to 0'))\}]$,

(c) $x \vee y \approx x \to (0 \to y')$,

(d) $0 \vee 0 \approx 0$,

(e) $0' \approx 0$,

(f) $x \to x' \approx x$,

(g) $x \to [y \to (x \to y')] \approx x \to y$,

(h) $x \to y' \approx y \to x'$ (CP),

(i) $x \to (y \to x') \approx y \to x$,

(j) $x' \approx x$.

Proof. (a): $x \to (x \to 0') \overset{(A)}{\approx} x \to [(x \to 0) \to 0] \overset{(A)}{\approx} (x \to x') \to 0$
$\overset{\text{def of } \wedge}{\approx} x \wedge x \overset{(MID)}{\approx} x$.

(b): $x \vee y \approx (x' \to y'')'' \overset{(A)}{\approx} [x' \to (y \to 0')]''$

$\overset{(A)}{\approx} [x \to \{0 \to (y \to 0')\}]'' \overset{(A)}{\approx} [x \to \{0 \to (y \to 0')\}']'$

$\overset{(A)}{\approx} [x \to \{0 \to (y \to 0')'\}]' \overset{(A)}{\approx} [x \to \{0 \to (y \to 0'')\}]'$

$\overset{(A)}{\approx} x \to [0 \to (y \to 0'')]' \overset{(A)}{\approx} x \to [0 \to (y \to 0'')']$

$\overset{(A)}{\approx} x \to [0 \to (y \to 0''')] \overset{(A)}{\approx} x \to [0 \to \{y \to (0 \to (0 \to 0'))\}]$.

(c): This is immediate from (a) and (b).

(d): From (c), we have $0 \vee 0 = 0 \to (0 \to 0')$. Now use (a).

(e): From associativity, $0 \to 0' = 0$. Hence, $0' = 0 \to (0 \to 0') = 0$ by (a).

(f): This is immediate from (a) and (e).

(g): $x \to [y \to (x \to y')] \overset{(A)}{\approx} (x \to y) \to (x \to y)' \overset{(f)}{\approx} x \to y$.

(h): $x \to y' \overset{(e)}{\approx} x \to (y \to 0') \overset{(A)}{\approx} x \to y'' \overset{(A)}{\approx} (x \to y')' \approx x \wedge y \overset{(MC)}{\approx} y \wedge x$.
Hence, $x \to y' \overset{(A)}{\approx} y \to x'' \overset{(A)}{\approx} y \to (x \to 0') \overset{(e)}{\approx} y \to x'$.

(i): $x \to (y \to x') \overset{(A)}{\approx} x \to (y \to x)' \overset{(h)}{\approx} (y \to x) \to x' \overset{(A)}{\approx} y \to (x \to x')$
$\overset{(f)}{\approx} y \to x$.

(j): $x' \overset{(g)}{\approx} x \to [0 \to (x \to 0')] \overset{(i)}{\approx} x \to (x \to 0) \overset{(f)}{\approx} x$. $\qquad\square$

Theorem 11.2. $\mathcal{MC} \cap \mathcal{MID} \cap \mathcal{A} \subseteq \mathcal{C} \cap \mathcal{I}_{1,0} \cap \mathcal{CP}$.

Proof. Let $\mathbf{A} \in \mathcal{MC} \cap \mathcal{MID} \cap \mathcal{A}$. Then

$$x \to y \overset{11.1(j)}{\approx} x \to y' \overset{11.1(h)}{\approx} y \to x' \overset{11.1(j)}{\approx} y \to x.$$

In view of (h) and (j) of Lemma 11.1, the proof is now complete. $\qquad\square$

The following corollary is immediate from the preceding theorem and Corollary 10.4, and provides another axiomatization of $\mathcal{V}(\mathbf{2_s})$.

Corollary 11.3. $\mathcal{MC} \cap \mathcal{MID} \cap \mathcal{A} = \mathcal{SL} = \mathcal{V}(\mathbf{2_s}) \subset \mathcal{CP}$.

12. The varieties $\mathcal{MC}$, $\mathcal{ID}$, $\mathcal{C}$ and $\mathcal{I}_{1,0}$

The purpose of this section is to show that $\mathcal{MC} \cap \mathcal{ID} = \mathcal{C} \cap \mathcal{I}_{1,0}$.

Lemma 12.1. Let $\mathbf{A} \in \mathcal{MC} \cap \mathcal{ID}$. Then $\mathbf{A}$ satisfies

(a) $[(0 \to x) \to y'']' \approx (x \to y)'$,
(b) $(0 \to x')' \approx x''$,
(c) $(x \to x'')' \approx x''$,
(d) $x'' \approx x'$,
(e) $[(x \to y) \to z]' \approx (x \to y) \to z$.

Proof. (a) and (b): These are immediate from (I) and the identity (ID).

(c): Use (a), (b), and Lemma 2.5.

(d): $x'' \overset{(c) \text{ and } (I)}{\approx} [(0' \to x) \to x'''']' \overset{2.5}{\approx} [(0 \to x) \to x'']' \overset{(a)}{\approx} [x \to x]' \overset{(ID)}{\approx} x'$.

(e): $[(x \to y) \to z]' \overset{(I)}{\approx} [(z' \to x) \to (y \to z)']''$ $\overset{(d)}{\approx} [(z' \to x) \to (y \to z)']' \overset{(I)}{\approx} (x \to y) \to z$. This completes the proof. $\quad\square$

Theorem 12.2. $\mathcal{MC} \cap \mathcal{ID} = \mathcal{I}_{1,0} \cap \mathcal{C}$.

Proof. First, we wish to prove $\mathcal{MC} \cap \mathcal{ID} \subseteq \mathcal{I}_{1,0} \cap \mathcal{C}$. By (ID), we have that $x' \approx (x \to x)' \approx [(x \to x) \to x]'$. Hence, $x' \approx [(x \to x) \to x] \approx x$ by Lemma 12.1 (e) and (ID).

Next, from $x \to y \approx (x \to y)' \approx (x \to y')' \approx x \wedge y \approx y \wedge x \approx (y \to x')' \approx y \to x' \approx y \to x$, we conclude $\mathbf{A} \in \mathcal{C}$. Thus we have, in view of Corollary 10.4, that $\mathcal{MC} \cap \mathcal{ID} \subseteq \mathcal{I}_{1,0} \cap \mathcal{C} = \mathcal{V}(\mathbf{2_s}) \subseteq \mathcal{MC} \cap \mathcal{ID}$, completing the proof. $\quad\square$

The following corollary is immediate from the preceding theorem and Corollary 11.3.

Corollary 12.3. $\mathcal{MC} \cap \mathcal{ID} = \mathcal{V}(\mathbf{2_s}) = \mathcal{MC} \cap \mathcal{MID} \cap \mathcal{A}$.

13. Concluding remarks

In [4], [5], and [6], these investigations are continued. (Note that the implicator groupoids are referred to in those papers as "implication zroupoids".) In [4], it is proved that the variety $\mathcal{I}_{2,0}$ is a maximal subvariety of $\mathcal{I}$ with respect to the property that the relation $\leq$, which is defined as follows:

$$x \leq y \text{ if and only if } (x \to y')' = x, \text{ for } x, y \in \mathbf{A} \text{ and } \mathbf{A} \in \mathcal{I},$$

is a partial order. Furthermore, all the finite $\mathcal{I}_{2,0}$-chains, relative to this order, are determined. In [5], we describe all simple algebras in $\mathcal{I}$ and, consequently, we give a description of semisimple subvarieties of $\mathcal{I}$.

Paper [6] is a further addtion to the series [12], [4], [5], and the present paper. It studies the structure of the derived algebras $\mathbf{A^m} := \langle A, \wedge, 0 \rangle$ and $\mathbf{A^{mj}} := \langle A, \wedge, \vee, 0 \rangle$ of $\mathbf{A} \in \mathcal{I}$. It also introduces new subvarieties of $\mathcal{I}$ and determines their relationships with other subvarieties, both old and new. In fact, it is shown that for each $\mathcal{I}$-zroupoid $\mathbf{A}$, $\mathbf{A}^m$ is a semigroup, which, together with Theorem 7.3 of this paper implies that for $\mathbf{A} \in \mathcal{I}_{2,0} \cap \mathcal{MC}$, the derived algebra $\mathbf{A^{mj}}$ is both a distributive bisemilattice and a Birkhoff system. It is also shown in [6] that $\mathcal{CLD} \subset \mathcal{SRD} \subset \mathcal{RD}$, where the varieties $\mathcal{CLD}$, $\mathcal{SRD}$, and $\mathcal{RD}$ are respectively defined, relative to $\mathcal{I}$, by (CLD) $x \to (y \to z) \approx (x \to z) \to (y \to x)$, (SRD) $(x \to y) \to z \approx (z \to x) \to (y \to z)$, and (RD) $(x \to y) \to z \approx (x \to z) \to (y \to z)$. Furthermore, [6] shows the following relationship among some of of the varieties investigated in this paper and $\mathcal{CLD}$: $\mathcal{C} \subset \mathcal{CP} \cap \mathcal{A} \cap \mathcal{MC} \cap \mathcal{CLD}$. Both of the results just mentioned are much stronger than the ones that were announced in [12].

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