Order in Implication Zroupoids

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Abstract

The variety $I$ of implication zroupoids (using a binary operation $\rightarrow$ and a constant 0) was defined and investigated by Sankappanavar in [7], as a generalization of De Morgan algebras. Also, in [7], several new subvarieties of $I$ were introduced, including the subvariety $I_{2,0}$, defined by the identity: $x'' \approx x$, which plays a crucial role in this paper. Some more new subvarieties of $I$ are studied in [3] that includes the subvariety $SL$ of semilattices with a least element 0; and an explicit description of semisimple subvarieties of $I$ is given in [5].

It is a well known fact that there is a partial order (denote it by $\sqsubseteq$) induced by the operation $\land$, both in the variety $SL$ of semilattices with a least element and in the variety $DM$ of De Morgan algebras. As both $SL$ and $DM$ are subvarieties of $I$ and the definition of partial order can be expressed in terms of the implication and the constant, it is but natural to ask whether the relation $\sqsubseteq$ on $I$ is actually a partial order in some (larger) subvariety of $I$ that includes both $SL$ and $DM$.

The purpose of the present paper is two-fold: Firstly, a complete answer is given to the above mentioned problem. Indeed, our first main theorem shows that the variety $I_{2,0}$ is a maximal subvariety of $I$ with respect to the property that the relation $\sqsubseteq$ is a partial order on its members. In view of this result, one is then naturally led to consider the problem of determining the number of non-isomorphic algebras in $I_{2,0}$ that can be defined on an $n$-element chain (herein called $I_{2,0}$-chains), $n$ being a natural number. Secondly, we answer this problem in our second main theorem which says that, for each $n \in \mathbb{N}$, there are exactly $n$ nonisomorphic $I_{2,0}$-chains of size $n$.

1 Introduction

The widely known fact that Boolean algebras can be defined using only implication and a constant was extended to De Morgan algebras in [7]. The crucial role played by a certain identity, called (I), led Sankappanavar, in 2012, to define and investigate, the variety $I$ of implication zroupoids ($I$-zroupoids) generalizing De Morgan algebras. Also, in [7], he introduced several new subvarieties of $I$ and found some relationships among those subvarieties. One of the subvarieties of $I$, called $I_{2,0}$, defined by the identity: $x'' \approx x$ and studied in [7], plays a crucial role in this paper. In [3], we introduce several more new subvarieties of $I$, including the subvariety $SL$ which is term-equivalent to the (well known) variety of $\lor$-semilattices with a least element 0, and describe further relationships among the subvarieties of $I$. An explicit description of semisimple subvarieties of $I$ is given in [5].

It is also a well known fact that there is a partial order induced by the operation $\land$, both in the variety $SL$ of semilattices with a least element and in the variety $DM$ of De Morgan algebras. As both $SL$ and $DM$ are subvarieties of $I$ and the definition of partial order can be expressed in terms of the implication and constant, it is but natural to ask whether the relation $\sqsubseteq$ (now defined) on $I$ is actually a partial order in some (larger) subvariety of $I$ that includes both $SL$ and $DM$. 
The purpose of the present paper is two-fold: Firstly, a complete answer is given to the above mentioned problem. Indeed, our first main theorem shows that the variety $I_{2,0}$ is a maximal subvariety of $I$ with respect to the property that the relation $\sqsubseteq$, defined by:

$$x \sqsubseteq y \text{ if and only if } (x \rightarrow y')' = x,$$

for $x, y \in A$ and $A \in I$,

is a partial order. In view of this result, one is then naturally led to consider the problem of determining the number of non-isomorphic algebras in $I_{2,0}$ ($I_{2,0}$-chains) that can be defined on an $n$-element set, $n$ being a natural number. Secondly, we answer this problem in our second main result which says that, for each $n \in \mathbb{N}$, there are exactly $n$ nonisomorphic $I_{2,0}$-chains of size $n$.

2 Preliminaries

In this section we recall some definitions and results from [3], [5] and [7] that will be needed for this paper. Basic references are [1] and [2].

Definition 2.1 [7] A groupoid with zero (zroupoid, for short) is an algebra $A = \langle A, \rightarrow, 0 \rangle$, where $\rightarrow$ is a binary operation and 0 is a constant. A zroupoid $A = \langle A, \rightarrow, 0 \rangle$ is an implication zroupoid ($I$-zroupoid, for short) if the following identities hold in $A$, where $x' := x \rightarrow 0$:

(I) $(x \rightarrow y) \rightarrow z \approx ((x' \rightarrow x) \rightarrow (y \rightarrow z))'$

(I$_0$) $0'' \approx 0$.

The variety of $I$-zroupoids is denoted by $I$.

In this paper we use the characterizations of De Morgan algebras, Kleene algebras and Boolean algebras (see [7]), and semilattices with least element 0 (see [3]), as definitions.

Definition 2.2 An implication zroupoid $A = \langle A, \rightarrow, 0 \rangle$ is a De Morgan algebra (DM-algebra, for short) if $A$ satisfies the axiom:

(DM) $(x \rightarrow y) \rightarrow x \approx x$.

A DM-algebra $A = \langle A, \rightarrow, 0 \rangle$ is a Kleene algebra (KL-algebra, for short) if $A$ satisfies the axiom:

(KL$_1$) $(x \rightarrow x) \rightarrow (y \rightarrow y)' \approx x \rightarrow x$

or, equivalently,

(KL$_2$) $(y \rightarrow y) \rightarrow (x \rightarrow x) \approx x \rightarrow x$.

A DM-algebra $A = \langle A, \rightarrow, 0 \rangle$ is a Boolean algebra (BA-algebra, for short) if $A$ satisfies the axiom:

(BA) $x \rightarrow x \approx 0'$.

An implication zroupoid $A = \langle A, \rightarrow, 0 \rangle$ is a semilattice with 0 (SL-algebra, for short) if $A$ satisfies the axioms:

(SM$_1$) $x' \approx x$

(SM$_2$) $x \rightarrow y \approx y \rightarrow x$. (Commutativity).
We denote by $\text{DM}$, $\text{KL}$, $\text{BA}$ and $\text{SL}$, respectively, the variety of $\text{DM}$-algebras, $\text{KL}$-algebras, $\text{BA}$-algebras, and $\text{SL}$-algebras.

We recall from [7] the definition of another subvariety of $\text{I}$, namely $\text{I}_{2,0}$, which plays a fundamental role in this paper.

**Definition 2.3** $\text{I}_{2,0}$ denotes the subvariety of $\text{I}$ defined by the identity:

\[ x'' \approx x. \]

We note that $\text{DM}$, $\text{KL}$, $\text{BA}$ and $\text{SL}$ are all subvarieties of $\text{I}_{2,0}$ (see [7] and [3]).

**Lemma 2.4** [7, Theorem 8.15] Let $A$ be an $\text{I}$-zroupoid. Then the following are equivalent:

(a) $0' \rightarrow x \approx x$

(b) $x'' \approx x$

(c) $(x \rightarrow x')' \approx x$

(d) $x' \rightarrow x \approx x$.

**Lemma 2.5** [7] Let $A \in \text{I}_{2,0}$. Then

(a) $x' \rightarrow 0' \approx 0 \rightarrow x$

(b) $0 \rightarrow x' \approx x \rightarrow 0'$.

Several identities true in $\text{I}_{2,0}$ are given in [3], [5] and [7]. Some of those that are needed for this paper are listed in the next lemma, which also presents some new identities of $\text{I}_{2,0}$ that will be useful later in this paper. The proof of the lemma is given in the Appendix.

**Lemma 2.6** Let $A \in \text{I}_{2,0}$. Then $A$ satisfies:

(1) $(x \rightarrow 0') \rightarrow y \approx (x \rightarrow y') \rightarrow y$

(2) $(0 \rightarrow x') \rightarrow (y \rightarrow x) \approx y \rightarrow x$

(3) $(y \rightarrow x)' \approx (0 \rightarrow x) \rightarrow (y \rightarrow x)'$

(4) $[x \rightarrow (y \rightarrow x)]' \approx (x \rightarrow y) \rightarrow x$

(5) $(y \rightarrow x) \rightarrow y \approx (0 \rightarrow x) \rightarrow y$

(6) $0 \rightarrow x \approx 0 \rightarrow (0 \rightarrow x)$

(7) $0 \rightarrow [(0 \rightarrow x) \rightarrow (0 \rightarrow y')] \approx 0 \rightarrow (x \rightarrow y)$

(8) $[x' \rightarrow (0 \rightarrow y)]' \approx (0 \rightarrow x) \rightarrow (0 \rightarrow y)'$

(9) $0 \rightarrow (0 \rightarrow x)' \approx 0 \rightarrow x'$

(10) $0 \rightarrow (x' \rightarrow y') \approx x \rightarrow (0 \rightarrow y')$

(11) $[(x \rightarrow 0') \rightarrow y]' \approx (0 \rightarrow x) \rightarrow y'$
(12) $0 \rightarrow [(0 \rightarrow x) \rightarrow y'] \approx x \rightarrow (0 \rightarrow y')$

(13) $0 \rightarrow (x \rightarrow y) \approx x \rightarrow (0 \rightarrow y)$

(14) $(x \rightarrow y) \rightarrow y' \approx y \rightarrow (x \rightarrow y)'$

(15) $(x' \rightarrow y) \rightarrow [(0 \rightarrow z) \rightarrow x'] \approx (0 \rightarrow y) \rightarrow [(0 \rightarrow z) \rightarrow x']$

(16) $0 \rightarrow (x \rightarrow y')' \approx 0 \rightarrow (x' \rightarrow y)$

(17) $x \rightarrow (y \rightarrow x') \approx y \rightarrow x'$

(18) $[(0 \rightarrow x) \rightarrow y] \rightarrow x \approx y \rightarrow x$

(19) $0 \rightarrow (x \rightarrow y)] \rightarrow x \approx (0 \rightarrow y) \rightarrow x$

(20) $(0 \rightarrow x) \rightarrow (0 \rightarrow y) \approx x \rightarrow (0 \rightarrow y)$

(21) $x \rightarrow y \approx x \rightarrow (x \rightarrow y)$

(22) $[(x \rightarrow (0 \rightarrow y)) \rightarrow z'] \approx z \rightarrow [(x \rightarrow y) \rightarrow z']$

(23) $0 \rightarrow (x \rightarrow y)] \rightarrow y' \approx y \rightarrow (x \rightarrow y)'$

(24) $x \rightarrow [(y \rightarrow z) \rightarrow x'] \approx (0 \rightarrow y) \rightarrow [x \rightarrow (z \rightarrow x')]$

(25) $0 \rightarrow [(0 \rightarrow x) \rightarrow y] \approx x \rightarrow (0 \rightarrow y)$

(26) $x \rightarrow (y \rightarrow x') \approx (y \rightarrow 0') \rightarrow x'$

(27) $[(x' \rightarrow (y \rightarrow (z \rightarrow x))] \rightarrow [(y \rightarrow z) \rightarrow x] \approx (y \rightarrow z) \rightarrow x$

(28) $[[0 \rightarrow (x \rightarrow y')] \rightarrow (0 \rightarrow y')]' \approx 0 \rightarrow (x \rightarrow y')$

(29) $[(0 \rightarrow \{(x \rightarrow y) \rightarrow z\}] \rightarrow \{0 \rightarrow (y \rightarrow z)\}' \approx 0 \rightarrow \{(x \rightarrow y) \rightarrow z\}$

(30) $[x \rightarrow (0 \rightarrow y')]' \approx x' \rightarrow (y \rightarrow 0')'$

(31) $[0 \rightarrow x) \rightarrow y'] \approx y \rightarrow (x \rightarrow y')$

(32) $[x \rightarrow (y \rightarrow 0')]' \approx x' \rightarrow (0 \rightarrow y')$

(33) $(x \rightarrow y)' \rightarrow (0 \rightarrow x)' \approx y' \rightarrow x'$

(34) $(0 \rightarrow x)' \rightarrow (0 \rightarrow y)' \approx 0 \rightarrow (x' \rightarrow y')$

(35) $[(x \rightarrow y)' \rightarrow \{y \rightarrow (x \rightarrow y)\}' \approx (x \rightarrow y)'$

(36) $[[0 \rightarrow x) \rightarrow y] \rightarrow (x \rightarrow y)' \approx (0 \rightarrow x) \rightarrow y$

(37) $[x \rightarrow (y \rightarrow x')] \rightarrow x' \approx x \rightarrow (y \rightarrow x)'$. 

4
3 Partial order in Implication Zroupoids

Let \( A = \langle A; \to, 0 \rangle \in I \). We define the operations \( \wedge \) and \( \vee \) on \( A \) by:

- \( x \wedge y := (x \to y)' \),
- \( x \vee y := (x' \wedge y')' \).

Note that the above definition of \( \wedge \) is a simultaneous generalization of the \( \wedge \) operation of De Morgan algebras and that of \( \mathbf{SL} (= \text{semilattices with least element } 0) \). It is, of course, well known that the meet operation induces a partial order on both \( \mathbf{DM} \) and \( \mathbf{SL} \), which naturally leads us to the following definition of a binary relation \( \sqsubseteq \) on algebras in \( I \).

**Definition 3.1** Let \( A \in I \). We define the relation \( \sqsubseteq \) on \( A \) as follows:

\[
x \sqsubseteq y \text{ if and only if } x \wedge y = x \quad \text{(equivalently, } (x \to y)' = x)\.
\]

For \( a, b \in A \), we write

- \( a \subseteq b \text{ if } a \sqsubseteq b \text{ and } a \neq b \),
- \( a \sqsupseteq b \text{ if } b \sqsubseteq a \), and
- \( a \sqsupseteq b \text{ if } a \sqsubseteq b \text{ and } a \neq b \).

We already know from [3] that \( \langle A; \wedge, \vee \rangle \) is a lattice if and only if \( A \) is a De Morgan Algebra, implying that \( \sqsubseteq \) is a partial order on \( A \). We know (see [3]) that \( \sqsubseteq \) is also a partial order on algebras in \( \mathbf{SL} \). This fact led us naturally to consider the possibility of the existence of a subvariety \( V \) of \( I \), containing both \( \mathbf{SL} \) and \( \mathbf{DM} \), such that, for every algebra \( A \) in \( V \), the relation \( \sqsubseteq \) on \( A \) is actually a partial order.

In this section we will prove our first main result which says that the subvariety \( I_{2.0} \) is a maximal subvariety of \( I \) with respect to the property that the relation \( \sqsubseteq \) is a partial order on every member of that variety. To achieve this end, we need to, first, prove that \( \sqsubseteq \) is indeed a partial order on every member of \( I_{2.0} \), which will be done using the following lemmas.

**Lemma 3.2** Let \( A \in I_{2.0} \). Then the relation \( \sqsubseteq \) is antisymmetric on \( A \).

**Proof** Let \( a, b \in A \) such that \( a \sqsubseteq b \) and \( b \sqsubseteq a \). Let \( c \in A \) be arbitrary. Then, using (I) and the hypothesis, one observes that \( (c \to a) \to b' = [(b \to c) \to (a \to b')]' = [(b \to c) \to a]' \).

Consequently,

\[
\begin{align*}
(3.1) \quad (c \to a) \to b' &= [(b \to c) \to a]', \text{ where } c \in A.
\end{align*}
\]

Hence,

\[
\begin{align*}
a' &= (a \wedge b)' \quad \text{by hypothesis} \\
    &= (a \to b')'' \quad \text{by definition of } \wedge \\
    &= a \to b' \quad \text{using Lemma [2.3][2]} \\
    &= (a' \to a) \to b' \quad \text{from (3.1) with } c = a' \\
    &= [(b \to a') \to a]' \quad \text{by hypothesis,}
\end{align*}
\]
and, therefore,

\[(3.2) \ a' = (b' \to a'). \]

Now,\[
\begin{align*}
b' &= \left[ b \to a' \right]' \quad \text{by hypothesis} \\
&= b \to a' \\
&= (0 \to a'') \to (b \to a') \quad \text{by Lemma 2.6 (2) with } x = a', y = b \\
&= (0 \to a) \to (b \to a'') \quad \text{by hypothesis}.
\end{align*}
\]

Thus,
\[(3.3) \ b' = (0 \to a) \to b'. \]

Therefore,
\[
\begin{align*}
a' &= \left[ b' \to a \right]' \quad \text{from (3.2)} \\
&= \left[ (b \to 0) \to a \right]' \\
&= (0 \to a) \to b' \quad \text{from (3.1) with } c = 0 \\
&= b' \quad \text{by (3.3)}.
\end{align*}
\]

Consequently, we have that \( a = a'' = b'' = b \), thus proving that \( \sqsubseteq \) is antisymmetric on \( A \). \( \Box \)

Now, we turn to proving the transitivity of the relation \( \sqsubseteq \). For this, we need the following lemmas. The proof of the following (technical) lemma is given in the Appendix.

**Lemma 3.3** Let \( A \in I_{2,0} \) with \( a, b \in A \) such that \( a \sqsubseteq b \). Let \( d \in A \) be arbitrary. Then

\[
\begin{align*}
(1) \quad (0 \to a') \to b &= a' \to b \\
(2) \quad b \to a' &= (0 \to b) \to a' \\
(3) \quad b \to a' &= a' \\
(4) \quad 0 \to (a' \to b) &= 0 \to a \\
(5) \quad [(b \to d) \to a]' &= (d \to a) \to b' \\
(6) \quad (0 \to d) \to a' &= \left[ \{ d \to (0 \to b') \} \right] \to a' \\
(7) \quad a \to [a' \to d] \to \{ (0 \to a) \to b' \} &= (0 \to d) \to a' \\
(8) \quad a \to [d \to a \to b'] &= a \to (d \to a)' \\
(9) \quad [0 \to (b \to d)] \to a &= (0 \to d) \to a \\
(10) \quad [b \to (a \to d)] \to a &= (0 \to d) \to a \\
(11) \quad b \to (0 \to a') &= 0 \to a' \\
(12) \quad [(d \to a) \to b]' &= (b \to d) \to a \\
(13) \quad a' \to b &= b' \to a \\
(14) \quad (d \to a') \to b &= (d \to 0') \to (a' \to b)
\end{align*}
\]
Lemma 3.4 Let $A \in I_{2,0}$ and let $a, b, e \in A$ such that $(a \to b')' = a$ and $(0 \to e') \to b = b$, and let $d \in A$ be arbitrary. Then

(a) $b \to d = (0 \to (d \to e)) \to (b \to d)$

(b) $(0 \to e) \to a' = a'$

(c) $(0 \to e') \to a = a$.

(d) $(0 \to e) \to [(a \to (a \to d)] = a \to d$.

Proof

(a) $b \to d = [(0 \to e') \to b] \to d$

(by hypothesis)

$= [(d' \to (0 \to e')) \to (b \to d')] \to [(0 \to e') \to b] \to d$

(by Lemma 2.6 (27))

using $x = d, y = 0 \to e', z = b$

$= [(d' \to (0 \to e')) \to (b \to d')] \to (b \to d)$

(by hypothesis)

$= [0 \to [(0 \to e') \to (0 \to e')]'] \to (b \to d)$

(by Lemma 2.6 (11))

$= [0 \to [(0 \to d) \to (0 \to e')']'] \to (b \to d)$

(by Lemma 2.6 (8))

$= [0 \to (d \to e)] \to (b \to d)$

(by Lemma 2.6 (37)).

Using Lemma 3.3 (3) (twice), and (a) with $d = a'$, we obtain $[0 \to (a' \to e)] \to a' = [0 \to (a' \to e)] \to (b \to a') = b \to a' = a'$. Hence,

(3.4) $[0 \to (a' \to e)] \to a' = a'$.

Then,

$(0 \to e) \to a' = [0 \to (a' \to e)] \to a'$ by Lemma 2.6 (19) using $x = a', y = e$

by (3.4).

(b) $(0 \to e') \to a = [0 \to (0 \to e')] \to a$

(by Lemma 2.6 (9))

$= [(0 \to e) \to 0'] \to a$

(by Lemma 2.5 (30))

$= [(0 \to e) \to a'] \to a$

(by Lemma 2.6 (11))

$= a' \to a$

(by (15))

$= a$

(by Lemma 2.4 (4)).
\[(a \rightarrow d) = [(0 \rightarrow c') \rightarrow a] \rightarrow d \]
\[= [(d' \rightarrow (0 \rightarrow c')) \rightarrow (a \rightarrow d')] \rightarrow [(0 \rightarrow c') \rightarrow (a \rightarrow d)] \]
\[= [(d' \rightarrow (0 \rightarrow c')) \rightarrow (a \rightarrow d')] \rightarrow (a \rightarrow d) \]
\[= [0 \rightarrow ((0 \rightarrow d) \rightarrow (0 \rightarrow c'))] \rightarrow (a \rightarrow d) \]
\[= [0 \rightarrow (d \rightarrow e)] \rightarrow (a \rightarrow d) \]

Thus,

\[(3.5) \quad a \rightarrow d = [0 \rightarrow (d \rightarrow e)] \rightarrow (a \rightarrow d). \]

Now,

\[(0 \rightarrow e) \rightarrow [a \rightarrow (a \rightarrow d)] = [0 \rightarrow [(a \rightarrow (a \rightarrow d)) \rightarrow e]] \rightarrow [a \rightarrow (a \rightarrow d)] \]
\[= [0 \rightarrow [(a \rightarrow (a \rightarrow d)) \rightarrow e]] \rightarrow [a \rightarrow (a \rightarrow (a \rightarrow d))] \]
\[= a \rightarrow [a \rightarrow (a \rightarrow d)] \]
\[= a \rightarrow d \]

Thus, \((d)\) is proved and the proof of the lemma is complete. \(\square\)

Each of the next three lemmas prove a crucial step in the proof of transitivity of \(\sqsubseteq\).

**Lemma 3.5** Let \(A \in I_{2,0}\) and let \(a, b, c \in A\) such that \(a \sqsubseteq b\) and \(b \sqsubseteq c\). Let \(d, e, f \in A\) be arbitrary. Then

1. \((0 \rightarrow c') \rightarrow b = b\)
2. \((0 \rightarrow c) \rightarrow [a \rightarrow (a \rightarrow d)] = a \rightarrow d\)
3. \((0 \rightarrow c) \rightarrow (a \rightarrow d) = a \rightarrow d\)
4. \([0 \rightarrow ((0 \rightarrow b) \rightarrow c')] \rightarrow b = b\)
5. \([d' \rightarrow [0 \rightarrow ((0 \rightarrow b) \rightarrow c')]] \rightarrow (b \rightarrow d)' = (b \rightarrow d)'\)
6. \([b \rightarrow d] \rightarrow [e \rightarrow (b \rightarrow d)'] = [e \rightarrow 0'] \rightarrow (b \rightarrow d)'\)
7. \([b \rightarrow (a \rightarrow c')] \rightarrow a = a\)
8. \((0 \rightarrow b) \rightarrow (a \rightarrow d) = a \rightarrow d\)
(9) \(0 \to [b \to (a \to d)] = 0 \to (a \to d)\)

(10) \(0 \to [(b \to (a \to d)) \to c] = 0 \to [(a \to d) \to c]\)

(11) \([0 \to (d' \to c)] \to (0 \to b)' = (0 \to d) \to (0 \to b)'\)

(12) \(0 \to (a' \to c) \sqsubseteq 0 \to b\)

(13) \((0 \to a) \to (0 \to b) = (0 \to a)'\)

(14) \(0 \to (a' \to c) = 0 \to a\)

(15) \((d \to e) \to [(b \to (a \to f))'] \to (0 \to a)' = (d \to e) \to [(a' \to b) \to (f' \to a')]\).

**Proof** By hypothesis, we have \((a \to b)' = a\) and \((b \to c)' = b\).

(1) \[
(0 \to c') \to b = (c \to 0') \to b \quad \text{by Lemma 2.3 (a)}
\]
(2) \[
= [(b' \to c) \to (0' \to b')]' \quad \text{by (I)}
\]
(3) \[
= [(b' \to c) \to b']' \quad \text{by Lemma 2.4 (a)}
\]
(4) \[
= [(0 \to c) \to b']' \quad \text{by Lemma 2.6 (5)}
\]
(5) \[
= [(c' \to 0') \to b']' \quad \text{by Lemma 2.5 (a)}
\]
(6) \[
= [(b'' \to c') \to (0' \to b')]' \quad \text{from (I)}
\]
(7) \[
= (b'' \to c') \to (0' \to b')'
\]
(8) \[
= (b \to c') \to (0' \to b')'
\]
(9) \[
= (b \to c') \to b'' \quad \text{by Lemma 2.4 (a)}
\]
(10) \[
= (b \to c') \to b \quad \text{by hypothesis}
\]
(11) \[
= b' \to b \quad \text{by hypothesis}
\]
(12) \[
= b \quad \text{by Lemma 2.4 (d)}.
\]

(2) This is immediate from (1) and Lemma 3.4 (1) with \(e = c\).

(3) Using Lemma 2.5 (21) and (2) we have that \((0 \to c) \to (a \to d) = (0 \to c) \to [(a \to (a \to d))] = a \to d\), implying (3).

(4) \[
[0 \to ((0 \to b) \to c')] \to b = \{ (b' \to 0) \to [(0 \to b) \to c'] \to b]' \} \quad \text{by (I)}
\]
(5) \[
= \{ b \to [(0 \to b) \to c'] \to b'] \} \quad \text{by (I)}
\]
(6) \[
= \{ b \to (c' \to b)' \} \quad \text{by Lemma 2.4 (18)}
\]
(7) \[
= \{ (b' \to b) \to (c' \to b)' \} \quad \text{by Lemma 2.4 (11)}
\]
(8) \[
= (b \to c') \to b \quad \text{by (I)}
\]
(9) \[
= (b \to c')'' \to b \quad \text{by hypothesis}
\]
(10) \[
= b' \to b \quad \text{by Lemma 2.4 (d)}.
\]
(11) \[
= b \quad \text{by (I)}
\].
\[(b \to d) \to [e \to (b \to d)]' = [e \to (b \to d)] \to (b \to d)' \text{ by Lemma 2.6 (14) with } x = e, y = b \to d\]
\[= [e \to 0'] \to (b \to d)' \text{ by Lemma 2.6 (1)}.\]

\[(b \to (a \to c')) \to a = [(a' \to b) \to \{(a \to c') \to a\}']' \text{ by (1)}\]
\[= [(a' \to b) \to \{(0 \to c') \to a\}']' \text{ by Lemma 2.6 (5)}\]
\[= [(a' \to b) \to a']' \text{ by (1) and Lemma 3.3 (4)}\]
\[= (a \to 0) \to (b \to a')' \text{ by (1)}\]
\[= a' \to (b \to a')' \text{ by Lemma 3.3 (3)}\]
\[= a' \to a'' \text{ by Lemma 3.3 (4)}\]
\[= a' \to a\]
\[= a \text{ by Lemma 2.4 (4)}.\]

\[(0 \to b') \to b = (0 \to 0') \to b \text{ by Lemma 2.1 (1)}\]
\[= (0'' \to 0') \to b \text{ by Lemma 2.4 (4)}\]
\[= 0' \to b \text{ by Lemma 2.4 (4)}\]
\[= b \text{ by Lemma 2.4 (4)}.\]

Hence, by the hypothesis, together with Lemma 3.3 (4), we obtain that \((0 \to b) \to \{a \to (a \to d)\} = a \to d\). Hence, by Lemma 2.6 (21), we have \((0 \to b) \to (a \to d) = a \to d\).

\[0 \to (a \to d) = 0 \to [(0 \to b) \to (a \to d)] \text{ by (5)}\]
\[= b \to [0 \to (a \to d)] \text{ by Lemma 2.6 (25) with } x = b, y = a \to d\]
\[= 0 \to [b \to (a \to d)] \text{ by Lemma 2.6 (13)}.\]

\[0 \to [b \to (a \to d)] \to e = [b \to (a \to d)] \to (0 \to e) \text{ by Lemma 2.6 (13)}\]
\[= 0 \to [(0 \to \{b \to (a \to d)\}] \to e \text{ by Lemma 2.6 (20)}\]
\[= 0 \to [(0 \to (a \to d)] \to e \text{ by (6)}\]
\[= (a \to d) \to (0 \to e) \text{ by Lemma 2.6 (20)}\]
\[= 0 \to [(a \to d) \to e] \text{ by Lemma 2.6 (13)}.\]

\[0 \to (d' \to c) \to (0 \to b)' = [0 \to (d' \to c)] \to (b' \to 0')' \text{ by Lemma 2.5 (4)}\]
\[= [\{(d' \to c) \to b'\} \to 0']' \text{ by (1)}\]
\[= [\{(d \to d') \to (c \to b')\}' \to 0']' \text{ by (1)}\]
\[= [(d \to b') \to b']' \to 0')' \text{ by Lemma 3.3 (3)}\]
\[= [(d \to d') \to b'] \to 0')' \text{ by (1)}\]
\[= [(0 \to d') \to b'] \to 0')' \text{ by Lemma 2.6 (5)}\]
\[= [0 \to \{(0 \to d') \to b\}']' \text{ by Lemma 2.5 (4)}\]
\[= [(0 \to d') \to (0 \to b)']' \text{ by Lemma 2.6 (13)}\]
\[= [(d \to d') \to (0 \to b)']' \text{ by Lemma 2.6 (13)}\]
\[= (0 \to d) \to (0 \to b)' \text{ by Lemma 2.6 (11)}.\]
(12) 
\[0 \to (a' \to c) = 0 \to [(a \to b')'' \to c] \quad \text{by hyphotesis}\]
\[\subseteq 0 \to \left( b' \to c \right) \quad \text{by Lemma 2.6 (29)}\]
\[= 0 \to b \quad \text{by hyphotesis and Lemma 3.3 (1)}\]

(13) 
\[0 \to (a \to b)' = \left[ a' \to (0 \to b) \right]' \quad \text{by Lemma 2.6 (8)}\]
\[= \left[ 0 \to (a' \to b) \right]'' \quad \text{by Lemma 2.6 (13)}\]
\[= (0 \to a)' \quad \text{by hyphotesis and Lemma 3.3 (4)}\]

(14) 
\[0 \to (a' \to c) = \left[ \{0 \to (a' \to c) \} \to (0 \to b) \right]' \quad \text{by (12)}\]
\[= \left[ (0 \to a) \to (0 \to b)' \right]' \quad \text{by (11) with } d = a\]
\[= (0 \to a)'' \quad \text{by (13)}\]
\[= 0 \to a. \]

(15) 
\[(d \to c) \to [(a' \to b) \to (f' \to a')]\]
\[= (d \to e) \to \left[ \{(0 \to a') \to b \} \to (f' \to a') \right] \quad \text{by Lemma 3.3 (1)}\]
\[= (d \to e) \to \left[ \{(0 \to a') \to b \} \to \{(f \to 0) \to a' \} \right] \quad \text{by (I)}\]
\[= (d \to e) \to \left[ \{(b \to (a \to f)) \to (0 \to a') \}' \right] \quad \text{by (I)}\]
\[= (d \to e) \to \left[ \{(b \to (a \to f))' \to (a' \to 0) \}' \right] \quad \text{by (30) with } x = b \to (a \to f) \text{ and } y = a'\]
\[= (d \to e) \to \left[ \{(b \to (a \to f))' \to (0 \to a) \}' \right] \quad \text{by Lemma 2.3 (a)}\]

Hence, we have \((d \to e) \to \left[ \{b \to (a \to f) \}' \to (0 \to a) \}' \right] = (d \to e) \to \left[ \{a' \to b) \to (f' \to a') \right].\]

□

**Lemma 3.6** Let \(A \in I_{2,0}\) and let \(a, b, c \in A\) such that \(a \subseteq b\) and \(b \subseteq c\). Let \(d \in A\) be arbitrary. Then

(a) \([c \to (b \to a')] \to b = (0 \to a') \to b\)

(b) \((c \to a') \to b = a' \to b\)

(c) \((a' \to b) \to (c \to a') = c \to a'\)

(d) \(c \to a' = a \to [b \to (a \to c')]\)

(e) \(0 \to (a \to d) = 0 \to [c \to (a \to d)]\)
(f) \((d \to a) \to d \sqsubseteq (a' \to b) \to d\)

(g) \((a' \to b) \to c' = (0 \to a) \to b'\)

(h) \(0 \to (a \to c') \sqsubseteq 0 \to a'\)

(i) \(0 \to (a \to c') = 0 \to a'.\)

(j) \(c \to (a \to c') \sqsubseteq 0 \to (a \to c')\)

(k) \(c \to (a \to c') \sqsubseteq 0 \to a'\)

(l) \((c \to (a \to c'))' \to (0 \to a)' = c \to (a \to c')\)

(m) \(a \to [b \to (a \to c')] = a \to c'\)

(n) \(c \to a' = a \to c'.\)

Proof

(a) Since \((b \to c')' = b\), by Lemma \ref{lem:lemma} \((10)\) with \(d = a'\), we have \((c \to (b \to a')) \to b = (0 \to a') \to b\).

(b) \[
(c \to a') \to b = [c \to (b \to a')] \to b \quad \text{by Lemma } \ref{lem:lemma} \((8)\)
= (0 \to a') \to b \quad \text{by } (a),
\]
from which we get \((c \to a') \to b = (0 \to a') \to b\), which, together with Lemma \ref{lem:lemma} \((1)\), implies \((c \to a') \to b = a' \to b\).

(c) \[
c \to a' = (0 \to a) \to (c \to a') \quad \text{by Lemma } \ref{lem:lemma} \((2)\) with \(x = a', y = c\)
= [0 \to (a' \to b)] \to (c \to a') \quad \text{by Lemma } \ref{lem:lemma} \((1)\)
= [0 \to [(c \to a') \to b]] \to (c \to a') \quad \text{by } (i)
= [[c \to a'] \to (0 \to b)] \to (c \to a') \quad \text{by Lemma } \ref{lem:lemma} \((13)\)
= [0 \to (0 \to b)] \to (c \to a') \quad \text{by Lemma } \ref{lem:lemma} \((15)\)
= (0 \to b) \to (c \to a') \quad \text{by Lemma } \ref{lem:lemma} \((5)\)
= [[(c \to a') \to b] \to (c \to a')] \quad \text{by Lemma } \ref{lem:lemma} \((15)\)
= (a' \to b) \to (c \to a') \quad \text{by } (i).
(d) 

\[
c \rightarrow a' = (0 \rightarrow a) \rightarrow (c \rightarrow a')
\]

by Lemma 2.6 (2)

\[
= (0 \rightarrow a) \rightarrow [(a' \rightarrow b) \rightarrow (c \rightarrow a')]
\]

by Lemma 2.5 (1)

\[
= (0 \rightarrow a) \rightarrow [(a' \rightarrow b) \rightarrow (c'' \rightarrow a')]
\]

by Lemma 3.5 (15) with \(d = 0, e = a, f = c'\)

\[
= (0 \rightarrow a) \rightarrow \{(b \rightarrow (a \rightarrow c'))' \rightarrow \{0 \rightarrow (a' \rightarrow c')\}'\}
\]

by Lemma 3.5 (14)

\[
= (0 \rightarrow a) \rightarrow \{(b \rightarrow (a \rightarrow c'))' \rightarrow \{0 \rightarrow (a \rightarrow c')\}'\}
\]

by Lemma 2.6 (10)

\[
= (0 \rightarrow a) \rightarrow \{(b \rightarrow (a \rightarrow c'))' \rightarrow \{0 \rightarrow \{b \rightarrow (a \rightarrow c')\}'\}\}
\]

by Lemma 3.5 (10) with \(d = c', e = 0\)

\[
= [b \rightarrow (a \rightarrow c')]' \rightarrow [(a \rightarrow 0) \rightarrow \{b \rightarrow (a \rightarrow c')\}]'
\]

by Lemma 2.6 (24) with \(x = [b \rightarrow (a \rightarrow c')]'\), \(y = a, z = 0\)

\[
= \{(0 \rightarrow (a \rightarrow 0)) \rightarrow b \rightarrow (a \rightarrow c')\}'
\]

by Lemma 2.6 (1) and (24) with \(x = a \rightarrow 0,\ y = [b \rightarrow (a \rightarrow c')]'\)

\[
= [a \rightarrow (0 \rightarrow 0)] \rightarrow b \rightarrow (a \rightarrow c')
\]

by Lemma 2.6 (13)

\[
= [[a \rightarrow (0 \rightarrow 0)] \rightarrow b \rightarrow (a \rightarrow c')]''
\]

by Lemma 3.5 (13)

\[
= [(a \rightarrow 0') \rightarrow (b \rightarrow (a \rightarrow c'))]''
\]

by Lemma 2.6 (13)

\[
= [(b \rightarrow (a \rightarrow c')) \rightarrow (a \rightarrow \{b \rightarrow (a \rightarrow c')\}]'
\]

by Lemma 3.5 (6) with \(c = a, d = a \rightarrow c'\)

\[
= [\{b \rightarrow (a \rightarrow c')\} \rightarrow a] \rightarrow [b \rightarrow (a \rightarrow c')]
\]

by Lemma 2.6 (1)

\[
= a \rightarrow [b \rightarrow (a \rightarrow c')]
\]

by Lemma 3.5 (7).

(e) 

\[
0 \rightarrow (a \rightarrow d) = 0 \rightarrow \{(0 \rightarrow c) \rightarrow (a \rightarrow d)\]
\]

by Lemma 3.5 (3)

\[
= 0 \rightarrow [c \rightarrow (a \rightarrow d)]
\]

by Lemma 2.6 (25)

\[
= 0 \rightarrow [c \rightarrow (a \rightarrow d)]
\]

by Lemma 2.6 (13).

(f) 

\[
(d \rightarrow a) \rightarrow d = (0 \rightarrow a) \rightarrow d
\]

by Lemma 2.6 (14)

\[
= [0 \rightarrow (a' \rightarrow b)] \rightarrow d
\]

by Lemma 3.3 (11)

\[
\sqsubseteq (a' \rightarrow b) \rightarrow d
\]

by Lemma 2.6 (36).

(g) 

\[
(a' \rightarrow b) \rightarrow c' = [(c \rightarrow a') \rightarrow (b \rightarrow c')]'
\]

by (1)

\[
= [(c \rightarrow a') \rightarrow b']
\]

by hypothesis

\[
= [(c \rightarrow (b \rightarrow a')) \rightarrow b']
\]

by Lemma 3.3 (11)

\[
= [0 \rightarrow (a' \rightarrow b)]
\]

by Lemma 3.3 (10) with \(d = a'\) since \(b \sqsubseteq c\)

\[
= [(b \rightarrow a') \rightarrow b']
\]

by Lemma 2.6 (15)

\[
= [(b \rightarrow a') \rightarrow b''']
\]

by Lemma 2.4 (15)

\[
= [(b \rightarrow a') \rightarrow (0' \rightarrow b')]'
\]

by Lemma 2.4 (16)

\[
= (a' \rightarrow 0') \rightarrow b'
\]

by (1)

\[
= (0 \rightarrow a) \rightarrow b'.
\]

by Lemma 2.5 (16).

Hence, one has \((a' \rightarrow b) \rightarrow c' = (0 \rightarrow a) \rightarrow b'\).

(h) From Lemma 3.5 (11), we have \((0 \rightarrow c') \rightarrow b = b\). Hence, we can use Lemma 3.3. Therefore, we have
0 \to (a \to c') = 0 \to [(0 \to c') \to a] \to c'] \quad \text{by Lemma 3.4 (14) and Lemma 3.5 (1)}
= [(0 \to c') \to a] \to (0 \to c') \quad \text{by Lemma 2.6 (13)}
\subseteq (a' \to b) \to (0 \to c') \quad \text{by (1) with } d = 0 \to c'
= 0 \to [(a' \to b) \to c'] \quad \text{by Lemma 2.6 (13)}
= 0 \to [(0 \to a) \to b'] \quad \text{by (10)}
= 0 \to [(b \to 0) \to (a \to b')]' \quad \text{by (1)}
= 0 \to [b' \to (a \to b')]' \quad \text{by hypothesis}
= 0 \to (b' \to a)' \quad \text{by Lemma 2.6 (10)}
= 0 \to a' \quad \text{by Lemma 3.3 (3)}.

(i) $0 \to a' = 0 \to (a \to 0)$
= $0 \to [c \to (a \to 0)]$ \quad \text{by (1)}
= $0 \to (c \to a')$ \quad \text{by Lemma 2.6 (33)}
= $0 \to [(a \to c')' \to (0 \to a)']$ \quad \text{by Lemma 2.6 (33) and Lemma 2.6 (6)}
= $0 \to (a \to c')' \to (a' \to 0)'$ \quad \text{by Lemma 2.5 (a)}
= $[(0 \to (a \to c')) \to (0 \to a')]' \quad \text{by Lemma 2.5 (30) with } x = 0 \to (a \to c'), y = a'$
= $0 \to (a \to c')$ \quad \text{by (1)}.

(j) $\{c \to (a \to c') \to 0 \to (a \to c')\}' = \{c \to [(a \to c') \to 0] \to (a \to c')\}' \quad \text{by (1)}$
= $\{0 \to (a \to c') \to c \to [(a \to c') \to 0] \to (a \to c')\}' \quad \text{by (1)}$
= $\{0 \to (a \to c') \to [(a \to c') \to 0] \to (a \to c')\}' \quad \text{by Lemma 2.6 (13)}$
= $\{0 \to (a \to c') \to [(a \to c') \to (a \to c')]\} \quad \text{by Lemma 2.6 (13)}$
= $c \to (a \to c') \quad \text{by Lemma 2.6 (13) with } x = a \to c', y = c.$

(k) From (j) we have that $c \to (a \to c') \subseteq 0 \to (a \to c')$. Then using (1) we get $c \to (a \to c') \subseteq 0 \to a'$.

(l) $[c \to (a \to c')]' \to (0 \to a)' = [c \to (a \to c')]' \to (a' \to 0)' \quad \text{by Lemma 2.6 (33)}$
= $\{c \to (a \to c') \to (0 \to a)\}' \quad \text{by Lemma 2.6 (30)}$
= $c \to (a \to c') \quad \text{by (1)}.$

(m) $a \to [b \to (a \to c')] = c \to a'$ \quad \text{by (1)}
= $c'' \to a'$ \quad \text{by (1)}
= $(a \to c')' \to (0 \to a)' \quad \text{by Lemma 2.6 (33)}$
= $[c \to (a \to c')]' \to (0 \to a)' \quad \text{by Lemma 2.6 (10)}$
= $c \to (a \to c') \quad \text{by (1)}$
= $a \to c' \quad \text{by Lemma 2.6 (17)}.$
(n) From (m) and (n), we get \( c \to a' \equiv a \to c' \).

\[ \square \]

**Lemma 3.7** Let \( A \in I_{2,0} \) and let \( a, b, c \in A \) such that \( a \sqsubseteq b \) and \( b \sqsubseteq c \). Then

(a) \( c' \to [(c \to d) \to b] \sqsubseteq c \)

(b) \( 0 \to a' = c \to (0 \to a') \)

(c) \( c' \to (a' \to b) \sqsubseteq c \)

(d) \( (0 \to a') \to b = (c \to a') \to b \)

(e) \( c' \to (a' \to b) \sqsubseteq 0 \to c \)

(f) \( [(0 \to a) \to b] \to c = c' \to (a' \to b) \)

(g) \( a' \to c = c' \to (a' \to b) \)

(h) \( a' \to c \sqsubseteq c \)

(i) \( a' \to c = (0 \to a') \to c \).

**Proof**

(a)

\[
\begin{align*}
  c' \to [(c \to d) \to b] & = c' \to [(c \to d) \to (b \to c')]' \\
  & = c' \to [(d \to b) \to c']' \\
  & \sqsubseteq c'' \\
  & = c.
\end{align*}
\]

(b)

\[
\begin{align*}
  0 \to a' & = b \to (0 \to a') \\
  & = [0 \to \{(0 \to a') \to c\}] \to [b \to (0 \to a')] \\
  & = [0 \to \{(0 \to a') \to c\}] \to (0 \to a') \\
  & = [(0 \to a') \to (0 \to c)] \to (0 \to a') \\
  & = (0 \to c) \to (0 \to a') \\
  & = c \to (0 \to a')
\end{align*}
\]

(c)

\[
\begin{align*}
  c' \to (a' \to b) & = c' \to [(0 \to a') \to b] \\
  & = c' \to [(c \to (0 \to a')) \to b] \\
  & \sqsubseteq c
\end{align*}
\]

with \( d = 0 \to a' \).
(d) \[
(0 \to a') \to b = [(c \to (0' \to a')) \to b]
= [(b' \to c) \to ((0 \to a') \to b)']
= [(b' \to c) \to ((0 \to a') \to (b' \to b')']
= [(b' \to c) \to (b \to (a' \to b')']
= [(b' \to c) \to ((a' \to 0') \to b')']
= [(b' \to c) \to ((0 \to a) \to b')']
= [(b' \to c) \to (a' \to b')']
= (c \to a') \to b
\]
by (I) and \(x'' \approx x\).

(e) \[
c' \to (a' \to b) = c' \to [(0 \to a') \to b]
\]
by Lemma 3.3 (15) and Lemma 3.3 (16)
\[
\sqsubseteq 0 \to c.
\]
by Lemma 3.3 (17) with \(d = a'\).

(f) \[
c' \to (a' \to b) = [(c' \to (a' \to b)) \to (0 \to c)']
\]
by (e)
\[
= [(a' \to b) \to 0 \to c]
\]
by (I)
\[
= (a' \to b') \to c
\]
by (I)
\[
= [(0 \to a) \to b'] \to c
\]
by Lemma 3.3 (16).

(g) \[
c' \to (a' \to b) = ((0 \to a) \to b') \to c
\]
by (I)
\[
= [(0 \to a) \to 0'] \to (b' \to c)
\]
by Lemma 3.3 (14) with \(d = 0 \to a\)
\[
= [(a' \to 0') \to 0] \to (b' \to c)
\]
by Lemma 2.6 (13)
\[
= [(a' \to 0) \to 0'] \to (b' \to c)
\]
by Lemma 2.6 (1)
\[
= [a'' \to 0'] \to (b' \to c)
\]
\[
= (a \to b') \to c
\]
by Lemma 3.3 (14) with \(d = a\)
\[
= a' \to c
\]
by hypothesis.

(h) This is immediate from (e) and (e).

(i) \[
(0 \to a') \to c = (c \to a') \to c
\]
by Lemma 2.6 (5)
\[
= [c \to (a' \to c)']
\]
by Lemma 2.6 (4)
\[
= [(a' \to c) \to c']
\]
by Lemma 2.6 (14)
\[
= a' \to c
\]
by (I).

\(\square\)

We are now ready to complete the proof of transitivity of \(\sqsubseteq\).

**Theorem 3.8** \(\sqsubseteq\) is transitive.
Proof Let \( a, b, c \in A \) such that \( a \sqsubseteq b \) and \( b \sqsubseteq c \). Observe that
\[
\begin{align*}
a' & = a \rightarrow 0 \\
 & = (0 \rightarrow c) \rightarrow (a \rightarrow 0) \text{ by Lemma } 3.5 (3) \text{ with } d = 0 \\
 & = (0 \rightarrow c) \rightarrow a' \text{ by Lemma } 2.6 (5) \\
& = ((0 \rightarrow a') \rightarrow c) \rightarrow a' \text{ by Lemma } 3.7 (1) \\
& = c \rightarrow a' \text{ by Lemma } 2.6 (18) \\
& = a \rightarrow c' \text{ by Lemma } 3.6 (n).
\end{align*}
\]
Consequently,
\[
a = a'' = (a \rightarrow c')',
\]
implying \( a \sqsubseteq c \). Hence, \( \sqsubseteq \) is transitive on \( A \).
\( \square \)

We are now prepared to present our first main theorem.

**Theorem 3.9** The variety \( I_{2,0} \) is a maximal subvariety of \( I \) with respect to the property that the relation \( \sqsubseteq \) introduced in Definition 3.1 is a partial order.

**Proof** Let \( A \in I_{2,0} \). The relation \( \sqsubseteq \) is a partial order on \( A \) in view of Lemma 2.4 (c), Lemma 3.2, and Theorem 3.8.

Next, let \( V \) be a subvariety of \( I \) such that \( \sqsubseteq \) is a partial order on every algebra in \( V \). Now let \( A \in V \). Reflexivity of \( \sqsubseteq \) implies that \( A \models (x \rightarrow x')' \approx x \). Therefore, by Lemma 2.4, we conclude that \( A \in I_{2,0} \), and hence, \( V \subseteq I_{2,0} \), completing the proof.
\( \square \)

4 A method to construct finite \( I_{2,0} \)-chains

Now that we know the relation \( \sqsubseteq \) is a partial order on algebras in \( I_{2,0} \), it is natural to consider those algebras in \( I_{2,0} \), in which \( \sqsubseteq \) is a total order.

**Definition 4.1** Let \( A \in I \). We say that \( A \) is an \( I_{2,0} \)-chain (chain, for short) if \( A \in I_{2,0} \) and the relation \( \sqsubseteq \) (see Definition 3.1) is totally ordered on \( A \).

In this section we describe a method of constructing finite \( I_{2,0} \)-chains. But, first, we will present some examples of \( I_{2,0} \)-chains that will foreshadow the method to construct finite \( I_{2,0} \)-chains. We note that, in these examples, the number 0 is the constant element.

It is easy to see that the only 2-element \( I_{2,0} \)-chains, up to isomorphism, are
\[
\begin{array}{ccc}
\rightarrow: & 0 & 1 \\
0 & 1 & 1 \quad \text{with } 0 \sqsubseteq 1. \\
1 & 0 & 1
\end{array}
\]
and the only 3-element \( I_{2,0} \)-chains, up to isomorphism, are
\[
\begin{array}{ccc}
\rightarrow: & 0 & 1 & 2 \\
0 & 2 & 2 & 2 \quad \text{with } 0 \sqsubseteq 1 \sqsubseteq 2, \\
1 & 1 & 1 & 2 \\
2 & 0 & 1 & 2
\end{array}
\]
\[
\begin{array}{ccc}
\rightarrow: & -1 & 0 \\
-1 & -1 & -1 \quad \text{with } -1 \sqsubseteq 0 \sqsubseteq 1, \\
0 & -1 & 1 \\
1 & -1 & 0 & 1
\end{array}
\]
Note that, henceforth, we will use the symbol $\leq$ to denote the natural order in $\mathbb{Z}$. Recall that $\sqsubseteq$ is being used for the order (see Definition 3.1).

The next definition describes a general method to construct a class of finite $I_{2,0}$-chains, generalizing the above examples. In the next section, we will show that, this method, in fact, yields, up to isomorphism, all finite $I_{2,0}$-chains.

**Definition 4.2** Let $k \in \mathbb{N}$. Let $m, n \in \omega$ be such that the interval $[-n, m] \subseteq \mathbb{Z}$ with $|[-n, m]| = k$ and $0 \leq n, m \leq k - 1$. The (auxiliary) functions $p$ (predecessor) and $\ast$ are defined on $[-n, m]$ as follows:

$$p(x) = \begin{cases} x - 1 & \text{if } x > -n \\ -n & \text{if } x = -n, \end{cases}$$

and

$$x^\ast = \begin{cases} m & \text{if } x = 0 \\ x & \text{if } x < 0 \\ p(p(x))^\ast & \text{if } x > 0. \end{cases}$$

For convenience, we write $p(p(x)^\ast)$ for $p((p(x))^\ast)$. (Notice that the function $\ast$ is defined recursively for $x \geq 0$.)

Define the algebra $[-n, m]$ as follows:

$$[-n, m] := ([n, m]; \Rightarrow, 0),$$

where $0 \in [-n, m]$ is the constant and $\Rightarrow$ is defined by

$$x \Rightarrow y = \begin{cases} \max(x^\ast, y) & \text{if } x, y \geq 0 \\ \min(x, y) & \text{otherwise}. \end{cases}$$

We set $x' := x \Rightarrow 0$.

We shall now illustrate the method described in the above definition by applying it to construct a 6-element $I_{2,0}$-chain.

Let $k = 6$, and consider the interval $A = [-2, 3] = \{-2, -1, 0, 1, 2, 3\}$. Since $0 \Rightarrow 0 = \max(0^\ast, 0) = \max(3, 0) = 3$ and $a \Rightarrow b = \min(a, b)$ if $a < 0$ or $b < 0$, we arrive at the following partial table for $\Rightarrow$:

$$\begin{array}{cccccc}
\Rightarrow & -2 & -1 & 0 & 1 & 2 & 3 \\
-1 & -2 & -1 & -1 & -1 & -1 & -1 \\
0 & -2 & -1 & 3 & ? & ? & ? \\
\end{array}$$

Next, we determine the operations $p$ and $\ast$:
\[
\begin{array}{c|c}
\infty & x^* \\
\hline
0 & 3 \\
1 & p(p(1)^*) = p(0^*) = p(3) = 2 \\
2 & p(p(2)^*) = p(1^*) = p(2) = 1 \\
3 & p(p(3)^*) = p(2^*) = p(1) = 0
\end{array}
\]

Hence the table for \( \Rightarrow \) becomes:

\[
\begin{array}{ccccccc}
\Rightarrow & -2 & -1 & 0 & 1 & 2 & 3 \\
-1 & -2 & -1 & -1 & -1 & -1 & -1 \\
0 & -2 & -1 & 3 & ? & ? & ? \\
3 & -2 & -1 & 0 & ? & ? & ?
\end{array}
\]

Observe that \( 0 \Rightarrow 1 = \max(0^*, 1) = \max(3, 1) = 3, 1 \Rightarrow 1 = \max(1^*, 1) = \max(2, 1) = 2, 2 \Rightarrow 1 = \max(2^*, 1) = \max(1, 1) = 1 \) and \( 3 \Rightarrow 1 = \max(3^*, 1) = \max(0, 1) = 1 \). Then we get

\[
\begin{array}{ccccccc}
\Rightarrow & -2 & -1 & 0 & 1 & 2 & 3 \\
-1 & -2 & -1 & -1 & -1 & -1 & -1 \\
0 & -2 & -1 & 3 & 3 & ? & ? \\
1 & -2 & -1 & 2 & 2 & ? & ? \\
2 & -2 & -1 & 1 & 1 & ? & ? \\
3 & -2 & -1 & 0 & 1 & ? & ?
\end{array}
\]

Iterating this process we obtain the following complete table for \( \Rightarrow \):

\[
\begin{array}{ccccccc}
\Rightarrow & -2 & -1 & 0 & 1 & 2 & 3 \\
-1 & -2 & -1 & -1 & -1 & -1 & -1 \\
0 & -2 & -1 & 3 & 3 & 3 & 3 \\
1 & -2 & -1 & 2 & 2 & 2 & 3 \\
2 & -2 & -1 & 1 & 1 & 2 & 3 \\
3 & -2 & -1 & 0 & 1 & 2 & 3
\end{array}
\]

Thus we have constructed the algebra \([-n, m] \). Observe that \(-2 \sqsubseteq -1 \sqsubseteq 0 \sqsubseteq 1 \sqsubseteq 2 \sqsubseteq 3 \) and \( x'' = x^{**} = x \). Also, it is routine to verify \([-n, m] \in I_{2,0} \). Hence it is an \( I_{2,0} \)-chain.

Returning to the general method, we now aim to prove that \([-n; m] \) is an \( I_{2,0} \)-chain. To prove this, we will need the following lemmas.

**Lemma 4.3** If \( x \in [-n, m] \) and \( 0 \leq x \leq m \) then \( x^* = m - x \) and, consequently, \( x^* \in [0, m] \).

**Proof** We prove this lemma by induction on the element \( x \). Assume that \( x = 0 \). Then \( 0^* = m = m - 0 \).

Next, suppose \( x > 0 \). Since \(-n \leq 0 < x\), we have \( p(x) = x - 1 \). Hence, by inductive hypothesis, we have

\[
(4.1) \quad p(x)^* = m - p(x) = m - (x - 1) = m - x + 1.
\]
From $x > 0$, we can conclude that $m - x + 1 \leq m$. Also, since $x \leq m$, we obtain $0 \leq m - x$, thus $-n - 1 < 0 \leq m - x$, implying $m - x + 1 > -n$. Then we get $p(m - x + 1) = m - x + 1 - 1$. By Lemma 4.3, \( x^* = p((p(x))^*) = p(m - x + 1) = m - x \), completing the induction. It is clear that \( x^* \in [0, m] \).

**Corollary 4.4** If \( x \in [-n, m] \) then \( x' = x^* \).

**Proof** If \( x < 0 \) we have that \( x' = x \Rightarrow 0 = \min(x, 0) = x^* \). If \( x > 0 \), then by Lemma 4.3 \( x^* \geq 0 \), and hence \( x' = x \Rightarrow 0 = \max(x^*, 0) = x^* \). \( \square \)

**Lemma 4.5** If \( x \in [-n, m] \) then \( x'' = x \).

**Proof** We consider the following cases:

- If \( x < 0 \), then \( x^* = x \), and hence \( x'' = x \).
- If \( x \geq 0 \),
  \[
  x'' = (m - x)^* \quad \text{by Lemma 4.3 since } 0 < x \leq m
  = m - (m - x) \quad \text{by Lemma 4.3 since } 0 \leq m - x \leq m
  = x.
  \]

Consequently, by Corollary 4.4, \( x'' = x \). \( \square \)

**Lemma 4.6** If \( x, y \in [-n, m] \) and \( 0 \leq x \leq y \) then \( x^* \geq y^* \).

**Proof** We prove this lemma by induction on the element \( x \). If \( x = 0 \), \( x^* = 0^* = m \geq y^* \) by Lemma 4.3.

Now assume that \( x > 0 \). Since \( 0 < x \leq y \), we have that \( x^* = p(p(x)^*) \) and \( y^* = p(p(y)^*) \). Note that \( 0 \leq p(x) \leq p(y) \). Then, by induction hypothesis, we get \( p(y)^* \leq p(x)^* \). Hence \( x^* = p(p(x)^*) \geq p(p(y)^*) = y^* \). \( \square \)

**Lemma 4.7** Let \( k \in \mathbb{N} \). Let \( m, n \in \omega \) be such that the interval \([−n, m]\) ⊆ \( \mathbb{Z} \) with \(|[−n, m]| = k \) and \( 0 \leq n, m \leq k - 1 \). Then, \([−n, m] \in \mathcal{I}_{2,0}\).

**Proof** The proof that \( [−n; m] : \Rightarrow 0 \) satisfies the identity (I) is long and computational, but routine. Hence we leave the verification to the reader with the recommendation that the following cases be considered, where \( i, j, k \in [−n; m] \):

1. \( i, j, k \geq 0, i^* \geq j, i \geq k \)
2. \( i, j, k \geq 0, i^* \geq j, i < k \)
3. \( i, j, k \geq 0, i^* \geq j, k \geq i \)
4. \( i, j, k \geq 0, i^* < j, k < i, j^* \leq k \)
5. \( i, j, k \geq 0, i^* < j, k < i, j^* > k \)
6. \( i, j \geq 0 \) and \( k < 0 \)
7. \( i \geq 0, j < 0 \) and \( k \geq 0 \)
8. \( i \geq 0, j < 0 \) and \( k < 0 \)
9. \( i < 0, j \geq 0 \) and \( k \geq 0 \)
10. \( i < 0, j \geq 0 \) and \( k < 0 \)
11. \( i < 0, j < 0 \) and \( k \geq 0 \)
12. \( i, j, k \geq 0 \)

Observe that, if \( x \in [−n, m] \), then, from Corollary 4.4 we have \( x' = x^* \), and from Lemma 4.5 we have that \( x'' = x \); and in particular \( 0'' = 0 \). Thus, we conclude that \( [−n, m] : \Rightarrow 0 \) \( \in \mathcal{I}_{2,0} \). \( \square \)
In view of the above lemma and Theorem 3.8, the relation defined by
\[ x \sqsubseteq y \quad \text{if and only if} \quad (x \Rightarrow y)' = x \]
is a partial order on \([-n, m]\). We now wish to show that \(\sqsubseteq\) is indeed a total order.

**Lemma 4.8** Let \([-n, m]\) be the algebra, as defined in Definition 4.2. Then
\[ \langle [-n, m]; \sqsubseteq \rangle \cong \langle [-n, m]; \leq \rangle. \]

**Proof** Let \(x, y \in [-n, m]\). It is enough to prove that \(x \leq y\) if and only if \(x \sqsubseteq y\).
Assume that \(x \leq y\). We will consider the following cases:

- **Case 1**: \(x < 0\). Then
  \[ (x \Rightarrow y)' = (x \Rightarrow y^*) = \left[ \min(x, y^*) \right]^*. \]
  We consider further the following subcases:
  - **Case 1.1**: \(y < 0\).
    \[ (x \Rightarrow y)' = \left[ \min(x, y^*) \right]^* \quad \text{by (4.2)} \]
    \[ = \left[ \min(x, y) \right]^* \quad \text{since } y < 0 \]
    \[ = x^* \quad \text{since } x \leq y \]
    \[ = x. \quad \text{since } x < 0 \]
  - **Case 1.2**: \(y \geq 0\).
    \[ (x \Rightarrow y)' = \left[ \min(x, y^*) \right]^* \quad \text{by (4.2)} \]
    \[ = x^* \quad \text{since } y^* \geq 0 \text{ by Lemma 4.3, and } x < 0 \]
    \[ = x. \]

- **Case 2**: \(x \geq 0\). Therefore \(y \geq 0\). In this case
  \[ (x \Rightarrow y)' = (x \Rightarrow y^*) = \left[ \max(x^*, y^*) \right]^* \]
  \[ = x^{**} \quad \text{by Lemma 4.6} \]
  \[ = x. \]

Thus, in all these cases, \(x \sqsubseteq y\).

For the converse, suppose \(x \sqsubseteq y\).

- **Case 1**: \(x < 0\). If \(y \geq 0\) then \(x < y\). So, we can assume \(y < 0\). Then
  \[ x = x' \quad \text{since } x < 0 \]
  \[ = (x \Rightarrow y)''' \quad \text{by hypothesis} \]
  \[ = x \Rightarrow y' \quad \text{by Lemma 4.6} \]
  \[ = x \Rightarrow y \]
  \[ = \min(x, y). \]
  Hence \(x \leq y\).
Case 2: $x \geq 0$. Suppose $y < 0$. Then
\[
x = (x \Rightarrow y)' \quad \text{by hypothesis}
\]
\[
= (x \Rightarrow y)'
\]
\[
= \min(x, y)'
\]
\[
= y',
\]
a contradiction. Hence $y \geq 0$. Consequently,
\[
x' = (x \Rightarrow y)''
\]
\[
= x \Rightarrow y' \quad \text{by Lemma 4.5}
\]
\[
= \max(x', y'),
\]
so, $x' \geq y'$. Then, by Lemma 4.3 and Lemma 4.6, $x = x'' \leq y'' = y$.

In view of Lemma 4.7 and Lemma 4.8, we have proved the following

Theorem 4.9 \([-n, m]\) is an $I_{2,0}$-chain, where
\[-n \sqsubset -n + 1 \sqsubset \ldots \sqsubset -1 \sqsubset 0 \sqsubset 1 \sqsubset 2 \sqsubset \ldots \sqsubset m.\]

5 Characterization of finite $I_{2,0}$-chains

In this section we are going to prove our second main result. The following lemmas will be useful later in this section.

Lemma 5.1 Let $A \in I_{2,0}$. Then $0'$ is the greatest element in $A$, relative to $\sqsubset$.

Proof Let $a \in A$. Since $(a \rightarrow (0 \rightarrow 0))' = (a \rightarrow 0) = a$, we have $a \sqsubseteq 0'$. \qed

Lemma 5.2 Let $A \in I_{2,0}$ and let $a, b \in A$ with $0 \sqsubseteq a \sqsubseteq b$. Then $b' \sqsubseteq a'$.

Proof
\[
(b' \rightarrow a'')' = (b' \rightarrow a')'
\]
\[
= (b' \rightarrow (a \rightarrow b'))' \quad \text{by hypothesis}
\]
\[
= ((a \rightarrow 0') \rightarrow b')' \quad \text{by Lemma 2.6 (2b)}
\]
\[
= ((a \rightarrow 0') \rightarrow b')
\]
\[
= ((0 \rightarrow a') \rightarrow b)' \quad \text{by Lemma 2.5 (a)}
\]
\[
= (0' \rightarrow b)'
\]
\[
= b' \quad \text{by hypothesis}
\]
\[
= b' \quad \text{by Lemma 2.4 (a).}
\]

\qed

Lemma 5.3 Let $A \in I_{2,0}$ and let $a \in A$. If $0 \sqsubseteq a$ then $0 \rightarrow a = 0'$.
Proof First notice that, since \(0 \subseteq a\), \(0' = (0 \to a)' = 0 \to a'\). Consequently,

\[(5.1) \quad 0' = 0 \to a'.\]

Then

\[
\begin{align*}
0' &= 0' \to 0' & & \text{by Lemma 2.4 (a)} \\
&= (0 \to a') \to 0' & & \text{(5.1)} \\
&= (0' \to a') \to 0' & & \text{by Lemma 2.5 (a)} \\
&= a' \to 0' & & \text{by Lemma 2.4 (a)} \\
&= 0 \to a. & & \text{by Lemma 2.5 (a)}
\end{align*}
\]

□

**Lemma 5.4** Let \(A \in I_{2,0}\) and let \(a, b \in A\). If \(0 \subseteq a\) and \(0 \subseteq b\) then \(0 \subseteq a \to b\).

**Proof**

\[
\begin{align*}
[0 \to (a \to b)']' &= [(a \to b) \to 0']' & & \text{by Lemma 2.5 (a)} \\
&= (0 \to a) \to (b \to 0')' & & \text{by (I)} \\
&= (0 \to a) \to (0 \to b')' & & \text{by Lemma 2.5 (a)} \\
&= (0 \to a) \to 0 & & \text{since } 0 \subseteq b \\
&= 0' \to 0 & & \text{by Lemma 5.3 since } 0 \subseteq a \\
&= 0. & & \text{by Lemma 2.4 (a)}
\end{align*}
\]

□

**Corollary 5.5** Let \(A \in I_{2,0}\) and \(a \in A\). If \(a \supseteq 0\) then \(a' \supseteq 0\).

**Lemma 5.6** Let \(A\) be an \(I_{2,0}\)-chain and let \(a, b \in A\). Then \(a' \to b' = b \to a\).

**Proof** Since \(A\) is a chain, we can assume that \(b' \subset a\) or \(a \subset b'\).

If \(b' \subset a\), \((b' \to a)' = b', \text{ then } b' \to a' = b\). Hence \(b \to a = (b' \to a') \to a = [(a' \to b') \to (a' \to a)']'\), using (I). By Lemma 2.4 (a), \([(a' \to b') \to (a' \to a)']' = [(a' \to b') \to a']' = (((a \to a') \to (b' \to a')')')' = (a \to a') \to (b' \to a')' = (a'' \to a') \to (b' \to a')' = a' \to b'.

If \(a \subset b'\) then we have \(a' = (a \to b')'' = a \to b\), and the rest of the argument is similar to the previous case.

□

**Lemma 5.7** Let \(A\) be a \(I_{2,0}\)-chain with \(|A| \geq 2\) and let \(a \in A\) such that \(a \subset 0\). Then

(a) \(0 \to a' = a'\)
(b) \(0 \to a = a\)
(c) \((a \to a) \to a = a \to a\)
(d) \(a \to a = a'\)
(e) \(a \to a = a\)
(f) \(a = a'\).

**Proof**

(a) Since \(a \subset 0\), we have that \(a = (a \to 0)'\). Therefore, \(a' = (a \to 0)''' = a \to 0' = 0 \to a'\) by Lemma 2.5 (b).
Since \( a \subseteq 0 \), we have

\((5.2)\) \( a = (a \rightarrow 0)' \).

Then we get

\[
(0 \rightarrow a) \rightarrow 0' = \left[ (0 \rightarrow 0) \rightarrow (a \rightarrow 0)' \right]' \quad \text{by (I)} \\
= \left[ (0 \rightarrow 0) \rightarrow a \right]' \quad \text{by (5.2)} \\
= \left[ 0' \rightarrow a \right]' \\
= a' \quad \text{by lemma 2.4 (a)}
\]

Using Lemma 2.5 (b), we obtain

\((5.3)\) \( a' = 0 \rightarrow (0 \rightarrow a)' \).

Since \( A \) is a chain, \( 0 \subseteq 0 \rightarrow a \) or \( 0 \rightarrow a \subseteq 0 \). Suppose that \( 0 \subseteq 0 \rightarrow a \). Then \( (0 \rightarrow (0 \rightarrow a)')' = 0 \) Therefore, by (5.3), \( a = a'' = (0 \rightarrow (0 \rightarrow a)')' = 0 \), a contradiction, since \( a \neq 0 \). Consequently, \( 0 \rightarrow a \subseteq 0 \). Hence, we have

\[
0 \rightarrow a = \left( (0 \rightarrow a) \rightarrow 0 \right)' \quad \text{since } 0 \rightarrow a \subseteq 0 \\
= (0 \rightarrow (0 \rightarrow a)')' \quad \text{by lemma 2.5 (b)} \\
= a'' \quad \text{by (5.3)} \\
= a.
\]

\((c)\)

\[
a \rightarrow a = (0 \rightarrow a) \rightarrow a \quad \text{by item (b)} \\
= (a' \rightarrow 0') \rightarrow a \quad \text{by lemma 2.5 (a)} \\
= \left[ (a' \rightarrow a') \rightarrow (0' \rightarrow a) \right]' \quad \text{by (I)} \\
= \left[ (a \rightarrow a) \rightarrow (0' \rightarrow a) \right]' \quad \text{by Lemma 5.6} \\
= \left[ (a \rightarrow a) \rightarrow a' \right]' \quad \text{by lemma 2.5 (a)} \\
= \left[ [(a'' \rightarrow a) \rightarrow (a \rightarrow a')]' \right]' \quad \text{by (I)} \\
= (a \rightarrow a) \rightarrow (a \rightarrow a')' \\
= (a \rightarrow a) \rightarrow a'' \quad \text{by lemma 2.5 (b)} \\
= (a \rightarrow a) \rightarrow a.
\]

\((d)\)

Since \( A \) is a chain, \( 0 \rightarrow a' \subseteq a \) or \( a \subseteq 0 \rightarrow a' \).

First, we assume that \( 0 \rightarrow a' \subseteq a \). Then

\[
a \rightarrow a = (a \rightarrow a) \rightarrow a \quad \text{by (c)} \\
= (a' \rightarrow a') \rightarrow a \quad \text{by Lemma 5.6} \\
= (a' \rightarrow (a' \rightarrow a')' \quad \text{by Lemma 5.6} \\
= (a \rightarrow 0) \rightarrow (a' \rightarrow a')' \\
= \left[ (a \rightarrow 0) \rightarrow (a' \rightarrow a') \right]'' \\
= \left[ (0 \rightarrow a') \rightarrow a' \right]' \quad \text{using (I)} \\
= 0 \rightarrow a' \quad \text{since } 0 \rightarrow a' \subseteq a \\
= a' \quad \text{using (b)}.
\]

Next, we assume \( a \subseteq 0 \rightarrow a' \), i.e., \( (a \rightarrow (0 \rightarrow a')')' = a \). Then, from (b), we have \( a \rightarrow a = (a \rightarrow a'')'' = [(a \rightarrow (0 \rightarrow a'))']' = a' \).
Using the items (ii), (iv) and Lemma 2.5 (ii), we have \( a \rightarrow a = (a \rightarrow a) \rightarrow a = a' \rightarrow a = a \).

This follows immediately from the two preceding items.

**Remark 5.10**

**Lemma 5.8** Let \( A \) be an \( \mathbf{I}_{2,0} \)-chain with \( |A| \geq 2 \), and let \( a, b \in A \). If \( 0 \sqsubseteq a \) and \( b \sqsubseteq 0 \) then \( b \rightarrow a = b \) and \( a \rightarrow b = b \).

**Proof** Since 0 \( \sqsubseteq a \) and \( b \sqsubseteq 0 \), we have that \( (0 \rightarrow a' \prime) = 0 \) and \( (b \rightarrow 0') = b \). Therefore, using Lemma 5.6, \( b = b'' = b' \rightarrow 0 = (b \rightarrow 0)' \rightarrow (0 \rightarrow a') = (b \rightarrow 0') \rightarrow (0 \rightarrow a')' = (b \rightarrow b') \rightarrow (a \rightarrow 0')' = [(b' \rightarrow a) \rightarrow 0'] \). Hence,

\[
(5.4) \quad b = [(b' \rightarrow a) \rightarrow 0'].
\]

From the hypothesis and Lemma 5.7 (ii), we have

\[
(5.5) \quad b' = b.
\]

Suppose that \( 0 \sqsubseteq b' \rightarrow a \). Then \( 0 = [0 \rightarrow (b' \rightarrow a)'] = [(b' \rightarrow a) \rightarrow 0'] \) by Lemma 5.6, implying \( 0 = b \), which is a contradiction in view of (5.4). Consequently, \( b' \rightarrow a \sqsubseteq 0 \), since \( A \) is a chain. Hence,

\[
(5.6) \quad b' \rightarrow a = [(b' \rightarrow a) \rightarrow 0'].
\]

From (5.4), (5.5) and (5.6) we conclude \( b = b \rightarrow a \), proving the first half of the conclusion of the lemma. From

\[
\begin{align*}
b &= (b \rightarrow a')' \quad \text{since } b \sqsubseteq a, \text{ as } 0 \sqsubseteq a \text{ and } b \sqsubseteq 0 \\
 &= (a'' \rightarrow b')' \quad \text{by Lemma 5.6} \\
 &= (a \rightarrow b')' \quad \text{by (5.5)}
\end{align*}
\]

we conclude that \( a \rightarrow b = b' = b \) in view of (5.5), completing the second half. \( \square \)

**Definition 5.9** Let \( A = \langle A; \rightarrow, 0 \rangle \) be a finite \( \mathbf{I}_{2,0} \)-chain. We let \( A^+ := \{ a \in A : a \sqsubseteq 0 \} \) and \( A^- := \{ a \in A : a \sqsubseteq 0 \} \). Observe that \( A = A^+ \cup \{0\} \cup A^- \). Henceforth, without loss of generality, we will represent \( A = [-n, m] \) with \( 0 \leq n, m \leq |A| - 1 \), such that

\[
-n \sqsubseteq -n + 1 \sqsubseteq \ldots \sqsubseteq -1 \sqsubseteq 0 \sqsubseteq 1 \sqsubseteq 2 \sqsubseteq \ldots \sqsubseteq m.
\]

**Remark 5.10** In view of the above definition, we can use the functions \( * \) and \( p \) of Definition 4.2 as functions on the domain \([-n, m]\) of \( A \) as well.

Now, we wish to prove that \( \langle A; \rightarrow, 0 \rangle = \langle [-n, m]; \Rightarrow, 0 \rangle \). To achieve this, we need the following lemmas.

**Lemma 5.11** Let \( A = \langle A; \rightarrow, 0 \rangle \) be a finite \( \mathbf{I}_{2,0} \)-chain with \( |A| \geq 2 \). If \( a \sqsubseteq 0 \) then \( a' = p(p(a)) \).

**Proof** By hypothesis we have that \( a \sqsubseteq 0 \). Then \( p(a) \sqsubseteq 0 \). Hence \( 0 \sqsubseteq p(a) \sqsubseteq a \). Then, by Lemma 5.2

\[
(5.7) \quad a' \sqsubseteq p(a)'.
\]
Since $a \equiv 0$, by Corollary $5.5$ $a' \equiv 0$. Therefore, by $5.7$,

(5.8) $0 \equiv p(a')$. 

If $a' = p(a)$ then $a = p(a)$ and, consequently, $a = -n$, a contradiction, so $a' \subset p(a)'$, and hence, $0 \subset a' \subset p(p(a)) \subset p(a)$. By lemma $5.2$ $a \equiv [p(p(a))]' \equiv p(a)$. Thus

(5.9) $[p(p(a))]' = \{a, p(a)\}$. 

If $[p(p(a))]' = p(a)$, we have that $p(p(a)) = [p(p(a))]'' = p(a)'$, a contradiction, since $p(a)' \equiv 0$ by (5.3). Therefore $[p(p(a))]' = a$ and therefore, $p(p(a)) = a'$. □

**Lemma 5.12** Let $A = \langle A; \to, 0 \rangle$ be a finite $I_{2,0}$-chain. If $a \in A$ then $a^* = a'$.

**Proof** The statement $0' = m = 0^*$ follows from Lemma $5.1$. If $a \equiv 0$ then $a' = a$ by Lemma $5.7$.[1] and $a = a^*$ by definition, implying $a = a^*$.

Now assume that $a \equiv 0$. We will verify that $a' = a^*$ by induction on $a$. If $a = 1$, then, as $0' = 0^*$, we have, by Lemma $5.11$ that $1' = p(1)' = p(0') = p(0^*) = p(1^*) = 1^*$. The inductive hypothesis is that $p(1) = p(0)$. Hence, we have, by Lemma $5.11$ $a' = p(p(a)) = p(p(a)^*) = a^*$. □

The following theorem shows that the general method described in Definition $4.2$ essentially gives all finite $I_{2,0}$-chains.

**Theorem 5.13** Let $A$ be a finite $I_{2,0}$-chain. Then $A \cong \langle [-n, m]; \to, 0 \rangle$ for some $0 \leq n, m \leq |A| - 1$.

**Proof** We will use the notation of Definition $5.9$. Let $i, j \in A$. From Lemma $5.12$ $i' = i^*$ and $j' = j^*$. It suffices to verify that

$$i \to j = \begin{cases} \max(i', j) & \text{if } i, j \equiv 0 \\ \min(i, j) & \text{otherwise} \end{cases}$$

with $0' = m$. We consider the following cases:

- **Case 1**: $j > 0$.

  We need the following subcases:

  - **Case 1.1**: $i > 0$.

    We make the following further subcases:

    * **Case 1.1.1**: $i' \geq j$.

      Since $i' \equiv j$, we observe that

      (5.10) $(j \to i')' = j$.

      Hence

      $$i \to j = i \to (j \to i')'$$ by (5.10)

      $$= i \to (j \to i)'$$

      $$= [(i \to j) \to i]'$$ by Lemma $2.6$[4]

      $$= [(0 \to j) \to i]'$$ by Lemma $2.8$[6]

      $$= [0' \to i]'$$ by Lemma $5.3$ since $j \equiv 0$

      $$= i'$$ by Lemma $2.4$[13]

      $$= \max(i', j)$$ since $i' \equiv j$
Case 1.1.2: $i' < j$.

Since $i' \subseteq j$, we have

$$(5.11) \ (i' \rightarrow j')' = i'.$$

Therefore,

$$i \rightarrow j = i'' \rightarrow j$$
$$= (i' \rightarrow j')'' \rightarrow j \quad \text{by (5.11)}$$
$$= (i' \rightarrow j') \rightarrow j$$
$$= (i' \rightarrow 0') \rightarrow j \quad \text{by Lemma 2.6 (1)}$$
$$= (0 \rightarrow i) \rightarrow j \quad \text{by Lemma 2.3 (3)}$$
$$= 0' \rightarrow j \quad \text{by Lemma 5.3 since } i \not\subset 0$$
$$= j \quad \text{by Lemma 2.3 (1)}$$
$$= \max(i', j) \quad \text{since } i' \subseteq j$$

– Case 1.2: $i = 0$.

Using Lemma 5.3 and Lemma 5.1, $0 \rightarrow j = 0' = \max(0', j)$.

– Case 1.3: $i < 0$.

Therefore,

$$i \rightarrow j = (0 \rightarrow i) \rightarrow j$$
$$= (i' \rightarrow 0') \rightarrow j$$
$$= (i \rightarrow 0') \rightarrow j \quad \text{by Lemma 5.7 (1)}$$
$$= [(j' \rightarrow i) \rightarrow (0' \rightarrow j)']' \quad \text{by (1)}$$
$$= [(j' \rightarrow i) \rightarrow j']'$$
$$= [(0 \rightarrow i) \rightarrow j']' \quad \text{by Lemma 2.6 (3)}$$
$$= (i \rightarrow j')' \quad \text{by Lemma 5.7 (3)}$$
$$= i \quad \text{since } i \subset j$$
$$= \min(i, j)$$

Case 2: $j < 0$.

It is useful to consider the following subcases:

– Case 2.1: $i > 0$

$$i \rightarrow j = i \rightarrow j'$$
$$= i \rightarrow (j \rightarrow i')'' \quad \text{by Lemma 5.7 (1)}$$
$$= i \rightarrow (j \rightarrow i')$$
$$= j \rightarrow i' \quad \text{by Lemma 2.6 (17)}$$
$$= (j \rightarrow i')''$$
$$= j' \quad \text{since } j \subset i$$
$$= j \quad \text{by Lemma 5.7 (1)}$$
$$= \min(i, j)$$

– Case 2.2: $i = 0$.

$$i \rightarrow j = 0 \rightarrow j$$
$$= j \quad \text{by Lemma 5.7 (1)}$$
$$= \min(i, j)$$

– Case 2.3: $i < 0$. 

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* Case 2.3.1: $i \leq j$.
As $i \sqsubseteq j$, we have

\[ (5.12) \quad (i \rightarrow j')' = i. \]

Observe

\[
\begin{align*}
  i \rightarrow j &= i \rightarrow j' & \text{by Lemma 5.7 (f)} \\
  &= (i \rightarrow j')'' & \text{by (5.12)} \\
  &= i' & \text{by Lemma 5.7 (f)} \\
  &= i & \text{by Lemma 5.7 (f)} \\
  &= \min(i, j).
\end{align*}
\]

* Case 2.3.2: $i > j$. We have

\[ (5.13) \quad (j \rightarrow i')' = j. \]

as $j \sqsubseteq i$. Hence

\[
\begin{align*}
  i \rightarrow j &= j' \rightarrow i' & \text{by Lemma 5.6} \\
  &= j \rightarrow i' & \text{by Lemma 5.7 (f)} \\
  &= (j \rightarrow i')'' & \text{by (5.13)} \\
  &= j' & \text{by Lemma 5.7 (f)} \\
  &= j & \text{by Lemma 5.7 (f)} \\
  &= \min(j, i)
\end{align*}
\]

• Case 3: $j = 0$.
  
  – Case 3.1: $i \geq 0$.
  
  By Corollary 5.5 as $i \sqsupseteq 0$, we have that $i' = i \rightarrow 0 \sqsupseteq 0$. Hence $i \rightarrow 0 = i' = \max(i', 0)$.
  
  – Case 3.2: $i < 0$. We have that

\[
\begin{align*}
  i \rightarrow j &= i \rightarrow 0 \\
  &= i' & \text{by Lemma 5.7 (f)} \\
  &= \min(i, j)
\end{align*}
\]

Hence $A \cong \langle [-n, m]; \Rightarrow, 0 \rangle$.

The following theorem, our second main result, is now immediate from the preceding results.

**Theorem 5.14** There are $n$ non-isomorphic $I_{2,0}$-chains of size $n$, for $n \in \mathbb{N}$.

### A Appendix: Proofs

We would like to mention here that the identity: $x'' \approx x$ is used in these proofs frequently without explicit mention.

**Proof of Lemma 2.6** Items (1) to (17) are proved in [3]. The proofs of (18) to (26) are given in [5]. Let $a, b, c, d \in A$.
(27) \[ (b \rightarrow c) \rightarrow a \ = \ [(b \rightarrow c) \rightarrow a]' \rightarrow [(b \rightarrow c) \rightarrow a] \] by Lemma 2.4 (3)
\[ = \ [(a' \rightarrow b) \rightarrow (c \rightarrow a)'']' \rightarrow [(b \rightarrow c) \rightarrow a] \] from (1)
\[ = \ [(a' \rightarrow b) \rightarrow (c \rightarrow a)'] \rightarrow [(b \rightarrow c) \rightarrow a] \]

(28) \[ [0 \rightarrow (a \rightarrow b)''] \rightarrow (0 \rightarrow b)''' \rightarrow [0 \rightarrow (a \rightarrow b)''] \] by Lemma 2.5 (11)
\[ = \ [(a \rightarrow b)' \rightarrow b] \rightarrow 0' \] by (1)
\[ = 0 \rightarrow [(a \rightarrow b)' \rightarrow b]' \] by Lemma 2.5 (11)
\[ = (a \rightarrow b) \rightarrow (0 \rightarrow b)' \] by (10)
\[ = 0 \rightarrow [(a \rightarrow b) \rightarrow b] \] by (13)
\[ = 0 \rightarrow [(a \rightarrow 0') \rightarrow b'] \] by (11)
\[ = (a \rightarrow 0') \rightarrow (0 \rightarrow b') \] by (13)
\[ = (0 \rightarrow a') \rightarrow (0 \rightarrow b') \] by Lemma 2.4 (11)
\[ = a' \rightarrow (0 \rightarrow b') \] by (20)
\[ = [(0 \rightarrow a) \rightarrow (0 \rightarrow b)'']' \] by (8)
\[ = [(0 \rightarrow a) \rightarrow (b \rightarrow 0')]'' \] by Lemma 2.4 (11)
\[ = (a \rightarrow b) \rightarrow 0' \] by (1)
\[ = 0 \rightarrow (a \rightarrow b)' \] by Lemma 2.4 (11)

(29) \[ 0 \rightarrow [(a \rightarrow b) \rightarrow c] \ = \ 0 \rightarrow [(c' \rightarrow a) \rightarrow (b \rightarrow c)']' \] by (1)
\[ \subset \ = 0 \rightarrow (b \rightarrow c)'' \] by 28
\[ = 0 \rightarrow (b \rightarrow c) \]

(30) \[ a' \rightarrow (b \rightarrow 0')' \ = \ (a \rightarrow 0) \rightarrow (b \rightarrow 0')' \]
\[ = \ [\{(b \rightarrow 0') \rightarrow a\} \rightarrow \{0 \rightarrow (b \rightarrow 0')\}']' \] by (1)
\[ = \ [\{(b \rightarrow 0') \rightarrow a\} \rightarrow \{0 \rightarrow (0 \rightarrow b')\}']' \] by Lemma 2.5 (11)
\[ = \ [\{(b \rightarrow 0') \rightarrow a\} \rightarrow (0 \rightarrow b')']' \] by 49
\[ = \ [\{(0 \rightarrow b') \rightarrow a\} \rightarrow (0 \rightarrow b')']' \] by Lemma 2.5 (11)
\[ = \ [\{(0 \rightarrow 0 \rightarrow b')\} \rightarrow a] \rightarrow (0 \rightarrow b')']' \] by 49
\[ = \ [a \rightarrow (0 \rightarrow b')']' \] by 18

(31) \[ [(0 \rightarrow a) \rightarrow b'] = \ [(b \rightarrow a) \rightarrow b]' \] by 51
\[ = \ [b \rightarrow (a \rightarrow b)']' \] by 11
\[ = b \rightarrow (a \rightarrow b) \]

(32) \[ [a \rightarrow (b \rightarrow 0')]' = \ [a \rightarrow (0 \rightarrow b')]' \] by Lemma 2.5 (11)
\[ = \ a' \rightarrow (b' \rightarrow 0')' \] by 30
\[ = \ a' \rightarrow (0 \rightarrow b')' \] by Lemma 2.5 (11)

(33) \[ b' \rightarrow a' = \ (b \rightarrow 0) \rightarrow a' \]
\[ = \ [(a \rightarrow b) \rightarrow (0 \rightarrow a')]'' \] by (1)
\[ = \ [(a \rightarrow b) \rightarrow (a \rightarrow 0')]' \] by Lemma 2.5 (11)
\[ = \ (a \rightarrow b)' \rightarrow (0 \rightarrow a)' \] by 32 with \( x = a \rightarrow b, y = a \)

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\[(0 \to a)' \to (0 \to b)' = \left[ (0 \to a) \to 0 \to (0 \to b)' \right] \]
\[= \left[ \{(0 \to b) \to (0 \to a)\} \to \{0 \to (0 \to b)'\}' \right] \]
\[\text{by (I)} \]
\[= \left[ \{(0 \to b) \to (0 \to b)'\} \to (0 \to b) \right] \to \left[ (0 \to a) \to \{0 \to (0 \to b)'\}' \right] \]
\[\text{by (I)} \]
\[= \left[ \{(0 \to b) \to (0 \to b)'\} \to (0 \to b) \right] \to \left[ (0 \to a) \to \{0 \to (0 \to b)'\}' \right] \]
\[\text{by (3)} \]
\[= \left[ \{0 \to b\}' \to (0 \to b) \right] \to \left[ (0 \to a) \to \{0 \to (0 \to b)'\}' \right] \]
\[\text{by Lemma 2.4 (i)} \]
\[= \left[ \{(0 \to a) \to \{0 \to (0 \to b)'\}' \right] \]
\[\text{by Lemma 2.4 (i)} \]
\[= \left[ (0 \to b) \to [(0 \to a) \to (0 \to b)'\]' \right] \]
\[\text{by (3)} \]
\[= \left[ (0 \to b) \to [(0 \to a) \to (b \to 0')\]' \right] \]
\[\text{by Lemma 2.5 (i)} \]
\[= \left[ (0 \to b) \to [0 \to (a \to b)'] \right] \]
\[\text{by (1)} \]
\[= \left[ (0 \to b) \to (a \to b)' \right] \]
\[\text{by Lemma 2.5 (i)} \]
\[= 0 \to [(0 \to b) \to (a \to b)'] \]
\[\text{by (13)} \]
\[= 0 \to (a \to b)' \]
\[\text{by (3)} \]
\[= (a \to b) \to 0' \]
\[\text{by Lemma 2.5 (i)} \]
\[= [(0 \to a) \to (b \to 0')]' \]
\[\text{by (1)} \]
\[= [(0 \to a) \to (0 \to b)'\]' \]
\[\text{by Lemma 2.5 (i)} \]
\[= [a' \to (0 \to b')]', \]
\[\text{by (8)} \]
\[= \left[ \left[ (0 \to a) \to (b \to 0') \right] \to \left[ (a \to b)' \right] \right] \]
\[\text{by (3)} \]
\[= \left[ (a \to b)' \right] \]
\[\text{by (3)} \]
\[= \left[ (0 \to a) \to (a \to b)' \right] \]
\[\text{by (13)} \]
\[= (b \to a) \to b \]
\[\text{by (3)} \]
\[= [b \to (a \to b)', \]
\[\text{by (4)} \]
\[= (a \to b)' \to [b \to (a \to b)'], \]
\[\text{by (35) with } x = b, y = (a \to b)' \]
\[= \left[ \{(a \to b)' \to b \right] \to (a \to b)' \]
\[\text{by (4)} \]
\[= \left[ (0 \to b) \to (a \to b)' \right] \]
\[\text{by (5)} \]
\[= (a \to b)' \]
\[\text{by (5)} \]
\[= a \to b \]
\[\text{since } x'' \approx x \]
Proof of Lemma 3.3

(1) Observe that by Lemma 2.5 (a), Lemma 2.6 (1) and the hypothesis we have that \((0 \rightarrow a') \rightarrow b = (a \rightarrow 0') \rightarrow b = (a \rightarrow b') \rightarrow b = (a \rightarrow b')'' \rightarrow b = a' \rightarrow b.

(2) 
\[
\begin{align*}
  b \rightarrow a' &= [(0 \rightarrow a') \rightarrow b] \rightarrow a' \quad \text{by Lemma 2.6 (18)} \\
  &= (a' \rightarrow b) \rightarrow a' \quad \text{from (1)} \\
  &= (0 \rightarrow b) \rightarrow a' \quad \text{by Lemma 2.6 (5)}.
\end{align*}
\]

(3) 
\[
\begin{align*}
  b \rightarrow a' &= (0 \rightarrow b) \rightarrow a' \quad \text{from (2)} \\
  &= (0 \rightarrow b) \rightarrow (a \rightarrow b')'' \quad \text{by hypothesis} \\
  &= (0 \rightarrow b) \rightarrow (a \rightarrow b') \\
  &= (0 \rightarrow b') \rightarrow (a \rightarrow b') \quad \text{by Lemma 2.6 (2)} \\
  &= a \rightarrow b' \quad \text{by hypothesis} \\
  &= (a \rightarrow b')'' \quad \text{by Lemma 2.6 (2)} \\
  &= a' \quad \text{by hypothesis}
\end{align*}
\]

(4) 
\[
\begin{align*}
  0 \rightarrow (a' \rightarrow b) &= a' \rightarrow (0 \rightarrow b) \quad \text{by Lemma 2.6 (13)} \\
  &= 0 \rightarrow (a \rightarrow b')' \quad \text{by Lemma 2.6 (10)} \\
  &= 0 \rightarrow a \quad \text{by hypothesis}
\end{align*}
\]

(5) By hypothesis and (1) we have that \((d \rightarrow a) \rightarrow b' = [(b \rightarrow d) \rightarrow (a \rightarrow b')]' = [(b \rightarrow d) \rightarrow a']'.

(6) 
\[
\begin{align*}
  \{d \rightarrow (0 \rightarrow b')\} \rightarrow a' &= (a' \rightarrow d) \rightarrow \{0 \rightarrow b' \rightarrow a'\}' \quad \text{by (1)} \\
  &= (a' \rightarrow d) \rightarrow \{(a \rightarrow b') \rightarrow a'\}' \quad \text{by Lemma 2.6 (5)} \\
  &= (a' \rightarrow d) \rightarrow \{(a \rightarrow b')'' \rightarrow a'\}' \\
  &= (a' \rightarrow d) \rightarrow (a' \rightarrow a)' \quad \text{by hypothesis} \\
  &= (a' \rightarrow d) \rightarrow a' \quad \text{by Lemma 2.6 (4)} \\
  &= (a' \rightarrow d) \rightarrow (0' \rightarrow a)' \quad \text{by Lemma 2.6 (4a)} \\
  &= [(d \rightarrow 0') \rightarrow a]' \quad \text{by (1)} \\
  &= (0 \rightarrow d) \rightarrow a' \quad \text{by Lemma 2.6 (11)}
\end{align*}
\]

\[
\square
\]
(7)

\[ a \rightarrow [(a' \rightarrow d) \rightarrow \{(0 \rightarrow a) \rightarrow b'] = a \rightarrow [(a' \rightarrow d) \rightarrow \{(b \rightarrow 0) \rightarrow (a \rightarrow b')'] \]

by (I)

\[ = a \rightarrow [(a' \rightarrow d) \rightarrow \{(b \rightarrow 0) \rightarrow a'] \]

by hypothesis

\[ = a \rightarrow [\{d \rightarrow (b \rightarrow 0)\} \rightarrow a']' \]

by (I)

\[ = [[d \rightarrow (0 \rightarrow (b \rightarrow 0))] \rightarrow a']' \]

by Lemma 2.6 (22) with \( x = d, y = b \rightarrow 0, z = a \)

\[ = [[d \rightarrow (0 \rightarrow b')] \rightarrow a']' \]

by (6)

(8)

\[ a \rightarrow ((d \rightarrow a) \rightarrow b') = a \rightarrow ((b \rightarrow d) \rightarrow a')' \]

by (5)

\[ = a'' \rightarrow ((b \rightarrow d) \rightarrow a')' \]

by Lemma 2.6 (1)

\[ = (a' \rightarrow 0) \rightarrow ((b \rightarrow d) \rightarrow a')' \]

by Lemma 2.6 (1)

\[ = \{0 \rightarrow (b \rightarrow 0)\} \rightarrow a']' \]

by Lemma 2.6 (1)

\[ = [\{(b \rightarrow 0) \rightarrow 0') \rightarrow 0'\} \rightarrow a']' \]

by Lemma 2.6 (1)

\[ = [\{d \rightarrow (d \rightarrow a) \rightarrow 0') \rightarrow a']' \]

by Lemma 2.6 (1)

\[ = [\{d \rightarrow (d \rightarrow a) \rightarrow 0') \rightarrow a']' \]

by Lemma 2.6 (1)

\[ = [\{0 \rightarrow ((d \rightarrow a) \rightarrow b')] \rightarrow a']' \]

by (5)

\[ = [\{a \rightarrow ((d \rightarrow a) \rightarrow b')\} \rightarrow a']' \]

by Lemma 2.6 (5)

\[ = (a' \rightarrow a) \rightarrow [\{(d \rightarrow a) \rightarrow b') \rightarrow a']' \]

by Lemma 2.6 (1)

\[ = a \rightarrow [\{(d \rightarrow a) \rightarrow b') \rightarrow a']' \]

by Lemma 2.6 (1)

\[ = a \rightarrow [\{a' \rightarrow (d \rightarrow a)\} \rightarrow (b' \rightarrow a')'] \]

by (I)

\[ = a \rightarrow [\{a' \rightarrow (d \rightarrow a)\} \rightarrow \{(b \rightarrow 0) \rightarrow a')'] \]

by hypothesis

\[ = a \rightarrow [\{a' \rightarrow (d \rightarrow a)\} \rightarrow \{(0 \rightarrow a) \rightarrow b')\}] \]

by (I)

\[ = [0 \rightarrow (d \rightarrow a)] \rightarrow a' \]

by (7) with \( d := d \rightarrow a \)

\[ = a \rightarrow (d \rightarrow a)' \]

by Lemma 2.6 (23)

(9)

\[ [0 \rightarrow (b \rightarrow d)] \rightarrow a = [(a' \rightarrow 0) \rightarrow ((b \rightarrow d) \rightarrow a')']' \]

by (1)

\[ = [a \rightarrow ((b \rightarrow d) \rightarrow a')']' \]

by Lemma 2.6 (1)

\[ = [a \rightarrow ((d \rightarrow a) \rightarrow b')'] \]

by (5)

\[ = [a \rightarrow (d \rightarrow a)'] \]

by (8)

\[ = (a \rightarrow d) \rightarrow a \]

by Lemma 2.6 (1)

\[ = (0 \rightarrow d) \rightarrow a \]

by Lemma 2.6 (1)
(10) \[(b \rightarrow (a \rightarrow d)) \rightarrow a = \left[ (a' \rightarrow b) \rightarrow \{a \rightarrow d\} \right]' \] by (I)
\[= \left[ (a' \rightarrow b) \rightarrow \{0 \rightarrow a\} \right]' \] by Lemma 2.6 (5)
\[= [b \rightarrow (0 \rightarrow d)] \rightarrow a \] by (I)
\[= [0 \rightarrow (b \rightarrow d)] \rightarrow a \] by Lemma 2.6 (13)
\[= (0 \rightarrow d) \rightarrow a \] by (9)

(11) \[b \rightarrow (0 \rightarrow a') = (0 \rightarrow b) \rightarrow (0 \rightarrow a') \] by Lemma 2.6 (20)
\[= 0 \rightarrow [(0 \rightarrow b) \rightarrow a'] \] by Lemma 2.6 (13)
\[= 0 \rightarrow [(a \rightarrow 0) \rightarrow (b \rightarrow a')'] \] by (I)
\[= 0 \rightarrow [a' \rightarrow (b \rightarrow a')'] \] by (3)
\[= 0 \rightarrow (a' \rightarrow a')' \] by Lemma 2.6 (4)
\[= 0 \rightarrow (0 \rightarrow (b \rightarrow a')') \] by (I)
\[= 0 \rightarrow (b \rightarrow d) \] by Lemma 2.4 (d)

(12) From (1) and by hypothesis we have that \[[d \rightarrow a] \rightarrow b]' = (b \rightarrow d) \rightarrow (a \rightarrow b)' = (b \rightarrow d) \rightarrow a.

(13) \[a' \rightarrow b = (a \rightarrow b') \rightarrow b \] by hypothesis
\[= [(b' \rightarrow a) \rightarrow (b' \rightarrow b)']' \] using (I)
\[= [(b' \rightarrow a) \rightarrow b']' \] by Lemma 2.4 (d)
\[= b' \rightarrow (a \rightarrow b')' \] by Lemma 2.6 (3)
\[= b' \rightarrow a \] by hypothesis
\[(d \rightarrow 0') \rightarrow (a' \rightarrow b) = (a' \rightarrow b)' \rightarrow [(d \rightarrow 0') \rightarrow (a' \rightarrow b)]\]
= \((a' \rightarrow b)' \rightarrow [(d \rightarrow 0') \rightarrow (b' \rightarrow a)]\)
by Lemma 2.6 [17]
= \((a' \rightarrow b)' \rightarrow [(d \rightarrow 0') \rightarrow \{(0 \rightarrow a) \rightarrow b'\}']\)
by [13]
= \((a' \rightarrow b)' \rightarrow [\{(0 \rightarrow a) \rightarrow b'\} \rightarrow \{d \rightarrow (0 \rightarrow a) \rightarrow b'\}']\)
by Lemma 2.6 [20] with \(d = 0\)
= \((a' \rightarrow b)' \rightarrow [\{(a' \rightarrow b)' \rightarrow \{d \rightarrow (0 \rightarrow a) \rightarrow b'\}']\)
by [12] with \(d = 0\)
= \((a' \rightarrow b)' \rightarrow \{d \rightarrow (0 \rightarrow a) \rightarrow b'\}']\)
by Lemma 2.6 [21]
= \((b' \rightarrow a)' \rightarrow \{d \rightarrow (0 \rightarrow a) \rightarrow b'\}']\)
by [13]
= \([(0 \rightarrow a) \rightarrow b'] \rightarrow \{d \rightarrow (0 \rightarrow a) \rightarrow b'\}']\)
by [12] with \(d = 0\)
= \([(0 \rightarrow d) \rightarrow \{(0 \rightarrow a) \rightarrow b'\}']\)
by Lemma 2.6 [31] with \(x = a, y = d, z = a\)
= \([(b' \rightarrow d) \rightarrow \{(0 \rightarrow a) \rightarrow b'\}']\)
by Lemma 2.6 [15] with \(x = b, y = d, z = a\)
= \([(b' \rightarrow d) \rightarrow (d \rightarrow a')]\)
by [12] with \(d = 0\)
= \([(b' \rightarrow d) \rightarrow (a' \rightarrow b)']\)
by Lemma 2.6 [13]
= \((d \rightarrow a') \rightarrow b\)
by (1).

\[(0 \rightarrow a') \rightarrow b) = [(a \rightarrow 0') \rightarrow b]'\]
by Lemma 2.6 [31]
= \((0 \rightarrow a) \rightarrow b\)
by Lemma 2.6 [11]

\[(a' \rightarrow b)' = [(0 \rightarrow a') \rightarrow b]\]
by (1)
= \((0 \rightarrow a) \rightarrow b\)
by (16)
\[ ([b' \rightarrow ((b \rightarrow d) \rightarrow a)] \rightarrow (0 \rightarrow b)')' = \left\{ ([b' \rightarrow ((d \rightarrow a) \rightarrow b')'] \rightarrow (0 \rightarrow b)\right\}' \]

by (12)

\[ = \left\{ ([d \rightarrow a] \rightarrow b')' \rightarrow 0 \right\} \rightarrow b \]

by (I)

\[ = \left\{ ([d \rightarrow a] \rightarrow b') \right\} \rightarrow b \]

\[ = b' \rightarrow ([d \rightarrow a] \rightarrow b')' \]

by Lemma 2.6 (14) with \( x = d \rightarrow a, y = b' \)

\[ = b' \rightarrow ([b \rightarrow d] \rightarrow a) \]

by (12).

\[ \square \]

References


