

Langlands duality in Liouville- H_3^+ WZNW correspondenceGaston Giribet ¹, Yu Nakayama ², Lorena Nicolás ³¹ Department of Physics, Universidad de Buenos Aires and CONICET
*Ciudad Universitaria, Pabellón I, 1428. Buenos Aires, Argentina.*² Berkeley Center for Theoretical Physics and Department of Physics
*University of California, Berkeley, California 94720-7300.*³ Instituto de Astronomía y Física del Espacio, CONICET
*Ciudad Universitaria, C.C. 67 Suc. 28, 1428, Buenos Aires, Argentina.***Abstract**

We show a physical realization of the Langlands duality in correlation functions of H_3^+ WZNW model. We derive a dual version of the Stoyanovky-Ribault-Teschner (SRT) formula that relates the correlation function of the H_3^+ WZNW and the dual Liouville theory to investigate the level duality $k-2 \rightarrow (k-2)^{-1}$ in the WZNW correlation functions. Then, we show that such a dual version of the H_3^+ -Liouville relation can be interpreted as a particular case of a biparametric family of non-rational CFTs based on the Liouville correlation functions, which was recently proposed by Ribault. We study symmetries of these new non-rational CFTs and compute correlation functions explicitly by using the free field realization to see how a generalized Langlands duality manifests itself in this framework. Finally, we suggest an interpretation of the SRT formula as realizing the Drinfeld-Sokolov Hamiltonian reduction. Again, the Hamiltonian reduction reveals the Langlands duality in the H_3^+ WZNW model. Our new identity for the correlation functions of H_3^+ WZNW model may yield a first step to understand quantum geometric Langlands correspondence yet to be formulated mathematically.

1 Introduction

Two-dimensional non-rational conformal field theories (CFTs) have many applications both in physics and mathematics, from quantum (stringy) black hole in physics to the geometric Langlands program in mathematics. Most of what we currently know about these theories, however, is based on our understanding of Liouville field theory (LFT) [1]. In fact, LFT is by far the best understood theory among non-rational CFTs, which turns out to be the prototypical model to establish their exact quantization. A clear example is the $H_3^+ = SL(2, \mathbb{C})/SU(2)$ Wess-Zumino-Novikov-Witten (WZNW) theory, whose structure was actually understood by resorting to the analogy with LFT [2, 3, 4].

The story took a new direction three years ago when S. Ribault and J. Teschner showed that the relation between LFT and WZNW model could be pushed forward, beyond the level of a mere *analogy*, to the level of *correspondence* in correlation functions. In [5], they proved that arbitrary correlation functions of the H_3^+ WZNW model admit simple expressions in terms of correlation functions of LFT. More precisely, any n -point function of the H_3^+ WZNW theory on the topology of the sphere can be written in terms of a $2n - 2$ -point functions of LFT. This correspondence between observables of these two non-rational CFTs follows from a previous result of A. Stoyanovky, who proved in [6] a surprising functional relation between solutions to the Knizhnik-Zamolodchikov (KZ) equation and to the Belavin-Polyakov-Zamolodchikov (BPZ) equation. In this paper, we refer to the formula that connects WZNW correlation functions and Liouville correlation functions as Stoyanovsky-Ribault-Teschner (SRT) formula.

The primary aim of this paper is to further investigate the SRT H_3^+ -Liouville correspondence and its generalizations, especially in order to understand the Langlands level duality in correlation functions of H_3^+ WZNW model and study its physical applications. We, thus, begin with the review of recent development in this direction.

1.1 The H_3^+ -Liouville correspondence

The H_3^+ -Liouville correspondence has several interesting applications in string theory. For example, it can be straightforwardly adapted to describe the $SL(2, \mathbb{R})_k/U(1)$ coset model, so that string amplitudes in the two-dimensional black hole background can be described by Liouville correlation functions [7]. This correspondence is also relevant to study string theory in three-dimensional Anti-de Sitter space (AdS_3), the dynamics of inhomogeneous tachyon condensation in closed string theory, the six-dimensional little string theory, and many other scenarios (see [8, 9, 10, 11, 12] and references therein). Some of these applications were investigated in [13], where it was pointed out that in order to fully describe the tree-level string amplitudes in AdS_3 , the result of [5] needed to be generalized to include the spectral flowed sectors¹ of $SL(2, \mathbb{R})_k$. In [15], S. Ribault achieved to incorporate such spectral flowed sectors by extending the results of [5]. The key point was to generalize the KZ equation to the case of WZNW correlation functions that involve spectral flowed fields. In particular, it was shown that if in a given WZNW

¹In this paper, we do not make a clear distinction between the $SL(2, \mathbb{R})$ WZNW model and H_3^+ model because we assume that the analytic continuation of correlation functions describes the former from the latter. See for instance [14] for a similar treatment.

correlation function the conservation of the spectral flow number is violated in $\Delta\omega$ units, then such a correlation function can always be written in terms of a $2n - 2 - \Delta\omega$ -point correlation function of LFT. This new correspondence increased the set of WZNW correlation functions that admit a representation in terms of LFT (see formula (59) of Appendix B).

After the formulation of the SRT H_3^+ -Liouville correspondence on the worldsheet sphere, further generalizations were accomplished. First, its extension to the case of worldsheet geometry with boundaries was worked out in [16, 17, 18, 19], which can be regarded as a worldsheet description of the D-brane in the string theory context.

The second generalization was the extension to the case of higher genus correlation functions: in [20] Y. Hikida and V. Schomerus proved that any n -point correlation functions of the H_3^+ WZNW model at genus g can be written in terms of $2n + 2g - 2$ -point functions of LFT. This higher genus generalization was done by employing a path integral derivation of the H_3^+ -Liouville correspondence (see also [21]).

Very recently, following the path integral approach of [20], S. Ribault proposed a novel generalization of H_3^+ -Liouville correspondence, arguing that LFT may provide a representation of observables of a wider set of CFTs [22]. According to this proposal, SRT H_3^+ -Liouville correspondence could be merely a particular example of a more general correspondence. The statement [22] is that $2n - 2$ -point correlation functions of LFT on the sphere can be regarded as generators of n -point correlation functions of a biparametric family of non-rational CFTs. Each member of this family of theories is characterized by two continuous parameters, b and m , and the parameterization is such that LFT corresponds to the particular case $m = 0$, having central charge $c_L = 1 + 6(b + b^{-1})^2$.

We here make a following preliminary observation, on which we elaborate more in this paper. On one hand, among members of the above biparametric family, the H_3^+ WZNW model corresponds to the case $m = 1$, where the WZNW level is given by $k = b^{-2} + 2$ and its central charge by $c_{SL(2)} = 3 + 6b^2$. On the other hand, as we will show, the case $m = b^2$ also corresponds to the H_3^+ WZNW theory whose central charge $c_{SL(2)} = 3 + 6b^{-2}$, but with level $k = b^2 + 2$. This implies that the H_3^+ model is represented by two curves in the space of parameters (m, b) of [22]. Fixing the level k then corresponds to fixing a point on each curve, where the one curve turns out to be related to the other by the Langlands level duality $k - 2 \rightarrow (k - 2)^{-1}$. In this paper, we try to reveal the manifestation of Langlands duality in the H_3^+ WZNW model, and we will also argue that this could be seen as an example the more general duality. Specifically, more members of the biparametric family of CFTs proposed in [22] could actually appear twice in the space of parameters defined by the (m, b) plane. This idea is suggested by the structure of the conformal Ward identities and it appears naturally when discussing the current algebra that generates the symmetries of the theories.

1.2 Langlands duality and WZNW theory

The relation between the H_3^+ WZNW model (or $SL(2, \mathbb{R})_k$ WZNW model) and LFT reveals its significance not only in physics, but also in mathematics. Long before the advent of the SRT relation, the connection between these two CFTs had been studied within the context of the

geometric Langlands correspondence² (see e.g. [25, 26, 28, 29]). It turns out that CFTs with affine Kac-Moody symmetry (let us call the corresponding algebra \hat{g}) give a natural realization of the geometric Langlands program, as they provide a natural way of realizing the so-called Hecke eigensheaves. In this context, Hecke eigensheaf is an object closely related to the chiral conformal blocks of the G_k WZNW model, and they are \mathcal{D} -modules on the moduli space of the G -bundle on the worldsheet that are attached to the ${}^L G$ -bundle with holomorphic connection on the worldsheet, being ${}^L G$ the Langlands dual of the Lie group G associated to the Lie algebra g ; see [25].

One of the simplest but still highly nontrivial examples of the geometric Langlands correspondence appears in the case of $\hat{g} = sl(2)_k$ at the critical level $k \rightarrow 2$ (i.e. where the level k takes the value of the Coxeter number). On one hand, we have the sheaf of coinvariants³ (roughly speaking, \mathcal{D} -modules associated to the chiral correlation functions or conformal blocks) of the $sl(2)_{k=2}$ over the moduli space of holomorphic $SL(2)$ -bundle on the worldsheet. On the other hand, we have a flat holomorphic ${}^L SL(2)$ -bundle on the worldsheet, which generates a classical Virasoro algebra as its Poisson structure (upon Hamiltonian reduction). The Langlands correspondence predicts a correspondence between these two notions.

To study this connection, one first investigates the $sl(2)_k$ current algebra at its critical point $k \rightarrow 2$. First of all, the structure of chiral correlation functions depends on the $SL(2)$ -bundle by varying background $SL(2)$ gauge field in the action. In the Langlands correspondence, we identify this deformation of correlation functions with another deformation induced by changing the center of the $sl(2)_{k=2}$ current algebra which leads to a center-dependent representation of primary vertex operators. It turns out that the center of the affine algebra is generated by the un-normalized Sugawara current which yields the classical Virasoro algebra⁴ corresponding to a holomorphic ${}^L SL(2) = SL(2)/\mathbb{Z}_2$ connection on the worldsheet with the structure of an ${}^L sl(2)$ -oper (i.e. modulo gauge equivalence by its Borel subalgebra).

The geometric Langlands correspondence formulated on the $sl(2)_k$ current algebra in this way has an intimate connection with the Drinfeld-Sokolov (DS) Hamiltonian reduction, which reduces the $SL(2, \mathbb{R})_k$ WZNW model to LFT, [31, 32]. In the framework of geometric Langlands correspondence, the isomorphism between the center of the $sl(2)_k$ current algebra at the critical level and the classical Virasoro algebra can also be seen as a corollary of the quantum Hamiltonian reduction, namely, the isomorphism between the representations of Hamiltonian reduced chiral algebra and those of the Virasoro algebra at the quantum level⁵. Beyond the

²In physics literature, the terminology “Langlands duality” is often used to refer to the geometric Langlands correspondence [24] we are mentioning here. However, we would like to save the terminology “Langlands duality” to refer to a more specific duality; namely the duality under level transformation $k - 2 \rightarrow (k - 2)^{-1}$ of the $\hat{sl}(2)_k$ current algebra. The reason why this duality is referred to as “Langlands duality” is that it corresponds to the \mathcal{W} -algebra (Virasoro) duality isomorphism which plays an important role in the geometric Langlands correspondence (see section 8 of Ref. [25] and section 6 of Ref. [23] for discussions).

³They can be shown to be the Hecke eigensheaves.

⁴Notice that at the critical point $k = 2$ the Virasoro current (the stress-tensor $T(z)$ with appropriate normalization) commutes with the Kac-Moody currents $J^a(z)$.

⁵By using the Wakimoto free field construction, one can show that the BRST cohomology of the quantum Hamiltonian reduction is the center of the $sl(2)_k$ current algebra at the critical level, which completes the argument (see for instance [31]).

classical correspondence, the quantum Hamiltonian reduction also suggests a further mysterious duality (Langlands duality [32]) for the $sl(2)_k$ current algebra with $k \neq 2$. The quantum version of Hamiltonian reduction connects the $sl(2)_k$ current algebra of level k and the Virasoro algebra with the central charge $c = 1 + 6(b + b^{-1})$, where $b^{-2} = k - 2$. Notice that this is actually the point where the Langlands correspondence relates the $SL(2, \mathbb{R})_k$ WZNW model at the critical level ($k \rightarrow 2$) with the classical LFT ($b \rightarrow \infty$). The crucial observation is that the same Virasoro algebra is obtained with the Langlands dual $sl(2)_{\tilde{k}}$ algebra at the dual level \tilde{k} , where $b^2 = \tilde{k} - 2$. Thus, under Hamiltonian reduction, the $sl(2)_k$ current algebra manifests the intriguing level duality $k - 2 \rightarrow \tilde{k} - 2 = (k - 2)^{-1}$. From the viewpoint the Virasoro algebra the duality is nothing but the Liouville self-duality under $b \rightarrow b^{-1}$. However, from the viewpoint of the $sl(2)_k$ current algebra, it is quite mysterious: not only it relates the strongly coupled system with the weakly coupled system, but it also changes the central charge as $c_{SL(2)} = 3 + 6b^2 \rightarrow 3 + 6b^{-2}$.

In this paper, we attempt to shed more light on this level duality. More precisely, we would like to investigate a possible connection among the following three notions: the Langlands level duality $k - 2 \rightarrow \tilde{k} - 2$, the SRT H_3^+ -Liouville correspondence, and the DS Hamiltonian reduction from the viewpoint of the correlation functions. To do this, we will begin by deriving a generalization of the SRT relation that will manifest the level duality. Indeed, one can reformulate the SRT formula by essentially changing b with b^{-1} in the LFT side, and then we show that it leads to a surprising relation between the correlation functions in the H_3^+ WZNW theory with level k and those with the dual level \tilde{k} . Such a duality relation at finite values of k is actually envisaged also by mathematicians, as somehow it encodes the quantum version of the geometric Langlands correspondence. Mathematical understanding of the Langlands correspondence at the off-critical level is under lively investigation (e.g. [33], see also [34, 35] from a physical account). In this sense, our dual version of SRT formula can be seen as a physical intuition of quantum Langlands correspondence yet to be formulated mathematically at the level of the full correlation functions⁶.

In [20], Hikida and Schomerus discussed the relevance of the SRT H_3^+ -Liouville correspondence in the context of classical geometric Langlands correspondence on higher genus curves as it precisely gives the appropriate basis of the WZNW conformal blocks that can be expressed directly in terms of the conformal blocks of the Virasoro algebra (corresponding to the same Virasoro algebra obtained through DS Hamiltonian reduction). Our version of the quantum Langlands duality for the correlation functions can be straightforwardly generalized to higher genus correspondence by virtue of their results, so it will give more insight about the quantum Langlands duality on higher genus.

Before closing the introduction, a final word about the notation is in order. As mentioned, understanding of the Langlands level duality beyond the critical value is of significance both in physics and mathematics. Nevertheless, our motivations are entirely based on physical grounds, and consequently, our discussion will be in the language usually employed within the physics context.

⁶We emphasize that our approach only gives a relation between the full correlation functions and not between the chiral correlation functions.

1.3 Overview

The rest of the paper is organized as follows. In section 2, we derive the dual version of the H_3^+ -Liouville correspondence formula of [5]. We do this by using both the algebraic method and the path integral approach. In section 3, we show that the dual version of the H_3^+ -Liouville formula can be interpreted as a particular case of the Lagrangian representation of a biparametric family of non-rational CFTs recently proposed in [22]. We also compute the correlation functions in these CFTs by using the free field theory representations to give explicit expression for the three-point functions. Through the discussion, a generalization of the Langlands level duality will be manifested among these new non-rational CFTs. In section 4, we study the DS Hamiltonian reduction at the level of correlation functions and its relation to the SRT H_3^+ -Liouville correspondence. The Hamiltonian reduction interpretation of the (dual) SRT formula yields a surprising identity realizing the quantum Langlands duality. We conclude in section 5 with some remarks and open questions. In Appendix A, we collect some information of special functions used in the main text. In Appendix B, we generalize our discussion in the case of winding number violating correlation functions.

2 Dual version of SRT formula

Our first goal is to derive the dual version of the H_3^+ -Liouville correspondence of [5]. The formula relates n -point functions of H_3^+ WZNW model in the so-called μ basis (see section 2-2 for more details) and $(2n-2)$ -point functions of LFT. More precisely, we would like to propose the following dual version of SRT formula⁷

$$\left\langle \prod_{i=1}^n \Phi_{j_i}(\mu_i | z_i) \right\rangle_{H_3^+} = \frac{\pi}{2\tilde{b}} (-\pi)^n \delta^{(2)} \left(\sum_{i=1}^n \mu_i \right) |\Theta_n|^2 \left\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \prod_{t=1}^{n-2} V_{-\frac{\tilde{b}}{2}}(y_t) \right\rangle_{LFT}, \quad (1)$$

where the correlation function on the right hand side corresponds to a $2n-2$ -point function of LFT, which involves $2n-2$ exponential primary fields $V_\alpha(z) = e^{\sqrt{2}\alpha\varphi(z)}$. The central charge of LFT is given in terms of \tilde{b} as $c_L = 1 + 6Q^2$, $Q = \tilde{b} + \tilde{b}^{-1}$. We define LFT by the classical action

$$S_{LFT} = \frac{1}{2\pi} \int d^2z \left(\partial\varphi\bar{\partial}\varphi + 2\pi\mu_L e^{\sqrt{2}\tilde{b}\varphi} \right).$$

The interpolating function Θ_n is given by

$$\Theta_n(z_1, \dots, z_n | y_1, \dots, y_{n-2} | u) = \frac{u \prod_{r < s \leq n} (z_r - z_s)^{\frac{\tilde{b}^2}{2}} \prod_{t < l \leq n-2} (y_t - y_l)^{\frac{\tilde{b}^2}{2}}}{\prod_{r=1}^n \prod_{t=1}^{n-2} (z_r - y_t)^{\frac{\tilde{b}^2}{2}}}, \quad (2)$$

where y_i are related to μ_i and u (so-called Sklyanin's separation of variables) as follows:

$$u = \sum_{i=1}^n \mu_i z_i, \quad \sum_{i=1}^n \frac{\mu_i}{t - z_i} = u \frac{\prod_{j=1}^{n-2} (t - y_j)}{\prod_{i=1}^n (t - z_i)}. \quad (3)$$

⁷To avoid a confusion, we use the notation \tilde{b} for the Liouville exponent to emphasize we are discussing the dual SRT formula, while we eventually identify \tilde{b} with b when we study the Langlands duality.

The Liouville momenta α_i is related to the $SL(2, \mathbb{R})$ -spin variables j_i as

$$\alpha_i = \tilde{b}^{-1} (j_i + 1) + \tilde{b}/2 , \quad (4)$$

while the Liouville parameter \tilde{b} is related to the WZNW level k by⁸

$$\tilde{b}^2 = k - 2 , \quad (5)$$

which implies the relation between conformal dimensions as $\Delta_\alpha + \Delta_{-\tilde{b}/2} + \tilde{b}^2/2 = \Delta_\alpha - k/4 = -\tilde{b}^{-2}j(j+1) = \Delta_j$.

Expression (1) represents a dual version of the SRT formula, as it was presented in [5]. In fact, the original version of the formula in [5] is obtained from (1) by replacing $\tilde{b} \rightarrow b^{-1}$. This is actually the key point here: the fact that such a dual expression exists implies that the formula (1) and the one in [5], considered together, induce the duality under $b^{-2} = k - 2 \rightarrow (k - 2)^{-1}$ at the level of WZNW correlation functions, provided the self-duality under $b \rightarrow \tilde{b}$ of Liouville theory holds and a suitable transformation of the $SL(2, R)$ -spin variables as $b(j+1+b^{-2}/2) \rightarrow \tilde{b}(j_i+1+\tilde{b}^{-2}/2)$ is introduced. We return to this point in section 4.

Now, let us prove (1) by reviewing the analysis of [5] and [20].

2.1 Knizhnik-Zamolodchikov equation

Let us begin with the relation between reflection coefficients of both LFT and WZNW model. First, consider the Liouville two-point function

$$R^L(\alpha) = -(\pi\mu_L\gamma(\tilde{b}^2))^{\frac{Q-2\alpha}{\tilde{b}}} \frac{\Gamma(1+\tilde{b}(2\alpha-Q))\Gamma(1+\tilde{b}^{-1}(2\alpha-Q))}{\Gamma(1-\tilde{b}(2\alpha-Q))\Gamma(1-\tilde{b}^{-1}(2\alpha-Q))} . \quad (6)$$

and the $SL(2, \mathbb{R})_k$ two-point function

$$R^H(j) = - \left(\frac{\gamma(\frac{1}{k-2})}{\pi(k-2)} \right)^{-2j-1} \frac{\Gamma(2j+1)}{\Gamma(-2j-1)} \frac{\Gamma(\frac{2j+1}{k-2})}{\Gamma(-\frac{2j+1}{k-2})} , \quad (7)$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. It is straightforward to verify that reflection coefficients (6) and (7) are related as

$$R^L(\tilde{b}^{-1}(j+1) + \tilde{b}/2) = R^H(j) , \quad (8)$$

as long as

$$\left(\pi\mu_L\gamma(\tilde{b}^2) \right)^{\tilde{b}^{-2}} = \frac{\gamma(\frac{1}{k-2})}{\pi(k-2)} , \quad \text{or} \quad \tilde{\mu}_L = \frac{1}{\pi^2\tilde{b}^2} . \quad (9)$$

for the prefactor to match.⁹ This shows how the dual relation (1) holds for the simple case of the two-point function.

⁸Note that in the original formula, we have $b^2 = (k-2)^{-2}$.

⁹Recall the Liouville duality relation [37]: $(\pi\tilde{\mu}_L\gamma(\tilde{b}^{-2}))^{\tilde{b}} = (\pi\mu_L\gamma(b^2))^{1/\tilde{b}}$. In [5] the convention $\mu = b^2/\pi^2$ was used.

To go further, let us consider the Liouville four-point function,

$$F^L(\alpha_1, \alpha_2, -\tilde{b}/2, \alpha_3) = \left\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)V_{-\frac{\tilde{b}}{2}}(y_1)V_{\alpha_3}(z_3) \right\rangle_{LFT} , \quad (10)$$

which is given by

$$\begin{aligned} F^L(\alpha_1, \alpha_2, -\tilde{b}/2, \alpha_3) &= |z_{32}|^{2(\Delta_{\alpha_1}-\Delta_{\alpha_2}-\Delta_{\alpha_3}+\Delta_{-\tilde{b}/2})}|z_{31}|^{2(\Delta_{\alpha_2}-\Delta_{\alpha_1}-\Delta_{\alpha_3}+\Delta_{-\tilde{b}/2})} \times \\ &\times |z_{21}|^{2(\Delta_{\alpha_3}-\Delta_{\alpha_1}-\Delta_{\alpha_2}-\Delta_{-\tilde{b}/2})}|z_3 - y_1|^{-4\Delta_{-\tilde{b}/2}}|1 - z|^{2(j_1+1)+\tilde{b}^2} \times \\ &\times \sum_{\eta=\pm} \left(|z|^{2(\Delta_{\alpha_3}-\eta\tilde{b}/2-\Delta_{\alpha_3}+\Delta_{\eta\tilde{b}/2})} \tilde{C}_{\eta}^L(\alpha_3) C^L(\alpha_2, \alpha_1, \alpha_3 - \eta\tilde{b}/2) \right) \times \\ &\times {}_2F_1(-j_3^{\eta} + j_1 + j_2 + 1, -j_3^{\eta} + j_1 - j_2, -2j_3^{\eta}, z) , \end{aligned} \quad (11)$$

where

$$z_{ab} = z_a - z_b , \quad z = \frac{(z_1 - z_2)(y_1 - z_3)}{(z_1 - z_3)(y_1 - z_2)} ,$$

and $j^- = j, j^+ = -j - 1$. In (11), the function $C^L(\alpha_3, \alpha_2, \alpha_1)$ corresponds to Liouville structure constant

$$C^L(\alpha_3, \alpha_2, \alpha_1) = (\pi\mu_L\gamma(\tilde{b}^2)\tilde{b}^{2-2\tilde{b}^2})^s \frac{\Upsilon'(0)}{\Upsilon(\alpha_1 + \alpha_2 + \alpha_3 - Q)} \prod_{i=1}^3 \frac{\Upsilon(2\alpha_i)}{\Upsilon(\alpha_1 + \alpha_2 + \alpha_3 - 2\alpha_i)} ,$$

where $s = 1 + \tilde{b}^{-2} - \tilde{b}^{-1}(\alpha_1 + \alpha_2 + \alpha_3)$. See Appendix A for the definition and properties of $\Upsilon(x)$. The special structure constants $\tilde{C}_{\eta}^L(\alpha)$ in (11) are given by

$$\tilde{C}_{-}^L(\alpha) = (\pi\mu_L\gamma(\tilde{b}^2))^{\tilde{b}^2} \frac{\gamma(2\tilde{b}\alpha - 1 - \tilde{b}^2)}{\gamma(2\tilde{b}\alpha)} , \quad \tilde{C}_{+}^L(\alpha) = 1 .$$

It is relatively easy to show that (10) becomes

$$\begin{aligned} F^L(\alpha_1, \alpha_2, -\tilde{b}/2, \alpha_3) &= |z_{32}|^{2(\Delta_{\alpha_1}-\Delta_{\alpha_2}-\Delta_{\alpha_3}-\Delta_{-\tilde{b}/2})}|z_{31}|^{2(\Delta_{\alpha_2}-\Delta_{\alpha_1}-\Delta_{\alpha_3}-\Delta_{-\tilde{b}/2})} \times \\ &\times |z_{21}|^{2(\Delta_{\alpha_3}-\Delta_{\alpha_1}-\Delta_{\alpha_2}-\Delta_{-\tilde{b}/2})}|u|^{4\Delta_{-\tilde{b}/2}}|\mu_1|^{\tilde{b}^2}|\mu_2|^{\tilde{b}^2}|\mu_3|^{\tilde{b}^2} \times \\ &\times (-2\pi^2\tilde{b})C^H(j_3, j_2, j_1)D^H[j, \mu] \end{aligned} \quad (12)$$

with $C^H(j_3, j_2, j_1)D^H[j, \mu]$ being the H_3^+ WZNW structure constants found in [2, 38, 39] written in terms of the so-called μ -basis introduced in [5]. Then, if one multiplies (12) by Θ_3 , one finds the expected agreement, as stated in (1).

To complete the proof, we have to study higher-point functions. For this purpose, we first show the relation between the BPZ equation and the KZ equation, corresponding to the right and the left hand side of (1) respectively. The BPZ equation satisfied by LFT correlation functions that involve degenerate field $V_{-\tilde{b}/2}$ is given by

$$\left[\frac{1}{\tilde{b}^2} \frac{\partial^2}{\partial y_r^2} + \sum_{s \neq r} \left(\frac{1}{y_r - y_s} \frac{\partial}{\partial y_s} + \frac{\Delta_{-\tilde{b}/2}}{(y_r - y_s)^2} \right) + \sum_s \left(\frac{1}{y_r - z_s} \frac{\partial}{\partial z_s} + \frac{\Delta_{\alpha_s}}{(y_r - z_s)^2} \right) \right] \Omega_{2n-2}^L = 0 ,$$

where Ω_{2n-2}^L denotes the Liouville correlation function appearing in (1). On the other hand, the Sklyanin separation of variable yields the following form for the KZ equation

$$\left[\frac{1}{\tilde{b}^2} \frac{\partial^2}{\partial y_a^2} + \sum_{r=1}^n \frac{1}{y_a - z_r} \left(\frac{\partial}{\partial z_r} + \frac{\partial}{\partial y_a} \right) - \sum_{b \neq a} \frac{1}{y_a - y_b} \left(\frac{\partial}{\partial y_a} - \frac{\partial}{\partial y_b} \right) + \sum_{r=1}^n \frac{\Delta_{j_r}}{(y_a - z_r)^2} \right] \Omega_n^H = 0 ,$$

where Ω_n^H denotes the H_3^+ correlation function in (1).

Crucial observation is that these two equations agree with each other after twisting by Θ_n . Now, since the correlation functions of both theories satisfy the same linear differential equation, one can show (1) by taking a particular limit $z_{12} \rightarrow 0$ and following the same induction argument used in [5].

In summary, the same argument employed in [5] leads to the derivation of the dual relation (1). Our dual formulation clearly implies the close relation between the Liouville self duality under $\tilde{b} \rightarrow \tilde{b}^{-1}(= b)$ and the Langlands level duality under $k - 2 \rightarrow (k - 2)^{-1}$ in $SL(2, \mathbb{R})_k$ WZNW correlation functions. Generalization to the winding violating correlation functions and disk one-point functions should be straightforward (see Appendix B).

We stress that although our dual formula looks as if it were a mere rewriting of the original formula with the dual variable, it is not. It is rather a consequence of the Liouville duality under $b \rightarrow b^{-1}$. For example, we had to set the *dual* cosmological constant to a particular value and this relation is different from the original SRT formula, whose origin will be clarified further when we discuss the path integral derivation. Alternatively speaking, the inductive proof of the equivalence between the original SRT formulation and the dual formulation presented here can be thought of as a derivation of the self-duality of LFT at the level of n -point correlation functions. Because the Liouville self-duality is not trivial¹⁰ and not proven in general, our dual formula is indeed non-trivial. We will see that the existence of such a dual formula leads to a far-reaching consequence of the Langlands duality in the H_3^+ -correlation functions.

In the next subsection, we present an alternative derivation of (1) by using the path integral approach.

2.2 Path integral derivation

Dual SRT formula (1) can be obtained also from the path integral approach by using the so-called dual screening operator. The possibility was already mentioned in [20]. The starting point is the H_3^+ WZNW model represented by the free field action

$$S_0 = \frac{1}{2\pi} \int d^2z (\partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma}) , \quad (13)$$

where ϕ field has a background charge¹¹ $\hat{Q} = 1/\sqrt{k-2} = \tilde{b}^{-1}(= b)$, and by the addition of the *dual* screening operator

$$S_s = \frac{1}{2\pi} \int d^2z (-\beta\bar{\beta})^{\tilde{b}^2} e^{\sqrt{2}\tilde{b}\phi} . \quad (14)$$

¹⁰For instance, it is known to break down in the spherical partition function [40].

¹¹We formulate the path integral on the flat Euclidean space. A careful treatment of the curvature coupling can be found in [20].

The $sl(2)_k$ vertex operators in the so-called μ -basis can be realized as

$$\Phi_j(\mu|z) = |\mu|^{2j+2} e^{\mu\gamma - \bar{\mu}\bar{\gamma}} e^{\sqrt{2\tilde{b}^{-1}}(j+1)\phi} ,$$

whose conformal dimensions are $\Delta_j = -\tilde{b}^{-2}j(j+1)$.

To compute the left hand side of (1) and explicitly connect it with the right hand side, we would like to evaluate the path integral

$$\left\langle \prod_{i=1}^n \Phi_{j_i}(\mu_i|z_i) \right\rangle_{H_3^+} = \int \mathcal{D}\phi \mathcal{D}\gamma \mathcal{D}\bar{\gamma} \mathcal{D}\beta \mathcal{D}\bar{\beta} e^{-S_0 - S_s} \prod_{i=1}^n \Phi_{j_i}(\mu_i|z_i) .$$

The integration over field γ (with a suitable contour modification) yields the delta function constraint

$$\bar{\partial}\beta(w) = 2\pi \sum_{i=1}^n \mu_i \delta(w - z_i) ,$$

or, equivalently, the integrated condition

$$\beta(w) = \sum_{i=1}^n \frac{\mu_i}{w - z_i}$$

with $\sum_{i=1}^n \mu_i = 0$. Then, we can introduce y_j and u such that

$$\beta(w) = u \frac{\prod_{j=1}^{n-2} (w - y_j)}{\prod_{i=1}^n (w - z_i)} .$$

Integrating over field β gives

$$\begin{aligned} & |u|^2 \delta \left(\sum_{i=1}^n \mu_i \right) \int \mathcal{D}\phi e^{-\frac{1}{2\pi} \int d^2w \left(\partial\phi \bar{\partial}\phi + |u|^{2\tilde{b}^2} \left| \prod_{t=1}^{n-2} (w - y_t) \prod_{i=1}^n (w - z_i)^{-1} \right|^{2\tilde{b}^2} e^{\sqrt{2\tilde{b}}\phi} \right)} \times \\ & \times \prod_{i=1}^n |\mu_i|^{2(j_i+1)} e^{\sqrt{2\tilde{b}^{-1}}(j_i+1)\phi} . \end{aligned}$$

Now, to remove the prefactor in front of the interaction, we define the new field

$$\varphi(w) = \phi(w) + \sqrt{2\tilde{b}} \log |u|^2 + \sqrt{2\tilde{b}} \left(\sum_{j=1}^{n-2} \log |w - y_j|^2 - \sum_{i=1}^n \log |w - z_i|^2 \right) .$$

This yields the path integral representation

$$\begin{aligned} \left\langle \prod_{i=1}^n \Phi_{j_i}(\mu_i|z_i) \right\rangle_{H_3^+} &= |\Theta_n|^2 \delta \left(\sum_{i=1}^n \mu_i \right) \int \mathcal{D}\varphi e^{-\frac{1}{2\pi} \int d^2w \left(\frac{1}{2} \partial\varphi \bar{\partial}\varphi + e^{\sqrt{2\tilde{b}}\varphi} \right)} \times \\ & \times \prod_{i=1}^n e^{(\sqrt{2\tilde{b}^{-1}}(j_i+1) + \frac{\sqrt{2\tilde{b}}}{2})\varphi(z_i)} \prod_{l=1}^{n-2} e^{-\frac{\sqrt{2\tilde{b}}}{2}\varphi(y_l)} , \end{aligned}$$

where the background charge of φ is given by $Q = \tilde{b} + \tilde{b}^{-1}$. The shift $\hat{Q} = \tilde{b}^{-1} \rightarrow Q = \tilde{b} + \tilde{b}^{-1}$ of the background charge could be understood by keeping track of the curvature coupling as

in [20]. Notice that the right hand side of the equation above corresponds to the expected $2n - 2$ -point function of LFT. In this way, we have obtained the path integral derivation of the dual SRT formula.

In retrospect of the path integral derivation, we could interpret the dual formula from the original variable b instead of \tilde{b} . If we did this, we would end up with the Liouville action with the *dual* Liouville interaction $e^{\sqrt{2}b^{-1}\phi}$. This is the reason why we have to set the dual cosmological constant (and not the original cosmological constant) to a particular value in (9). This clearly shows that the dual screening charge in H_3^+ model corresponds to the dual Liouville interaction.

3 A biparametric family of CFTs

In this section, we will study relation (1) in the context of the generalization of SRT correspondence recently proposed in [22]. We will analyze the biparametric family of non-rational CFTs there to show that the dual version of the SRT H_3^+ -Liouville formula discussed in section 2 can be interpreted as a particular case of the theories described [22].

Let us consider the following quantity [22]

$$\Omega_n^{(m)} = \delta^{(2)} \left(\sum_{i=1}^n \mu_i \right) |\Theta_n^{(m)}|^{2m^2} \left\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \prod_{t=1}^{n-2} V_{-\frac{m}{2b}}(y_t) \right\rangle_{LFT}, \quad (15)$$

where m and b are two (real-valued) continuous parameters. The coordinates u , z_i , y_r and μ_i are related through the Sklyanin change of variable (3), and the function $\Theta_n^{(m)}$ is defined by

$$\Theta_n^{(m)}(z_1, \dots, z_n | y_1, \dots, y_{n-2} | u) = \frac{u^{\left(\frac{1}{m} + b^{-2}\left(\frac{1}{m} - 1\right)\right)} \prod_{r < s \leq n} (z_r - z_s)^{\frac{1}{2b^2}} \prod_{t < l \leq n-2} (y_t - y_l)^{\frac{1}{2b^2}}}{\prod_{r=1}^n \prod_{t=1}^{n-2} (z_r - y_t)^{\frac{1}{2b^2}}}. \quad (16)$$

In (15), as in (1), the correlation function in the right hand side corresponds to a $2n - 2$ -point function in LFT. This correlation function involves $2n - 2$ exponential primary fields $V_\alpha(z) = e^{\sqrt{2}\alpha\varphi(z)}$, with $n - 2$ of these fields having momentum $\alpha = -m/2b$.

In [22], it was argued that $\Omega_n^{(m)}$ defined as in (15) could be interpreted as a correlation function of a certain CFT, which is characterized by m and b . This means that $\Omega_n^{(m)}$ could be written as

$$\Omega_n^{(m)} = \left\langle \prod_{i=1}^n \Phi_{j_i}(\mu_i | z_i) \right\rangle_{CFT}, \quad (17)$$

where $\Phi_j(\mu|z)$ would correspond to primary operators of a CFT. This CFT is conjectured to exist, and it is considered “solvable” in the sense that its correlation functions are known provided the LFT representation (15) is given. In turn, (15) is thought of as a definition of a biparametric family of new non-rational CFTs. The H_3^+ WZNW theory corresponds to the particular case $m = 1$ as (15) reduces to the H_3^+ -Liouville correspondence of [5]. On the other hand, LFT is obtained in the trivial case $m = 0$. It is worth noticing that the dual version of the SRT formula we derived in section 2 is obtained at $m = b^2$ in (15). In fact, in this case, the right hand side of (15) coincides with the right hand side of (1).

3.1 Free field realization

In [22], a Lagrangian representation of this family of CFTs was given by generalizing the path integral approach of [20]. The Lagrangian for the CFT of which (17) are its correlation functions is given by the action

$$S[\lambda] = \frac{1}{2\pi} \int d^2z \left(\partial\phi\bar{\partial}\phi + \beta\bar{\partial}\gamma + \bar{\beta}\partial\bar{\gamma} + \frac{Q_m}{2\sqrt{2}}R\phi + 2\pi\lambda(-\beta\bar{\beta})^m e^{\sqrt{2}b\phi} \right) \quad (18)$$

with the background charge $Q_m = b + b^{-1}(1 - m)$. Here, λ represents a real coupling constant whose specific value is controlled by the zero mode of ϕ . It is easy to verify that the interaction term $(-\beta\bar{\beta})^m e^{\sqrt{2}b\phi}$ has a conformal dimension $(1, 1)$. Realization (18) is actually reminiscent of the Lagrangian representation of the $SL(2, \mathbb{R})_k$ WZNW model. In fact, (18) does agree with the Wakimoto free field representation of the $SL(2, \mathbb{R})_k$ model in the particular case(s) $m = 1$ (and $m = b^2 (= \tilde{b}^2 = k - 2)$), where the WZNW level k is given by $k = b^{-2} + 2$ (resp. $k = b^2 + 2$).

Lagrangian realization (18) enabled us to study the symmetry algebra underlying the solvable CFT [22], which is generated by the stress tensor

$$T(z) = -\beta(z)\partial\gamma(z) - \frac{1}{2}(\partial\phi(z))^2 + (b + b^{-1}(1 - m))\partial^2\phi(z) \quad (19)$$

and the Borel subalgebra of the following representation of the affine algebra $\hat{sl}(2)_k$

$$J^+(z) = \beta(z) , \quad (20)$$

$$J^-(z) = \beta(z)\gamma^2(z) - \sqrt{2}mb^{-1}\gamma(z)\partial\phi(z) + (m^2b^{-2} + 2)\partial\gamma(z) , \quad (21)$$

$$J^3(z) = -\beta(z)\gamma(z) + \frac{1}{\sqrt{2}}mb^{-1}\partial\phi(z) . \quad (22)$$

Here, as usual, fields β and γ form a commuting ghost system, while field ϕ is a free boson with background charge $Q_m = (b + b^{-1}(1 - m))$. These fields have non-vanishing propagators given by

$$\langle\beta(z)\gamma(w)\rangle = (z - w)^{-1} , \quad \langle\phi(z, \bar{z})\phi(w, \bar{w})\rangle = -\log|z - w|^2 .$$

The central charge associated to stress tensor (19) is given by $c = 3 + 6Q_m^2$. It is worth noticing that interaction term $(-\beta\bar{\beta})^m e^{\sqrt{2}b\phi}$ in (18) commutes with (19) for any value of m but it commutes with the currents (20)-(22) only for $m = 1$ and $m = b^2$. In particular, the operator product expansion (OPE) with $J^-(z)$ yields

$$\begin{aligned} J_{(z)}^- \beta_{(w)}^m e^{\sqrt{2}b\phi(w)} &\sim \frac{m}{(z - w)^2} \left((m^2b^{-2} - m + 1)\beta^{m-1} e^{\sqrt{2}b\phi} + \right. \\ &\quad \left. + (z - w)(\sqrt{2}mb^{-1}\partial\phi\beta + (m - 1)\partial\beta)\beta^{m-2} e^{\sqrt{2}b\phi} \right) + \dots , \end{aligned}$$

which in the cases $m = 1$ and $m = b^2$ yields total derivatives

$$J_{(z)}^- \beta_{(w)} e^{\sqrt{2}b\phi(w)} \sim b^{-2} \partial_w \frac{e^{\sqrt{2}b\phi(w)}}{(z - w)} + \dots$$

$$J_{(z)}^- \beta_{(w)}^{b^2} e^{\sqrt{2}b\phi(w)} \sim b^{+2} \partial_w \frac{\beta_{(w)}^{b^2-1} e^{\sqrt{2}b\phi(w)}}{(z - w)} + \dots$$

respectively. As a result, for generic values of m and b , the symmetries of theory (18) turns out to be generated by the Virasoro current $T(z)$ and the subalgebra generated by $J^+(z)$ and $J^3(z)$.

Lagrangian realization (18) also provides the explicit form of the primary operators $\Phi_j(\mu|z)$, which read

$$\Phi_j(\mu|z) = |\mu|^{2m(j+1)} e^{\mu\gamma(z) - \bar{\mu}\bar{\gamma}(\bar{z})} e^{\sqrt{2}b(j+1)\phi(z, \bar{z})} .$$

These are Virasoro primary fields of dimension

$$\Delta_j = -(j+1)(b^2j + m - 1) \quad (23)$$

with respect to the stress tensor (19). Notice that momenta j_i and Liouville momenta α_i in (15) are related by $\alpha_i = b(j_i + 1 + mb^{-2}/2)$.

It is natural to consider the following representation for the vertex operators,

$$\Phi_{j,p,\bar{p}}(z) \sim \gamma_{(z)}^{p-m(j+1)} \bar{\gamma}_{(\bar{z})}^{\bar{p}-m(j+1)} e^{\sqrt{2}b(j+1)\phi(z, \bar{z})} . \quad (24)$$

Again, this is reminiscent of the Wakimoto free field representation of the $SL(2, \mathbb{R})_k$ WZNW model, and yields the relation

$$\Phi_{j,p,\bar{p}}(z) = \frac{\Gamma(1+p-m(j+1))}{\Gamma(m(j+1)-\bar{p})} \int d^2\mu \mu^{-p-1} \bar{\mu}^{-\bar{p}-1} \Phi_j(\mu|z) . \quad (25)$$

The relation between basis $\Phi_{j,p,\bar{p}}(z)$ and $\Phi_j(\mu|z)$ follows from the functional relations $\Gamma(n)\Gamma(1-n) = (-)^n\Gamma(0)$ and $\int ds e^{-st} s^{x-1} = t^{-x}\Gamma(x)$. Operators (24) obey the following OPE with respect to the current algebra (20)-(22)

$$\begin{aligned} J^\pm(z)\Phi_{j,p,\bar{p}}(w) &\sim \frac{p_i \mp m(j_i + 1)}{(z-w)} \Phi_{j,p\mp 1,\bar{p}}(w) + \dots \\ J^3(z)\Phi_{j,p,\bar{p}}(w) &\sim \frac{-p_i}{(z-w)} \Phi_{j,p,\bar{p}}(w) + \dots \end{aligned}$$

so that, in particular, these are Kac-Moody primaries under the Borel subalgebra generated by $J^+(z)$ and $J^3(z)$, which are symmetry of the system.

In the next subsection, we will compute three-point functions of vertex operators (24) by employing the Lagrangian realization (18) for a generic member of the biparametric family of CFTs, but, first, let us discuss a particular case: note that if we specify $m = b^2 (= \bar{b}^2 = k - 2)$ in (16), and compare it with the definition (2), we get the relation¹²

$$|\Theta_n^{(b^2)}|^{2b^4} = |\Theta_n|^2 . \quad (26)$$

This implies that $m = b^2$ in (18) yields an alternative representation of the H_3^+ WZNW theory. We note that this representation was the one employed in [36] to explicitly compute WZNW three-point functions. When $m = b^2$ the interaction term in (18) corresponds to the dual

¹²In particular, in the case $m = 1$ function $\Theta_n^{(1)}$ coincides with the function Θ_n of [5].

screening charge (14), namely $S_s \sim \int d^2z \beta^{k-2} \bar{\beta}^{k-2} e^{\sqrt{2k-4}\phi}$, where the relation with the WZNW level is now $m = b^2 = k - 2$. This has to be compared with the case $m = 1$ in (18), which corresponds to the standard Wakimoto representation with the screening $S_s \sim \int d^2z \beta \bar{\beta} e^{\sqrt{\frac{2}{k-2}}\phi}$, with $b^2 = (k - 2)^{-1}$. Notice that the relation between the Liouville parameter b and the WZNW level k in each case is different; one is related to each other by $k - 2 \rightarrow (k - 2)^{-1}$. This shows that in the framework of H_3^+ -Liouville correspondence Langlands duality turns out to be induced by the Liouville self-duality under $b \rightarrow b^{-1}$.

In summary, the dual version of the H_3^+ -Liouville correspondence we discussed in section 2 corresponds to a particular case of the Lagrangian representation (18), that with $m = b^2 = k - 2$. This implies the H_3^+ WZNW theory turns out to be double-represented within the family of CFTs proposed in [22]. The H_3^+ model is represented by two different curves in the space of parameters, and fixing the level k corresponds to fixing a point on each curve. One curve is related to the other by the level duality $k - 2 \rightarrow (k - 2)^{-1}$, and this agrees with free field realizations considered in the literature. The idea we would like to suggest is that, presumably, this double-representation of CFTs within the biparametric family of [22] is a more general feature, and not only happens to the H_3^+ WZNW theory. In fact, the structure of the conformal Ward identities suggests that the CFT corresponding to the case $m = n$ (for a positive integer number $n \in \mathbb{Z}_{>0}$) coincides with that corresponding to the case $m = nb^2$. Moreover, notice that function $|\Theta_n^{(m)}|^{2m^2}$ in (15) is such that the change $m \rightarrow mb^2$ can be always reinterpreted as the inversion $b \rightarrow b^{-1}$ but keeping m fixed; and the same happens with the auxiliary fields $V_{-m/2b}$ in the right hand side of (15). Thus, assuming Liouville self-duality, one is led to conclude that both cases $m = n$ and $m = nb^2$ do correspond to the same CFT. It would be interesting to explore this aspect, as it would yield a generalization of what Langlands level duality is for the WZNW theory.

Before going into the explicit computation of correlation functions, we would like to mention one open question about the CFTs described by (18). This is the question of identifying such CFTs. These theories likely correspond to actual CFTs; but, which CFTs are those? We have just commented that the particular case $m = b^2$ also corresponds to the H_3^+ WZNW; however, analyzing in detail other particular cases seems to be a more difficult problem. What we certainly know about the CFTs proposed in [22] is that they likely exist, and that a subset of their observables are given by (15). Nevertheless, it remains a hard task to attempt a classification, or to identify more particular cases. We could also ask whether additional correlation functions other than those in (15)-(17) are required to fully characterize the set of observables. For instance, we know that this is actually the case for $m = 1$ and $m = b^2$, where the spectral flowed sectors require a different amount of Liouville fields on the right hand side of (15). Since spectral flow symmetry is still an automorphism of the remnant algebra generated by $J^{3,-}(z)$ and $T(z)$, it is likely that a different amount of Liouville insertions in (15) would also correspond to well-defined correlation functions of the theories described by (18). This certainly deserves further analysis. In the next subsection, we study the explicit form of correlation functions to reveal some features of these hypothetical CFTs.

3.2 Correlation functions

Let us compute correlation functions of the CFTs defined by (18). We study the n -point correlation functions (17) in the p -basis

$$\Omega_n^{(m)} = \left\langle \prod_{i=1}^n \Phi_{j_i, p_i, \bar{p}_i}(z_i) \right\rangle_{CFT} , \quad (27)$$

which are defined by

$$\Omega_n^{(m)} = \int \mathcal{D}\phi \mathcal{D}\gamma \mathcal{D}\bar{\gamma} \mathcal{D}\beta \mathcal{D}\bar{\beta} e^{-S[\lambda]} \prod_{i=1}^n \gamma_{(z_i)}^{p_i - m(j_i+1)} \bar{\gamma}_{(\bar{z}_i)}^{\bar{p}_i - m(j_i+1)} e^{\sqrt{2}b(j_i+1)\phi(z_i, \bar{z}_i)} .$$

After integrating out the zero-modes, the correlation function can be written as

$$\begin{aligned} \Omega_n^{(m)} &= (-1)^{ms} \lambda^s b^{-1} \Gamma(-s) \delta \left(\sum_{i=1}^n j_i + n + s - 1 - b^{-2}(1-m) \right) \times \\ &\times \int \prod_{r=1}^s d^2 w_r \int \tilde{\mathcal{D}}\phi \mathcal{D}\gamma \mathcal{D}\bar{\gamma} \mathcal{D}\beta \mathcal{D}\bar{\beta} e^{-S[\lambda=0]} \prod_{r=1}^s \beta_{(w_r)}^m e^{\sqrt{2}b\phi(w_r)} \times \\ &\times \prod_{i=1}^n \gamma_{(z_i)}^{p_i - m(j_i+1)} e^{\sqrt{2}b(j_i+1)\phi(z_i)} \times c.c. , \end{aligned} \quad (28)$$

where $c.c.$ refers to the complex conjugate contribution, and the path integral measure $\tilde{\mathcal{D}}\phi$ in the second line has to be understood as excluding the zero mode. The integration over the zero mode of ϕ yields the first line in (28), implying the condition

$$\sum_{r=1}^n j_r + n + s = 1 + b^{-2}(1-m) , \quad (29)$$

which, combined with the Riemann-Roch theorem, yields

$$\sum_{r=1}^n \bar{p}_r = \sum_{r=1}^n p_r = (mb^{-2} - 1)(1-m) . \quad (30)$$

As usually happens in non-rational theories, expression (28) has to be understood just formally: since the kinematical configurations in (29) can yield non-integer values s of screening charges, the integrals and products in (28) (and consequently the Wick contractions, see also [41]) generally do not seem to be well-defined. Nevertheless, these are usual features in non-rational CFTs and fortunately the analytic continuation of such expressions is under control. Besides, for positive integer values of s , the overall factor $\Gamma(-s)$ in (28) diverges. This divergence is interpreted as due to the non-compactness of the target space as in the case of LFT [42].

Performing the Wick contractions in (28), we find the following expression

$$\begin{aligned} \Omega_n^{(m)} &= (-1)^{ms} \lambda^s b^{-1} \Gamma(-s) \delta \left(\sum_{i=1}^n j_i + n + s - 1 - b^{-2}(1-m) \right) \times \\ &\times \prod_{a < b \leq n} |z_{ab}|^{-4b^2(j_a+1)(j_b+1)} \int \prod_{r=1}^s d^2 w_r \prod_{r < t \leq s} |w_r - w_t|^{-4b^2} \times \\ &\times \prod_{r=1}^s \prod_{a=1}^n |z_a - w_r|^{-4b^2(j_a+1)} \times \sum_{(i,r)} \prod_{r=1}^s \prod_{i=1}^n w_i^r (z_i - w_r)^{-c_i^r} \times c.c. , \end{aligned}$$

where the sum $\sum_{(i,r)}$ runs over the different ways of choosing pairs (i, r) among $ms - 1$, and with $i = 1, \dots, n$ while $r = 1, \dots, s$. Such contributions correspond to the different combinations when performing the Wick contractions of β - γ fields, and coefficients w_i^r are the multiplicity factors that count the different ways of contracting c_i^r fields, piking those up among $r \leq s$ and $i \leq p_i - m(j_i + 1)$.

Now, let us focus on the three-point functions, which generically take the form

$$\left\langle \prod_{i=1}^{n=3} \Phi_{j_i, p_i, \bar{p}_i}(z_i) \right\rangle_{CFT} = |z_{12}|^{2(\Delta_{j_3} - \Delta_{j_1} - \Delta_{j_2})} |z_{13}|^{2(\Delta_{j_2} - \Delta_{j_3} - \Delta_{j_1})} |z_{23}|^{2(\Delta_{j_1} - \Delta_{j_2} - \Delta_{j_3})} C_{(j_1, j_2, j_3 | p_1, p_2, p_3)}^{(m)},$$

where $C_{(j_1, j_2, j_3 | p_1, p_2, p_3)}^{(m)}$ represent the structure constants. Here, we will concentrate on the case $p_2 = \bar{p}_2 = m(1 + j_2)$ because the computation becomes drastically simpler: the Wick contraction of the γ - β system can be carried out without resorting to abstruse combinatorics. Nevertheless, the computation for the generic case can be done by a suitable adaptation of the results of [44]. See for instance Eq. (2.15) of [44], and cf. Eq. (3.20) in [45]. We will not address the case $p_2 \neq m(1 + j_2)$ here.

Let us compute $C_{(j_1, j_2, j_3 | p_1, m(1+j_2), p_3)}^{(m)}$. To do this, first we need to compute the Wick contractions

$$W_{(j_1, j_2, j_3 | p_1, m(1+j_2), p_3)}^{(m)} = \int \prod_{r=1}^s d^2 w_r \left\langle e^{\sqrt{2}b(j_1+1)\phi(0)} e^{\sqrt{2}b(j_2+1)\phi(1)} e^{\sqrt{2}b(j_3+1)\phi(\infty)} \prod_{r=1}^s e^{\sqrt{2}b\phi(w_r, \bar{w}_r)} \right\rangle_{\lambda=0} \\ \times \left\langle \gamma_{(0)}^{p_1 - m(j_1+1)} \gamma_{(\infty)}^{p_3 - m(j_3+1)} \prod_{r=1}^s \beta_{(w_r)}^m \right\rangle_{\lambda=0} \left\langle \bar{\gamma}_{(0)}^{\bar{p}_1 - m(j_1+1)} \bar{\gamma}_{(\infty)}^{\bar{p}_3 - m(j_3+1)} \prod_{r=1}^s \bar{\beta}_{(\bar{w}_r)}^m \right\rangle_{\lambda=0}, \quad (31)$$

where the subscript $\lambda = 0$ refers to the fact that these correlation functions are defined in terms of the free action $S[\lambda = 0]$. Standard free field techniques enable us to write

$$W_{(j_1, j_2, j_3 | p_1, m(1+j_2), p_3)}^{(m)} = \frac{\Gamma(1 - m(j_1 + 1) + \bar{p}_1) \Gamma(1 - m(j_3 + 1) + p_3)}{\Gamma(m(j_1 + 1) - p_1) \Gamma(m(j_3 + 1) - \bar{p}_3)} \times \\ \times \int \prod_{r=1}^s d^2 w_r \prod_{r=1}^s |w_r|^{-4b^2(j_1+1) - 2m} |1 - w_r|^{-4b^2(j_2+1)} \prod_{r < t \leq s} |w_r - w_t|^{-4b^2}. \quad (32)$$

The Γ -functions in the first line come from the multiplicity factor when contracting the fields of β - γ system. This contribution can be obtained as in [36] by generalizing the procedure in [45]. This yields

$$\left\langle \gamma_{(0)}^{p_1 - m(j_1+1)} \gamma_{(\infty)}^{p_3 - m(j_3+1)} \prod_{r=1}^s \beta_{(w_r)}^m \right\rangle = \lim_{w_i^{(t)} \rightarrow w_i^{(1)} = w_i} \mathcal{P}^{-1} \frac{\partial^{ms} \mathcal{P}}{\partial w_1^{(1)} \dots \partial w_1^{(m)} \dots \partial w_s^{(1)} \dots \partial w_s^{(m)}} \quad (33)$$

with

$$\mathcal{P} = \prod_{r=1}^s \prod_{t=1}^m (w_r^{(t)})^{m(j_1+1) - p_1} \prod_{r < l} (w_r^{(t)} - w_l^{(t)}). \quad (34)$$

Here, we regulate the correlation function by point-splitting method for the insertion points of the screening operators, as $\beta_{(w_r)}^m \rightarrow \prod_{t=1}^m \beta_{(w_r^{(t)})}$, taking ms different points as $w_r^{(t)}$, with

$r = 1, \dots, s$ and $t = 1, \dots, m$, and then taking the coincidence limit $w_r^{(t)} \rightarrow w_r^{(t-1)} \rightarrow \dots w_r^{(2)} \rightarrow w_r^{(1)} = w_r$. Accordingly, for a particular case $p_i = \bar{p}_i$, one obtains

$$\left| \left\langle \gamma_{(0)}^{p_1 - m(j_1 + 1)} \gamma_{(\infty)}^{p_3 - m(j_3 + 1)} \prod_{r=1}^s \beta_{(w_r)}^m \right\rangle \right|^2 = (-1)^{ms} \gamma(1 - m(j_1 + 1) + p_1) \times \\ \times \gamma(1 - m(j_3 + 1) + p_3) \prod_{r=1}^s |w_r|^{-2m} .$$

On the other hand, the generalized Selberg integral in the second line of (32) can be computed by using the formula (58) of Appendix A, developed by Dotsenko and Fateev in [43]. The result takes the form

$$C_{(j_1, j_2, j_3 | p_1, m(1+j_2), p_3)}^{(m)} = b^{-1} \lambda^s \pi^s \Gamma(-s) \Gamma(s+1) \gamma^s (1 + b^2) \delta \left(\sum_{i=1}^n j_i + n + s - 1 - b^{-2}(1 - m) \right) \times \\ \times \frac{\Gamma(1 - m(j_1 + 1) + \bar{p}_1) \Gamma(1 - m(j_3 + 1) + p_3)}{\Gamma(m(j_1 + 1) - p_1) \Gamma(m(j_3 + 1) - \bar{p}_3)} \prod_{r=0}^{s-1} \frac{\gamma(-(r+1)b^2) \gamma(b^2(j_1 - j_3 + j_2 + 1 + r))}{\gamma(b^2(2j_1 + 2 + r) + m) \gamma(b^2(2j_2 + 2 + r))} . \quad (35)$$

Now, analytic continuation of this expression is needed in order to find the general result, incorporating also the configurations yielding non-integer s . Such analytic continuation is done by requiring the residue of the exact expression evaluated at $s = -2 + b^{-2}(1 - m) - j_1 - j_2 - j_3 \in \mathbb{Z}_{\geq 0}$ to coincide with (35). The analytic continuation yields

$$C_{(j_1, j_2, j_3 | p_1, m(1+j_2), p_3)}^{(m)} = (\pi \lambda b^{-2} \gamma(b^2))^s \frac{\Gamma(1 - m(j_1 + 1) + \bar{p}_1) \Gamma(1 - m(j_3 + 1) + p_3)}{b \Gamma(m(j_1 + 1) - p_1) \Gamma(m(j_3 + 1) - \bar{p}_3)} \times \\ \times \frac{G(1 + j_1 + j_2 + j_3 + (m - 2)b^{-2}) G(j_1 - j_2 + j_3 + (m - 1)b^{-2})}{G(2j_1 + 1 + (m - 1)b^{-2}) G(2j_3 + 1 + (m - 1)b^{-2})} \times \\ \times \frac{G(-j_1 + j_2 + j_3 - b^{-2}) G(j_1 + j_2 - j_3 - b^{-2})}{G(-1) G(2j_2 + 1 - b^{-2})} , \quad (36)$$

where $s = -2 - j_1 - j_2 - j_3 + (1 - m)b^{-2}$, and the definition of the G -function can be found in Appendix A. To derive this expression one can use the functional relation

$$\gamma(-rb^2) = \frac{\Gamma(-rb^2)}{\Gamma(1 + rb^2)} = \frac{G(r)}{G(r - 1)} . \quad (37)$$

Expression (36) gives the three-point function for the case¹³ $p_2 = \bar{p}_2 = m(j_2 + 1)$, for generic values of m and b . Of course, when $m = 1$ this expression coincides with that for the $SL(2, \mathbb{R})_k$ WZNW model for the case where one of the vertex operators is given by the highest-weight representation of $SL(2, \mathbb{R})$ (with the identification $k = b^{-2} + 2$). In [36], it was shown how it reproduces the Melin transform of the three-point functions of [2, 38, 39]. More precisely, considering $m = 1$ in (36) and the functional relation $G(x) = \gamma(-x)(k - 2)^{(2x+1)} G(x + 2 - k)$, one finds Eqs. (60)-(62) of [36], which yields the expected result. More remarkably, the same

¹³Understanding the kinematic configuration $p_i = m(j_i + 1)$ in general would require the analysis of the spectrum of the CFT corresponding to generic values of m and b .

result is obtained for the case $m = b^2 = k - 2$, as it was discussed in [36], see Eq. (50) there. As a simpler consistency check, one can also verify that $m = 0$ in (36) yields the correlation function of three exponential fields $e^{\sqrt{2}\alpha_i\varphi(z_i)}$ of Liouville field theory, with $\alpha_i = b(j_i + 1)$; namely $C_{(j_1, j_2, j_3)}^{(m=0)} = C^L(\alpha_1, \alpha_2, \alpha_3)$ by writing $G(x)$ in terms of $\Upsilon(x)$ by using (54) of Appendix A (upon neglecting β - γ contribution).

The pole structure of (36) can be analyzed as follows: the function $G(x)$ develops single poles at

$$x = p + q b^{-2}, \quad x = -(p + 1) - (q + 1) b^{-2} \quad (38)$$

for any pair of positive integers $p \in \mathbb{Z}_{\geq 0}$ and $q \in \mathbb{Z}_{\geq 0}$. This implies that expression (36) presents poles at

$$\begin{aligned} -j_1 + j_2 + j_3 &= p + (q + 1) b^{-2} & -j_1 + j_2 + j_3 &= -(p + 1) - q b^{-2} \\ j_1 + j_2 - j_3 &= p + (q + 1) b^{-2} & j_1 + j_2 - j_3 &= -(p + 1) - q b^{-2} \end{aligned}$$

and at

$$\begin{aligned} j_1 - j_2 + j_3 &= p + (q + 1 - m) b^{-2}, & j_1 - j_2 + j_3 &= -(p + 1) - (q + m) b^{-2} \\ j_1 + j_2 + j_3 &= p - 1 + (q + 2 - m) b^{-2} & j_1 + j_2 + j_3 &= -(p + 2) - (q + m - 1) b^{-2}. \end{aligned}$$

It is worth noticing that these pole conditions remain unchanged if one first performs the changes $m \rightarrow -mb^2$, $j_i \rightarrow -b^{-2}j_i$, and then replaces b^2 by b^{-2} . This is actually a manifestation of the level duality under $k - 2 \rightarrow (k - 2)^{-1}$. The properties of the three-point function under the level duality can be understood by introducing the dual function $\tilde{G}(x)$, see (57) of Appendix A, which presents poles at $x = p + q b^2$ and at $x = -(p + 1) - (q + 1) b^2$, instead of (38). Thus, taking into account the functional relation $\tilde{G}(xb^2) = b^{2b^2x(x+b^{-2}+1)}G(x)$, one finds that expression (36) can be written in terms of the analogous quantity but for the inverse parameter b^{-1} with an appropriate redefinition of the spin variables j_i .

On the other hand, the pole contributions coming from the Γ -functions in the first line of (36) are of a different sort, since these depend on momenta p_i . The p_i -dependent pole conditions depend on the specific power of fields γ in the functional form of the vertex operators (24), while the pole conditions written down above do not depend on those specific powers $p_i - m(j_i + 1)$.

The two-point function can be also obtained from (36) by using the functional relation

$$\lim_{j_2 \rightarrow -1} \frac{G(-j_1 + j_3 + j_2 - b^{-2}) G(j_1 - j_3 + j_2 - b^{-2})}{G(-1)G(2j_2 + 1 - b^{-2})} = 2\pi b^{-2} \delta(j_1 - j_3). \quad (39)$$

In the limit $j_2 \rightarrow -1$, operator $\Phi_{j_2, p_2, p_2}(z_2)$ approaches the identity operator, and so we can write

$$\begin{aligned} \langle \Phi_{j_1, p_1, \bar{p}_1}(z_1) \Phi_{j_3, p_3, \bar{p}_3}(z_3) \rangle_{CFT} &\sim |z_{13}|^{-2\Delta_{j_1}} (\pi \lambda b^2 \gamma(b^2))^s \gamma(-s) \gamma(1 + b^2 s) \delta(j_1 - j_3) \times \\ &\times \frac{\Gamma(1 - m(j_1 + 1) + \bar{p}_1) \Gamma(mb^{-2}(1 - m) - mj_1 - p_1)}{\Gamma(m(j_1 + 1) - p_1) \Gamma(1 + mb^{-2}(m - 1) + mj_1 + \bar{p}_1)}, \end{aligned} \quad (40)$$

where now $s = -1 - 2j_1 + b^{-2}(1 - m)$, and, according to (30), we have $p_3 = (mb^{-2} - 1)(1 - m) - p_1$. The limit $j_2 \rightarrow -1$ of the three-point function is known to agree with the two-point function up to a b -dependent (j -independent) factor [45, 36, 40]. Besides, notice that we could also take the limit $j_2 \rightarrow b^{-2}(1 - m)$ in (36), and this would also yields a two-point function. This is because operator $\Phi_{j_2=b^{-2}(1-m)}$ also corresponds to a dimension-zero operator that can be considered as a (Weyl reflected) conjugate representation of the identity operator. This is analogous to what happens in the $SL(2, \mathbb{R})_k$ WZNW model with the operators $\Phi_{j_2=0}$ and $\Phi_{j_2=-1}$; while inserting one of these operators in the three-point function leads to the reflection coefficient $\sim R^H(j_1)\delta(j_1 - j_2)$, inserting the other leads¹⁴ to the “unreflected term” $\sim \delta(j_1 + j_2 + 1)$. In the general case (namely generic m), the generalization of $SL(2, \mathbb{R})$ Weyl reflection is given by $j \rightarrow -1 - j - b^{-2}(m - 1)$, which leaves the conformal dimension (23) unchanged. Thus, the limit $j_2 \rightarrow b^{-2}(1 - m)$ gives $s = 0$ and consequently yields the expression

$$\langle \Phi_{j_1, p_1, \bar{p}_1}(z_1) \Phi_{j_3, p_3, \bar{p}_3}(z_3) \rangle_{CFT} \sim |z_{13}|^{-2\Delta_{j_1}} \delta(j_1 + j_3 + 1 + b^{-2}(m - 1)) . \quad (41)$$

This expression can be obtained from (40) by replacing $j_3 \rightarrow -1 - j_3 - b^{-2}(m - 1)$ before evaluating the δ -function $\delta(j_1 - j_3)$. Summarizing, (40) and (41) give all the contributions to the two-point function.

To conclude this section, let us comment on an alternative free field representation. Presumably, some of the CFTs corresponding to certain values of m and b could also be realized by coupling Liouville theory in a non-minimal way to a free boson η plus a linear dilaton theory for a field χ with background charge $Q_\chi^{(m)} = (mb^{-2}(m - 2) - 2m)^{1/2}$, and then perturbing the whole theory by introducing an operator $\Phi_\kappa = e^{\kappa\chi}$ with κ satisfying $\kappa(Q_\chi - \kappa) + \Delta_{\alpha=-m/2b} = -\kappa^2 + Q_\chi^{(m)}\kappa - m/2 - mb^{-2}(1 + m/2)/2 = 1$, and eventually dressing Φ_κ with the appropriate Liouville field $V_{-m/2b} = e^{-m\varphi/\sqrt{2b}}$. For the case $m = 1$, this was done in [12]. For this case, as well as for $m = b^2$, the background charge corresponds to $Q_\chi^{(1)} = Q_\chi^{(b^2)} = -i\sqrt{k}$. In particular, this realization reproduces the correct value of the central charge as $c_{CFT} = c_L + 2 + 6(Q_\chi^{(m)})^2 = 3 + 6(b + b^{-1}(1 - m))^2$. Correlation functions in terms of such a Coulomb-gas representation could be computed by similar means.

4 Relation to Hamiltonian reduction

In this section, we discuss the relation between Hamiltonian reduction and SRT formula. Eventually, this relation gives us a concrete realization of the Langlands duality in correlation functions of H_3^+ WZNW model.

4.1 Reviewing Drinfeld-Sokolov Hamiltonian reduction

Hamiltonian reduction yields a way of reducing $SL(2, \mathbb{R})_k$ WZNW model to LFT. This can be regraded as a reduction of the degrees of freedom of WZNW by imposing constraints on $SL(2, \mathbb{R})$ momenta. This results in LFT that governs the remnant dynamics.

¹⁴After the Melin-Fourier transform of expression (36) we also have the same contribution in the x -basis as well; see for instance [36] and references therein.

The procedure is as follows: first, impose the gauge

$$J^+(z) = k , \quad (42)$$

and its anti-holomorphic partner. This is implemented by means of the BRST method: we introduce a b - c ghost system with central charge $c_g = -2$ and define the BRST charge as¹⁵

$$Q_{BRST} = \frac{1}{2\pi i} \oint dw (J^+(w) - k) c(w) .$$

This is analogous to the BRST implementation of the $SL(2, \mathbb{R})_k/U(1)$ WZNW coset construction, see for instance [45] and references therein.

Current $J^+(z)$ originally corresponds to a primary field of conformal dimension +1 with respect to the Sugawara stress tensor. Therefore, in order to impose (42) in a coordinate invariant way, one has to perform a change in the stress tensor to turn J^+ into a dimension-zero operator. This change in the stress tensor is usually referred to as an *improvement* or *topological twist*, which is defined by the shifting

$$T(z) \rightarrow \hat{T}(z) = T(z) - \partial J^3(z) .$$

Taking into account Wakimoto representation

$$\begin{aligned} T(z) &= -\beta(z)\partial\gamma(z) - \frac{1}{2}(\partial\phi)^2 - \frac{1}{\sqrt{2k-4}}\partial^2\phi(z) , \\ J^3(z) &= -\beta(z)\gamma(z) + \sqrt{\frac{k-2}{2}}\partial\phi(z) , \end{aligned}$$

the *improved* stress tensor takes the form

$$\hat{T}(z) = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{\sqrt{2(k-2)}}\partial^2\phi(z) + \partial\beta(z)\gamma(z) - \sqrt{\frac{k-2}{2}}\partial^2\phi(z) . \quad (43)$$

One can verify that the OPE $\hat{T}(z)\beta(w)$ is consistent with treating $J^+(z)$ as a zero-dimension field

$$\hat{T}(z)\beta(w) \sim \frac{\partial\beta(z)}{(z-w)} + \dots .$$

Besides, from the Wakimoto realization $J^+(z) = \beta(z)$ the implementation of constraint $J^+(z) = k$ yields $\partial\beta(z) = 0$ in (43); namely

$$\hat{T}(z) = -\frac{1}{2}(\partial\phi)^2 + \frac{Q}{\sqrt{2}}\partial^2\phi(z) , \quad (44)$$

where $Q = b + b^{-1}$ and $b^{-2} = k - 2$. This actually corresponds to the Liouville stress-tensor. Thus, we see how the WZNW theory reduces to LFT by implementing constraint (42). Computing the OPE $\hat{T}(z)\hat{T}(w)$ one also verifies that the central charge of LFT is given by

$$\hat{c} = 1 + 6Q^2 = c_{SL(2)} + 6k - 2 ,$$

¹⁵See also [46] for a very interesting discussion.

where $c_{SL(2)} = 3 + 6/(k - 2)$ is the central charge of the $SL(2, \mathbb{R})_k$ WZNW theory.

Now, we should specify how the spectrum of WZNW theory relates to the spectrum of LFT, which are represented by the exponential primary fields $V_\alpha(z) = e^{\sqrt{2}\alpha\phi(z)}$. First, recall the formula for the conformal dimension of these fields:

$$\Delta_\alpha = \alpha(Q - \alpha) . \quad (45)$$

On the other hand, in the WZNW side it is convenient to focus on the fields belonging to highest-weight representations of $SL(2, \mathbb{R})$, namely those fields satisfying $j = -p = -\bar{p}$ (or its Weyl reflected counterpart $j + 1 = p = \bar{p}$). These are primary fields with respect to the improved stress-tensor $\hat{T}(z)$, as it can be checked by computing the OPE $\hat{T}(z)\Phi_{j,p,\bar{p}}(w)$, whose conformal dimension is

$$\hat{\Delta}_j = -\frac{j(j+1)}{k-2} - j = -bj(Q - (-bj)) . \quad (46)$$

Thus, comparing (45) with (46) we see that it is natural to identify the Liouville momentum α and the WZNW momentum j by the simple relation

$$\alpha = -bj , \quad (47)$$

or its Weyl reflected counterpart $\alpha = b(j + 1)$, depending on the conventions. Namely, Hamiltonian reduction induces the following identification $V_{\alpha=-bj}(z) \leftrightarrow \Phi_{j,-j,-j}(z)$ between vertex operators of both theories. According to this correspondence, we expect that the Hamiltonian reduction would be realized at the level of correlation functions through the form

$$\left\langle \prod_{i=1}^n V_{-bj_i}(z_i) \right\rangle_{LFT} \sim f(b) \left\langle \prod_{i=1}^n \Phi_{j_i}^{(0)}(z_i) \right\rangle_{H_3^+} \quad (48)$$

with the notation

$$\Phi_j^{(0)}(z) = \mathcal{N}(j, b) \Phi_{j,p=-j,\bar{p}=-j}(z) ,$$

where $p_i + j_i = \bar{p}_i + j_i = 0$, $f(b)$ is some b -dependent numerical factor, and $\mathcal{N}(j, b)$ is some normalization of the vertex operators. In the following, we will discuss a realization of (48) in terms of SRT correspondence.

4.2 Implementing the reduction as the $\mu_i \rightarrow 0$ limit

We would like to attempt interpretation of the SRT formula from the viewpoint of the quantum DS Hamiltonian reduction. As discussed above, the conventional way of implementing the reduction is by imposing the gauge condition (42), which in the Wakimoto representation reads $\beta = k$. On the other hand, in the path integral formulation discussed in section 2.2, integration over γ has given the condition

$$\bar{\partial}\beta(w) = 2\pi \sum_{i=1}^n \mu_i \delta(w - z_i) ,$$

or, equivalently,

$$\beta(w) = \text{const} + \sum_{i=1}^n \frac{\mu_i}{w - z_i} . \quad (49)$$

As a 1-form, the constant should vanish in principle, but after the *improvement* of the energy momentum tensor, the constant term would be allowed¹⁶.

Now, let us take the limit $\mu_i \rightarrow 0$ (while keeping $\sum_{i=1}^n \mu_i = 0$) in the (dual) SRT formula. More precisely, we first take $\mu_i \rightarrow 0$ for $i = 2, 3, \dots, n-1$, and then we take the further limit $\mu_1 = -\mu_n \rightarrow 0$. This limit is in harmony with the spirit of the Hamiltonian reduction because (49) suggests that in order to fix β to be a constant number, $\mu_i \rightarrow 0$ limit seems unavoidable. In this limit, the parameter y_i appearing in the SRT formula becomes $y_i \rightarrow z_{i+1}$ for $i = 1 \dots n-2$ through the relation

$$\mu_i = u \frac{\prod_{j=1}^{n-2} (z_i - y_j)}{\prod_{j \neq i} (z_i - z_j)} .$$

The crucial observation is that in this limit we can essentially remove $n-2$ extra vertex operators¹⁷ in the Liouville side of the (dual) SRT formula (1). This is because we take the limit $y_i \rightarrow z_{i+1}$ and degenerate field $V_{-\tilde{b}/2}$ collide with $n-2$ fields V_α in the Liouville correlation functions. Since $V_{-\tilde{b}/2}$ is a degenerate field, it takes the very simple OPE

$$V_\alpha(z)V_{-\tilde{b}/2}(w) = \frac{V_{\alpha-\tilde{b}/2}(w)}{|z-w|^{-2\alpha\tilde{b}}} + \tilde{C}_-^L(\alpha) \frac{V_{\alpha+\tilde{b}/2}(w)}{|z-w|^{-2(Q-\alpha)\tilde{b}}} + (\text{descendants}) .$$

Thus, for $\alpha < Q/2$, which is in the so-called Seiberg bound (corresponding to $j < -1/2$), only the first term in the OPE survives because it dominates over the second term in the $w \rightarrow z$ limit, and the Liouville side of the SRT formula then is given by the n -point correlation function

$$\left\langle V_{\alpha_1}(z_1) \prod_{i=2}^{n-1} V_{\tilde{\alpha}_i}(z_i) V_{\alpha_n}(z_n) \right\rangle_{LFT} ,$$

where $\alpha_i = \tilde{b}^{-1}(j_i + 1 + \tilde{b}^2/2) = \tilde{b}^{-1}(j_i + k/2)$ and $\tilde{\alpha}_i = \tilde{b}^{-1}(j_i + 1)$. The factor Θ_n becomes singular in the limit $y_i \rightarrow z_{i+1}$, but it is not difficult to remove this singularity, and one can see (up to some normalization constant \mathcal{C}) that

$$\left\langle \prod_{i=1}^n \Phi_{j_i}(0|z_i) \right\rangle = \mathcal{C} |\tilde{\Theta}_n|^2 \left\langle V_{\alpha_1}(z_1) \prod_{i=2}^{n-1} V_{\tilde{\alpha}_i}(z_i) V_{\alpha_n}(z_n) \right\rangle_{LFT} , \quad (50)$$

where $\tilde{\Theta}_n = \prod_{1 < i < j < n-1} (z_i - z_j)^{2\tilde{b}^2}$ is the regulated version of Θ_n in the SRT formula. A similar singular normalization factor appeared in the earlier attempt of Hamiltonian reduction by setting $x_i = z_i$ [47].

¹⁶As mentioned, this improvement can be obtained by shifting the Sugawara stress-tensor $T(z)$ as $T(z) \rightarrow T(z) - \partial J^3(z)$. Alternatively, one can add a piece $\bar{\omega}J$ in the action, where ω is a worldsheet connection, which leads to the equation of motion $(\bar{\partial} + \bar{\omega})\beta(w) = 2\pi \sum_{i=1}^n \mu_i \delta(w - z_i)$.

¹⁷The other two vertex operators seem special as in the H_3^+ side. In the H_3^+ side, we do not necessarily take $\mu \rightarrow 0$ limit for two vertex operators. In the Liouville side, the relation between j and α is not modified. These vertex operators can be put at 0 and ∞ and can be regarded as *in* and *out* vacua instead.

Notice that the relation between $\tilde{\alpha}_i$ and j_i in (50) agrees with the Weyl reflected version of (47), namely performing $j_i \rightarrow -1 - j_i$ there. Besides, the relation between α_i and j_i corresponds to performing the change $j_i \rightarrow -k/2 - j_i$ in (47). We further discuss this point in Appendix B and argue this is consistent with what is expected from Hamiltonian reduction. We propose that the formula (50) yields the Hamiltonian reduction interpretation of the SRT formula.

4.3 Realizing Langlands level duality

Now, let us make a remark on level duality in WZNW theory. As we did in (1), it is possible to derive a dual version of formula of (50) by replacing \tilde{b} and b^{-1} . Since LFT does not have any extra insertion in the particular limit we have considered (i.e. $\mu_i \rightarrow 0$), by equating the two expressions (the one in the SRT formula and the other in the dual SRT formula) with the crucial identification $b = \tilde{b}$, then we find the following surprising identity¹⁸

$$\left\langle \prod_{i=1}^n \Phi_{j_i}(0|z_i) \right\rangle_k = \tilde{C} \left\langle \prod_{i=1}^n \Phi_{\tilde{j}_i}(0|z_i) \right\rangle_{\tilde{k}}, \quad (51)$$

where the levels of the WZNW model in the both side of this expression are related though the Langlands duality

$$\tilde{k} - 2 = (k - 2)^{-1}, \quad (52)$$

and we introduced a numerical coefficient \tilde{C} that regularizes the $\mu_i \rightarrow 0$ limit. The spin of the vertex operators on each side obey the relation $\tilde{j}_i = (j + 1)/(k - 2) - 1$ for $i = 1, n$, and $\tilde{j}_i = (j + 1)/(k - 2) - 1/2 - (k - 2)^{-1}/2$ for $i = 2, \dots, n - 1$. One can explicitly check that this identity is true for the two-point functions and the three-point functions (see Appendix A for useful identities such as (57)).

This identity can be regarded as a manifestation of the quantum Langlands duality at the level of correlation function. In physical applications, it shows a strong/weak coupling duality between different CFTs even with different central charges. In the context of string theory, it might relate the scattering amplitudes of the completely different string compactifications even with different target space dimensionality. For example, if we embed the $SL(2, \mathbb{R})/U(1)$ coset model in the superstring compactification, the supersymmetric version of the Langlands duality is given by $\hat{k} \rightarrow 1/\hat{k}$, where $\hat{k} = k - 2$ is the supersymmetric level of the current algebra. This gives a non-trivial duality between the scattering amplitudes in the two-dimensional black hole for $\hat{k} = 1/2$ and those in the A_1 type singularity (Eguchi-Hanson space) for $\hat{k} = 2$. From the viewpoint of the AdS₃/CFT₂ correspondence, it predicts a strong-weak duality in the boundary conformal field theory as well, whose origin is rather mysterious.

From the mathematical point of view, an identity like (51) could give a clue to understand the quantum version of the geometric Langlands correspondence, which is yet to be fully formulated in precise mathematical language. See e.g. [33] for an interesting discussion on an attempt.

¹⁸Strictly speaking, since the cosmological constant of the LFT is different between the original formula and the dual formula, we have to adjust the screening parameter in H_3^+ model (λ in the Wakimoto realization) so that the both sides show the same scaling behavior.

5 Discussion and outlook

In this paper, we have investigated the interrelationship among the following three notions: the Langlands level duality $k - 2 \rightarrow \tilde{k} - 2 = 1/(k - 2)$, the SRT H_3^+ -Liouville correspondence, and the DS Hamiltonian reduction. First, we have derived a dual version of H_3^+ -Liouville correspondence formula (1) induced by the Liouville self-duality under $b \rightarrow b^{-1}$. We have also discussed how the dual formula can be interpreted as a particular case of the non-rational CFTs recently proposed in [22]. By using the free field realization, we have confirmed that the H_3^+ WZNW model is actually double-represented within the space of parameters (m, b) . The free field techniques have also enabled us to compute three-point functions for these non-rational CFTs.

The dual formula (1), together with the original formula of [5], show how Langlands level duality of H_3^+ WZNW model manifests itself at the level of n -point correlation functions. This is particularly realized in (51) (see also (65) of Appendix B). We have argued how such equations can be regarded as a realization of Hamiltonian reduction at the level of correlation functions. More precisely, we have proposed its realization in terms of a limit of the (dual) SRT relation. This is represented in (50) derived in the limit $\mu_i \rightarrow 0$, which corresponds to the Hamiltonian reduction in μ basis.

Studying the relation between the SRT H_3^+ -Liouville correspondence and the DS Hamiltonian reduction could be relevant in the context of the geometric Langlands correspondence. In fact, (50) (see also (62) in Appendix B) can be seen as a quantum non-chiral version of the geometric Langlands correspondence, in the sense that this formula selects precise basis of the WZNW n -point correlation functions that admit to be expressed in terms of n -point correlation functions of the Virasoro algebra. We emphasize that the advantage of (50) (see also (62) in Appendix B) over the other expressions is that it provides a map between n -point functions in both sides, without involving additional degenerate fields.

Last but not least, we would like to advocate that understanding of the precise relation between the quantum DS Hamiltonian reduction and the SRT H_3^+ -Liouville correspondence would be important for its higher rank generalization. It is commonly believed that an analogous correspondence should exist between the $SL(N, \mathbb{R})_k$ WZNW model and the affine Toda¹⁹ field theory (corresponding to SRT formula for $N = 2$). Since the quantum version of the Hamiltonian reduction admits a generalization to $N > 2$ [31], this approach could be a natural way to tackle this open question to find its feasible extensions in higher ranks.

Acknowledgement

G.G. thanks M. Porrati and S. Ribault for interesting comments. The work of G.G. is supported by UBA, CONICET and ANPCyT, through grants UBACyT X861, PIP6160, PICT34557. The research of Y. N. is supported in part by NSF grant PHY-0555662 and the UC Berkeley Center for Theoretical Physics.

¹⁹Recently, [48] computed certain classes of correlation functions in conformal Toda field theories.

A Special functions

Here we summarize some useful formulae on special functions.

The function $\Upsilon(x)$ was introduced by Zamolodchikov and Zamolodchikov in Ref. [37], and it is defined by

$$\Upsilon(x) = \exp \left(\int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2 \left(\frac{Q}{2} - x \right) \frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right] \right) \quad (53)$$

for $0 < \Re e(x) < Q$, and by its analytic continuation outside the strip. It satisfies the shift equations

$$\Upsilon(x+b) = \gamma(bx)b^{1-2x}\Upsilon(x), \quad \Upsilon(x+b^{-1}) = \gamma(xb^{-1})b^{2xb^{-1}-1}\Upsilon(x),$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$, which obeys $\gamma(x)\gamma(1-x) = 1$, $\gamma(x)\gamma(-x) = -x^{-2}$. Notice also that definition (53) is invariant under $b \rightarrow b^{-1}$, and this fact yields further interesting functional relations.

On the other hand, the special function $G(x)$ is defined in terms of $\Upsilon(x)$ by

$$G(x) = \Upsilon^{-1}(-bx)b^{-b^2x^2-(b^2+1)x}, \quad (54)$$

and consequently satisfies the shift equations

$$G(x) = \gamma(-x)b^{-2(2x+1)}G(x-b^{-2}), \quad G(x+1) = \gamma(-(1+x)b^2)G(x). \quad (55)$$

Function $G(x)$ can be also defined in terms of Barnes' Γ_2 -function, as follows

$$G(x) = b^{x(xb^2-1-b^2)}\Gamma_2(-x|1, b^{-2})\Gamma_2(b^{-2}-1+x|1, b^{-2}), \quad (56)$$

with

$$\log \Gamma_2(x|1, y) = \lim_{\varepsilon \rightarrow 0} \partial_\varepsilon \left(\sum_{n,m} (x+n+my)^{-\varepsilon} - \sum_{n,m} (n+my)^{-\varepsilon} \right),$$

where the first sum runs over positive integers $n \in \mathbb{Z}_{\geq 0}$ and $m \in \mathbb{Z}_{\geq 0}$, while the second sum excludes the step $n = m = 0$. Function $G(x)$ develops single poles at

$$x = n + mb^{-2}, \quad x = -(1+n) - (m+1)b^{-2}.$$

It is also useful to introduce the *dual* function $\tilde{G}(x)$ that is defined as in (56) by replacing $b \rightarrow b^{-1}$. A manifestation of the invariance of (53) under $b \rightarrow b^{-1}$ is the following identity

$$G(x) = (k-2)^{(x^2+(k-1)x)/(k-2)} \tilde{G}(x/(k-2)), \quad (57)$$

where $b^{-2} = k-2$. Relation (57) can be seen from (54).

In the main text, we have also used the following integral formula, which is known as Dotsenko-Fateev integral

$$\begin{aligned} & \frac{1}{m!} \int d^2 z_i \prod_{i=1}^m |z_i|^{2\alpha} |1-z_i|^{2\beta} \prod_{i<j}^m |z_i-z_j|^{4\rho} = \pi^m (\gamma(1-\rho))^m \times \\ & \times \prod_{i=1}^m \gamma(i\rho) \gamma(1+\alpha+(i-1)\rho) \gamma(1+\beta+(i-1)\rho) \gamma(-1-\alpha-\beta-(m-2+i)\rho). \end{aligned} \quad (58)$$

B Spectral flowed correlation functions

In this appendix, we discuss an alternative way of realizing the Hamiltonian reduction at the level of correlation functions in terms of SRT formula. In particular, we will study the WZNW correlation functions that involve spectral flowed states. That is, we will consider the correlation functions that violate the so-called winding number, as defined in [15].

As mentioned in the introduction, the SRT formula of [5] was generalized in [15] to the case with spectral flowed (winding) states of $SL(2, \mathbb{R})_k$ WZNW theory. The general result states that n -point functions of the H_3^+ WZNW theory that violate the winding conservation in $\Delta\omega$ units is given by a $2n - 2 - \Delta\omega$ -point functions in LFT (where $n - 2 - \Delta\omega$ Liouville vertex operators correspond degenerate fields $V_{-1/2b}$). Explicitly, we have

$$\begin{aligned}
\left\langle \prod_{i=1}^n \Phi_{j_i, p_i, \bar{p}_i}^{\omega_i}(z_i) \right\rangle_{H_3^+} &= \frac{2\pi^{3-2n} b c_k^{\Delta\omega}}{\Gamma(n-1-\Delta\omega)} \prod_{i=1}^n \frac{\Gamma(-j_i + p_i)}{\Gamma(j_i + 1 - \bar{p}_i)} \times \\
&\times \prod_{1 \leq l < t \leq n} (z_l - z_t)^{\frac{k}{2} - \frac{k}{2}\omega_l\omega_t + \omega_l p_t + p_l \omega_t + p_l + p_t} (\bar{z}_l - \bar{z}_t)^{\frac{k}{2} - \frac{k}{2}\omega_l\omega_t + \omega_l \bar{p}_t + \bar{p}_l \omega_t + \bar{p}_l + \bar{p}_t} \times \\
&\times \int \prod_{t=1}^{n-2-\Delta\omega} d^2 y_t \prod_{l=1}^n \prod_{t=1}^{n-2-\Delta\omega} (z_l - y_t)^{\frac{k}{2} - p_t} (\bar{z}_l - \bar{y}_t)^{\frac{k}{2} - \bar{p}_t} \times \\
&\times \prod_{1 \leq a < b \leq n-2-\Delta\omega} |y_a - y_b|^k \left\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \prod_{t=1}^{n-2-\Delta\omega} V_{-\frac{1}{2b}}(y_t) \right\rangle_{LFT}, \tag{59}
\end{aligned}$$

where $b^{-2} = k - 2$, $\sum_{i=1}^n \omega_i = -\Delta\omega$, $\alpha_i = b(j_i + k/2)$, $\sum_{i=1}^n p_i = \sum_{i=1}^n \bar{p}_i = -k\Delta\omega/2$, and c_k is a k -dependent factor; see [15] for details. Expression (59) is the most general version of the H_3^+ -Liouville correspondence involving spectral flowed states $\Phi_{j,p,\bar{p}}^\omega(z)$ of the WZNW theory. In particular, for $\Delta\omega = 0$ one recovers the SRT relation between n -point WZNW functions and $2n - 2$ -point Liouville functions.

On the other hand, in the case of maximally violating amplitudes (i.e. $\Delta\omega = n - 2$), all the degenerate field $V_{-1/2b}$ in (59) disappear and the formula actually yields the correspondence between n -point WZNW functions and n -point LFT functions. In such a case, we have $\Delta\omega = -\sum_{i=1}^n \omega_i = n - 2$, and (59) takes the form

$$\begin{aligned}
\left\langle \prod_{i=1}^n \Phi_{j_i, p_i, \bar{p}_i}^{\omega_i}(z_i) \right\rangle_{H_3^+} &= 2\pi^{3-2n} b c_k^{n-2} \prod_{i=1}^n \frac{\Gamma(-j_i + p_i)}{\Gamma(j_i + 1 - \bar{p}_i)} \left\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \right\rangle_{LFT} \times \\
&\times \prod_{1 \leq l < t \leq n} (z_l - z_t)^{\frac{k}{2} - \frac{k}{2}\omega_l\omega_t + \omega_l p_t + p_l \omega_t + p_l + p_t} (\bar{z}_l - \bar{z}_t)^{\frac{k}{2} - \frac{k}{2}\omega_l\omega_t + \omega_l \bar{p}_t + \bar{p}_l \omega_t + \bar{p}_l + \bar{p}_t}, \tag{60}
\end{aligned}$$

where $\sum_{i=1}^n p_i = \sum_{i=1}^n \bar{p}_i = k(n - 2)/2$. Notice that the notation here is such that spectral flowed fields $\Phi_{j_i, p_i, \bar{p}_i}^{\omega_i}$ have conformal dimension given by

$$\Delta_j^{\omega, p} = -b^2 j(j + 1) + p\omega - k\omega^2/4.$$

Now, let us investigate two particular cases of (60), which are relevant for the Hamiltonian reduction.

Case 1: Firstly, consider two states of spectral flow sector $\omega = 0$, and $n - 2$ states of sector $\omega = -1$. Suppose $\omega_1 = \omega_n = 0$, $p_1 = p_1 = j_i$, $p_n = p_n = -j_n$, while $\omega_2 = \omega_3 = \dots \omega_{n-1} = -1$, $p_2 = p_3 = \dots p_{n-1} = \bar{p}_2 = \bar{p}_3 = \dots \bar{p}_{n-1} = -k/2$. In this case, (60) can be written as follows

$$\left\langle \prod_{i=1}^n V_{\alpha_i}(z_i) \right\rangle_{LFT} = \frac{\pi}{2b} |z_1 - z_n|^{2j_1 + 2j_n - k} \left\langle \Phi_{j_1}^{(0)}(z_1) \prod_{i=2}^{n-1} \Phi_{j_i}^{(-)}(z_i) \Phi_{j_n}^{(0)}(z_n) \right\rangle_{H_3^+}, \quad (61)$$

where we have defined

$$\Phi_j^{(0)}(z) = \frac{1}{\gamma(-2j)} \Phi_{j,-j,-j}^{\omega=0}(z), \quad \Phi_j^{(-)}(z) = \frac{c_k \pi^2}{\gamma(k/2 - j)} \Phi_{j,-\frac{k}{2},-\frac{k}{2}}^{\omega=-1}(z).$$

Expression (61) is certainly similar to (50). As in (50), it would be convenient to fix the inserting points as $z_1 = 0$ and $z_n = \infty$ by using projective invariance. This would make the overall factor $|z_1 - z_n|^{2j_1 + 2j_n - k}$ to disappear, yielding a correspondence between n -point functions of both theories.

Case 2: Secondly, consider one state of the spectral flow sector $\omega = +1$, and $n - 1$ states of sector $\omega = -1$. Now suppose $\omega_1 = +1$, $p_1 = p_1 = +k/2$, while $\omega_2 = \omega_3 = \dots \omega_{n-1} = \omega_n = -1$, $p_2 = p_3 = \dots p_{n-1} = p_n = \bar{p}_2 = \bar{p}_3 = \dots \bar{p}_{n-1} = \bar{p}_n = -k/2$. In this case, we find

$$\left\langle \prod_{i=1}^n V_{-bj_i}(z_i) \right\rangle_{LFT} = \frac{\pi}{2b} \left\langle \Phi_{-\frac{k}{2}-j_1}^{(+)}(z_1) \prod_{i=2}^n \Phi_{-\frac{k}{2}-j_i}^{(-)}(z_i) \right\rangle_{H_3^+}, \quad (62)$$

where we have introduced

$$\Phi_{-\frac{k}{2}-j}^{(-)}(z) = \frac{c_k \pi^2}{\gamma(j+k)} \Phi_{-\frac{k}{2}-j,-\frac{k}{2},-\frac{k}{2}}^{-1}(z), \quad \Phi_{-\frac{k}{2}-j}^{(+)}(z) = \frac{1}{c_k \pi^2 \gamma(j)} \Phi_{-\frac{k}{2}-j,+\frac{k}{2},+\frac{k}{2}}^{+1}(z),$$

and

$$\alpha = b(j + k/2). \quad (63)$$

Remarkably, in expression (62), all the dependence on $|z_i - z_j|$ dropped out without fixing the insertion points z_i . Consequently, this can be actually thought of as a direct correspondence between n -point WZNW correlation functions and n -point LFT correlation functions. Operator $\Phi_{-\frac{k}{2}-j_1}^{(+)}$ in (62) should be regarded as the one defining the *out* vacuum state, while operators $\Phi_{-\frac{k}{2}-j_{i>1}}^{(-)}$ act on the *in* vacuum creating worldsheet string states. Also notice that field $\Phi_{-\frac{k}{2}-j}^{(+)}$ has the following conformal dimension with respect to the stress tensor $T(z)$,

$$\Delta_{-\frac{k}{2}-j}^{\omega=-1,p=-k/2} = \frac{(j+k/2)(-j-k/2+1)}{k-2} + \frac{k}{4} = -\frac{j(j+1)}{k-2} - j,$$

which certainly agrees with the formula (46) for the conformal dimension of fields $\Phi_{j,-j,-j}^0$ with respect to the improved stress tensor $\hat{T}(z) = T(z) - \partial J^3(z)$. In turn, we have

$$\hat{\Delta}_j = \Delta_{-\frac{k}{2}-j}^{\omega=\pm 1,p=\pm k/2}.$$

Hence, fields $\Phi_{j,-j,-j}^0$ and $\Phi_{-\frac{k}{2}-j,\pm k/2,\pm k/2}^{\pm 1}$ have the same conformal dimension, and these also represent same value of Liouville momentum $\alpha = -bj$, as in (47). In this sense, we

can associate fields as $\Phi_j^{(0)}(z) \leftrightarrow \Phi_{-k/2-j}^{(\pm)}(z)$. This manifestly shows the parallelism between realizations (48) and (62). Moreover, it is also consistent with the relation between (47) and (63). We conclude that (62) can be thought of as a realization of Hamiltonian reduction at the level of correlation functions.

To understand the emergence of spectral flowed sector $\omega = -1$ in (62), one has to take into account that implementing the condition $J^+ = \beta = k$ induces a shifting²⁰ of the modes $J_n^{\pm,3}$ of Kac-Moody currents $J_{(z)}^{\pm,3} = \sum_n J_n^{\pm,3} z^{-n-1}$, and such shifting can be interpreted as a spectral flow transformation with parameter $\omega = -1$, which yields the flow $J_n^{\pm} \rightarrow J_{n \mp 1}^{\pm}$, $J_0^3 \rightarrow J_0^3 + k/2$. Again, this is related to the fact that the conformal dimension of a field $\Phi_j^{(0)} \sim \Phi_{j,j,j}^{(\omega=0)}$ with respect to the improved stress tensor $T(z) - \partial_z J^3(z)$ agrees with that of a flowed field $\Phi_j^{(-)} \sim \Phi_{-j-k/2,k/2,k/2}^{(\omega=-1)}$ with respect to $T(z)$.

Let us notice that, as in the case of the μ -basis, it turns out that performing the change $b \rightarrow \tilde{b}^{-1}$ in (62) yields a dual version of such a formula. In fact, we can write down the dual formula as

$$\left\langle \prod_{t=1}^n V_{\alpha_t}(z_t) \right\rangle_{LFT} = \frac{\tilde{b}}{2\pi^3} \left\langle \Phi_{j_1}^{(+)}(z_1) \prod_{r=2}^n \Phi_{j_r}^{(-)}(z_r) \right\rangle_{H_3^+}, \quad (64)$$

where now $\alpha = \tilde{b}^{-1}(j+1) + b/2$, $\tilde{b}^2 = k-2$. Then, from (62) and (64) we obtain the duality relation

$$\left\langle \Phi_{j_1}^{(+)}(z_1) \prod_{r=2}^n \Phi_{j_r}^{(-)}(z_r) \right\rangle_k = \hat{C} \left\langle \Phi_{\tilde{j}_1}^{(+)}(z_1) \prod_{r=2}^n \Phi_{\tilde{j}_r}^{(-)}(z_r) \right\rangle_{\tilde{k}}, \quad (65)$$

where $\hat{C} = (k-2)^{-2}$, and $\tilde{j} = (k-2)(j+1/2) - 1/2$ and $\tilde{k} - 2 = (k-2)^{-1}$. This identity again can be regarded as a manifestation of the quantum Langlands duality at the level of correlation functions by relating the strongly coupled system with the weakly coupled system.

To conclude this appendix, let us make a remark on the generalization of the formula (59) to the case of higher genus correlation functions. In fact, it would be very interesting to extend the H_3^+ -Liouville correspondence to the case of higher-genus correlation functions involving spectral flowed sectors (winding sectors of string theory in AdS_3). An intriguing result for $SL(2, \mathbb{R})_k$ WZNW correlation functions on the sphere is the existence of an upper bound for the violation of the winding number conservation. It turns out that winding number conservation in a given tree-level n -point function can be violated up to $n-2$ units²¹.

In the context of the H_3^+ -Liouville correspondence, this upper bound $\Delta\omega \leq n-2$ for the winding violation is nicely realized as follows: according to (59) the n -point WZNW correlation functions that violate the winding conservation in $\Delta\omega$ units are related to $2n-2-\Delta\omega$ -point functions of LFT, where $n-2-\Delta\omega$ Liouville vertex operators represent degenerate fields $V_{-1/2b}$. In other words, violating the winding conservation in $\Delta\omega$ units on the WZNW side corresponds to removing $\Delta\omega$ degenerate fields on the Liouville side of the original formula in [5]. Then, a natural question is how such a picture is generalized to the case of higher genus correlation functions. Presumably, for a genus- g correlation function the general story remains the same, and one could associate the units of winding violation to the amount of degenerate Liouville

²⁰G.G. thanks M. Porrati for pointing it out.

²¹This bound can be understood in terms of the $sl(2)_k$ symmetry of the theory; see appendix D of [14] for a discussion.

fields $V_{-1/2b}$. Actually, one is tempted to conjecture that the upper bound for the violation of the winding number in a genus- g n -point correlation function is given by $\Delta\omega \leq n + 2g - 2$. On one hand, this is the amount of Liouville degenerate fields in the case of winding number conserved amplitudes [20]; on the other hand, it numerically matches with the expected number if one thinks the maximally violating genus- g correlation function as factorized in terms of several maximally violating genus-zero correlation functions. It would be very interesting to confirm that such a bound is obeyed.

References

- [1] Yu Nakayama, *Liouville Field Theory - A decade after the revolution*, Int.J.Mod.Phys. **A19** (2004) 2771, [arXiv:hep-th/0402009].
- [2] J. Teschner, *On structure constants and fusion rules in the $SL(2, C)/SU(2)$ WZNW model*, Nucl. Phys. **B546** (1999) 390, [arXiv:hep-th/9712256].
- [3] J. Teschner, *Crossing Symmetry in the H_3^+ WZNW model*, Phys. Lett. **B521** (2001) 127, [arXiv:hep-th/0108121].
- [4] O. Andreev, *Operator algebra of the $SL(2)$ conformal field theories*, Phys. Lett. **B363** (1995) 166, [arXiv:hep-th/9504082].
- [5] S. Ribault and J. Teschner, *H_3^+ correlation functions from Liouville theory*, JHEP **0506** (2005) 014, [arXiv:hep-th/0502048].
- [6] A. Stoyanovsky, *A relation between the Knizhnik-Zamolodchikov and Belavin-Polyakov-Zamolodchikov systems of partial differential equations*, [arXiv:math-ph/0012013].
- [7] V. Fateev, *Relation between Sine-Liouville and Liouville correlation functions*, unpublished.
- [8] T. Takayanagi, *$c < 1$ String from Two Dimensional Black Holes*, JHEP **07** (2005) 050; [arXiv:hep-th/0503237].
- [9] S. Nakamura and V. Niarchos, *Notes On The S-Matrix Of Bosonic And Topological Non-Critical Strings*, JHEP **10** (2005) 025; [arXiv:hep-th/0507252].
- [10] D. Sahakyan and T. Takayanagi, *On the Connection between $N = 2$ Minimal String and $(1, n)$ it Bosonic Minimal String*, JHEP **06** (2006) 027; [arXiv:hep-th/0512112].
- [11] V. Niarchos, *On Minimal $N = 4$ Topological Strings And The $(1, k)$ Minimal Bosonic String*, JHEP **03** (2006) 045; [arXiv:hep-th/0512222].
- [12] G. Giribet, *The String Theory on AdS_3 as a Marginal Deformation of a Linear Dilaton Background*, Nucl. Phys. **B737** (2006) 209; [arXiv:hep-th/0511252].

- [13] G. Giribet and Y. Nakayama, The Stoyanovsky-Ribault-Teschner Map and String Scattering Amplitudes, *Int. J. Mod. Phys.* **A21** (2006) 4003; [arXiv:hep-th/0505203].
- [14] J. Maldacena and H. Ooguri, *Strings in AdS₃ and the SL(2, R) WZW Model. Part 3: Correlation Functions*, *Phys.Rev.* **D65** (2002) 106006, [arXiv:hep-th/0111180].
- [15] S. Ribault, *Knizhnik-Zamolodchikov equations and spectral flow in AdS₃ string theory*, *JHEP* **0509** (2005) 045, [arXiv:hep-th/0507114].
- [16] S. Ribault, *Discrete D-branes in AdS₃ and in the 2d black hole*, *JHEP* **06** (2006) 015; [arXiv:hep-th/0512238].
- [17] K. Hosomichi and S. Ribault, *Solution of the H₃⁺ model on a disc*, *JHEP* **01** (2007) 057; [arXiv:hep-th/0610117].
- [18] S. Ribault, *Boundary three-point function on AdS₂ D-branes*, *JHEP* **01** (2008) 004; [arXiv:0708.3028].
- [19] V. Fateev and S. Ribault, *Boundary action of the H₃⁺ model*, *JHEP* **02** (2008) 024; [arXiv:0710.2093].
- [20] Y. Hikida and V. Schomerus, *H₃⁺ WZNW model from Liouville field theory*, [arXiv:0706.1030].
- [21] Y. Hikida and V. Schomerus, *Structure constants of the OSP(1|2) WZNW model*; [arXiv:0711.0338].
- [22] S. Ribault, *A family of solvable non-rational conformal field theories*, [arXiv:0803.2099].
- [23] P. Bouwknegt and K. Schoutens, *W-symmetry in Conformal Field Theory*, *Phys. Rept.* **223** (1993) 183, [arXiv:hep-th/9210010].
- [24] D. Gaitsgory, *Informal introduction to geometric Langlands*, Chapter 6 in *An introduction to the Langlands program*, Ed. J. Bernstein and S. Gelbart, Birkhäuser, Boston, 2004.
- [25] E. Frenkel, *Lectures on the Langlands Program and Conformal Field Theory*, Lectures delivered at the Les Houches School on Number Theory and Physics, March, 2003, and at the DARPA Workshop on Langlands Program and Physics, at the Institute for Advanced Study, March, 2004, [arXiv:hep-th/0512172].
- [26] E. Frenkel, *Ramifications of the geometric Langlands Program*, Proceedings of the CIME Summer School on Representation Theory and Complex Analysis, Venice, June 2004, [arXiv:math/0611294].
- [27] E. Frenkel, *Lectures on Wakimoto modules, opers and the center at the critical level*, [arXiv:math/0210029].

- [28] M. Tan, *Gauging Spacetime Symmetries on the Worldsheet and the Geometric Langlands program*, JHEP **0803**, 033 (2008), [arXiv:0710.5796 [hep-th]].
- [29] M. Tan, *Gauging Spacetime Symmetries on the Worldsheet and the Geometric Langlands Program - II*, [arXiv:0804.0804 [hep-th]].
- [30] B. Feigin, E. Frenkel and N. Reshetikhin, *Gaudin Model, Bethe Ansatz and Critical Level*, Commun. Math. Phys. **166** (1994) 27, [arXiv:hep-th/9402022].
- [31] M. Bershadsky and H. Ooguri, *Hidden $Sl(N)$ Symmetry In Conformal Field Theories*, Commun. Math. Phys. **126**, 49 (1989).
- [32] B. Feigin and E. Frenkel, *Affine Kac-Moody Algebras At The Critical Level And Gelfand-Dikiĭ Algebras*, Int. J. Mod. Phys. A **7S1A**, 197 (1992).
- [33] A. V. Stoyanovsky, *Quantum Langlands duality and conformal field theory*, [arXiv:math/0610974].
- [34] I. Bakas and C. Sourdis, *Aspects of WZW models at critical level*, Fortsch. Phys. **53** (2005) 409, [arXiv:hep-th/0501127].
- [35] I. Bakas and C. Sourdis, *On the tensionless limit of gauged WZW models*, JHEP **0406** (2004) 049, [arXiv:hep-th/0403165] .
- [36] G. Giribet and C. Núñez, *Correlators in AdS_3 string theory*, JHEP **0106** (2001) 010, [arXiv:hep-th/0105200].
- [37] A. Zamolodchikov and Al. Zamolodchikov, *Structure Constants and Conformal Bootstrap in Liouville Field Theory*, Nucl. Phys. **B477** (1996) 577, [arXiv:hep-th/9506136].
- [38] J. Teschner, *The Mini-Superspace Limit of the $SL(2, C)/SU(2)$ -WZNW Model*, Nucl. Phys. **B546** (1999) 369, [arXiv:hep-th/9712258].
- [39] J. Teschner, *Operator product expansion and factorization in the H_3^+ -WZNW model*, Nucl. Phys. **B571** (2000) 555, [arXiv:hep-th/9906215].
- [40] Al. Zamolodchikov, *Perturbed Conformal Field Theory on Fluctuating Sphere*, Contribution to the Balkan Workshop BW2003, “Mathematical, Theoretical and Phenomenological Challenges Beyond Standard Model”, Vrnjancka Banja, Serbia, [arXiv:hep-th/0508044].
- [41] J.L. Petersen, J. Rasmussen and M. Yu, *Free Field Realizations of 2D Current Algebras, Screening Currents and Primary Fields*, Nucl. Phys. **B502** (1997) 649, [arXiv:hep-th/9704052].
- [42] P. Di Francesco and D. Kutasov, *World Sheet and Space Time Physics in Two Dimensional (Super) String Theory*, Nucl. Phys. **B375** (1992) 119, [arXiv:hep-th/9109005].

- [43] V. Dotsenko and V. Fateev, *four point correlation functions and the operator algebra in the two-dimensional conformal Invariant theories with the central charge $c < 1$* , Nucl. Phys. **B251** (1985) 691.
- [44] Y. Satoh, *Three-point function and operator product expansion in $SL(2)$ conformal field theory*, Nucl. Phys. **B629** (2002) 188., [arXiv:hep-th/0109059].
- [45] K. Becker and M. Becker, *Interactions in the $SL(2, R)/U(1)$ Black Hole Background*, Nucl. Phys. **B418** (1994) 206, [arXiv:hep-th/9310046].
- [46] F. Nitti and M. Porrati, *Hidden $sl(2, R)$ Symmetry in 2D CFTs and the Wave Function of 3D Quantum Gravity*, JHEP **0401** (2004) 028, [arXiv:hep-th/0311069].
- [47] J. Petersen, J. Rasmussen and M. Yu, *Hamiltonian Reduction of $SL(2)$ -theories at the Level of Correlators*, Nucl. Phys. **B457** (1995) 343, [arXiv:hep-th/9506180].
- [48] V. A. Fateev and A. V. Litvinov, *Correlation functions in conformal Toda field theory I*, JHEP **0711**, 002 (2007) [arXiv:0709.3806 [hep-th]].