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FOUNDATIONS

On derived algebras and subvarieties of implication zroupoids

Juan M. Cornejo[1] · **Hanamantagouda P. Sankappanavar**[2]

Abstract In 2012, the second author introduced and studied in Sankappanavar (Sci Math Jpn 75(1):21–50, 2012) the variety $\mathcal{I}$ of algebras, called implication zroupoids, that generalize De Morgan algebras. An algebra $\mathbf{A} = \langle A, \to, 0\rangle$, where $\to$ is binary and 0 is a constant, is called an *implication zroupoid* ($\mathcal{I}$-zroupoid, for short) if $\mathbf{A}$ satisfies: $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$ and $0'' \approx 0$, where $x' := x \to 0$. The present authors devoted the papers, Cornejo and Sankappanavar (Alegbra Univers, 2016a; Stud Log 104(3):417–453, 2016b. doi:10.1007/s11225-015-9646-8; and Soft Comput: 20:3139–3151, 2016c. doi:10.1007/s00500-015-1950-8), to the investigation of the structure of the lattice of subvarieties of $\mathcal{I}$, and to making further contributions to the theory of implication zroupoids. This paper investigates the structure of the derived algebras $\mathbf{A^m} := \langle A, \wedge, 0\rangle$ and $\mathbf{A^{mj}} := \langle A, \wedge, \vee, 0\rangle$ of $\mathbf{A} \in \mathcal{I}$, where $x \wedge y := (x \to y')'$ and $x \vee y := (x' \wedge y')'$, as well as the lattice of subvarieties of $\mathcal{I}$. The varieties $\mathcal{I}_{2,0}$, $\mathcal{RD}$, $\mathcal{SRD}$, $\mathcal{C}$, $\mathcal{CP}$, $\mathcal{A}$, $\mathcal{MC}$, and $\mathcal{CLD}$ are defined relative to $\mathcal{I}$, respectively, by: ($\mathrm{I}_{2,0}$) $x'' \approx x$, (RD) $(x \to y) \to z \approx (x \to z) \to (y \to z)$, (SRD) $(x \to y) \to z \approx (z \to x) \to (y \to z)$, (C) $x \to y \approx y \to x$, (CP) $x \to y' \approx y \to x'$, (A) $(x \to y) \to z \approx x \to (y \to z)$, (MC) $x \wedge y \approx y \wedge x$, (CLD) $x \to (y \to z) \approx (x \to z) \to (y \to x)$. The purpose of this paper is two-fold. Firstly, we show that, for each $\mathbf{A} \in \mathcal{I}$, $\mathbf{A^m}$ is a semigroup. From this result, we deduce that, for $\mathbf{A} \in \mathcal{I}_{2,0} \cap \mathcal{MC}$, the derived algebra $\mathbf{A^{mj}}$ is a distributive bisemilattice and is also a Birkhoff system. Secondly, we show that $\mathcal{CLD} \subset \mathcal{SRD} \subset \mathcal{RD}$ and $\mathcal{C} \subset \mathcal{CP} \cap \mathcal{A} \cap \mathcal{MC} \cap \mathcal{CLD}$, both of which are much stronger results than were announced in Sankappanavar (Sci Math Jpn 75(1):21–50, 2012).

Keywords Implication zroupoid · Derived algebras · Distributive bisemilattice · Birkhoff system · Subvarieties · Left distributive law · Right distributive law · Semigroup

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✉ Hanamantagouda P. Sankappanavar
sankapph@newpaltz.edu

Juan M. Cornejo
jmcornejo@uns.edu.ar

1 INMABB - CONICET, Departamento de Matemática, Universidad Nacional del Sur, Alem 1253, Bahía Blanca, Argentina

2 Department of Mathematics, State University of New York, New Paltz, New York 12561, USA

1 Introduction

Bernstein (1934) gave a system of axioms for Boolean algebras in terms of implication only; however, his original axioms were not equational. A quick look at his axioms would reveal that, with an additional constant, they could easily be translated into equational ones. In 2012, the second author of this paper extended this modified Bernstein's theorem to De Morgan algebras in Sankappanavar (2012). Indeed, it is shown in Sankappanavar (2012) that the varieties of De Morgan algebras, Kleene algebras, and Boolean algebras are term-equivalent, to varieties whose defining axioms use only the implication $\to$ and the constant 0.

The primary role played by the identity (I): $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$, where $x' := x \to 0$, which occurs as an axiom in the definition of each of those new varieties motivated the second author of this paper to introduce a new (equational) class of algebras called "implication zroupoids" in Sankappanavar (2012).

An algebra $\mathbf{A} = \langle A, \to, 0\rangle$, where $\to$ is binary and 0 is a constant, is called a *zroupoid*. Let $x' := x \to 0$. A zroupoid $\mathbf{A} = \langle A, \to, 0\rangle$ is an *implication zroupoid* ($\mathcal{I}$-zroupoid, for short) if $\mathbf{A}$ satisfies:

(I) $\quad (x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$,
(I_0) $\quad 0'' \approx 0$.

Throughout this paper $\mathcal{I}$ denotes the variety of implication zroupoids.

It is proved in Sankappanavar (2012) that the variety $\mathcal{I}$ is a generalization of the variety of De Morgan algebras. It also exhibits several interesting properties of $\mathcal{I}$; for example, the identity $x''' \to y \approx x' \to y$ holds in $\mathcal{I}$. Several new and interesting subvarieties of $\mathcal{I}$ are also introduced and investigated in Sankappanavar (2012). The (still largely unexplored) lattice of subvarieties of $\mathcal{I}$ seems to be fairly complex. Problem 6 of Sankappanavar (2012) asks for the investigation of the structure of the lattice of subvarieties of $\mathcal{I}$.

The varieties $\mathcal{I}_{1,0}, \mathcal{I}_{2,0}, \mathcal{I}_{3,1}, \mathcal{ID}, \mathcal{Z}, \mathcal{MID}, \mathcal{JID}, \mathcal{MC}, \mathcal{C}, \mathcal{CP}, \mathcal{SCP}, \mathcal{A}, \mathcal{RD}, \mathcal{LAP}, \mathcal{SRD}, \mathcal{TII}, \mathcal{CLD}, \mathcal{WCP}, \mathcal{DM}, \mathcal{KL}$, and $\mathcal{BA}$ are defined relative to $\mathcal{I}$, respectively, as follows, where $x \wedge y := (x \to y')'$ and $x \vee y := (x' \wedge y')'$:

($I_{1,0}$) $\quad x' \approx x$, $\quad$ ($I_{2,0}$) $\quad x'' \approx x$, $\quad$ ($I_{3,1}$) $\quad x''' \approx x'$,
(ID) $\quad x \to x \approx x$,
(Z) $\quad x \to y \approx 0$, $\quad$ (MID) $\quad x \wedge x \approx x$,
(JID) $\quad x \vee x \approx x$,
(MC) $\quad x \wedge y \approx y \wedge x$. $\quad$ (C) $\quad x \to y \approx y \to x$,
(CP) $\quad x \to y' \approx y \to x'$,
(SCP) $\quad x \to y \approx y' \to x'$,
(A) $\quad (x \to y) \to z \approx x \to (y \to z)$,
(RD) $\quad (x \to y) \to z \approx (x \to z) \to (y \to z)$,
(LAP) $\quad (x \to x) \to x \approx x$,
(SRD) $\quad (x \to y) \to z \approx (z \to x) \to (y \to z)$,
(TII) $\quad 0' \to (x \to y) \approx (x \to y)$,
(CLD) $\quad x \to (y \to z) \approx (x \to z) \to (y \to x)$,
(WCP) $\quad x' \to y \approx y' \to x$,
(DM) $\quad (x \to y) \to x \approx x$ $\quad$ (De Morgan Algebras),
(KL) $\quad (x \to x) \to (y \to y) \approx (y \to y)$ $\quad$ (Kleene algebras), and
(BA) $\quad x \to x \approx 0'$ $\quad$ (Boolean algebras).

The reader can see the interrelationships among these varieties given in the Hasse diagram at the end of Sect. 5.

The paper (Cornejo and Sankappanavar 2016a) is a continuation of Sankappanavar (2012) and presents further relationships among some of the varieties mentioned above. (We should point out here that the algebras in $\mathcal{I}$ are referred to in Cornejo and Sankappanavar (2016a) as "implicator groupoids".) It is proved there that $\mathcal{I}_{2,0} = \mathcal{MID} = \mathcal{JID}$ and $\mathcal{SCP} \subset \mathcal{MC}$, and the varieties of Boolean algebras and Kleene algebras are characterized as suitable subvarieties of $\mathcal{I}_{2,0}$. It is shown that a Glivenko-like theorem holds for implication zroupoids. It is also proved that $\mathcal{Z} \subset \mathcal{C} \subset \mathcal{A} \subset \mathcal{I}_{3,1}$ and $\mathcal{I}_{1,0} = \mathcal{ID} \cap \mathcal{A}$. The varieties generated by the three 2-element implication zroupoids are characterized. It turns out that the congruence lattices of implication zroupoids do not satisfy any nontrivial lattice identities. It is also shown that $\mathcal{MC} \cap \mathcal{ID} = \mathcal{MC} \cap \mathcal{MID} \cap \mathcal{A} = \mathcal{C} \cap \mathcal{I}_{1,0} = \mathcal{SL}$. For an implication zroupoid $\mathbf{A}$, the following are equivalent: (i) the derived algebra $\mathbf{A^{mj}} = \langle A, \wedge, \vee, 0\rangle$ is a lattice with 0, (ii) the absorption identity holds in $\mathbf{A^{mj}}$, (iii) $\mathbf{A}$ is a De Morgan algebra, and (iv) $\mathbf{A}$ satisfies the identities $x \wedge 0 \approx 0$ and $x'' \approx x$.

Cornejo and Sankappanavar (2016b) is a further contribution to the theory of implication zroupoids, continuing the work of Sankappanavar (2012) and Cornejo and Sankappanavar (2016a). The importance of the variety $\mathcal{I}_{2,0}$, which contains the varieties $\mathcal{SL}$ and $\mathcal{DM}$, is highlighted by the fact that the variety $\mathcal{I}_{2,0}$ is a maximal subvariety of $\mathcal{I}$ with respect to the property that the relation $\sqsubseteq$, defined by:

$$x \sqsubseteq y \text{ if and only if } (x \to y')' = x, \text{ for } x, y \in \mathbf{A} \text{ and } \mathbf{A} \in \mathcal{I},$$

is a partial order. The problem of determining the number of nonisomorphic chains in $\mathcal{I}_{2,0}$ ($\mathcal{I}_{2,0}$-chains) that can be defined on an n-element set, n being a natural number, is then answered by proving that there are exactly n nonisomorphic $\mathcal{I}_{2,0}$-chains of size n, for each $n \in \mathbb{N}$.

Continuing the investigations done in Sankappanavar (2012), Cornejo and Sankappanavar (2016a, b), the paper (Cornejo and Sankappanavar 2016c) describes the simple algebras and semisimple subvarieties of $\mathcal{I}$. It is shown that there are, up to isomorphism, five (nontrivial) simple algebras in $\mathcal{I}$, namely the 2-element trivial implication zroupoid $\mathbf{2_z}$, where $x \to y := 0$, the 2-element $\vee$-semilattice $\mathbf{2_s}$ with the least element 0, the 2-element Boolean algebra $\mathbf{2_b}$, the 3-element Kleene algebra $\mathbf{3_k}$, and the 4-element De Morgan algebra $\mathbf{4_d}$. From this description it follows that the semisimple subvarieties of $\mathcal{I}$ are precisely the subvarieties of the variety $\mathbb{V}(\mathbf{2_z}, \mathbf{2_s}, \mathbf{4_d})$ and hence are locally finite. It also follows that the lattice of semisimple varieties of implication zroupoids is isomorphic to the direct product of a 4-element Boolean lattice and a 4-element chain.

Given an $\mathcal{I}$-zroupoid $\mathbf{A}$, there are naturally induced operations $\wedge$ and $\vee$ on A as follows:

- $x \wedge y := (x \to y')'$, and
- $x \vee y := (x' \wedge y')'$.

With each implication zroupoid $\mathbf{A}$, we associate the following algebras, referred to as "derived algebras":

- $\mathbf{A^m} := \langle A, \wedge, 0\rangle$,
- $\mathbf{A^j} := \langle A, \vee, 0\rangle$,

- $\mathbf{A^{mj}} := \langle A, \wedge, \vee, 0\rangle$.

The present paper is a further addition to the series (Sankappanavar 2012; Cornejo and Sankappanavar 2016a, b, c) and studies the structure of the derived algebras $\mathbf{A^m}$ and $\mathbf{A^{mj}}$, as well as some of the subvarieties of $\mathcal{I}$ mentioned above. More specifically, the purpose of this paper is twofold. First, we show that, for each $\mathcal{I}$-zroupoid $\mathbf{A}$, $\mathbf{A}^m$ is a semigroup. From this result, using Cornejo and Sankappanavar (2016a, Theorem 7.3), we deduce that, for $\mathbf{A} \in \mathcal{I}_{2,0} \cap \mathcal{MC}$, the derived algebra $\mathbf{A^{mj}}$ is both a distributive bisemilattice and a Birkhoff system. Second, we show that $\mathcal{CLD} \subset \mathcal{SRD} \subset \mathcal{RD}$ and $\mathcal{C} \subset \mathcal{CP} \cap \mathcal{A} \cap \mathcal{MC} \cap \mathcal{CLD}$, both of which are much stronger results than were announced in Sankappanavar (2012).

We would like to acknowledge that the software "Prover 9/Mace 4" developed by McCune (2005–2010) have been useful to us in some of our findings presented in this paper. We have used them to find examples and to check some conjectures.

2 Preliminaries

We refer the reader to the textbooks Balbes and Dwinger (1974), Burris and Sankappanavar (1981), and Rasiowa (1974) for the concepts and results assumed in this paper. In this section we give results (some old and some new) useful in the rest of the paper. To start, we wish to note that, in a De Morgan algebra, one defines $x \to y := x' \vee y$.

Lemma 2.1 Sankappanavar (2012, **Theorem 8.15**) *Let* $\mathbf{A}$ *be an* $\mathcal{I}$*-zroupoid and* $a \in A$. *Then the following are equivalent:*

(a) $0' \to a = a$,
(b) $a'' = a$,
(c) $(a \to a')' = a$,
(d) $a' \to a = a$.

Lemma 2.2 Sankappanavar (2012, **Lemma 8.13**) *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then* $\mathbf{A}$ *satisfies:*

(a) $x' \to 0' \approx 0 \to x$,
(b) $0 \to x' \approx x \to 0'$.

Lemma 2.3 Sankappanavar (2012, **Lemma 7.5(b)**) *Let* $\mathbf{A}$ *be an* $\mathcal{I}$*-zroupoid. Then* $\mathbf{A}$ *satisfies* $(x \to y'')' \approx (x \to y)'$.

Lemma 2.4 Cornejo and Sankappanavar (2016a, **Lemma 2.8(2)**) *Let* $\mathbf{A}$ *be an I-zroupoid. Then* $\mathbf{A}$ *satisfies:*

(a) $(x \to y) \to z \approx [(x \to y) \to z]''$,
(b) $(x \to y)' \approx (x'' \to y)'$.

Lemma 2.5 Sankappanavar (2012, **Corollary 7.7**) *Let* $\mathbf{A}$ *be an I-zroupoid. Then* $\mathbf{A}$ *satisfies* $x'''' \approx x''$.

Theorem 2.6 Cornejo and Sankappanavar (2016a, **Theorem 4.2(a)**) *Let* $\mathbf{A} = \langle A, \to, 0\rangle \in \mathcal{I}$ *and let* $A'' := \{x'' : x \in A\}$. *Then* $\langle A'', \to, 0\rangle \in \mathcal{I}_{2,0}$.

Lemma 2.7 *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then* $\mathbf{A}$ *satisfies:*

(a) $(x \to 0') \to y \approx (x \to y') \to y$,
(b) $x \to (0 \to x)' \approx x'$,
(c) $(x \to y) \to (0 \to y)' \approx (x \to y)'$,
(d) $[(0 \to x) \to y] \to x \approx y \to x$,
(e) $[x \to (y \to x)']' \approx (x \to y) \to x$,
(f) $(y \to x) \to y \approx (0 \to x) \to y$,
(g) $0 \to x \approx 0 \to (0 \to x)$,
(h) $(0 \to x) \to (x \to y) \approx x \to (x \to y)$,
(i) $(0 \to x) \to (0 \to y) \approx x \to (0 \to y)$,
(j) $x \to y \approx x \to (x \to y)$,
(k) $[x' \to (0 \to y)]' \approx (0 \to x) \to (0 \to y)'$,
(l) $0 \to (0 \to x)' \approx 0 \to x'$,
(m) $0 \to (x' \to y)' \approx x \to (0 \to y')$,
(n) $0 \to (x \to y) \approx x \to (0 \to y)$,
(o) $(x \to y) \to y' \approx y \to (x \to y)'$,
(p) $0 \to [(0 \to x) \to y] \approx x \to (0 \to y)$,
(q) $0 \to (x \to y')' \approx 0 \to (x' \to y)$,
(r) $[(0 \to x) \to y]' \approx y \to (x \to y)'$,
(s) $[(x \to y) \to x] \to [(y \to x) \to y] \approx x \to y$,
(t) $x \to (y \to x') \approx y \to x'$,
(u) $(z \to x) \to (y \to z) \approx (0 \to x) \to (y \to z)$,
(v) $0 \to [(x \to y)' \to z] \approx 0 \to [x \to (y' \to z)]$,
(w) $[(0 \to x) \to y] \to (z \to x) \approx y \to (z \to x)$,
(x) $[(x \to y) \to (y \to z)]' \approx (0 \to x) \to (y \to z)'$.

Proof For items (a), (b), (c), (e), (f), (g), (k), (l), (m), (n), (o), (q), (s), (t), (u) we refer the interested reader to the appendix of the arxiv version, arXiv:1509.03774v2 [math.LO] 9 Jun 2016, of Cornejo and Sankappanavar (2016a) which is available online at http://www.arxiv.org, where detailed proofs are given. The proof of items (d), (i), (j), (p), (r) are in Cornejo and Sankappanavar (2016a) and of items (h), (v), (w), (x) are in Cornejo and Sankappanavar (2016c). □

The following lemma is proved in Appendix.

Lemma 2.8 *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then* $\mathbf{A}$ *satisfies:*

(1) $(x \to y)' \to y \approx x \to y$,
(2) $(0 \to y) \to (x' \to u) \approx [x \to (y \to x)'] \to u$,
(3) $(x \to y) \to (y \to z) \approx (0 \to x') \to (y \to z)$,
(4) $[(x \to y) \to z] \to (z \to u) \approx (0 \to x) \to [(y \to z) \to (z \to u)]$,
(5) $[y \to (0 \to z)] \to x \approx [y \to (x \to z)] \to x$,
(6) $(0 \to x) \to [y \to (x \to z)] \approx x \to [y \to (x \to z)]$,

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(7) $x \to [y \to (x \to z)] \approx y \to (x \to z)$,
(8) $(x \to y) \to (0 \to y') \approx (x \to y) \to 0'$,
(9) $y \to (0 \to x)' \approx [y' \to (0 \to x')']'$,
(10) $x \to [y \to (0 \to x)'] \approx y \to x'$,
(11) $(x' \to y) \to z \approx [(x \to z) \to y] \to z$,
(12) $(x' \to y) \to (x \to z) \approx (0 \to y) \to (x \to z)$,
(13) $x \to [(0 \to x) \to y] \approx x \to y$,
(14) $(x \to 0') \to (y \to z) \approx [(0 \to x) \to y] \to z$,
(15) $[(x \to y) \to (z \to x)] \to u \approx (y \to 0') \to [(z \to x) \to u]$,
(16) $(0 \to x) \to [(y \to x) \to z] \approx (y \to x) \to z$,
(17) $(0 \to [(x \to y) \to z)] \to [(u \to x) \to y] \approx (0 \to x) \to [(y \to 0') \to ((0 \to z) \to ((u \to x) \to y))]$,
(18) $[x \to ((0 \to y) \to z)]' \approx (x \to z) \to [(y \to (0 \to z))']$,
(19) $[0 \to ((x \to y) \to z)] \to u \approx (0 \to x) \to [(0 \to (y \to z)) \to u]$,
(20) $[x \to ((0 \to y) \to z)] \to y \approx (x \to z) \to y$,
(21) $(x \to 0') \to [y \to ((x \to z) \to u)] \approx y \to [(x \to z) \to u]$,
(22) $(x \to 0') \to (y \to z) \approx y \to [(x \to (0 \to y)) \to z]$,
(23) $[x \to (0 \to y)] \to (y \to z) \approx (x \to y) \to (y \to z)$,
(24) $[(x \to y) \to (z \to x)] \to y \approx (z \to x) \to y$.

3 $\wedge$-Associativity in $\mathcal{I}_{2,0}$

In this section our goal is to prove the $\wedge$-associativity in $\mathcal{I}_{2,0}$. To achieve this goal, we need the following lemmas.

Lemma 3.1 *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then* $\mathbf{A}$ *satisfies* $(x \to y')' \to (y \to z) \approx x \to (y \to z)$.

Proof Let $a, b, c \in A$. Since

$$\begin{aligned}(0 \to a) &\to [b \to (a \to c)]\\ &= a \to [b \to (a \to c)] \quad \text{by Lemma 2.8 (6)}\\ &= b \to (a \to c) \quad \text{by Lemma 2.8 (7)},\end{aligned}$$

it follows that A satisfies

$$(0 \to x) \to [y \to (x \to z)] \approx y \to (x \to z). \tag{3.1}$$

Also, we get

$$[(x \to y) \to z]' \to x \approx (0 \to y) \to (z' \to x), \tag{3.2}$$

from

$$\begin{aligned}(0 \to b) &\to (c' \to a)\\ &= [c \to (b \to c)'] \to a \quad \text{by Lemma 2.8 (2)}\\ &= [c'' \to (b \to c)'] \to a\\ &= [(c' \to a) \to (b \to c)'] \to a \quad \text{by Lemma 2.8 (11)}\\ &= [(c' \to a) \to (b \to c)']'' \to a\\ &= [(a \to b) \to c]' \to a \quad \text{by (I)}.\end{aligned}$$

We see that the identity

$$[(x \to (0 \to y)) \to z] \to u \approx (0 \to x) \to [(0 \to y') \to (z \to u)], \tag{3.3}$$

holds in $\mathbf{A}$, since

$$\begin{aligned}(0 \to a) &\to [(0 \to b') \to (c \to d)]\\ &= (a' \to 0') \to [(0 \to b') \to (c \to d)] \quad \text{by Lemma 2.2 (b)}\\ &= [(0 \to a') \to (0 \to b')] \to (c \to d) \quad \text{by Lemma 2.8 (14)}\\ &= [0 \to (a' \to b')] \to (c \to d) \quad \text{by Lemma 2.7 items (i) and (n)}\\ &= [0 \to (a \to b)'] \to (c \to d) \quad \text{by Lemma 2.7 items (m) and (n)}\\ &= [(a \to b) \to 0'] \to (c \to d) \quad \text{by Lemma 2.2 (b)}\\ &= [\{0 \to (a \to b)\} \to c] \to d \quad \text{by Lemma 2.8 (14)}\\ &= [\{a \to (0 \to b)\} \to c] \to d \quad \text{by Lemma 2.7 (n)}.\end{aligned}$$

Observe that

$$\begin{aligned}(a \to b) &\to [(0 \to b) \to c]\\ &= [\{((0 \to b) \to c)' \to a\} \to \{b \to ((0 \to b) \to c)\}']' \quad \text{by (I)}\\ &= [\{((0 \to b) \to c)' \to a\} \to (b \to c)']' \quad \text{by Lemma 2.8 (13)}\\ &= [\{((c \to b) \to c)' \to a\} \to (b \to c)']' \quad \text{by Lemma 2.7 (f)}\\ &= [\{(c \to (b \to c)') \to a\} \to (b \to c)']' \quad \text{by Lemma 2.7 (e)}\\ &= [(b \to c) \to \{c \to (b \to c)'\}] \to [a \to (b \to c)']' \quad \text{by (I)}\\ &= [c \to (b \to c)'] \to [a \to (b \to c)']' \quad \text{by Lemma 2.7 (t)}\\ &= [(b \to c) \to c'] \to [a \to (b \to c)']' \quad \text{by Lemma 2.7 (o)}\\ &= [(c' \to a) \to (b \to c)']' \quad \text{by (I)}\\ &= (a \to b) \to c \quad \text{by (I)},\end{aligned}$$

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and, consequently, **A** satisfies

$$(x \to y) \to ((0 \to y) \to z) \approx (x \to y) \to z. \tag{3.4}$$

From

$$\begin{aligned}
&[b \to (0 \to c)] \to [d \to (a \to b)]\\
&\quad = [(0 \to b) \to (0 \to c)] \to [d \to (a \to b)]\\
&\qquad \text{by Lemma 2.7 (i)}\\
&\quad = [0 \to \{(0 \to b) \to c\}] \to [d \to (a \to b)]\\
&\qquad \text{by Lemma 2.7 (n)}\\
&\quad = [(a \to b) \to \{(0 \to b) \to c\}] \to [d \to (a \to b)]\\
&\qquad \text{by Lemma 2.7 (u)}\\
&\quad = [(a \to b) \to c] \to [d \to (a \to b)]\\
&\qquad \text{by (3.4),}
\end{aligned}$$

we conclude that the identity

$$\begin{aligned}
&[(x \to y) \to z] \to [u \to (x \to y)]\\
&\quad \approx [y \to (0 \to z)] \to [u \to (x \to y)]
\end{aligned} \tag{3.5}$$

is true in **A**. From Lemma 2.7 (u) and (3.5) we see that **A** satisfies

$$\begin{aligned}
[x \to (0 \to y)] \to [z \to (u \to x)] &\approx (0 \to y)\\
&\to [z \to (u \to x)].
\end{aligned} \tag{3.6}$$

From

$$\begin{aligned}
&b' \to (a \to c)\\
&\quad = (0 \to a) \to [b' \to (a \to c)] \quad \text{by (3.1)}\\
&\quad = [\{(a \to c) \to a\} \to b]' \to (a \to c) \quad \text{by (3.2)}\\
&\quad = [\{(0 \to c) \to a\} \to b]' \to (a \to c)\\
&\qquad \text{by Lemma 2.7 (f)}\\
&\quad = [(c \to 0') \to (a \to b)]' \to (a \to c)\\
&\qquad \text{by Lemma 2.8 (14)}\\
&\quad = [\{c \to (0 \to 0)\} \to (a \to b)]' \to (a \to c)\\
&\quad = [(0 \to c) \to \{(0 \to 0') \to (a \to b)'\}] \to (a \to c)\\
&\qquad \text{by (3.3)}\\
&\quad = [(0 \to c) \to \{0' \to (a \to b)'\}] \to (a \to c)\\
&\qquad \text{by Lemma 2.1 (d)}\\
&\quad = [(0 \to c) \to (a \to b)'] \to (a \to c)\\
&\qquad \text{by Lemma 2.1 (a)}\\
&\quad = (c \to 0') \to [(a \to b)' \to (a \to c)]\\
&\qquad \text{by Lemma 2.8 (15)}\\
&\quad = (a \to b)' \to (a \to c) \quad \text{by (3.6) and by Lemma 2.1 (a),}
\end{aligned}$$

we have that **A** satisfies

$$(x \to y)' \to (x \to z) \approx y' \to (x \to z). \tag{3.7}$$

Also, the identity

$$[x \to (y \to z)'] \to z \approx (x \to y) \to z \tag{3.8}$$

holds in **A**, since

$$\begin{aligned}
&[a \to (b \to c)'] \to c\\
&\quad = [(c' \to a) \to \{(b \to c)' \to c\}']' \quad \text{by (I)}\\
&\quad = [(c' \to a) \to (b \to c)']' \quad \text{by Lemma 2.8 (1)}\\
&\quad = (a \to b) \to c \quad \text{by (I).}
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&(a \to b')' \to (b \to c)\\
&\quad = [b \to \{a \to (0 \to b)'\}]' \to (b \to c)\\
&\qquad \text{by Lemma 2.8 (10)}\\
&\quad = [a \to (0 \to b)']' \to (b \to c)\\
&\qquad \text{by (3.7) with } x = b,\ y = a \to (0 \to b)'\\
&\quad = [a' \to (0 \to b')']'' \to (b \to c) \quad \text{by Lemma 2.8 (9)}\\
&\quad = [a' \to (0 \to b')'] \to (b \to c)\\
&\quad = [a' \to (b \to 0')'] \to (b \to c) \quad \text{by Lemma 2.2 (b)}\\
&\quad = [a' \to \{(b \to 0')' \to (b \to c)\}'] \to (b \to c)\\
&\qquad \text{by (3.8) with } x = a',\ y = (b \to 0')',\ z = b \to c\\
&\quad = [a' \to \{0'' \to (b \to c)\}'] \to (b \to c)\\
&\qquad \text{by (3.7) with } y = 0'\\
&\quad = [a' \to \{0 \to (b \to c)\}'] \to (b \to c)\\
&\quad = (a' \to 0) \to (b \to c) \quad \text{by (3.8)}\\
&\quad = a \to (b \to c).
\end{aligned}$$

This completes the proof. □

Lemma 3.2 *Let* **A** $\in \mathcal{I}_{2,0}$. *Then* **A** *satisfies* $(x \to y) \to (y \to z) \approx y \to ((x \to y) \to z)$.

Proof Let $a, b, c \in A$. Then

$$\begin{aligned}
&b \to [(a \to b) \to c]\\
&\quad = b \to [c' \to \{(a \to b) \to c\}] \quad \text{by Lemma 2.7 (t)}\\
&\quad = (b \to c'')' \to [c' \to \{(a \to b) \to c\}]\\
&\qquad \text{by Lemma 3.1 with } y = c' \text{ and } z = (a \to b) \to c\\
&\quad = (b \to c)' \to [c' \to \{(a \to b) \to c\}]\\
&\quad = (b \to c)' \to \{(a \to b) \to c\}\\
&\qquad \text{by Lemma 2.7 (t)}\\
&\quad = (b \to c)' \to [(c' \to a) \to (b \to c)']' \quad \text{by (I)}
\end{aligned}$$

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$= [\{(b \to c)'' \to 0\} \to \{(c' \to a) \to (b \to c)'\}']''$

$= [\{0 \to (c' \to a)\} \to (b \to c)']'$ by (I)

$= [(0 \to c) \to [(0 \to 0') \to \{(0 \to a) \to (b \to c)'\}]]'$

by Lemma 2.8 (17) with $x = c$, $y = 0$, $u = b$.

$= [(0 \to c) \to \{(0 \to a) \to (b \to c)'\}]'$

since $0 \to 0' \approx 0'$ and $0' \to x \approx x$

$= [[\{(0 \to a) \to (b \to c)'\}' \to 0]$

$\to [c \to \{(0 \to a) \to (b \to c)'\}]']''$ by (I)

$= [(0 \to a) \to (b \to c)']$

$\to [c \to \{(0 \to a) \to (b \to c)'\}]'$ using $x \approx x''$

$= [(0 \to a) \to (b \to c)']$

$\to [c \to [(0 \to c) \to \{(0 \to a) \to (b \to c)'\}]]'$

by Lemma 2.8 (13) with $x = c$ and y

$= (0 \to a) \to (b \to c)'$

$= [(0 \to a) \to (b \to c)']$

$\to [\{c \to ((c' \to 0') \to \{(0 \to a) \to (b \to c)')\}\}']$

by Lemma 2.2 (b)

$= [(0 \to a) \to (b \to c)']$

$\to [\{c \to (((0 \to c') \to (0 \to a)) \to (b \to c)')\}']$

by Lemma 2.8 (14)

$= [(0 \to a) \to (b \to c)']$

$\to [(c \to ((c' \to (0 \to a)) \to (b \to c)'))']$

by Lemma 2.7 (i)

$= [(0 \to a) \to (b \to c)']$

$\to [\{c \to (((0 \to a) \to b) \to c)'\}']$ by (I)

$= [(0 \to a) \to (b \to c)']$

$\to [\{(0 \to ((0 \to a) \to b)) \to c\}'']$

by Lemma 2.7 (r)

$= [(0 \to a) \to (b \to c)']$

$\to [\{0 \to ((0 \to a) \to b)\} \to c]$

$= [(0 \to a) \to (b \to c)']$

$\to [\{(0 \to a) \to (0 \to b)\} \to c]$

by Lemma 2.7 (n)

$= [(0 \to a) \to (b \to c)']$

$\to [\{a \to (0 \to b)\} \to c]$

by Lemma 2.7 (i)

$= (a \to 0') \to [(b \to c)'$

$\to [\{a \to (0 \to b)\} \to c]]$

by Lemma 2.8 (15) with $x = 0$, $y = a$, $z = b \to c$ and $u = (a \to (0 \to b)) \to c$

$= (b \to c)' \to [\{a \to (0 \to b)\} \to c]$

by Lemma 2.8 (21) with $x = b$, $y = (b \to c)'$, $z = 0 \to b$ and $u = c$

$= [(b \to c) \to [0 \to \{(a \to (0 \to b)) \to c\}]']$

$\to [\{a \to (0 \to b)\} \to c]$

by Lemma 2.8 (20) with $x = b \to c$, $y = (a \to (0 \to b)) \to c$, $z = 0$

$= [(b \to c) \to [0 \to \{(0 \to (a \to b)) \to c\}]']$

$\to [\{a \to (0 \to b)\} \to c]$

by Lemma 2.7 (n)

$= [(b \to c) \to \{(0 \to (a \to b)) \to (0 \to c)\}']$

$\to [\{a \to (0 \to b)\} \to c]$

by Lemma 2.7 (n)

$= [(b \to c) \to \{(a \to b) \to (0 \to c)\}']$

$\to [\{a \to (0 \to b)\} \to c]$

by Lemma 2.7 (i)

$= [b \to \{(0 \to (a \to b)) \to c\}]'$

$\to [\{a \to (0 \to b)\} \to c]$

by Lemma 2.8 (18) with $x = b$, $y = a \to b$, $z = c$

$= \{b \to [(a \to (0 \to b)) \to c]\}'$

$\to [(a \to (0 \to b)) \to c]$

by Lemma 2.7 (n)

$= b \to [\{a \to (0 \to b)\} \to c]$

by Lemma 2.8 (1)

$= b \to [\{(a \to (0 \to b))' \to (0 \to b)\} \to c]$

by Lemma 2.8 (1)

$= [\{a \to (0 \to b)\}' \to 0'] \to (b \to c)$

by Lemma 2.8 (22)

$= [0 \to \{a \to (0 \to b)\}] \to (b \to c)$

by Lemma 2.2 (b)

$= [a \to (0 \to (0 \to b))] \to (b \to c)$

by Lemma 2.7 (n)

$= [a \to (0 \to b)] \to (b \to c)$

by Lemma 2.7 (g)

$= (a \to b) \to (b \to c)$

by Lemma 2.8 (23)

$= [(a \to b)' \to b] \to (b \to c)$

by Lemma 2.8 (1)

$= [0 \to (a \to b)] \to [(0 \to b) \to (b \to c)]$

by Lemma 2.8 (4) with $y = 0$, $x = a \to b$, $z = b$, $u = c$

$= [\{(0 \to b) \to (b \to c)\} \to (a \to b)]$

$\to [(0 \to b) \to (b \to c)]$

by Lemma 2.7 (f)

$= [\{b \to ((0 \to b) \to (b \to c))\} \to (a \to b)]$

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$\to [(0 \to b) \to (b \to c)]$
by Lemma 2.8 (7)
$= (a \to b) \to [(0 \to b) \to (b \to c)]$
by Lemma 2.8 (24) with $y = (0 \to b) \to (b \to c)$
$= (a \to b) \to [b \to (b \to c)]$
by Lemma 2.7 (h)
$= (a \to b) \to (b \to c)$
by Lemma 2.7 (j). □

Lemma 3.3 *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then* $\mathbf{A}$ *satisfies* $(x \to y')' \to z \approx x \to (y \to z)$.

Proof Let $a, b, c \in A$. Then

$a \to (b \to c)$
$= a \to [(0 \to a) \to (b \to c)]$ by Lemma 2.8 (13)
$= a \to [b \to \{(0 \to a) \to (b \to c)\}]$
by Lemma 2.8 (7)
$= (a \to b')' \to [b \to \{(0 \to a) \to (b \to c)\}]$
by Lemma 3.1
$= (a \to b')' \to [(0 \to a) \to (b \to c)]$
by Lemma 2.8 (7)
$= (a \to b')' \to [(0 \to a) \to (b'' \to c)]$
$= (a \to b')' \to [\{b' \to (a \to b')'\} \to c]$
by Lemma 2.8 (2) with $x = b', y = a, u = c$
$= [b' \to (a \to b')'] \to [(a \to b')' \to c]$
by Lemma 3.2 with $x = b', y = (a \to b')', z = c$
$= [(a \to b') \to b] \to [(a \to b')' \to c]$
by Lemma 2.7 (o)
$= [(a \to b')'' \to b] \to [(a \to b')' \to c]$
$= (0 \to b) \to [(a \to b')' \to c]$ by Lemma 2.8 (12)
$= (b' \to 0') \to [(a \to b')' \to c]$ by Lemma 2.2 (b)
$= [(0 \to b') \to (a \to b')'] \to c$ by Lemma 2.8 (14)
$= [(0 \to b') \to (a \to b')']'' \to c$
$= [(b \to 0') \to (a \to b')']'' \to c$ by Lemma 2.2 (b)
$= [(0' \to a) \to b']' \to c$ by (I)
$= (a \to b')' \to c$ by Lemma 2.1 (a). □

Theorem 3.4 *Let* $\mathbf{A} \in \mathcal{I}_{2,0}$. *Then* $\mathbf{A}$ *satisfies the identity:*

$$(x \wedge y) \wedge z \approx x \wedge (y \wedge z).$$

Proof Let $a, b, c \in A$. Then

$(a \wedge b) \wedge c$
$= [(a \to b')' \to c']'$ by definition of $\wedge$
$= [a \to (b \to c')]'$ by Lemma 3.3
$= [a \to (b \to c')'']'$ by $x \approx x''$
$= a \wedge (b \wedge c)$ by definition of $\wedge$. □

4 $\wedge$-Associativity in $\mathcal{I}$

For a certain class of identities, in order to prove their validity in $\mathcal{I}$, it suffices to prove their validity in $\mathcal{I}_{2,0}$. To this effect, we will prove a Transfer Theorem in this section and give some applications of that theorem in this and the following sections.

Let $\overline{x}$ represent the n-sequence $x_1, x_2, \ldots, x_n$ of variables, $\overline{a} = a_1, a_2, \ldots, a_n \in A^n$, and let $\overline{a''} = a''_1, a''_2, \ldots, a''_n$.

Lemma 4.1 *Let* $\mathbf{A} \in \mathcal{I}$ *and* $t(\overline{x})$ *a term in the language of* $\mathcal{I}$*-zroupoids, Then*

$$\mathbf{A} \models (t^A(\overline{a}))'' \approx t^A(\overline{a''}).$$

Proof We will proceed by induction on the term $t(\overline{x})$.

- If $t(\overline{x}) = 0$, then $t^A(\overline{a''}) = 0'' = 0 = (t^A(\overline{a}))''$.
- If $t(\overline{x}) = x_i$ with $1 \le i \le n$, then $(t^A(\overline{a}))'' = a''_i = t^A(\overline{a''})$.
- If $t(\overline{x}) = t_1(\overline{x}) \to t_2(\overline{x})$ then

$(t^A(\overline{a}))''$
$= [(t_1^A(\overline{a}) \to t_2^A(\overline{a})]''$
$= [(t_1^A(\overline{a}) \to (t_2^A(\overline{a}))'']''$ by Lemma 2.4
$= [(t_1^A(\overline{a}))'' \to (t_2^A(\overline{a}))'']''$ by Lemma 2.4 (b)
$= [\{(t_1^A(\overline{a}))' \to 0\} \to (t_2^A(\overline{a}))'']''$
$= [((t_1^A(\overline{a}))' \to 0] \to (t_2^A(\overline{a}))''$ by Lemma 2.4 (a)
$= (t_1^A(\overline{a}))'' \to (t_2^A(\overline{a}))''$
$= t_1^A(\overline{a''}) \to t_2^A(\overline{a''})$ by induction
$= t^A(\overline{a''})$,

proving the lemma. □

Theorem 4.2 *(Transfer Theorem) Let* $t_i(\overline{x})$, $i = 1, \ldots, 6$ *be terms and* $\mathcal{V}$ *a subvariety of* $\mathcal{I}$. *If*

$$\mathcal{V} \cap \mathcal{I}_{2,0} \models [t_1(\overline{x}) \to t_2(\overline{x})] \to t_3(\overline{x}) \approx [t_4(\overline{x}) \to t_5(\overline{x})] \to t_6(\overline{x}),$$

then

$$\mathcal{V} \models [t_1(\overline{x}) \to t_2(\overline{x})] \to t_3(\overline{x}) \approx [t_4(\overline{x}) \to t_5(\overline{x})] \to t_6(\overline{x}).$$

Proof Let $\mathbf{A} \in \mathcal{V}$. Then

$$[t_1^A(\overline{a}) \to t_2^A(\overline{a})] \to t_3^A(\overline{a})$$
$$= [\{t_1^A(\overline{a}) \to t_2^A(\overline{a})\} \to t_3^A(\overline{a})]'' \quad \text{by Lemma 2.4 (a)}$$
$$= [t_1^A(\overline{a''}) \to t_2^A(\overline{a''})] \to t_3^A(\overline{a''}) \quad \text{by Lemma 4.1}$$

Using Lemma 2.5 we have that $a_1'', a_2'', \ldots, a_n'' \in A''$, and by Theorem 2.6, $\mathbf{A}'' \in \mathcal{V} \cap \mathcal{I}_{2,0}$. Then

$$[t_1^A(\overline{a}) \to t_2^A(\overline{a})] \to t_3^A(\overline{a})$$
$$= [t_1^A(\overline{a''}) \to t_2^A(\overline{a''})] \to t_3^A(\overline{a''})$$
$$\text{by the conclusion above}$$
$$= [t_4^A(\overline{a''}) \to t_5^A(\overline{a''})] \to t_6^A(\overline{a''})$$
$$\text{by hypothesis, since } \mathbf{A}'' \in \mathcal{V} \cap \mathcal{I}_{2,0}$$
$$= [\{t_4^A(\overline{a}) \to t_5^A(\overline{a})\} \to t_6^A(\overline{a})]'' \quad \text{by Lemma 4.1}$$
$$= [t_4^A(\overline{a}) \to t_5^A(\overline{a})] \to t_6^A(\overline{a}) \quad \text{by Lemma 2.4 (a).}$$

This completes the proof. □

Corollary 4.3 *Let $r_i(\overline{x})$, $i = 1, \ldots, 4$, be terms. If*

$$\mathcal{I}_{2,0} \models r_1(\overline{x}) \to r_2(\overline{x}) \approx r_3(\overline{x}) \to r_4(\overline{x}),$$

then

$$\mathcal{I} \models [r_1(\overline{x}) \to r_2(\overline{x})]' \approx [r_3(\overline{x}) \to r_4(\overline{x})]'.$$

Proof Let $\mathcal{I}_{2,0} \models r_1(\overline{x}) \to r_2(\overline{x}) \approx r_3(\overline{x}) \to r_4(\overline{x})$. Then,

$$\mathcal{I}_{2,0} \models [r_1(\overline{x}) \to r_2(\overline{x})]' \approx [r_3(\overline{x}) \to r_4(\overline{x})]',$$

which implies

$$\mathcal{I}_{2,0} \models [r_1(\overline{x}) \to r_2(\overline{x})] \to 0 \approx [r_3(\overline{x}) \to r_4(\overline{x})] \to 0.$$

Now we apply Theorem 4.2, using $\mathcal{V} := \mathcal{I}$, $t_1(\overline{x}) := r_1(\overline{x})$, $t_2(\overline{x}) := r_2(\overline{x})$, $t_3(\overline{x}) := 0$, $t_4(\overline{x}) := r_3(\overline{x})$, $t_5(\overline{x}) := r_4(\overline{x})$ and $t_6(\overline{x}) := 0$. Hence, we have that

$$\mathcal{I} \models [r_1(\overline{x}) \to r_2(\overline{x})] \to 0 \approx [r_3(\overline{x}) \to r_4(\overline{x})] \to 0,$$

proving the corollary. □

We are now ready to present our first main result.

Theorem 4.4 *Let $\mathbf{A} \in \mathcal{I}$. Then $\mathcal{A}^m$ is a semigroup.*

Proof By Theorem 3.4 we have that

$$\mathcal{I}_{2,0} \models x \wedge (y \wedge z) \approx (x \wedge y) \wedge z.$$

Then, using the definition of $\wedge$, we get

$$\mathcal{I}_{2,0} \models (x \to (y \wedge z)')' \approx ((x \wedge y) \to z')'.$$

Applying Corollary 4.3,

$$\mathcal{I} \models (x \to (y \wedge z)')' \approx ((x \wedge y) \to z')'.$$

Hence,

$$\mathcal{I} \models x \wedge (y \wedge z) \approx (x \wedge y) \wedge z,$$

proving the theorem. □

We remark here that the above theorem implies that $[x \to (y \to z')'']' \approx [(x \to y')' \to z']'$.

For $\mathbf{A}$ an $\mathcal{I}$-zroupoid, $\mathbf{A}^{\mathbf{mj}}$ is a *bisemigroup* if $\mathbf{A}^{\mathbf{m}}$ and $\mathbf{A}^{\mathbf{j}}$ are semigroups.

Theorem 4.5 *Let $\mathbf{A} \in \mathcal{I}$. Then $\mathbf{A}^{\mathbf{j}}$ is a semigroup.*

Proof Let $a, b, c \in A$.

$$a \vee (b \vee c)$$
$$= [a' \wedge (b' \wedge c')'']' \quad \text{by definition of } \vee$$
$$= [a' \to (b' \to c'')'''']'' \quad \text{by definition of } \wedge$$
$$= [a' \to (b' \to c'')'']'' \quad \text{by Lemma 2.5}$$
$$= [(a' \to b'')' \to c'']''$$
$$\text{by (the remark after) Theorem 4.4}$$
$$= [(a' \wedge b') \to c'']'' \quad \text{by definition of } \wedge$$
$$= [(a' \wedge b')'' \to c'']'' \quad \text{by Lemma 2.4 (b)}$$
$$= (a \vee b) \vee c \quad \text{by definition of } \vee.$$

□

Corollary 4.6 *Let $\mathbf{A} \in \mathcal{I}$. Then $\mathbf{A}^{\mathbf{mj}}$ is a bisemigroup.*

The following theorem is proved in Cornejo and Sankappanavar (2016a, Theorem 7.3).

Theorem 4.7 *Let $\mathbf{A} \in \mathcal{I}_{2,0} \cap \mathcal{MC}$. Then $\mathbf{A}^{\mathbf{mj}}$ satisfies:*

(a) $x \wedge x \approx x$,
(b) $x \vee x \approx x$,
(c) $x \vee y \approx y \vee x$,
(d) $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$,
(e) $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$,
(f) $x \wedge (x \vee y) \approx x \vee (x \wedge y)$.

In Plonka (1967), Plonka introduced the class of *distributive quasilattices*, which are now known as *distributive bisemilattices*. A *bisemilattice* is an algebra $\langle B, \wedge, \vee \rangle$ such that $\langle B, \wedge \rangle$ and $\langle B, \vee \rangle$ are both semilattices. A *distributive bisemilattice* (DBS) is a bisemilattice in which the distributive laws hold:

$$x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z);$$
$$x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z).$$

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The following corollaries are immediate from Corollary 4.6 and Theorem 4.7 and give interesting properties of the variety $\mathcal{I}_{2,0} \cap \mathcal{MC}$.

Corollary 4.8 *Let* $\mathbf{A}$ *in the variety* $\mathcal{I}_{2,0} \cap \mathcal{MC}$. *Then* $\mathbf{A}^{\mathbf{mj}}$ *is a distributive bisemilattice.*

A *Birkhoff system* is a bisemilattice satisfying the Birkhoff's identity:

$$(BR) \quad x \wedge (x \vee y) \approx x \vee (x \wedge y).$$

Corollary 4.9 *Let* $\mathbf{A} \in \mathcal{I}_{2,0} \cap \mathcal{MC}$. *Then* $\mathbf{A}^{\mathbf{mj}}$ *is a Birkhoff system.*

Thus, if $\mathbf{A} \in \mathcal{I}_{2,0} \cap \mathcal{MC}$, then $\mathbf{A}^{\mathbf{mj}}$ is both a distributive bisemilattice and a Birkhoff system.

5 Varieties $\mathcal{SRD}$, $\mathcal{RD}$, $\mathcal{C}$, $\mathcal{CP}$, and $\mathcal{CLD}$

Let $\mathbf{A}$ be an I-zroupoid. Recall that $\mathbf{A}$ is strong right distributive if the following condition holds in A:

$$(x \to y) \to z \approx (z \to x) \to (y \to z). \qquad \text{(SRD)}$$

$\mathbf{A}$ is right distributive if $\mathbf{A}$ satisfies:

$$(x \to y) \to z \approx (x \to z) \to (y \to z). \qquad \text{(RD)}$$

Recall also that $\mathcal{SRD}$ and $\mathcal{RD}$ denote the variety of strong right distributive and right distributive implication zroupoids, respectively.

Lemma 5.1 *Let* $\mathbf{A} \in \mathcal{I}_{2,0} \cap \mathcal{SRD}$ *then* $\mathcal{A}$ *satisfies the following identities:*

(a) $0' \approx 0$,
(b) $x' \approx x$,
(c) $(x \to y) \to z \approx (x \to z) \to (y \to z)$.

Proof Let $a, b, c \in A$.

(a)

$$\begin{aligned} 0 &= 0'' \\ &= (0 \to 0) \to 0 \\ &= (0 \to 0) \to (0 \to 0) \quad \text{by (SRD)} \\ &= 0' \to 0' \\ &= 0' \quad \text{by Lemma 2.1 (a).} \end{aligned}$$

(b)

$$\begin{aligned} a &= a'' \\ &= (a \to 0) \to 0 \\ &= (0 \to a) \to (0 \to 0) \quad \text{by (SRD)} \\ &= (0 \to a) \to 0' \\ &= (0 \to a) \to 0 \quad \text{by (a)} \\ &= (0' \to a) \to 0 \quad \text{by (a)} \\ &= a \to 0 \quad \text{by Lemma 2.1 (a).} \end{aligned}$$

(c)

$$\begin{aligned} &(a \to b) \to c \\ &= (c \to a) \to (b \to c) \quad \text{by (SRD)} \\ &= (0 \to a) \to (b \to c) \quad \text{by Lemma 2.7 (u)} \\ &= (a' \to 0') \to (b \to c) \quad \text{by Lemma 2.2 (b)} \\ &= (a \to 0) \to (b \to c) \quad \text{by (b)} \\ &= [(b \to c) \to a] \to [0 \to (b \to c)] \quad \text{by (SRD)} \\ &= [(b \to c) \to a] \to [(b \to c)' \to 0'] \\ &\quad \text{by Lemma 2.2 (b)} \\ &= [(b \to c) \to a] \to [(b \to c) \to 0] \quad \text{by (b)} \\ &= [(b \to c) \to a] \to (b \to c) \quad \text{by (b)} \\ &= [(b \to c) \to a] \to [c' \to (b \to c)] \\ &\quad \text{by Lemma 2.7 (t)} \\ &= [(b \to c) \to a] \to [c \to (b \to c)] \quad \text{by (b)} \\ &= (a \to c) \to (b \to c) \quad \text{by (SRD).} \end{aligned}$$

□

The following Theorem is immediate from Theorem 4.2 and Lemma 5.1 (c) and the example that follows.

Theorem 5.2 $\mathcal{SRD} \subset \mathcal{RD}$.

The following example, as can be easily verified, is in $\mathcal{RD}$ but fails to satisfy (SRD) (at $x = a$, $y = 0$, $z = 0$).

→:	0	a	b
0	0	a	b
a	b	a	b
b	a	a	b

Recall that an implication zroupoid $\mathbf{A}$ is

- *commutative* if the following condition holds in A:

$$x \to y \approx y \to x, \qquad \text{(C)}$$

- *contrapositive* if the following condition holds in A:

$$x \to y' \approx y \to x', \qquad \text{(CP)}$$

The variety $\mathcal{CLD}$ is defined, relative to $\mathcal{I}$, by

$$x \to (y \to z) \approx (x \to z) \to (y \to x). \qquad \text{(CLD)}$$

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[$\mathcal{CLD}$ was formerly referred to as $\mathcal{SLD}$ in Sankappanavar (2012).]

Recall that $\mathcal{C}$ and $\mathcal{CP}$ denote the varieties of commutative and contrapositive implication zroupoids, respectively.

Lemma 5.3 *Let* $\mathbf{A} \in \mathcal{C}$ *then* $\mathbf{A}$ *satisfies the following identities:*

(a) $(x \to y) \to z \approx x \to (y \to z)$
(b) $x \to y' \approx y \to x'$
(c) $x \wedge y \approx y \wedge x$.

Proof Let $a, b \in A$.

(a) It follows from Cornejo and Sankappanavar (2016a, Theorem 8.2).

(b)

$$\begin{aligned} a \to b' &= a \to (b \to 0) \\ &= (a \to b) \to 0 \quad \text{by (a)} \\ &= (b \to a) \to 0 \quad \text{by the identity } (C) \\ &= b \to (a \to 0) \quad \text{by (a)} \\ &= b \to a'. \end{aligned}$$

(c)

$$\begin{aligned} a \wedge b &= (a \to b')' \\ &= (b \to a')' \quad \text{by (b)} \\ &= b \wedge a. \end{aligned}$$

□

Lemma 5.4 *Let* $\mathbf{A} \in \mathcal{I}_{2,0} \cap \mathcal{C}$ *then* $\mathbf{A}$ *satisfies the following identities:*

(a) $0' \approx 0$,
(b) $x' \approx x$,
(c) $(x \to y) \to z \approx (z \to x) \to (y \to z)$.

Proof Let $a, b, c \in A$.

(a)

$$\begin{aligned} 0 &= 0'' \\ &= (0 \to 0) \to 0 \\ &= 0 \to (0 \to 0) \quad \text{by (C)} \\ &= 0 \to 0' \\ &= 0' \to 0 \quad \text{by (C)} \\ &= 0' \quad \text{by Lemma 2.1 (a).} \end{aligned}$$

((b))

$$\begin{aligned} a &= a'' \\ &= (a \to 0) \to 0 \\ &= (a \to 0') \to 0 \quad \text{by (a)} \\ &= (0' \to a) \to 0 \quad \text{by (C)} \\ &= a \to 0 \quad \text{by Lemma 2.1 (a).} \end{aligned}$$

(c)

$$\begin{aligned} (a \to b) \to c &= [(c' \to a) \to (b \to c)']' \quad \text{by (I)} \\ &= (c \to a) \to (b \to c) \quad \text{by ((b)).} \end{aligned}$$

□

Theorem 5.5 $\mathcal{C} \subset \mathcal{CP} \cap \mathcal{A} \cap \mathcal{MC} \cap \mathcal{CLD}$.

Proof By Lemma 5.3 we have that $\mathcal{C} \subset \mathcal{CP} \cap \mathcal{A} \cap \mathcal{MC}$. Using Theorem 4.2 and Lemma 5.4, we have

$$\mathcal{C} \subset \mathcal{SRD}. \qquad (*)$$

Let $\mathbf{A} \in \mathcal{C}$ and $a, b, c \in A$. Hence,

$$\begin{aligned} &a \to (b \to c) \\ &= (b \to c) \to a \quad \text{by (C)} \\ &= (c \to b) \to a \quad \text{by (C)} \\ &= (a \to c) \to (b \to a) \quad \text{by } (*). \end{aligned}$$

Thus, $\mathcal{C} \subseteq \mathcal{CLD}$. The following 4-element $\mathcal{I}$-zroupoid shows that the inclusion in the previous statement is proper. □

→:	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	c	0	0
c	0	0	0	0

Theorem 5.6 $\mathcal{CLD} \subset \mathcal{SRD}$.

Proof Let $\mathbf{A} \in \mathcal{CLD} \cap \mathcal{I}_{2,0}$ and let $a, b, c \in A$. Using Lemma 2.1 (a) and (CLD), we get $0' = 0 \to 0 = 0 \to (0' \to 0) = (0 \to 0) \to (0' \to 0) = 0' \to 0 = 0$. Hence,

$$0' = 0. \qquad (5.1)$$

So, $a' = a \to 0 = a \to 0'$. Then by (5.1), (CLD) and Lemma 2.1 (d), we have $a' = a \to 0 = a \to 0' = (a \to 0) \to (0 \to a) = a' \to (0' \to a) = a' \to a = a$, thus $\mathbf{A}$ satisfies:

$$x' \approx x. \qquad (5.2)$$

Now, using (5.1) and (5.2), and Lemma 2.1 (a), and (CLD), we obtain $b \to a = 0' \to (b \to a) = 0 \to (b \to a) = 0 \to (b' \to a) = (0 \to a) \to (b' \to 0) = (0' \to a) \to b'' = a \to b$. Thus, the following identity is true in $\mathbf{A}$:

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$$x \to y \approx y \to x. \tag{5.3}$$

Hence, we have

$$\begin{aligned}(a \to b) \to c = c \to (a \to b) & \\ = (c \to b) \to (a \to c) & \quad \text{by (CLD)} \\ = (a \to c) \to (c \to b) & \quad \text{by (5.3)} \\ = (c \to a) \to (b \to c) & \quad \text{by (5.3).}\end{aligned}$$

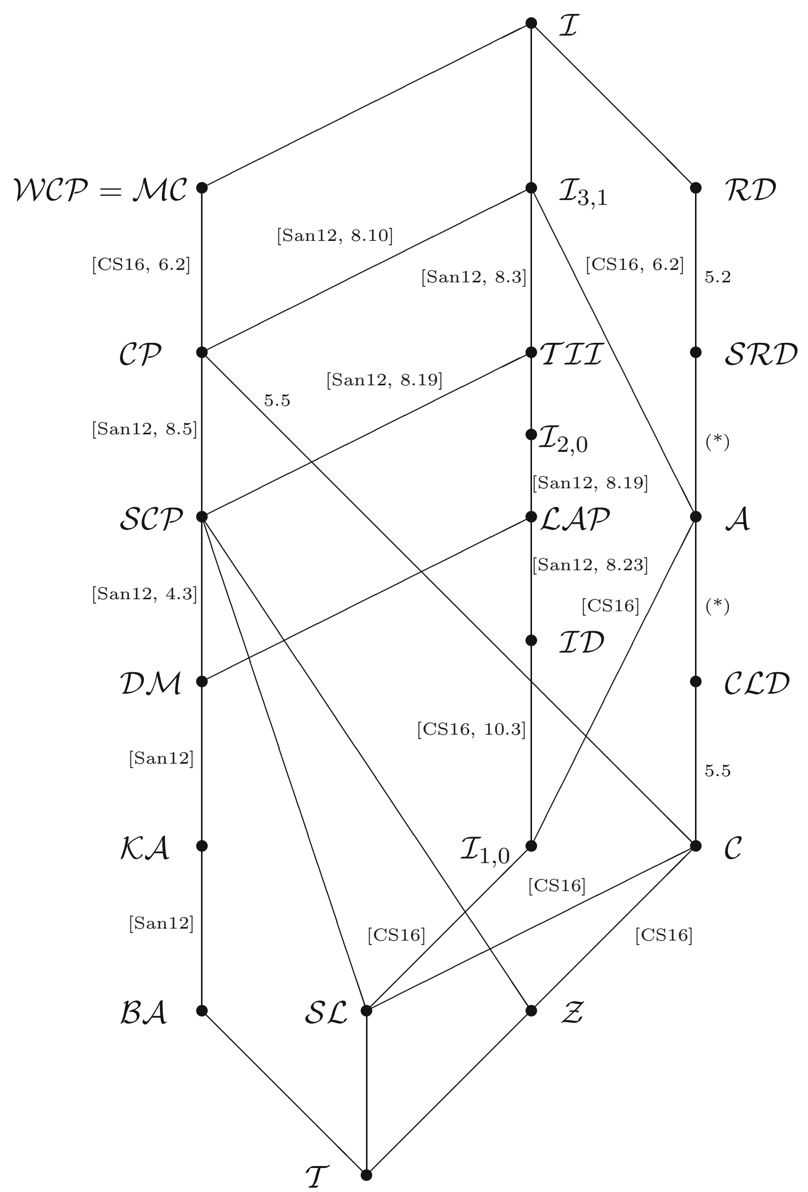
Thus, we have proved that if $\mathbf{A} \in \mathcal{CLD} \cap \mathcal{I}_{2,0}$, then $A \models (SRD)$. Now, apply Theorem 4.2 to finish off the proof. □

In view of Theorem 5.2 and Theorem 5.6 we have the following result.

Corollary 5.7 $\mathcal{CLD} \subset \mathcal{SRD} \subset \mathcal{RD}$.

The following picture describes the Hasse diagram of the poset of the subvarieties (known so far) of $\mathcal{I}$ under $\subseteq$. Each nonobvious link is augmented either by a reference (where it was first proved or where it is proved in this paper) or by the mark "(*)," in which case the proof will be presented in the forthcoming paper (Cornejo and Sankappanavar 2016e). The proof of the statement, $\mathcal{WCP} = \mathcal{MC}$, will also be presented in Cornejo and Sankappanavar (2016e). We note that T denotes the trivial variety.

POSET OF (KNOWN) SUBVARIETIES OF $\mathcal{I}$ under $\subseteq$



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Compliance with ethical standards

Conflict of interest Juan M. Cornejo declares that he has no conflict of interest. Hanamantagouda P. Sankappanavar declares that he has no conflict of interest.

Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

Appendix

Proof of Lemma 2.8 In the proofs below we sometimes use $\cdot$ for $\to$ for convenience.

Let $a, b, c, d, e \in A$.

(1)

$$(a \to b)' \to b$$
$$= [[b' \to (a \to b)] \to [0 \to b]']' \quad \text{by (I)}$$
$$= [a \to b] \to [0 \to b]']' \quad \text{by Lemma 2.7 (t)}$$
$$= (a \to b)'' \quad \text{by Lemma 2.7 (c)}$$
$$= a \to b.$$

(2) Since

$$[(a \to b') \to b] \to d$$
$$= [(a \to 0') \to b] \to d \quad \text{by Lemma 2.7 (a)}$$
$$= [(0 \to a') \to b] \to d \quad \text{by Lemma 2.2 (b)}$$
$$= [\{d' \to (0 \to a')\} \to (b \to d)']' \quad \text{by (I)}$$
$$= [\{0 \to (d' \to a')\} \to (b \to d)']'$$
by Lemma 2.7 (n)
$$= [\{(d' \to a')' \to 0'\} \to (b \to d)']'$$
by Lemma 2.2 (b)
$$= [\{(b \to d) \to (d' \to a')'\} \to \{0' \to (b \to d)'\}']''$$
by (I)
$$= [(b \to d) \to (d' \to a')'] \to [0' \to (b \to d)']'$$
$$= [(b \to d) \to (d' \to a')'] \to (b \to d)$$
by Lemma 2.1 (a) and $x'' \approx x$
$$= [0 \to (d' \to a')'] \to (b \to d) \quad \text{by Lemma 2.7 (f)}$$
$$= [d \to (0 \to a)] \to (b \to d) \quad \text{by Lemma 2.7 (m)}$$
$$= [\{(b \to d)' \to d\} \to [(0 \to a) \to (b \to d)]']' \quad \text{by (I)}$$
$$= [(b \to d) \to \{(0 \to a) \to (b \to d)\}']' \quad \text{by (1)}$$
$$= [(b \to d) \to (0 \to a)] \to (b \to d) \quad \text{by Lemma 2.7 (e)}$$
$$= [0 \to (0 \to a)] \to (b \to d) \quad \text{by Lemma 2.7 (f)}$$
$$= (0 \to a) \to (b \to d) \quad \text{by Lemma 2.7 (g),}$$

A satisfies

$$[(x \to y') \to y] \to u \approx (0 \to x) \to (y \to u). \tag{5.4}$$

Hence,

$$(0 \to b) \to (a' \to d)$$
$$= [(b \to a) \to a'] \to d \quad \text{by (5.4)}$$
$$= [a \to (b \to a)'] \to d \quad \text{by Lemma 2.7 (o).}$$

(3)

$$(a \to b) \to (b \to c)$$
$$= [\{(b \to c)' \to a\} \to \{b \to (b \to c)\}']'$$
$$= [\{(b \to c)' \to a\} \to (b \to c)']' \quad \text{by Lemma 2.7 (j)}$$
$$= [\{(b \to c)' \to a\} \to \{0' \to (b \to c)\}']'$$
by Lemma 2.1 (a)
$$= (a \to 0') \to (b \to c)$$
$$= (0 \to a') \to (b \to c) \quad \text{by Lemma 2.2 (b).}$$

(4) From

$$(0 \to b) \to (c \to d)'$$
$$= [(b \to c) \to (c \to d)]' \quad \text{by Lemma 2.7 (x)}$$
$$= [(0 \to b') \to (c \to d)]' \quad \text{by (3),}$$

we can conclude that A satisfies

$$(0 \to y) \to (z \to u)' \approx [(0 \to y') \to (z \to u)]'. \tag{5.5}$$

Hence,

$$[(a \cdot b) \cdot c] \cdot (c \cdot d)$$
$$= [0 \cdot (a \cdot b)'] \cdot (c \cdot d) \quad \text{by (3)}$$
$$= [a' \cdot (0 \cdot b')] \cdot (c \cdot d) \quad \text{by Lemma 2.7 (m)}$$
$$= [(0 \cdot a') \cdot (0 \cdot b')] \cdot (c \cdot d) \quad \text{by Lemma 2.7 (i)}$$
$$= [\{(c \cdot d)' \cdot (0 \cdot a')\} \cdot \{(0 \cdot b') \cdot (c \cdot d)\}']'$$
$$= [[\{(0 \cdot b') \cdot (c \cdot d)\} \cdot (c \cdot d)'] \cdot [(0 \cdot a') \cdot \{(0 \cdot b')$$
$$\cdot (c \cdot d)\}']']'' \quad \text{by (I)}$$
$$= [[(0 \cdot b') \cdot (c \cdot d)] \cdot (c \cdot d)'] \cdot [(0 \cdot a') \cdot \{(0 \cdot b')$$
$$\cdot (c \cdot d)\}']'$$
$$= [[(0 \cdot b') \cdot (c \cdot d)''] \cdot (c \cdot d)'] \cdot [(0 \cdot a') \cdot [(0 \cdot b')$$
$$\cdot (c \cdot d)]']'$$
$$= [[(0 \cdot b') \cdot 0'] \cdot (c \cdot d)'] \cdot [(0 \cdot a') \cdot [(0 \cdot b')$$
$$\cdot (c \cdot d)]']'$$
by Lemma 2.7 (a)

$$= [[(b \cdot 0') \cdot 0'] \cdot (c \cdot d)'] \cdot [(0 \cdot a') \cdot [(0 \cdot b') \cdot (c \cdot d)]']' \quad \text{by Lemma 2.2 (b)}$$
$$= [[(b \cdot 0'') \cdot 0'] \cdot (c \cdot d)'] \cdot [(0 \cdot a') \cdot [(0 \cdot b') \cdot (c \cdot d)]']' \quad \text{by Lemma 2.7 (a)}$$
$$= [[(b \cdot 0) \cdot 0'] \cdot (c \cdot d)'] \cdot [(0 \cdot a') \cdot [(0 \cdot b') \cdot (c \cdot d)]']'$$
$$= [[b' \cdot 0'] \cdot (c \cdot d)'] \cdot [(0 \cdot a') \cdot [(0 \cdot b') \cdot (c \cdot d)]']'$$
$$= [(0 \cdot b) \cdot (c \cdot d)'] \cdot [(0 \cdot a') \cdot [(0 \cdot b') \cdot (c \cdot d)]']' \quad \text{by Lemma 2.2 (b)}$$
$$= [(0 \cdot b') \cdot (c \cdot d)]' \cdot [(0 \cdot a') \cdot [(0 \cdot b') \cdot (c \cdot d)]']' \quad \text{by (5.5)}$$
$$= [[[(0 \cdot b') \cdot (c \cdot d)]' \cdot (0 \cdot a')] \cdot [(0 \cdot b') \cdot (c \cdot d)]']' \quad \text{by Lemma 2.7 (e)}$$
$$= [\{0 \cdot (0 \cdot a')\} \cdot \{(0 \cdot b') \cdot (c \cdot d)\}']' \quad \text{by Lemma 2.7 (f)}$$
$$= [(0 \cdot a') \cdot [(0 \cdot b') \cdot (c \cdot d)]']' \quad \text{by Lemma 2.7 (g)}$$
$$= [(0 \cdot a) \cdot \{(0 \cdot b') \cdot (c \cdot d)\}]'' \quad \text{by (5.5)}$$
$$= (0 \cdot a) \cdot [(0 \cdot b') \cdot (c \cdot d)]$$
$$= (0 \cdot a) \cdot [(b \cdot c) \cdot (c \cdot d)] \quad \text{by (3)}.$$

(5)

$$[b \to (0 \to c)] \to a$$
$$= [(a' \to b) \to \{(0 \to c) \to a\}']' \quad \text{by (I)}$$
$$= [(a' \to b) \to \{(a \to c) \to a\}']' \quad \text{by Lemma 2.7 (f)}$$
$$= [b \to (a \to c)] \to a \quad \text{by (I)}.$$

(6) Observe that A satisfies

$$[0 \to \{x \to (y \to z)\}] \to y \approx [x \to (y \to z)] \to y, \tag{5.6}$$

since

$$[0 \to \{a \to (b \to c)\}] \to b$$
$$= [a \to \{0 \to (b \to c)\}] \to b \quad \text{by Lemma 2.7 (n)}$$
$$= [(b' \to a) \to \{(0 \to (b \to c)) \to b\}']' \quad \text{by (I)}$$
$$= [(b' \to a) \to \{(b \to (b \to c)) \to b\}']' \quad \text{by Lemma 2.7 (f)}$$
$$= [a \to \{b \to (b \to c)\}] \to b \quad \text{by (I)}$$
$$= [a \to (b \to c)] \to b \quad \text{by Lemma 2.7 (j)}.$$

Then we have that

$$(0 \cdot a) \cdot [b \cdot (a \cdot c)]$$
$$= [\{b \cdot (a \cdot c)\} \cdot a] \cdot [b \cdot (a \cdot c)] \quad \text{by Lemma 2.7 (f)}$$
$$= [\{b \cdot (a \cdot c)\} \cdot a] \cdot [\{(b \cdot (a \cdot c)) \cdot a\} \cdot (b \cdot (a \cdot c))] \quad \text{by Lemma 2.7 (j)}$$
$$= ((b \cdot (a \cdot c)) \cdot a) \cdot [((b \cdot (0 \cdot c)) \cdot a) \cdot (b \cdot (a \cdot c))] \quad \text{by (5)}$$
$$= ([0 \cdot (b \cdot (a \cdot c))] \cdot a) \cdot [((b \cdot (0 \cdot c)) \cdot a) \cdot (b \cdot (a \cdot c))] \quad \text{by (5.6)}$$
$$= ([a \cdot (b \cdot (a \cdot c))] \cdot a) \cdot [((b \cdot (0 \cdot c)) \cdot a) \cdot (b \cdot (a \cdot c))] \quad \text{by Lemma 2.7 (f)}$$
$$= [\{a \cdot (b \cdot (a \cdot c))\} \cdot a] \cdot [((b \cdot (a \cdot c)) \cdot a) \cdot (b \cdot (a \cdot c))] \quad \text{by (5)}$$
$$= a \cdot [b \cdot (a \cdot c)] \quad \text{by Lemma 2.7 (s)}.$$

(7) The identity

$$((x \to (0 \to y)) \to z) \to (z \to (x \to y)) \approx z \to (x \to y) \tag{5.7}$$

follows from

$$((a \cdot (0 \cdot b)) \cdot c) \cdot (c \cdot (a \cdot b))$$
$$= (0 \cdot (a \cdot (0 \cdot b))') \cdot (c \cdot (a \cdot b)) \quad \text{by (3)}$$
$$= (0 \cdot (0 \cdot (a \cdot b))') \cdot (c \cdot (a \cdot b)) \quad \text{by Lemma 2.7 (n)}$$
$$= (0 \cdot (0' \cdot (a \cdot b)')) \cdot (c \cdot (a \cdot b)) \quad \text{by (m) and (n) of Lemma 2.7}$$
$$= (0 \cdot (a \cdot b)') \cdot (c \cdot (a \cdot b)) \quad \text{by Lemma 2.1 (a)}$$
$$= [[(c \cdot (a \cdot b))' \cdot 0] \cdot [(a \cdot b)' \cdot (c \cdot (a \cdot b))]']' \quad \text{by (I)}$$
$$= [(c \cdot (a \cdot b)) \cdot [(a \cdot b)' \cdot (c \cdot (a \cdot b))]']'$$
$$= [(c \cdot (a \cdot b)) \cdot (c \cdot (a \cdot b))']' \quad \text{by Lemma 2.7 (t)}$$
$$= (c \cdot (a \cdot b))'' \quad \text{by Lemma 2.1 (d)}$$
$$= c \cdot (a \cdot b).$$

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Hence, we have

$$
\begin{aligned}
&c \to (a \to b)\\
&\quad = ((a \to (0 \to b)) \to c) \to (c \to (a \to b))\\
&\qquad \text{by (5.7)}\\
&\quad = (0 \to a) \to [((0 \to b) \to c) \to (c \to (a \to b))]\\
&\qquad \text{by (4)}\\
&\quad = (0 \to a) \to [(0 \to 0) \to [(b \to c)\\
&\qquad \to (c \to (a \to b))]] \quad \text{by (4)}\\
&\quad = (0 \to a) \to [(b \to c) \to (c \to (a \to b))]\\
&\qquad \text{by Lemma 2.1. (a).}
\end{aligned}
$$

We, therefore, can conclude that the algebra **A** satisfies

$$
\begin{aligned}
(0 \to x) \ \to ((y \to z) \to (z \to (x \to y)))&\\
\approx z \to (x \to y).& \qquad (5.8)
\end{aligned}
$$

Also, from

$$
\begin{aligned}
&(a \to b) \to (b \to (c \to a))\\
&\quad = (0 \to 0) \to [(a \to b) \to (b \to (c \to a))]\\
&\qquad \text{by Lemma 2.1 (a)}\\
&\quad = [(0 \to a) \to b] \to (b \to (c \to a)) \quad \text{by (4)}\\
&\quad = [(0 \to a) \to b] \to (((0 \to a) \to b) \to (c \to a))\\
&\qquad \text{by Lemma 2.7 (w)}\\
&\quad = ((0 \to a) \to b) \to (c \to a)\\
&\qquad \text{by Lemma 2.7 (j)}\\
&\quad = b \to (c \to a) \quad \text{by Lemma 2.7 (w),}
\end{aligned}
$$

we see that A satisfies

$$(x \to y) \to (y \to (z \to x)) \approx y \to (z \to x). \qquad (5.9)$$

Consequently,

$$
\begin{aligned}
&c \to (a \to b)\\
&\quad = (0 \to a) \to ((b \to c) \to (c \to (a \to b)))\\
&\qquad \text{by (5.8)}\\
&\quad = (0 \to a) \to (c \to (a \to b)) \quad \text{by (5.9)}\\
&\quad = a \to (c \to (a \to b)) \quad \text{by (6).}
\end{aligned}
$$

(8)

$$
\begin{aligned}
&(a \to b) \to (0 \to b')\\
&\quad = [[(0 \to b')' \to a] \to [b \to (0 \to b')]']' \quad \text{by (I)}\\
&\quad = [[(0 \to b')' \to a] \to (0 \to b')']' \quad \text{by Lemma 2.7 (t)}\\
&\quad = [[0 \to a] \to (0 \to b')']' \quad \text{by Lemma 2.7 (f)}\\
&\quad = [[0 \to a] \to (b \to 0')']' \quad \text{by Lemma 2.2 (b)}\\
&\quad = (a \to b) \to 0'.
\end{aligned}
$$

(9)

$$
\begin{aligned}
&b \to (0 \to a)'\\
&\quad = (0' \to b) \to (0 \to a)' \quad \text{by Lemma 2.1 (a)}\\
&\quad = [[(0 \to a) \to 0'] \to [b \to (0 \to a)']']'\\
&\quad = [[0 \to (0 \to a)'] \to [b \to (0 \to a)']']'\\
&\qquad \text{by Lemma 2.2 (b)}\\
&\quad = [[0 \to a'] \to [b \to (0 \to a)']']' \quad \text{by Lemma 2.7 (l)}\\
&\quad = [[0 \to (a \to a')] \to [b \to (0 \to a)']']'\\
&\qquad \text{by Lemma 2.1 (d)}\\
&\quad = [[(0 \to a) \to (0 \to a')] \to [b \to (0 \to a)']']'\\
&\qquad \text{by Lemma 2.7 (i) and by Lemma 2.7 (n)}\\
&\quad = [(0 \to a') \to b] \to (0 \to a)' \quad \text{by (I)}\\
&\quad = [(0 \to a') \to b] \to (0 \to (0 \to a')')'\\
&\qquad \text{by Lemma 2.7 (l)}\\
&\quad = [[(0 \to a') \to b] \to (0 \to (0 \to a')')']''\\
&\quad = [b' \to (0 \to a')']' \quad \text{by (I).}
\end{aligned}
$$

(10)

$$
\begin{aligned}
&b \to a'\\
&\quad = b'' \to a'\\
&\quad = (b' \to 0) \to a'\\
&\quad = [(a \to b') \to (0 \to a')']' \quad \text{by (I)}\\
&\quad = [[(0 \to a') \to a] \to [b' \to (0 \to a')']']'' \quad \text{by (I)}\\
&\quad = [(0 \to a') \to a] \to [b' \to (0 \to a')']'\\
&\quad = [(0 \to a') \to a] \to [b \to (0 \to a)'] \quad \text{by (9)}\\
&\quad = [(a \to a') \to a] \to [b \to (0 \to a)']\\
&\qquad \text{by Lemma 2.7 (f)}\\
&\quad = [a' \to a] \to [b \to (0 \to a)'] \quad \text{by Lemma 2.1 (d)}\\
&\quad = a \to [b \to (0 \to a)'] \quad \text{by Lemma 2.1 (d).}
\end{aligned}
$$

(11) Since

$$
\begin{aligned}
&((0 \to b') \to c) \to (d \to b)'\\
&\quad = [[(d \to b) \to (0 \to b')] \to [c \to (d \to b)']']'\\
&\quad = [[(d \to b) \to 0'] \to [c \to (d \to b)']']' \quad \text{by (8)}\\
&\quad = (0' \to c) \to (d \to b)' \quad \text{by (I)}\\
&\quad = c \to (d \to b)' \quad \text{by Lemma 2.1 (a),}
\end{aligned}
$$

we have that the following identity holds in **A**:

$$((0 \to y') \to z) \to (u \to y)' \approx z \to (u \to y)'. \qquad (5.10)$$

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Then

$$\begin{aligned}
&[(a \cdot ((0 \cdot b') \cdot c)) \cdot d] \cdot b \\
&\quad = [[b' \cdot (a \cdot ((0 \cdot b') \cdot c))] \cdot [d \cdot b]']' \quad \text{by (I)} \\
&\quad = [[[d \cdot b] \cdot b'] \cdot [(a \cdot ((0 \cdot b') \cdot c)) \cdot [d \cdot b]]']'' \quad \text{by (I)} \\
&\quad = [[d \cdot b] \cdot b'] \cdot [(a \cdot ((0 \cdot b') \cdot c)) \cdot [d \cdot b]]' \\
&\quad = [[d \cdot b] \cdot b'] \cdot [[[d \cdot b]' \cdot a] \cdot [((0 \cdot b') \cdot c) \\
&\qquad \cdot [d \cdot b]]']'' \quad \text{by (I)} \\
&\quad = [[d \cdot b] \cdot b'] \cdot [[[d \cdot b]' \cdot a] \cdot [((0 \cdot b') \cdot c) \cdot [d \cdot b]]'] \\
&\quad = [[d \cdot b] \cdot b'] \cdot [[[d \cdot b]' \cdot a] \cdot [c \cdot (d \cdot b)']'] \\
&\qquad \text{by (5.10)} \\
&\quad = [[d \cdot b] \cdot b'] \cdot [[[d \cdot b]' \cdot a] \cdot [c \cdot (d \cdot b)']']'' \\
&\quad = [[d \cdot b] \cdot b'] \cdot [(a \cdot c) \cdot (d \cdot b)']' \quad \text{by (I)} \\
&\quad = [[[d \cdot b] \cdot b'] \cdot [(a \cdot c) \cdot (d \cdot b)']']'' \\
&\quad = [[b' \cdot (a \cdot c)] \cdot (d \cdot b)']' \quad \text{by (I)} \\
&\quad = ((a \cdot c) \cdot d) \cdot b \quad \text{by (I);}
\end{aligned}$$

and, consequently, A satisfies

$$\begin{aligned}
&[(x \to ((0 \to y') \to z)) \to u] \to y \\
&\quad \approx [(x \to z) \to u] \to y. \qquad (5.11)
\end{aligned}$$

Notice that

$$\begin{aligned}
&[(a' \to b) \to c] \to a \\
&\quad = [[a' \to (a' \to b)] \to [c \to a]']' \quad \text{by (I)} \\
&\quad = [[a' \to b] \to [c \to a]']' \quad \text{by Lemma 2.7 (j)} \\
&\quad = (b \to c) \to a.
\end{aligned}$$

So, the identity

$$[(x' \to y) \to z] \to x \approx (y \to z) \to x \qquad (5.12)$$

holds in A. Hence,

$$\begin{aligned}
&(a' \to b) \to c \\
&\quad = ((a \to 0) \to b) \to c \\
&\quad = [[a \to (0 \to c')'] \to b] \to c \quad \text{by (5.11)} \\
&\quad = [[c' \to [a \to (0 \to c')']] \to b] \to c \quad \text{by (5.12)} \\
&\quad = [(a \to c'') \to b] \to c \quad \text{by (10)} \\
&\quad = [(a \to c) \to b] \to c.
\end{aligned}$$

(12)

$$\begin{aligned}
&(a' \to b) \to (a \to c) \\
&\quad = [(a \to (a \to c)) \to b] \to (a \to c) \quad \text{by (11)} \\
&\quad = [(a \to c) \to b] \to (a \to c) \quad \text{by Lemma 2.7 (j)} \\
&\quad = (0 \to b) \to (a \to c) \quad \text{by Lemma 2.7 (f).}
\end{aligned}$$

(13)

$$\begin{aligned}
&a \cdot ((0 \cdot a) \cdot b) \\
&\quad = [[a \cdot ((0 \cdot a) \cdot b)] \cdot a] \cdot [[((0 \cdot a) \cdot b) \cdot a] \\
&\qquad \cdot ((0 \cdot a) \cdot b)] \quad \text{by Lemma 2.7 (s)} \\
&\quad = [[0 \cdot ((0 \cdot a) \cdot b)] \cdot a] \cdot [[((0 \cdot a) \cdot b) \cdot a] \\
&\qquad \cdot ((0 \cdot a) \cdot b)] \quad \text{by Lemma 2.7 (f)} \\
&\quad = [[(0 \cdot a) \cdot (0 \cdot b)] \cdot a] \cdot [[((0 \cdot a) \cdot b) \cdot a] \\
&\qquad \cdot ((0 \cdot a) \cdot b)] \quad \text{by Lemma 2.7 (n)} \\
&\quad = [[a \cdot (0 \cdot b)] \cdot a] \cdot [[((0 \cdot a) \cdot b) \cdot a] \\
&\qquad \cdot ((0 \cdot a) \cdot b)] \quad \text{by Lemma 2.7 (i)} \\
&\quad = [[0 \cdot (a \cdot b)] \cdot a] \cdot [[((0 \cdot a) \cdot b) \cdot a] \\
&\qquad \cdot ((0 \cdot a) \cdot b)] \quad \text{by Lemma 2.7 (n)} \\
&\quad = [[a \cdot (a \cdot b)] \cdot a] \cdot [[((0 \cdot a) \cdot b) \cdot a] \\
&\qquad \cdot ((0 \cdot a) \cdot b)] \quad \text{by Lemma 2.7 (f)} \\
&\quad = [[a \cdot b] \cdot a] \cdot [[((0 \cdot a) \cdot b) \cdot a] \\
&\qquad \cdot ((0 \cdot a) \cdot b)] \quad \text{by Lemma 2.7 (j)} \\
&\quad = [[a \cdot b] \cdot a] \cdot [[0 \cdot a] \\
&\qquad \cdot ((0 \cdot a) \cdot b)] \quad \text{by Lemma 2.7 (f)} \\
&\quad = [[a \cdot b] \cdot a] \cdot ((0 \cdot a) \cdot b) \quad \text{by Lemma 2.7 (j)} \\
&\quad = [[a \cdot b] \cdot a] \cdot ((b \cdot a) \cdot b) \quad \text{by Lemma 2.7 (f)} \\
&\quad = a \cdot b \quad \text{by Lemma 2.7 (s).}
\end{aligned}$$

(14)

$$\begin{aligned}
&(a \cdot 0') \cdot (b \cdot c) \\
&\quad = (0 \cdot a') \cdot (b \cdot c) \quad \text{by Lemma 2.2 (b)} \\
&\quad = [(b \cdot c) \cdot a'] \cdot (b \cdot c) \quad \text{by Lemma 2.7 (f)} \\
&\quad = [(b \cdot c) \cdot ((b \cdot c) \cdot a')] \cdot (b \cdot c) \quad \text{by Lemma 2.7 (j)} \\
&\quad = [(b \cdot c) \cdot ((b \cdot c) \cdot a')] \cdot (b \cdot c)'' \\
&\quad = [(b \cdot c) \cdot ((b \cdot c) \cdot a')] \cdot [0' \cdot (b \cdot c)']' \\
&\qquad \text{by Lemma 2.1 (a)} \\
&\quad = [[(b \cdot c) \cdot ((b \cdot c) \cdot a')] \cdot [0' \cdot (b \cdot c)']']'' \\
&\quad = [[((b \cdot c) \cdot a') \cdot 0'] \cdot (b \cdot c)']' \quad \text{by (I)} \\
&\quad = [[0 \cdot ((b \cdot c) \cdot a')'] \cdot (b \cdot c)']' \quad \text{by Lemma 2.2 (b)} \\
&\quad = [[0 \cdot ((b \cdot c)' \cdot a)] \cdot (b \cdot c)']' \\
&\qquad \text{by (m) and (n) of Lemma 2.7} \\
&\quad = [[(0 \cdot (b \cdot c)') \cdot (0 \cdot a)] \cdot (b \cdot c)']' \\
&\qquad \text{by Lemma 2.7 (n) and by Lemma 2.7 (i)} \\
&\quad = [[(b \cdot c) \cdot (0 \cdot (b \cdot c)')] \cdot [(0 \cdot a) \cdot (b \cdot c)']']'' \\
&\qquad \text{by (I)} \\
&\quad = [(b \cdot c) \cdot (0 \cdot (b \cdot c)')] \cdot [(0 \cdot a) \cdot (b \cdot c)']' \\
&\quad = [(b \cdot c) \cdot ((b \cdot c) \cdot 0')] \cdot [(0 \cdot a) \cdot (b \cdot c)']' \\
&\qquad \text{by Lemma 2.2 (b)}
\end{aligned}$$

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$$
\begin{aligned}
&= [(b \cdot c) \cdot ((b \cdot c) \cdot (0 \cdot c'))] \cdot [(0 \cdot a) \cdot (b \cdot c)']' \\
&\quad \text{by (8)} \\
&= [(b \cdot c) \cdot (0 \cdot c')] \cdot [(0 \cdot a) \cdot (b \cdot c)']' \\
&\quad \text{by Lemma 2.7 (j)} \\
&= [[(b \cdot c) \cdot (0 \cdot c')] \cdot [(0 \cdot a) \cdot (b \cdot c)']']'' \\
&= [[(0 \cdot c') \cdot (0 \cdot a)] \cdot (b \cdot c)']' \quad \text{by (I)} \\
&= [\{c' \cdot (0 \cdot a)\} \cdot (b \cdot c)']' \quad \text{by Lemma 2.7 (i)} \\
&= [(0 \cdot a) \cdot b] \cdot c \quad \text{by (I).}
\end{aligned}
$$

(15)

$$
\begin{aligned}
&((a \to b) \to (c \to a)) \to d \\
&\quad = [[(c \to a)' \to a] \to [b \to (c \to a)']']' \to d \\
&\quad\quad \text{by (I)} \\
&\quad = [(c \to a) \to [b \to (c \to a)']']' \to d \quad \text{by (1)} \\
&\quad = [[(c \to a) \to b] \to (c \to a)] \to d \\
&\quad\quad \text{by Lemma 2.7 (e)} \\
&\quad = [[0 \to b] \to (c \to a)] \to d \quad \text{by Lemma 2.7 (f)} \\
&\quad = (b \to 0') \to [(c \to a) \to d] \quad \text{by (14).}
\end{aligned}
$$

(16)

$$
\begin{aligned}
&(0 \to a) \to ((b \to a) \to c) \\
&\quad = (a' \to 0') \to ((b \to a) \to c) \quad \text{by Lemma 2.2 (b)} \\
&\quad = [(0 \to a') \to (b \to a)] \to c \quad \text{by (14)} \\
&\quad = [(a \to a') \to (b \to a)] \to c \quad \text{by Lemma 2.7 (u)} \\
&\quad = [a' \to (b \to a)] \to c \quad \text{by Lemma 2.1 (d)} \\
&\quad = (b \to a) \to c \quad \text{by Lemma 2.7 (t).}
\end{aligned}
$$

(17) Since

$$
\begin{aligned}
&[0 \to ((a \to b) \to c)] \to e \\
&\quad = [(a \to b) \to (0 \to c)] \to e \quad \text{by Lemma 2.7 (n)} \\
&\quad = [(a \to b) \to (c' \to 0')] \to e \quad \text{by Lemma 2.2 (a)} \\
&\quad = [(0' \to (a \to b)) \to (c' \to 0')] \to e \\
&\quad\quad \text{by Lemma 2.1 (a)} \\
&\quad = [(a \to b) \to 0'] \to [(c' \to 0') \to e] \quad \text{by (15)} \\
&\quad = [(a \to b) \to 0'] \to [(0 \to c) \to e] \\
&\quad\quad \text{by Lemma 2.2 (b)} \\
&\quad = [0 \to (a \to b)'] \to [(0 \to c) \to e] \\
&\quad\quad \text{by Lemma 2.2 (b),}
\end{aligned}
$$

we see that **A** satisfies

$$
\begin{aligned}
&[0 \to ((x \to y) \to z)] \to t \\
&\quad \approx [0 \to (x \to y)'] \to [(0 \to z) \to t]. \qquad (5.13)
\end{aligned}
$$

Also,

$$
\begin{aligned}
&(0 \to a) \to [(b \to 0') \to e] \\
&\quad = (a' \to 0') \to [(b \to 0') \to e] \quad \text{by Lemma 2.2 (b)} \\
&\quad = [(0' \to a') \to (b \to 0')] \to e \quad \text{by (15)} \\
&\quad = [a' \to (b \to 0')] \to e \quad \text{by Lemma 2.1 (a)} \\
&\quad = [a' \to (0 \to b')]'' \to e \\
&\quad = [(0 \to a) \to (0 \to b')']' \to e \quad \text{by Lemma 2.7 (k)} \\
&\quad = [(0 \to a) \to (b \to 0')']' \to e \\
&\quad = [(a \to b) \to 0'] \to e \quad \text{by (I)} \\
&\quad = [0 \to (a \to b)'] \to e \quad \text{by Lemma 2.2 (b).}
\end{aligned}
$$

Hence, the identity

$$
(0 \to x) \to [(y \to 0') \to t] \approx [0 \to (x \to y)'] \to t \qquad (5.14)
$$

holds in **A**. Therefore,

$$
\begin{aligned}
&(0 \to a) \to [(b \to 0') \to [(0 \to c) \\
&\quad \to ((d \to a) \to b)]] \\
&\quad = [0 \to (a \to b)'] \to [(0 \to c) \to ((d \to a) \to b)] \\
&\quad\quad \text{by (5.14) with } t = (0 \to c) \to ((d \to a) \to b) \\
&\quad = [0 \to ((a \to b) \to c)] \to ((d \to a) \to b) \\
&\quad\quad \text{by (5.13) with } t = (d \to a) \to b.
\end{aligned}
$$

(18) Since

$$
\begin{aligned}
&[(a \to b) \to b')] \to b \\
&\quad = [(a \to 0') \to b')] \to b \quad \text{by Lemma 2.7 (a)} \\
&\quad = [(a \to 0') \to 0')] \to b \quad \text{by Lemma 2.7 (a)} \\
&\quad = [(a \to 0) \to 0')] \to b \quad \text{by Lemma 2.7 (a)} \\
&\quad = [a' \to 0')] \to b \\
&\quad = (0 \to a) \to b \quad \text{by Lemma 2.2 (b),}
\end{aligned}
$$

A satisfies

$$
[(x \to y) \to y')] \to y \approx (0 \to x) \to y. \qquad (5.15)
$$

Also, the identity

$$
y' \to [(x \to y) \to 0']' \approx (0 \to x) \to y \qquad (5.16)
$$

holds in **A**, because

$$
\begin{aligned}
&b' \to [(a \to b) \to 0']' \\
&\quad = [[\{(a \to b) \to 0'\} \to b] \\
&\quad\quad \to [0 \to \{(a \to b) \to 0'\}']']' \quad \text{by (I)}
\end{aligned}
$$

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$$
\begin{aligned}
&= [[\{(a \to b) \to b'\} \to b] \\
&\quad \to [0 \to \{(a \to b) \to 0'\}']]' \quad \text{by Lemma 2.7 (a)} \\
&= [\{(0 \to a) \to b\} \\
&\quad \to [0 \to \{(a \to b) \to 0'\}']]' \quad \text{by (5.15)} \\
&= [\{(0 \to a) \to b\} \\
&\quad \to [0 \to \{0 \to (a \to b)'\}']]' \quad \text{by Lemma 2.2 (b)} \\
&= [\{(0 \to a) \to b\} \\
&\quad \to \{0 \to (a \to b)''\}']' \quad \text{by Lemma 2.7 (l)} \\
&= [\{(0 \to a) \to b\} \\
&\quad \to \{0 \to (a \to b)\}']' \\
&= [\{(0 \to a) \to b\} \\
&\quad \to \{a \to (0 \to b)\}']' \quad \text{by Lemma 2.7 (n)} \\
&= [\{(0 \to a) \to b\} \\
&\quad \to \{0 \to ((0 \to a) \to b)\}']' \quad \text{by Lemma 2.7 (p)} \\
&= [(0 \to a) \to b]'' \quad \text{by Lemma 2.7 (b).} \\
&= (0 \to a) \to b.
\end{aligned}
$$

Then

$$
\begin{aligned}
&(0 \cdot (a \cdot b)) \cdot [c \cdot [b' \cdot [(0 \cdot a) \cdot (b \cdot 0')']]]' \\
&= (0 \cdot (a \cdot b)) \cdot [c \cdot [b' \cdot [(0 \cdot a) \cdot (b \cdot 0')']'']]' \\
&= (0 \cdot (a \cdot b)) \cdot [c \cdot [b' \cdot [(a \cdot b) \cdot 0']']]' \quad \text{by (I)} \\
&= (0 \cdot (a \cdot b)) \cdot [c \cdot [(0 \cdot a) \cdot b]]' \quad \text{by (5.16)} \\
&= (a \cdot (0 \cdot b)) \cdot [c \cdot [(0 \cdot a) \cdot b]]' \quad \text{by Lemma 2.7 (n)} \\
&= (0 \cdot ((0 \cdot a) \cdot b)) \cdot [c \cdot [(0 \cdot a) \cdot b]]' \\
&\quad \text{by Lemma 2.7 (p)} \\
&= (((0 \cdot a) \cdot b)' \cdot 0') \cdot [c \cdot [(0 \cdot a) \cdot b]]' \\
&\quad \text{by Lemma 2.2 (b)} \\
&= [(((0 \cdot a) \cdot b)' \cdot 0') \cdot [c \cdot [(0 \cdot a) \cdot b]]']'' \\
&= [(0' \cdot c) \cdot ((0 \cdot a) \cdot b)]' \quad \text{by (I)} \\
&= [c \cdot ((0 \cdot a) \cdot b)]' \quad \text{by Lemma 2.1 (a).}
\end{aligned}
$$

Hence, **A** satisfies

$$
\begin{aligned}
&(0 \to (x \to y)) \to [z \to [y' \to [(0 \to x) \to (y \to 0')']]]' \\
&\approx [z \to ((0 \to x) \to y)]'. \qquad (5.17)
\end{aligned}
$$

Since

$$
\begin{aligned}
&(0 \cdot d) \cdot ((0 \cdot (a \cdot b)) \cdot (c \cdot d)') \\
&= (0 \cdot d) \cdot [0' \cdot ((0 \cdot (a \cdot b)) \cdot (c \cdot d)')] \\
&\quad \text{by Lemma 2.1 (a)} \\
&= (0 \cdot d) \cdot [(0 \cdot 0') \cdot ((0 \cdot (a \cdot b)) \cdot (c \cdot d)')] \\
&\quad \text{by Lemma 2.1 (d)} \\
&= [0 \cdot ((d \cdot 0) \cdot (a \cdot b))] \cdot (c \cdot d)' \quad \text{by (17)} \\
&= [0 \cdot (d' \cdot (a \cdot b))] \cdot (c \cdot d)' \\
&= [d' \cdot (0 \cdot (a \cdot b))] \cdot (c \cdot d)' \quad \text{by Lemma 2.7 (n)} \\
&= [[d' \cdot (0 \cdot (a \cdot b))] \cdot (c \cdot d)']'' \\
&= [[(0 \cdot (a \cdot b)) \cdot c] \cdot d]' \quad \text{by (I)} \\
&= [[(a \cdot (0 \cdot b)) \cdot c] \cdot d]' \quad \text{by Lemma 2.7 (n),}
\end{aligned}
$$

we can conclude that

$$
\begin{aligned}
&(((x \to (0 \to y)) \to z) \to u)' \\
&\approx (0 \to u) \to ((0 \to (x \to y)) \to (z \to u)') \qquad (5.18)
\end{aligned}
$$

is valid in the algebra. Also, the identity

$$
(0 \to x) \to (y \to (z \to x)') \approx y \to (z \to x)' \qquad (5.19)
$$

is valid in **A**, since

$$
\begin{aligned}
&b \to (c \to a)' \\
&= (c \to a) \to [b \to (c \to a)'] \quad \text{by Lemma 2.7 (t)} \\
&= (0 \to a) \to [(c \to a) \to [b \to (c \to a)']] \\
&\quad \text{by (16)} \\
&= (0 \to a) \to [b \to (c \to a)'] \quad \text{by Lemma 2.7 (t).}
\end{aligned}
$$

Hence, from (5.18) and (5.19), it follows that **A** satisfies

$$
\begin{aligned}
&(((x \to (0 \to y)) \to z) \to u)' \\
&\approx (0 \to (x \to y)) \to (z \to u)'. \qquad (5.20)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
&a' \to ((0 \to b) \to (c \to 0')') \\
&= a' \to ((0 \to b) \to (c \to 0')')'' \\
&= a' \to [(b \to c) \to 0']' \\
&= a' \to [0 \to (b \to c)']' \quad \text{by Lemma 2.2 (b)} \\
&= [a \to (0 \to (b \to c))']' \quad \text{by (9)} \\
&= [a \to (b \to (0 \to c))']' \quad \text{by Lemma 2.7 (n).}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&(x \to (y \to (0 \to z))')' \\
&\approx x' \to ((0 \to y) \to (z \to 0')') \qquad (5.21)
\end{aligned}
$$

holds in **A**. Since

$$
\begin{aligned}
&(c \cdot d) \cdot (a \cdot (0 \cdot b))' \\
&= [[(a \cdot (0 \cdot b))'' \cdot c] \cdot [d \cdot (a \cdot (0 \cdot b))']']' \quad \text{by (I)} \\
&= [[(a \cdot (0 \cdot b)) \cdot c] \cdot [d \cdot (a \cdot (0 \cdot b))']']' \\
&= (0 \cdot (a \cdot b)) \cdot [c \cdot [d \cdot (a \cdot (0 \cdot b))']']' \\
&\quad \text{by (5.20) with } u = [d \cdot (a \cdot (0 \cdot b))']'
\end{aligned}
$$

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$$= (0 \cdot (a \cdot b)) \cdot [c \cdot [d' \cdot [(0 \cdot a) \cdot (b \cdot 0')']]]'$$
by (5.21),

the identity

$$(0 \to (x \to y)) \to [z \to [u' \to [(0 \to x) \to (y \to 0')']]]' \approx (z \to u) \to (x \to (0 \to y))' \tag{5.22}$$

is true in **A**. Hence, from (5.17) and (5.22) the conclusion holds.

(19)

$$(0 \to a) \to [\{0 \to (b \to c)\} \to d]$$
$$= (a' \to 0') \to [(0 \to (b \to c)) \to d]$$
by Lemma 2.2 (b)
$$= [(0 \to a') \to (0 \to (b \to c))] \to d \quad \text{by (14)}$$
$$= [0 \to (a' \to (b \to c))] \to d$$
by (i) and (n) of Lemma 2.7
$$= [0 \to [(a' \to b')' \to c]] \to d \quad \text{by Lemma 2.7 (v)}$$
$$= [(0 \to (a' \to b')') \to (0 \to c)] \to d$$
by (n) and (i) of Lemma 2.7
$$= [(0 \to (a \to b)) \to (0 \to c)] \to d$$
by Lemma 2.7 (q)
$$= (0 \to ((a \to b) \to c)) \to d$$
by (i) and (n) of Lemma 2.7.

(20)

$$[a \to ((0 \to b) \to c)] \to b$$
$$= [(b' \to a) \to [\{(0 \to b) \to c\} \to b]']' \quad \text{by (I)}$$
$$= [(b' \to a) \to [c \to b]']' \quad \text{by Lemma 2.7 (d)}$$
$$= (a \to c) \to b \quad \text{by (I).}$$

(21) Notice that

$$[\{x \to (0 \to y)\} \to z] \to (x \to y)$$
$$\approx [\{0 \to (x \to y)\} \to z] \to (x \to y)$$
by Lemma 2.7 (n)
$$\approx z \to (x \to y) \quad \text{by Lemma 2.7 (d).} \tag{5.23}$$

Also, we have that

$$0 \to [(x \to (y \to z)) \to u] \approx 0 \to [x' \to \{(y \to z) \to u\}], \tag{5.24}$$

since

$$0 \to [a' \to ((b \to c) \to d)]$$
$$= 0 \to [(a' \to (b \to c)')' \to d]$$
by Lemma 2.7 (v)
$$= [0 \to (a' \to (b \to c)')'] \to (0 \to d)$$
by (n) and (i) of Lemma 2.7
$$= [0 \to (a \to (b \to c))] \to (0 \to d)$$
by Lemma 2.7 (q)
$$= 0 \to [\{a \to (b \to c)\} \to d]$$
by (n) and (i) of Lemma 2.7.

Observe that

$$0 \to [a \to \{(0 \to b) \to c\}]$$
$$= a \to [0 \to \{(0 \to b) \to c\}] \quad \text{by Lemma 2.7 (n)}$$
$$= a \to [(0 \to b) \to (0 \to c)] \quad \text{by Lemma 2.7 (n)}$$
$$= a \to [0 \to (b \to c)]$$
by (n) and (i) of Lemma 2.7
$$= a \to [b \to (0 \to c)] \quad \text{by Lemma 2.7 (n).}$$

Hence, **A** satisfies

$$0 \to [x \to \{(0 \to y) \to z\}] \approx x \to [y \to (0 \to z)]. \tag{5.25}$$

Therefore, we have

$$u \cdot [(a \cdot b) \cdot c]$$
$$= u \cdot [\{a \cdot ((0 \cdot c) \cdot b)\} \cdot c] \quad \text{by (20)}$$
$$= [((a \cdot ((0 \cdot c) \cdot b)) \cdot (0 \cdot c)) \cdot u] \cdot [(a \cdot ((0 \cdot c) \cdot b)) \cdot c] \quad \text{by (5.23)}$$
$$= (((a \cdot (0 \cdot b)) \cdot (0 \cdot c)) \cdot u) \cdot ((a \cdot ((0 \cdot c) \cdot b)) \cdot c) \quad \text{by (5)}$$
$$= ((0 \cdot ((a \cdot (0 \cdot b)) \cdot c)) \cdot u) \cdot ((a \cdot ((0 \cdot c) \cdot b)) \cdot c) \quad \text{by Lemma 2.7 (n)}$$
$$= ((0 \cdot (a' \cdot ((0 \cdot b) \cdot c))) \cdot u) \cdot ((a \cdot ((0 \cdot c) \cdot b)) \cdot c) \quad \text{by (5.24)}$$
$$= ((a' \cdot (b \cdot (0 \cdot c))) \cdot u) \cdot ((a \cdot ((0 \cdot c) \cdot b)) \cdot c) \quad \text{by (5.25)}$$
$$= ((0 \cdot (a' \cdot (b \cdot c))) \cdot u) \cdot ((a \cdot ((0 \cdot c) \cdot b)) \cdot c) \quad \text{by Lemma 2.7 (n) (twice)}$$
$$= ((0 \cdot a) \cdot ((0 \cdot (0 \cdot (b \cdot c))) \cdot u)) \cdot ((a \cdot ((0 \cdot c) \cdot b)) \cdot c) \quad \text{by (19)}$$
$$= ((0 \cdot a) \cdot ((0 \cdot (b \cdot c)) \cdot u)) \cdot ((a \cdot ((0 \cdot c) \cdot b)) \cdot c) \quad \text{by Lemma 2.7 (j)}$$

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$$= [(0 \cdot a) \cdot \{(0 \cdot (b \cdot c)) \cdot u\}] \cdot [(a \cdot b) \cdot c] \quad \text{by (20);}$$

and, consequently, **A** satisfies

$$((0 \to x) \to ((0 \to (y \to z)) \to u)) \to ((x \to y) \to z) \approx u \to ((x \to y) \to z). \tag{5.26}$$

Also, **A** satisfies

$$(x \to y)' \to (0 \to x) \approx 0 \to (x \to y), \tag{5.27}$$

since

$$\begin{aligned}
&(a \to b)' \to (0 \to b)\\
&= [[(0 \to b)' \to (a \to b)] \to [0 \to (0 \to b)]']' \quad \text{by (I)}\\
&= [[(0 \to b)' \to (a \to b)] \to (0 \to b)']' \quad \text{by Lemma 2.7 (g)}\\
&= [[0 \to (a \to b)] \to (0 \to b)']' \quad \text{by Lemma 2.7 (f)}\\
&= [[a \to (0 \to b)] \to (0 \to b)']' \quad \text{by Lemma 2.7 (n)}\\
&= [[a \to 0'] \to (0 \to b)']' \quad \text{by Lemma 2.7 (a)}\\
&= [[0 \to a'] \to (b' \to 0')']' \quad \text{by Lemma 2.2 (b)}\\
&= (a' \to b') \to 0' \quad \text{by (I)}\\
&= 0 \to (a' \to b')' \quad \text{by Lemma 2.2 (b)}\\
&= 0 \to (a \to b) \quad \text{by (m) and (n) of Lemma 2.7.}
\end{aligned}$$

Therefore, from

$$\begin{aligned}
&((0 \to a) \to b) \to (c \to a)\\
&= [[(c \to a)' \to (0 \to a)] \to [b \to (c \to a)]']' \quad \text{by (I)}\\
&= [[0 \to (c \to a)] \to [b \to (c \to a)]']' \quad \text{by (5.27)}\\
&= [[(c \to a)' \to 0'] \to [b \to (c \to a)]']' \quad \text{by Lemma 2.2 (b)}\\
&= (0' \to b) \to (c \to a) \quad \text{by (I)}\\
&= b \to (c \to a) \quad \text{by Lemma 2.1 (a),}
\end{aligned}$$

it follows that the identity

$$((0 \to x) \to y) \to (z \to x) \approx y \to (z \to x) \tag{5.28}$$

is valid in the algebra. Hence, observe that

$$\begin{aligned}
&(a \to b) \to (c \to (d \to a))\\
&= [(0 \to (d \to a)) \to (a \to b)] \to (c \to (d \to a)) \quad \text{by (5.28)}\\
&= [((a \to b) \to (d \to a)) \to (a \to b)] \to (c \to (d \to a)) \quad \text{by Lemma 2.7 (f)}\\
&= [((0 \to b) \to (d \to a)) \to (a \to b)] \to (c \to (d \to a)) \quad \text{by Lemma 2.7 (u)}\\
&= [(d \to a) \to (a \to b)] \to (c \to (d \to a)) \quad \text{by (5.28)}\\
&= [0 \to (a \to b)] \to (c \to (d \to a))\\
&\quad \text{by Lemma 2.7 (u) with } c := d \to a,\ a := a \to b,\ b := c.
\end{aligned}$$

Therefore, **A** satisfies the identity

$$(0 \to (x \to y)) \to (z \to (u \to x)) \approx (x \to y) \to (z \to (u \to x)). \tag{5.29}$$

Also, from

$$\begin{aligned}
&b \to (c \to a)\\
&= b \to (a' \to (c \to a)) \quad \text{by Lemma 2.7 (t)}\\
&= a' \to [b \to (a' \to (c \to a))] \quad \text{by (7)}\\
&= a' \to (b \to (c \to a)) \quad \text{by Lemma 2.7 (t),}
\end{aligned}$$

it follows that A satisfies

$$x' \to (y \to (z \to x)) \approx y \to (z \to x). \tag{5.30}$$

Now notice that the identity

$$(0 \to x) \to (y \to ((z \to x) \to u)) \approx y \to ((z \to x) \to u) \tag{5.31}$$

is valid in A, since

$$\begin{aligned}
&b \to ((c \to a) \to d)\\
&= (c \to a) \to [b \to ((c \to a) \to d)] \quad \text{by (7)}\\
&= (0 \to a) \to [(c \to a) \to [b \to ((c \to a) \to d)]] \quad \text{by (16)}\\
&= (0 \to a) \to [b \to ((c \to a) \to d)] \quad \text{by (7).}
\end{aligned}$$

Hence,

$$\begin{aligned}
&b \cdot ((a \cdot c) \cdot d)\\
&= [(0 \cdot a) \cdot [(0 \cdot (c \cdot d)) \cdot b]] \cdot ((a \cdot c) \cdot d) \quad \text{by (5.26)}\\
&= (a \cdot 0') \cdot [[(0 \cdot (c \cdot d)) \cdot b] \cdot ((a \cdot c) \cdot d)] \quad \text{by (14)}\\
&= (a \cdot 0') \cdot [[(c \cdot d) \cdot 0'] \cdot [b \cdot ((a \cdot c) \cdot d)]] \quad \text{by (14)}\\
&= (a \cdot 0') \cdot [[0 \cdot (c \cdot d)'] \cdot [b \cdot ((a \cdot c) \cdot d)]] \quad \text{by Lemma 2.2 (b)}\\
&= (a \cdot 0') \cdot [(0 \cdot c) \cdot [(0 \cdot d') \cdot [b \cdot ((a \cdot c) \cdot d)]]] \quad \text{by (19) with } z = 0\\
&= (a \cdot 0') \cdot [(0 \cdot c) \cdot [d' \cdot [b \cdot ((a \cdot c) \cdot d)]]]
\end{aligned}$$

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$$\text{by (5.29) with } y = 0$$
$$= (a \cdot 0') \cdot [(0 \cdot c) \cdot [b \cdot ((a \cdot c) \cdot d)]] \quad \text{by (5.30)}$$
$$= (a \cdot 0') \cdot [b \cdot ((a \cdot c) \cdot d)] \quad \text{by (5.31).}$$

(22) From Lemma 2.7 (n) and (13) we have that **A** satisfies

$$(x \to y) \to [\{x \to (0 \to y)\} \to z] \approx (x \to y) \to z. \tag{5.32}$$

Therefore, we have

$$\begin{aligned}
& b \cdot [\{a \cdot (0 \cdot b)\} \cdot c] \\
&= (a \cdot 0') \cdot [b \cdot \{(a \cdot (0 \cdot b)) \cdot c\}] \\
&\quad \text{by (21)} \\
&= (a \cdot 0') \cdot [b \cdot \{((0 \cdot a) \cdot (0 \cdot b)) \cdot c\}] \\
&\quad \text{by Lemma 2.7 (i)} \\
&= [(0 \cdot a) \cdot b] \cdot [\{(0 \cdot a) \cdot (0 \cdot b)\} \cdot c] \quad \text{by (14)} \\
&= [(0 \cdot a) \cdot b] \cdot c \quad \text{by (5.32)} \\
&= (a \cdot 0') \cdot (b \cdot c) \quad \text{by (14).}
\end{aligned}$$

(23)

$$\begin{aligned}
& [a \to (0 \to b)] \to (b \to c) \\
&= [\{(b \to c)' \to a\} \to \{(0 \to b) \to (b \to c)\}']' \\
&\quad \text{by (I)} \\
&= [\{(b \to c)' \to a\} \to \{b \to (b \to c)\}']' \\
&\quad \text{by Lemma 2.7 (h)} \\
&= (a \to b) \to (b \to c) \quad \text{by (I).}
\end{aligned}$$

(24)

$$\begin{aligned}
& [(a \to b) \to (c \to a)] \to b \\
&= [(0 \to b) \to (c \to a)] \to b \quad \text{by Lemma 2.7 (u)} \\
&= [\{b' \to (0 \to b)\} \to \{(c \to a) \to b\}']' \quad \text{by (I)} \\
&= [(0 \to b) \to \{(c \to a) \to b\}']' \\
&\quad \text{by Lemma 2.7 (t)} \\
&= [(b' \to 0') \to \{(c \to a) \to b\}']' \\
&\quad \text{by Lemma 2.2 (b)} \\
&= [0' \to (c \to a)] \to b \quad \text{by (I)} \\
&= (c \to a) \to b \quad \text{by Lemma 2.1 (a).}
\end{aligned}$$

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