Order in Implication Zroupoids

Juan M. CORNEJO and Hanamantagouda P. SANKAPPANAVAR

Abstract

The variety $\mathbf{I}$ of implication zroupoids (using a binary operation $\to$ and a constant 0) was defined and investigated by Sankappanavar in [7], as a generalization of De Morgan algebras. Also, in [7], several new subvarieties of $\mathbf{I}$ were introduced, including the subvariety $\mathbf{I_{2,0}}$, defined by the identity: $x'' \approx x$, which plays a crucial role in this paper. Some more new subvarieties of $\mathbf{I}$ are studied in [3] that includes the subvariety $\mathbf{SL}$ of semilattices with a least element 0; and an explicit description of semisimple subvarieties of $\mathbf{I}$ is given in [5].

It is a well known fact that there is a partial order (denote it by $\sqsubseteq$) induced by the operation $\wedge$, both in the variety $\mathbf{SL}$ of semilattices with a least element and in the variety $\mathbf{DM}$ of De Morgan algebras. As both $\mathbf{SL}$ and $\mathbf{DM}$ are subvarieties of $\mathbf{I}$ and the definition of partial order can be expressed in terms of the implication and the constant, it is but natural to ask whether the relation $\sqsubseteq$ on $\mathbf{I}$ is actually a partial order in some (larger) subvariety of $\mathbf{I}$ that includes both $\mathbf{SL}$ and $\mathbf{DM}$.

The purpose of the present paper is two-fold: Firstly, a complete answer is given to the above mentioned problem. Indeed, our first main theorem shows that the variety $\mathbf{I_{2,0}}$ is a maximal subvariety of $\mathbf{I}$ with respect to the property that the relation $\sqsubseteq$ is a partial order on its members.. In view of this result, one is then naturally led to consider the problem of determining the number of non-isomorphic algebras in $\mathbf{I_{2,0}}$ that can be defined on an n-element chain (herein called $\mathbf{I_{2,0}}$-chains), n being a natural number. Secondly, we answer this problem in our second main theorem which says that, for each $n \in \mathbb{N}$, there are exactly n nonisomorphic $\mathbf{I_{2,0}}$-chains of size n.

1 Introduction

The widely known fact that Boolean algebras can be defined using only implication and a constant was extended to De Morgan algebras in [7]. The crucial role played by a certain identity, called (I), led Sankappanavar, in 2012, to define and investigate, the variety $\mathbf{I}$ of implication zroupoids (I-zroupoids) generalizing De Morgan algebras. Also, in [7], he introduced several new subvarieties of $\mathbf{I}$ and found some relationships among those subvarieties. One of the subvarieties of $\mathbf{I}$, called $\mathbf{I_{2,0}}$, defined by the identity: $x'' \approx x$ and studied in [7], plays a crucial role in this paper. In [3], we introduce several more new subvarieties of $\mathbf{I}$, including the subvariety $\mathbf{SL}$ which is term-equivalent to the (well known) variety of $\vee$-semilattices with a least element 0, and describe further relationships among the subvarities of $\mathbf{I}$. An explicit description of semisimple subvarieties of $\mathbf{I}$ is given in [5].

It is also a well known fact that there is a partial order induced by the operation $\wedge$, both in the variety $\mathbf{SL}$ of semilattices with a least element and in the variety $\mathbf{DM}$ of De Morgan algebras. As both $\mathbf{SL}$ and $\mathbf{DM}$ are subvarieties of $\mathbf{I}$ and the defintion of partial order can be expressed in terms of the implication and constant, it is but natural to ask whether the relation $\sqsubseteq$ (now defined) on $\mathbf{I}$ is actually a partial order in some (larger) subvariety of $\mathbf{I}$ that includes both $\mathbf{SL}$ and $\mathbf{DM}$.

The purpose of the present paper is two-fold: Firstly, a complete answer is given to the above mentioned problem. Indeed, our first main theorem shows that the variety $\mathbf{I_{2,0}}$ is a maximal subvariety of $\mathbf{I}$ with respect to the property that the relation $\sqsubseteq$, defined by:

$$x \sqsubseteq y \text{ if and only if } (x \to y')' = x, \text{ for } x, y \in \mathbf{A} \text{ and } \mathbf{A} \in \mathbf{I},$$

is a partial order. In view of this result, one is then naturally led to consider the problem of determining the number of non-isomorphic algebras in $\mathbf{I_{2,0}}$ ($\mathbf{I_{2,0}}$-chains) that can be defined on an n-element set, n being a nutural number. Secondly, we answer this problem in our second main result which says that, for each $n \in \mathbb{N}$, there are exactly n nonisomorphic $\mathbf{I_{2,0}}$-chains of size n.

2 Preliminaries

In this section we recall some definitions and results from [3], [5] and [7] that will be needed for this paper. Basic references are [1] and [2].

Definition 2.1 [7] *A* groupoid with zero (*zroupoid*, for short) is an algebra $\mathbf{A} = \langle A, \to, 0\rangle$, where $\to$ is a binary operation and 0 is a constant. A zroupoid $\mathbf{A} = \langle A, \to, 0\rangle$ is an *implication zroupoid* (I-zroupoid, for short) if the following identities hold in $\mathbf{A}$, where $x' := x \to 0$:

(I) $(x \to y) \to z \approx [(z' \to x) \to (y \to z)']'$

(I_0) $0'' \approx 0$.

The variety of I-zroupoids is denoted by $\mathbf{I}$.

In this paper we use the characterizations of De Morgan algebras, Kleene algebras and Boolean algebras (see [7]), and semilattices with least element 0 (see [3]), as definitions.

Definition 2.2 *An implication zroupoid* $\mathbf{A} = \langle A, \to, 0\rangle$ *is a De Morgan algebra* (**DM**-*algebra, for short*) *if* $\mathbf{A}$ *satisfies the axiom:*

(DM) $(x \to y) \to x \approx x$.

A **DM**-*algebra* $\mathbf{A} = \langle A, \to, 0\rangle$ *is a Kleene algebra* (**KL**-*algebra, for short*) *if* $\mathbf{A}$ *satisfies the axiom:*

(KL_1) $(x \to x) \to (y \to y)' \approx x \to x$

or, equivalently,

(KL_2) $(y \to y) \to (x \to x) \approx x \to x$.

A **DM**-*algebra* $\mathbf{A} = \langle A, \to, 0\rangle$ *is a Boolean algebra* (**BA**-*algebra, for short*) *if* $\mathbf{A}$ *satisfies the axiom:*

(BA) $x \to x \approx 0'$.

An implication zroupoid $\mathbf{A} = \langle A, \to, 0\rangle$ *is a semilattice with* 0 (**SL**-*algebra, for short*) *if* $\mathbf{A}$ *satisfies the axioms:*

(SM1) $x' \approx x$

(SM2) $x \to y \approx y \to x$. (Commutativity).

We denote by **DM**, **KL**, **BA** *and* **SL**, *respectively, the variety of* **DM**-*algebras,* **KL**-*algebras,* **BA**-*algebras, and* **SL**-*algebras.*

We recall from [7] the definition of another subvariety of $\mathbf{I}$, namely $\mathbf{I}_{2,0}$, which plays a fundamental role in this paper.

Definition 2.3 $\mathbf{I}_{2,0}$ *denotes the subvariety of* $\mathbf{I}$ *defined by the identity:*

$$x'' \approx x.$$

We note that **DM**, **KL**, **BA** and **SL** are all subvarieties of $\mathbf{I}_{2,0}$ (see [7] and [3]).

Lemma 2.4 [7, Theorem 8.15] *Let* $\mathbf{A}$ *be an I-zroupoid. Then the following are equivalent:*

(a) $0' \to x \approx x$

(b) $x'' \approx x$

(c) $(x \to x')' \approx x$

(d) $x' \to x \approx x.$

Lemma 2.5 [7] *Let* $\mathbf{A} \in \mathbf{I}_{2,0}$. *Then*

(a) $x' \to 0' \approx 0 \to x$

(b) $0 \to x' \approx x \to 0'.$

Several identities true in $\mathbf{I}_{2,0}$ are given in [3], [5] and [7]. Some of those that are needed for this paper are listed in the next lemma, which also presents some new identities of $\mathbf{I}_{2,0}$ that will be useful later in this paper. The proof of the lemma is given in the Appendix.

Lemma 2.6 *Let* $\mathbf{A} \in \mathbf{I}_{2,0}$. *Then* $\mathbf{A}$ *satisfies:*

(1) $(x \to 0') \to y \approx (x \to y') \to y$

(2) $(0 \to x') \to (y \to x) \approx y \to x$

(3) $(y \to x)' \approx (0 \to x) \to (y \to x)'$

(4) $[x \to (y \to x)']' \approx (x \to y) \to x$

(5) $(y \to x) \to y \approx (0 \to x) \to y$

(6) $0 \to x \approx 0 \to (0 \to x)$

(7) $0 \to [(0 \to x) \to (0 \to y')'] \approx 0 \to (x \to y)$

(8) $[x' \to (0 \to y)]' \approx (0 \to x) \to (0 \to y)'$

(9) $0 \to (0 \to x)' \approx 0 \to x'$

(10) $0 \to (x' \to y)' \approx x \to (0 \to y')$

(11) $[(x \to 0') \to y]' \approx (0 \to x) \to y'$

(12) $0 \to [(0 \to x) \to y'] \approx x \to (0 \to y')$

(13) $0 \to (x \to y) \approx x \to (0 \to y)$

(14) $(x \to y) \to y' \approx y \to (x \to y)'$

(15) $(x' \to y) \to [(0 \to z) \to x'] \approx (0 \to y) \to [(0 \to z) \to x']$

(16) $0 \to (x \to y')' \approx 0 \to (x' \to y)$

(17) $x \to (y \to x') \approx y \to x'$

(18) $[(0 \to x) \to y] \to x \approx y \to x$

(19) $[0 \to (x \to y)] \to x \approx (0 \to y) \to x$

(20) $(0 \to x) \to (0 \to y) \approx x \to (0 \to y)$

(21) $x \to y \approx x \to (x \to y)$

(22) $[\{x \to (0 \to y)\} \to z]' \approx z \to [(x \to y) \to z]'$

(23) $[0 \to (x \to y)] \to y' \approx y \to (x \to y)'$

(24) $x \to [(y \to z) \to x]' \approx (0 \to y) \to [x \to (z \to x)']$

(25) $0 \to [(0 \to x) \to y] \approx x \to (0 \to y)$

(26) $x \to (y \to x)' \approx (y \to 0') \to x'$

(27) $[(x' \to y) \to (z \to x)'] \to [(y \to z) \to x] \approx (y \to z) \to x$

(28) $[\{0 \to (x \to y)'\} \to (0 \to y')']' \approx 0 \to (x \to y)'$

(29) $[[0 \to \{(x \to y) \to z\}] \to \{0 \to (y \to z)\}']' \approx 0 \to \{(x \to y) \to z\}$

(30) $[x \to (0 \to y)']' \approx x' \to (y \to 0')'$

(31) $[(0 \to x) \to y]' \approx y \to (x \to y)'$

(32) $[x \to (y \to 0')']' \approx x' \to (0 \to y)'$

(33) $(x \to y)' \to (0 \to x)' \approx y' \to x'$

(34) $(0 \to x)' \to (0 \to y)' \approx 0 \to (x' \to y')$

(35) $[(x \to y)' \to \{y \to (x \to y)'\}']' \approx (x \to y)'$

(36) $[\{(0 \to x) \to y\} \to (x \to y)']' \approx (0 \to x) \to y$

(37) $[\{x \to (y \to x)'\} \to x]' \approx x \to (y \to x)'$.

3 Partial order in Implication Zroupoids

Let $\mathbf{A} = \langle A; \to, 0\rangle \in \mathbf{I}$. We define the operations $\wedge$ and $\vee$ on $\mathbf{A}$ by:

- $x \wedge y := (x \to y')'$,
- $x \vee y := (x' \wedge y')'$.

Note that the above definition of $\wedge$ is a simultaneous generalization of the $\wedge$ operation of De Morgan algebras and that of **SL** (= semilattices with least element 0). It is, of course, well known that the meet operation induces a partial order on both **DM** and **SL**, which naturally leads us to the following definition of a binary relation $\sqsubseteq$ on algebras in **I**.

Definition 3.1 *Let $\mathbf{A} \in \mathbf{I}$. We define the relation $\sqsubseteq$ on A as follows:*

$$x \sqsubseteq y \text{ if and only if } x \wedge y = x \quad (\text{equivalently}, (x \to y')' = x).$$

For $a, b \in A$, we write

- *$a \sqsubset b$ if $a \sqsubseteq b$ and $a \neq b$,*
- *$a \sqsupseteq b$ if $b \sqsubseteq a$, and*
- *$a \sqsupset b$ if $a \sqsupseteq b$ and $a \neq b$.*

We already know from [3] that $\langle A; \wedge, \vee\rangle$ is a lattice if and only if $\mathbf{A}$ is a De Morgan Algebra, implying that $\sqsubseteq$ is a partial order on A. We know (see [3]) that $\sqsubseteq$ is also a partial order on algebras in **SL**. This fact led us naturally to consider the possibility of the existence of a subvariety **V** of **I**, containing both **SL** and **DM**, such that, for every algebra **A** in **V**, the relation $\sqsubseteq$ on **A** is actually a partial order.

In this section we will prove our first main result which says that the subvariety $\mathbf{I_{2,0}}$, is a maximal subvariety of **I** with respect to the property that the relation $\sqsubseteq$ is a partial order on every member of that variety. To achieve this end, we need to, first, prove that $\sqsubseteq$ is indeed a partial order on every member of $\mathbf{I_{2,0}}$, which will be done using the following lemmas.

Lemma 3.2 *Let $\mathbf{A} \in \mathbf{I}_{2,0}$. Then the relation $\sqsubseteq$ is antisymmetric on $\mathbf{A}$.*

Proof Let $a, b \in A$ such that $a \sqsubseteq b$ and $b \sqsubseteq a$. Let $c \in A$ be arbitrary. Then, using (I) and the hypothesis, one observes that $(c \to a) \to b' = [(b \to c) \to (a \to b')']' = [(b \to c) \to a]'$. Consequently,

(3.1) $(c \to a) \to b' = [(b \to c) \to a]'$, where $c \in A$.

Hence,

$$\begin{array}{rcll} a' & = & (a \wedge b)' & \text{by hypothesis} \\ & = & (a \to b')'' & \text{by definition of } \wedge \\ & = & a \to b' & \\ & = & (a' \to a) \to b' & \text{using Lemma 2.4}(d) \\ & = & [(b \to a') \to a]' & \text{from (3.1) with } c = a' \\ & = & [(b \to a')'' \to a]' & \\ & = & (b' \to a)' & \text{by hypothesis,} \end{array}$$

and, therefore,

(3.2) $a' = (b' \to a)'$.

Now,

$$\begin{aligned} b' &= [b \to a']'' && \text{by hypothesis} \\ &= b \to a' \\ &= (0 \to a'') \to (b \to a') && \text{by Lemma 2.6 (2) with } x = a', y = b \\ &= (0 \to a) \to (b \to a')'' \\ &= (0 \to a) \to b' && \text{by hypothesis.} \end{aligned}$$

Thus,

(3.3) $b' = (0 \to a) \to b'$.

Therefore,

$$\begin{aligned} a' &= [b' \to a]' && \text{from (3.2)} \\ &= [(b \to 0) \to a]' \\ &= (0 \to a) \to b' && \text{from (3.1) with } c = 0 \\ &= b' && \text{by (3.3).} \end{aligned}$$

Consequently, we have that $a = a'' = b'' = b$, thus proving that $\sqsubseteq$ is antisymmetric on $\mathbf{A}$. □

Now, we turn to proving the transitivity of the relation $\sqsubseteq$. For this, we need the following lemmas. The proof of the following (technical) lemma is given in the Appendix.

Lemma 3.3 *Let $\mathbf{A} \in \mathbf{I}_{2,0}$ with $a, b \in A$ such that $a \sqsubseteq b$. Let $d \in A$ be arbitrary. Then*

(1) $(0 \to a') \to b = a' \to b$

(2) $b \to a' = (0 \to b) \to a'$

(3) $b \to a' = a'$

(4) $0 \to (a' \to b) = 0 \to a$

(5) $[(b \to d) \to a]' = (d \to a) \to b'$

(6) $(0 \to d) \to a' = [\{d \to (0 \to b')\} \to a]'$

(7) $a \to [(a' \to d) \to \{(0 \to a) \to b'\}] = (0 \to d) \to a'$

(8) $a \to [(d \to a) \to b'] = a \to (d \to a)'$

(9) $[0 \to (b \to d)] \to a = (0 \to d) \to a$

(10) $[b \to (a \to d)] \to a = (0 \to d) \to a$

(11) $b \to (0 \to a') = 0 \to a'$

(12) $[(d \to a) \to b']' = (b \to d) \to a$

(13) $a' \to b = b' \to a$

(14) $(d \to a') \to b = (d \to 0') \to (a' \to b)$

(15) $[(0 \to a') \to b]' = (0 \to a) \to b'$

(16) $(a' \to b)' = (0 \to a) \to b'$

(17) $b' \to [(b \to d) \to a] \sqsubseteq 0 \to b$.

Lemma 3.4 *Let $\mathbf{A} \in \mathbf{I}_{2,0}$ and let $a, b, e \in A$ such that $(a \to b')' = a$ and $(0 \to e') \to b = b$, and let $d \in A$ be arbitrary. Then*

(a) $b \to d = (0 \to (d \to e)) \to (b \to d)$

(b) $(0 \to e) \to a' = a'$

(c) $(0 \to e') \to a = a$.

(d) $(0 \to e) \to [a \to (a \to d)] = a \to d$.

Proof

(a)

$$
\begin{array}{rcll}
b \to d & = & [(0 \to e') \to b] \to d & \text{by hypothesis} \\
& = & [\{d' \to (0 \to e')\} \to (b \to d)'] \to [\{(0 \to e') \to b\} \to d] & \text{by Lemma 2.6 (27)} \\
& & \qquad \text{using } x = d, y = 0 \to e', z = b & \\
& = & [\{d' \to (0 \to e')\} \to (b \to d)'] \to (b \to d) & \text{by hypothesis} \\
& = & [\{d' \to (0 \to e')\} \to 0'] \to (b \to d) & \text{by Lemma 2.6 (1)} \\
& = & [0 \to \{d' \to (0 \to e')\}'] \to (b \to d) & \text{by Lemma 2.5 (a)} \\
& = & [0 \to \{(0 \to d) \to (0 \to e')'\}] \to (b \to d) & \text{by Lemma 2.6 (8)} \\
& = & [0 \to (d \to e)] \to (b \to d) & \text{by Lemma 2.6 (7).}
\end{array}
$$

(b) Using Lemma 3.3 (3) (twice), and (a) with $d = a'$, we obtain $[0 \to (a' \to e)] \to a' = [0 \to (a' \to e)] \to (b \to a') = b \to a' = a'$. Hence,

(3.4) $[0 \to (a' \to e)] \to a' = a'$.

Then,

$$
\begin{array}{rcll}
(0 \to e) \to a' & = & [0 \to (a' \to e)] \to a' & \text{by Lemma 2.6 (19) using } x = a', y = e \\
& = & a' & \text{by (3.4).}
\end{array}
$$

(c)

$$
\begin{array}{rcll}
(0 \to e') \to a & = & [0 \to (0 \to e)'] \to a & \text{by Lemma 2.6 (9)} \\
& = & [(0 \to e) \to 0'] \to a & \text{by Lemma 2.5 (a)} \\
& = & [(0 \to e) \to a'] \to a & \text{by Lemma 2.6 (1)} \\
& = & a' \to a & \text{by (b)} \\
& = & a & \text{by Lemma 2.4 (d).}
\end{array}
$$

(d)

$$
\begin{array}{rcll}
a \to d & = & [(0 \to e') \to a] \to d & \text{by item (c)} \\
& = & [\{d' \to (0 \to e')\} \to (a \to d)'] \to [\{(0 \to e') \to a\} \to d] & \text{by Lemma 2.6 (27) with} \\
& & & x = d, y = 0 \to e', z = a \\
& = & [\{d' \to (0 \to e')\} \to (a \to d)'] \to (a \to d) & \text{by item (c)} \\
& = & [\{d' \to (0 \to e')\} \to 0'] \to (a \to d) & \text{by Lemma 2.6 (1)} \\
& = & [0 \to \{d' \to (0 \to e')\}'] \to (a \to d) & \text{by Lemma 2.5 (a)} \\
& = & [0 \to \{(0 \to d) \to (0 \to e')'\}] \to (a \to d) & \text{by Lemma 2.6 (8) with} \\
& & & x = d, y = e' \\
& = & [0 \to (d \to e)] \to (a \to d) & \text{by Lemma 2.6 (7)}
\end{array}
$$

Thus,

(3.5) $a \to d = [0 \to (d \to e)] \to (a \to d)$.

Now,

$$
\begin{aligned}
(0 \to e) \to [a \to (a \to d)] &= [0 \to [\{a \to (a \to d)\} \to e]] \to [a \to (a \to d)] \\
&\quad \text{by Lemma 2.6 (19) with } x = a \to (a \to d), y = e \\
&= [0 \to [\{a \to (a \to d)\} \to e]] \to [a \to \{a \to (a \to d)\}] \\
&\quad \text{by Lemma 2.6 (21)} \\
&= a \to [a \to (a \to d)] \\
&\quad \text{by (3.5) replacing } d \text{ with } a \to (a \to d) \\
&= a \to d \\
&\quad \text{by Lemma 2.6 (21).}
\end{aligned}
$$

Thus, (d) is proved and the proof of the lemma is complete. □

Each of the next three lemmas prove a crucial step in the proof of transitivity of $\sqsubseteq$.

Lemma 3.5 *Let $\mathbf{A} \in \mathbf{I}_{2,0}$ and let $a, b, c \in A$ such that $a \sqsubseteq b$ and $b \sqsubseteq c$. Let $d, e, f \in A$ be arbitrary. Then*

(1) $(0 \to c') \to b = b$

(2) $(0 \to c) \to [a \to (a \to d)] = a \to d$

(3) $(0 \to c) \to (a \to d) = a \to d$

(4) $[0 \to ((0 \to b) \to c')] \to b = b$

(5) $\{d' \to [0 \to ((0 \to b) \to c')]\} \to (b \to d)' = (b \to d)'$

(6) $(b \to d) \to [e \to (b \to d)]' = [e \to 0'] \to (b \to d)'$

(7) $[b \to (a \to c')] \to a = a$

(8) $(0 \to b) \to (a \to d) = a \to d$

(9) $0 \to [b \to (a \to d)] = 0 \to (a \to d)$

(10) $0 \to [\{b \to (a \to d)\} \to e] = 0 \to [(a \to d) \to e]$

(11) $[0 \to (d' \to c)] \to (0 \to b)' = (0 \to d) \to (0 \to b)'$

(12) $0 \to (a' \to c) \sqsubseteq 0 \to b$

(13) $(0 \to a) \to (0 \to b)' = (0 \to a)'$

(14) $0 \to (a' \to c) = 0 \to a$

(15) $(d \to e) \to [\{b \to (a \to f)\}' \to (0 \to a)'] = (d \to e) \to [(a' \to b) \to (f' \to a')]$.

Proof By hypothesis, we have $(a \to b')' = a$ and $(b \to c')' = b$.

(1)

$$\begin{aligned}
(0 \to c') \to b &= (c \to 0') \to b && \text{by Lemma 2.5 (a)}\\
&= [(b' \to c) \to (0' \to b)']' && \text{by (I)}\\
&= [(b' \to c) \to b']' && \text{by Lemmaa 2.4 (a)}\\
&= [(0 \to c) \to b']' && \text{by Lemma 2.6 (5)}\\
&= [(c' \to 0') \to b']' && \text{by Lemma 2.5 (a)}\\
&= [(b'' \to c') \to (0' \to b')']'' && \text{from (I)}\\
&= (b'' \to c') \to (0' \to b')' &&\\
&= (b \to c') \to (0' \to b')' &&\\
&= (b \to c') \to b'' && \text{by Lemmaa 2.4 (a)}\\
&= (b \to c') \to b &&\\
&= b' \to b && \text{by hypothesis}\\
&= b && \text{by Lemma 2.4 (d).}
\end{aligned}$$

(2) This is immmediate from (1) and Lemma 3.4 (d) with $e = c$.

(3) Using Lemma 2.6 (21) and (2) we have that $(0 \to c) \to (a \to d) = (0 \to c) \to [a \to (a \to d)] = a \to d$, implying (3).

(4)

$$\begin{aligned}
[0 \to ((0 \to b) \to c')] \to b &= \big\{(b' \to 0) \to [((0 \to b) \to c') \to b]'\big\}' && \text{by (I)}\\
&= \big\{b \to [((0 \to b) \to c') \to b]'\big\}' &&\\
&= \{b \to (c' \to b)'\}' && \text{by Lemma 2.6 (18)}\\
&= \{(b' \to b) \to (c' \to b)'\}' && \text{by Lemma 2.4 (d)}\\
&= (b \to c') \to b && \text{by (I)}\\
&= (b \to c')'' \to b &&\\
&= b' \to b && \text{by hypothesis}\\
&= b && \text{by Lemma 2.4 (d).}
\end{aligned}$$

(5)

$$\begin{aligned}
\{d' \to [0 \to ((0 \to b) \to c')]\} \to (b \to d)' &= \{[[0 \to ((0 \to b) \to c')] \to b] \to d\}' && \text{by (I)}\\
&= (b \to d)' && \text{by (4).}
\end{aligned}$$

(6)

$$\begin{aligned}(b \to d) \to [e \to (b \to d)]' &= [e \to (b \to d)] \to (b \to d)' && \text{by Lemma 2.6 (14) with } x = e, y = b \to d \\ &= [e \to 0'] \to (b \to d)' && \text{by Lemma 2.6 (1).}\end{aligned}$$

(7)

$$\begin{aligned}[b \to (a \to c')] \to a &= [(a' \to b) \to \{(a \to c') \to a\}']' && \text{by (I)} \\ &= [(a' \to b) \to \{(0 \to c') \to a\}']' && \text{by Lemma 2.6 (5)} \\ &= [(a' \to b) \to a']' && \text{by (1) and Lemma 3.4 (c)} \\ &= [(0 \to b) \to a']' && \text{by Lemma 2.6 (5)} \\ &= (a \to 0) \to (b \to a')' && \text{by (I)} \\ &= a' \to (b \to a')' && \\ &= a' \to a'' && \text{by Lemma 3.3 (3)} \\ &= a' \to a && \\ &= a && \text{by Lemma 2.4 (d).}\end{aligned}$$

(8)

$$\begin{aligned}(0 \to b') \to b &= (0 \to 0') \to b && \text{by Lemma 2.6 (1)} \\ &= (0'' \to 0') \to b && \\ &= 0' \to b && \text{by Lemma 2.4 (d)} \\ &= b && \text{by Lemma 2.4 (a).}\end{aligned}$$

Hence, by the hypothesis, together with Lemma 3.4 (d), we obtain that $(0 \to b) \to \{a \to (a \to d)\} = a \to d$. Hence, by Lemma 2.6 (21), we have $(0 \to b) \to (a \to d) = a \to d$.

(9)

$$\begin{aligned}0 \to (a \to d) &= 0 \to [(0 \to b) \to (a \to d)] && \text{by (8)} \\ &= b \to [0 \to (a \to d)] && \text{by Lemma 2.6 (25) with } x = b, y = a \to d \\ &= 0 \to [b \to (a \to d)]. && \text{by Lemma 2.6 (13).}\end{aligned}$$

(10)

$$\begin{aligned}0 \to [\{b \to (a \to d)\} \to e] &= [b \to (a \to d)] \to (0 \to e) && \text{by Lemma 2.6 (13)} \\ &= 0 \to [[0 \to \{b \to (a \to d)\}] \to e] && \text{by Lemma 2.6 (25)} \\ &= 0 \to [\{0 \to (a \to d)\} \to e] && \text{by (9)} \\ &= (a \to d) \to (0 \to e) && \text{by Lemma 2.6 (25)} \\ &= 0 \to [(a \to d) \to e] && \text{by Lemma 2.6 (13).}\end{aligned}$$

(11)

$$\begin{aligned}[0 \to (d' \to c)] \to (0 \to b)' &= [0 \to (d' \to c)] \to (b' \to 0')' && \text{by Lemma 2.5 (a)} \\ &= [\{(d' \to c) \to b'\} \to 0']' && \text{by (I)} \\ &= [\{(b \to d') \to (c \to b')'\}' \to 0']' && \text{by (I)} \\ &= [\{(b \to d') \to b''\}' \to 0']' && \text{by Lemma 3.3 (3)} \\ &= [\{(b \to d') \to b\}' \to 0']' && \\ &= [\{(0 \to d') \to b\}' \to 0']' && \text{by Lemma 2.6 (5)} \\ &= [0 \to \{(0 \to d') \to b\}]' && \text{by Lemma 2.5 (a)} \\ &= [(0 \to d') \to (0 \to b)]' && \text{by Lemma 2.6 (13)} \\ &= [(d \to 0') \to (0 \to b)]' && \\ &= (0 \to d) \to (0 \to b)' && \text{by Lemma 2.6 (11).}\end{aligned}$$

(12)

$$\begin{array}{rcll} 0 \to (a' \to c) & = & 0 \to [(a \to b')'' \to c] & \text{by hyphotesis} \\ & = & 0 \to [(a \to b') \to c] & \\ & \sqsubseteq & 0 \to (b' \to c) & \text{by Lemma 2.6 (29)} \\ & = & 0 \to b & \text{by hyphotesis and Lemma 3.3 (4).} \end{array}$$

(13)

$$\begin{array}{rcll} (0 \to a) \to (0 \to b)' & = & [a' \to (0 \to b)]' & \text{by Lemma 2.6 (8)} \\ & = & [0 \to (a' \to b)]' & \text{by Lemma 2.6 (13)} \\ & = & (0 \to a)'. & \text{by hyphotesis and Lemma 3.3 (4).} \end{array}$$

(14)

$$\begin{array}{rcll} 0 \to (a' \to c) & = & [\{0 \to (a' \to c)\} \to (0 \to b)']' & \text{by (12)} \\ & = & [(0 \to a) \to (0 \to b)']' & \text{by (11) with } d = a \\ & = & (0 \to a)'' & \text{by (13)} \\ & = & 0 \to a. & \end{array}$$

(15)

$$\begin{array}{l} (d \to e) \to [(a' \to b) \to (f' \to a')] \\ \quad = (d \to e) \to [\{(0 \to a') \to b\} \to (f' \to a')] \\ \qquad \text{by Lemma 3.3 (1)} \\ \quad = (d \to e) \to [\{(0 \to a') \to b\} \to \{(f \to 0) \to a'\}] \\ \quad = (d \to e) \to [\{(0 \to a') \to b\} \to \{(a \to f) \to (0 \to a')'\}'] \\ \qquad \text{by (I)} \\ \quad = (d \to e) \to [\{b \to (a \to f)\} \to (0 \to a')']' \\ \qquad \text{by (I)} \\ \quad = (d \to e) \to [\{b \to (a \to f)\}' \to (a' \to 0')'] \\ \qquad \text{by (30) with } x = b \to (a \to f) \text{ and } y = a' \\ \quad = (d \to e) \to [\{b \to (a \to f)\}' \to (0 \to a)'] \\ \qquad \text{by Lemma 2.5 (a).} \end{array}$$

Hence, we have $(d \to e) \to [\{b \to (a \to f)\}' \to (0 \to a)'] = (d \to e) \to [(a' \to b) \to (f' \to a')]$.

□

Lemma 3.6 *Let $\mathbf{A} \in \mathbf{I}_{2,0}$ and let $a, b, c \in A$ such that $a \sqsubseteq b$ and $b \sqsubseteq c$. Let $d \in A$ be arbitrary. Then*

(a) $[c \to (b \to a')] \to b = (0 \to a') \to b$

(b) $(c \to a') \to b = a' \to b$

(c) $(a' \to b) \to (c \to a') = c \to a'$

(d) $c \to a' = a \to [b \to (a \to c')]$

(e) $0 \to (a \to d) = 0 \to [c \to (a \to d)]$

(f) $(d \to a) \to d \sqsubseteq (a' \to b) \to d$

(g) $(a' \to b) \to c' = (0 \to a) \to b'$

(h) $0 \to (a \to c') \sqsubseteq 0 \to a'$

(i) $0 \to (a \to c') = 0 \to a'$.

(j) $c \to (a \to c') \sqsubseteq 0 \to (a \to c')$

(k) $c \to (a \to c') \sqsubseteq 0 \to a'$

(l) $(c \to (a \to c'))' \to (0 \to a)' = c \to (a \to c')$

(m) $a \to [b \to (a \to c')] = a \to c'$

(n) $c \to a' = a \to c'$.

Proof

(a) Since $(b \to c')' = b$, by Lemma 3.3 (10) with $d = a'$, we have $(c \to (b \to a')) \to b = (0 \to a') \to b$.

(b)

$$
\begin{aligned}
(c \to a') \to b &= [c \to (b \to a')] \to b && \text{by Lemma 3.3 (3)} \\
&= (0 \to a') \to b && \text{by (a),}
\end{aligned}
$$

from which we get $(c \to a') \to b = (0 \to a') \to b$, which, together with Lemma 3.3 (1), implies $(c \to a') \to b = a' \to b$.

(c)

$$
\begin{aligned}
c \to a' &= (0 \to a) \to (c \to a') && \text{by Lemma 2.6 (2) with } x = a', y = c \\
&= [0 \to (a' \to b)] \to (c \to a') && \text{by Lemma 3.3 (4)} \\
&= [0 \to \{(c \to a') \to b\}] \to (c \to a') && \text{by (b)} \\
&= [(c \to a') \to (0 \to b)] \to (c \to a') && \text{by Lemma 2.6 (13)} \\
&= [0 \to (0 \to b)] \to (c \to a') && \text{by Lemma 2.6 (5)} \\
&= (0 \to b) \to (c \to a') && \text{by Lemma 2.6 (6)} \\
&= [(c \to a') \to b] \to (c \to a') && \text{by Lemma 2.6 (5)} \\
&= (a' \to b) \to (c \to a') && \text{by (b).}
\end{aligned}
$$

(d)

$$\begin{aligned}
c \to a' &= (0 \to a) \to (c \to a') && \text{by Lemma 2.6 (2)} \\
&= (0 \to a) \to [(a' \to b) \to (c \to a')] && \text{by (c)} \\
&= (0 \to a) \to [(a' \to b) \to (c'' \to a')] && \\
&= (0 \to a) \to [\{b \to (a \to c')\}' \to (0 \to a)'] && \text{by Lemma 3.5 (15) with} \\
& && d = 0, e = a, f = c' \\
&= (0 \to a) \to [\{b \to (a \to c')\}' \to \{0 \to (a' \to c)\}'] && \text{by Lemma 3.5 (14)} \\
&= (0 \to a) \to [\{b \to (a \to c')\}' \to \{0 \to (a \to c')'\}'] && \text{by Lemma 2.6 (16)} \\
&= (0 \to a) \to [\{b \to (a \to c')\}' \to [0 \to \{b \to (a \to c')\}']'] && \text{by Lemma 3.5(10) with} \\
& && d = c', e = 0 \\
&= [b \to (a \to c')]' \to [(a \to 0) \to \{b \to (a \to c')\}']' && \text{by Lemma 2.6 (24) with} \\
& && x = [b \to (a \to c')]', \\
& && y = a, z = 0 \\
&= [\{0 \to (a \to 0)\} \to \{b \to (a \to c')\}']' && \text{by Lemma 2.6 (4) and (5)} \\
& && \text{with } x = a \to 0, \\
& && y = [b \to (a \to c')]' \\
&= [\{a \to (0 \to 0)\} \to \{b \to (a \to c')\}']' && \text{by Lemma 2.6 (13)} \\
&= [(a \to 0') \to \{b \to (a \to c')\}']' && \\
&= [\{b \to (a \to c')\} \to [a \to \{b \to (a \to c')\}]']' && \text{by Lemma 3.5(6) with} \\
& && e = a, d = a \to c' \\
&= [\{b \to (a \to c')\} \to a] \to [b \to (a \to c')] && \text{by Lemma 2.6 (4)} \\
&= a \to [b \to (a \to c')] && \text{by Lemma 3.5(7).}
\end{aligned}$$

(e)

$$\begin{aligned}
0 \to (a \to d) &= 0 \to [(0 \to c) \to (a \to d)] && \text{by Lemma 3.5(3)} \\
&= c \to [0 \to (a \to d)] && \text{by Lemma 2.6 (25)} \\
&= 0 \to [c \to (a \to d)] && \text{by Lemma 2.6 (13).}
\end{aligned}$$

(f)

$$\begin{aligned}
(d \to a) \to d &= (0 \to a) \to d && \text{by Lemma 2.6 (5)} \\
&= [0 \to (a' \to b)] \to d && \text{by Lemma 3.3 (4)} \\
&\sqsubseteq (a' \to b) \to d && \text{by Lemma 2.6 (36).}
\end{aligned}$$

(g)

$$\begin{aligned}
(a' \to b) \to c' &= [(c \to a') \to (b \to c')']' && \text{by (I)} \\
&= [(c \to a') \to b]' && \text{by hypothesis} \\
&= [\{c \to (b \to a')\} \to b]' && \text{by Lemma 3.3 (3)} \\
&= [(0 \to a') \to b]' && \text{by Lemma 3.3 (10) with} \\
& && d = a' \text{ since } b \sqsubseteq c \\
&= [(b \to a') \to b]' && \text{by Lemma 2.6 (5)} \\
&= [(b \to a') \to b'']' && \\
&= [(b \to a') \to (0' \to b')']' && \text{by Lemma 2.4 (a)} \\
&= (a' \to 0') \to b' && \text{by (I)} \\
&= (0 \to a) \to b'. && \text{by Lemma 2.5 (a).}
\end{aligned}$$

Hence, one has $(a' \to b) \to c' = (0 \to a) \to b'$.

(h) From Lemma 3.5 (1), we have $(0 \to c') \to b = b$. Hence, we can use Lemma 3.3. Therefore, we have

$$
\begin{aligned}
0 \to (a \to c') &= 0 \to [\{(0 \to c') \to a\} \to c'] && \text{by Lemma 3.4 (c) and Lemma 3.5(1)}\\
&= [(0 \to c') \to a] \to (0 \to c') && \text{by Lemma 2.6 (13)}\\
&\sqsubseteq (a' \to b) \to (0 \to c') && \text{by (f) with } d = 0 \to c'\\
&= 0 \to [(a' \to b) \to c'] && \text{by Lemma 2.6 (13)}\\
&= 0 \to [(0 \to a) \to b'] && \text{by (g)}\\
&= 0 \to [(b \to 0) \to (a \to b')']' && \text{by (I)}\\
&= 0 \to [b' \to (a \to b')']' && \\
&= 0 \to (b' \to a)' && \text{by hypothesis}\\
&= 0 \to (b \to a') && \text{by Lemma 2.6 (16)}\\
&= 0 \to a' && \text{by Lemma 3.3 (3).}
\end{aligned}
$$

(i)

$$
\begin{aligned}
0 \to a' &= 0 \to (a \to 0) && \\
&= 0 \to [c \to (a \to 0)] && \text{by (e)}\\
&= 0 \to (c \to a') && \\
&= 0 \to [(a \to c')' \to (0 \to a)'] && \text{by Lemma 2.6 (33)}\\
&= [0 \to (a \to c')]' \to (0 \to a)' && \text{by Lemma 2.6 (34) and Lemma 2.6 (6)}\\
&= [0 \to (a \to c')]' \to (a' \to 0')' && \text{by Lemma 2.5 (a)}\\
&= [\{0 \to (a \to c')\} \to (0 \to a')']' && \text{by Lemma 2.5 (30) with } x = 0 \to (a \to c'), y = a'\\
&= 0 \to (a \to c') && \text{by (h).}
\end{aligned}
$$

(j)

$$
\begin{aligned}
[\{c \to (a \to c')\} \to \{0 \to (a \to c')\}']' &= [\{0 \to (a \to c')\} \to c] \to [(a \to c') \to \{0 \to (a \to c')\}']' \\
&\qquad \text{by (I)}\\
&= [\{0 \to (a \to c')\} \to c] \to [\{(a \to c') \to 0\} \to (a \to c')] \\
&\qquad \text{by Lemma 2.6 (4)}\\
&= [\{0 \to (a \to c')\} \to c] \to [(a \to c')' \to (a \to c')] \\
&= [\{0 \to (a \to c')\} \to c] \to (a \to c') \\
&\qquad \text{by Lemma 2.4 (d)}\\
&= c \to (a \to c') \\
&\qquad \text{by Lemma 2.6 (18) with}\\
&\qquad x = a \to c', y = c.
\end{aligned}
$$

(k) From (j) we have that $c \to (a \to c') \sqsubseteq 0 \to (a \to c')$. Then using (i) we get $c \to (a \to c') \sqsubseteq 0 \to a'$.

(l)

$$
\begin{aligned}
[c \to (a \to c')]' \to (0 \to a)' &= [c \to (a \to c')]' \to (a' \to 0')' && \text{by Lemma 2.5 (a)}\\
&= [\{c \to (a \to c')\} \to (0 \to a')']' && \text{by Lemma 2.6 (30)}\\
&= c \to (a \to c') && \text{by (k).}
\end{aligned}
$$

(m)

$$
\begin{aligned}
a \to [b \to (a \to c')] = &= c \to a' && \text{by (d)}\\
&= c'' \to a' && \\
&= (a \to c')' \to (0 \to a)' && \text{by Lemma 2.6 (33)}\\
&= [c \to (a \to c')]' \to (0 \to a)' && \text{by Lemma 2.6 (17)}\\
&= c \to (a \to c') && \text{by (l)}\\
&= a \to c' && \text{by Lemma 2.6 (17).}
\end{aligned}
$$

(n) From (d) and (m), we get $c \to a' = a \to c'$.

$\square$

Lemma 3.7 *Let $\mathbf{A} \in \mathbf{I}_{2,0}$ and let $a, b, c \in A$ such that $a \sqsubseteq b$ and $b \sqsubseteq c$. Then*

(a) $c' \to [(c \to d) \to b] \sqsubseteq c$

(b) $0 \to a' = c \to (0 \to a')$

(c) $c' \to (a' \to b) \sqsubseteq c$

(d) $(0 \to a') \to b = (c \to a') \to b$

(e) $c' \to (a' \to b) \sqsubseteq 0 \to c$

(f) $[(0 \to a) \to b'] \to c = c' \to (a' \to b)$

(g) $a' \to c = c' \to (a' \to b)$

(h) $a' \to c \sqsubseteq c$

(i) $a' \to c = (0 \to a') \to c.$

Proof

(a)

$$\begin{array}{rcll} c' \to [(c \to d) \to b] & = & c' \to [(c \to d) \to (b \to c')'] & \text{by hypothesis} \\ & = & c' \to [(d \to b) \to c']' & \text{by (I)} \\ & \sqsubseteq & c'' & \text{by Lemma 2.6 (37)} \\ & = & c. & \end{array}$$

(b)

$$\begin{array}{rcll} 0 \to a' & = & b \to (0 \to a') & \text{by Lemma 3.3 (11)} \\ & = & [0 \to \{(0 \to a') \to c\}] \to [b \to (0 \to a')] & \text{by Lemma 3.5 (1)} \\ & & & \text{and Lemma 3.4 (a)} \\ & & & \text{with } d = 0 \to a', e = c \\ & = & [0 \to \{(0 \to a') \to c\}] \to (0 \to a') & \text{by Lemma 3.3 (11)} \\ & = & [(0 \to a') \to (0 \to c)] \to (0 \to a') & \text{by Lemma 2.6 (13)} \\ & = & [0 \to (0 \to c)] \to (0 \to a') & \text{by Lemma 2.6 (5)} \\ & = & (0 \to c) \to (0 \to a') & \text{by Lemma 2.6 (21)} \\ & = & c \to (0 \to a') & \text{by Lemma 2.6 (20).} \end{array}$$

(c)

$$\begin{array}{rcll} c' \to (a' \to b) & = & c' \to [(0 \to a') \to b] & \text{by Lemma 3.3 (1)} \\ & = & c' \to [\{c \to (0 \to a')\} \to b] & \text{by (b)} \\ & \sqsubseteq & c & \text{by (a) with } d = 0 \to a'. \end{array}$$

(d)

$$\begin{aligned}
(0 \to a') \to b &= [c \to (0 \to a')] \to b && \text{by (b)} \\
&= [(b' \to c) \to \{(0 \to a') \to b\}']' && \text{by (I)} \\
&= [(b' \to c) \to \{(b' \to 0) \to (a' \to b)'\}]' && \text{by (I) and } x'' \approx x \\
&= [(b' \to c) \to \{b \to (a' \to b)'\}]' && \\
&= [(b' \to c) \to \{(a' \to 0') \to b'\}]' && \text{by Lemma 2.6 (26)} \\
&= [(b' \to c) \to \{(0 \to a) \to b'\}]' && \text{by Lemma 2.5 (a)} \\
&= [(b' \to c) \to (a' \to b)']' && \text{by Lemma 3.3 (16)} \\
&= (c \to a') \to b && \text{by (I).}
\end{aligned}$$

(e)

$$\begin{aligned}
c' \to (a' \to b) &= c' \to [(0 \to a') \to b] && \text{by Lemma 3.3 (15) and Lemma 3.3 (16)} \\
&= c' \to [(c \to a') \to b] && \text{by (d)} \\
&\sqsubseteq 0 \to c. && \text{by Lemma 3.3 (17) with } d = a'.
\end{aligned}$$

(f)

$$\begin{aligned}
c' \to (a' \to b) &= [\{c' \to (a' \to b)\} \to (0 \to c)']' && \text{by (e)} \\
&= [(a' \to b) \to 0] \to c && \text{by (I)} \\
&= (a' \to b)' \to c && \\
&= [(0 \to a) \to b'] \to c && \text{by Lemma 3.3 (16).}
\end{aligned}$$

(g)

$$\begin{aligned}
c' \to (a' \to b) &= ((0 \to a) \to b') \to c && \text{by (f)} \\
&= [(0 \to a) \to 0'] \to (b' \to c) && \text{by Lemma 3.3 (14) with } d = 0 \to a \\
&= [(a' \to 0') \to 0'] \to (b' \to c) && \text{by Lemma 2.5 (a)} \\
&= [(a' \to 0) \to 0'] \to (b' \to c) && \text{by Lemma 2.6 (1)} \\
&= [a'' \to 0'] \to (b' \to c) && \\
&= (a \to 0') \to (b' \to c) && \\
&= (a \to b') \to c && \text{by Lemma 3.3 (14) with } d = a \\
&= a' \to c && \text{by hypothesis.}
\end{aligned}$$

(h) This is immediate from (g) and (c).

(i)

$$\begin{aligned}
(0 \to a') \to c &= (c \to a') \to c && \text{by Lemma 2.6 (5)} \\
&= [c \to (a' \to c)']' && \text{by Lemma 2.6 (4)} \\
&= [(a' \to c) \to c']' && \text{by Lemma 2.6 (14)} \\
&= a' \to c && \text{by (h).}
\end{aligned}$$

□

We are now ready to complete the proof of transitivity of $\sqsubseteq$.

Theorem 3.8 $\sqsubseteq$ *is transitive.*

Proof Let $a, b, c \in A$ such that $a \sqsubseteq b$ and $b \sqsubseteq c$. Observe that

$$\begin{array}{rcll} a' & = & a \to 0 & \\ & = & (0 \to c) \to (a \to 0) & \text{by Lemma 3.5 (3) with } d = 0 \\ & = & (0 \to c) \to a' & \\ & = & (a' \to c) \to a' & \text{by Lemma 2.6 (5)} \\ & = & ((0 \to a') \to c) \to a' & \text{by Lemma 3.7 (i)} \\ & = & c \to a' & \text{by Lemma 2.6 (18)} \\ & = & a \to c' & \text{by Lemma 3.6 (n).} \end{array}$$

Consequently,

$$a = a'' = (a \to c')',$$

implying $a \sqsubseteq c$. Hence, $\sqsubseteq$ is transitive on $\mathbf{A}$. □

We are now prepared to present our first main theorem.

Theorem 3.9 *The variety* $\mathbf{I_{2,0}}$ *is a maximal subvariety of* $\mathbf{I}$ *with respect to the property that the relation* $\sqsubseteq$ *introduced in Definition 3.1 is a partial order.*

Proof Let $\mathbf{A} \in \mathbf{I}_{2,0}$. The relation $\sqsubseteq$ is a partial order on A in view of Lemma 2.4 (c), Lemma 3.2, and Theorem 3.8.

Next, let $\mathbf{V}$ be a subvariety of $\mathbf{I}$ such that $\sqsubseteq$ is a partial order on every algebra in $\mathbf{V}$. Now let $\mathbf{A} \in \mathbf{V}$. Reflexivity of $\sqsubseteq$ implies that $\mathbf{A} \models (x \to x')' \approx x$. Therefore, by Lemma 2.4, we conclude that $\mathbf{A} \in \mathbf{I}_{2,0}$, and hence, $\mathbf{V} \subseteq \mathbf{I}_{2,0}$, completing the proof. □

4 A method to construct finite $\mathbf{I}_{2,0}$-chains

Now that we know the relation $\sqsubseteq$ is a partial order on algebras in $\mathbf{I}_{2,0}$, it is natural to consider those algebras in $\mathbf{I}_{2,0}$, in which $\sqsubseteq$ is a total order.

Definition 4.1 *Let* $\mathbf{A} \in \mathbf{I}$. *We say that* $\mathbf{A}$ *is an* $\mathbf{I}_{2,0}$-chain *(chain, for short) if* $\mathbf{A} \in \mathbf{I}_{2,0}$ *and the relation* $\sqsubseteq$ *(see Definition 3.1) is totally ordered on* A.

In this section we describe a method of constructing finite $\mathbf{I}_{2,0}$-chains. But, first, we will present some examples of $\mathbf{I}_{2,0}$-chains that will foreshadow the method to construct finite $\mathbf{I}_{2,0}$-chains. We note that, in these examples, the number 0 is the constant element.

It is easy to see that the only 2-element $\mathbf{I}_{2,0}$-chains, up to isomorphism, are

$\to$:	0	1
0	1	1
1	0	1

with $0 \sqsubset 1$.

$\to$:	-1	0
-1	-1	-1
0	-1	0

with $-1 \sqsubset 0$

and the only 3-element $\mathbf{I}_{2,0}$-chains, up to isomorphism, are

$\to$:	0	1	2
0	2	2	2
1	1	1	2
2	0	1	2

with $0 \sqsubset 1 \sqsubset 2$,

$\to$:	-1	0	1
-1	-1	-1	-1
0	-1	1	1
1	-1	0	1

with $-1 \sqsubset 0 \sqsubset 1$,

→:	-2	-1	0
-2	-2	-2	-2
-1	-2	-1	-1
0	-2	-1	0

with $-2 \sqsubset -1 \sqsubset 0$.

Note that, henceforth, we will use the symbol $\leq$ to denote the natural order in $\mathbb{Z}$. Recall that $\sqsubseteq$ is being used for the order (see Definition 3.1).

The next definition describes a general method to construct a class of finite $\mathbf{I_{2,0}}$-chains, generalizing the above examples. In the next section, we will show that, this method, in fact, yields, up to isomorphism, all finite $\mathbf{I_{2,0}}$-chains.

Definition 4.2 *Let $k \in \mathbb{N}$. Let $m, n \in \omega$ be such that the interval $[-n, m] \subseteq \mathbb{Z}$ with $|[-n, m]| = k$ and $0 \leq n, m \leq k - 1$. The (auxiliary) functions p (predecessor) and $*$ are defined on $[-n, m]$ as follows:*

$$p(x) = \begin{cases} x - 1 & \text{if } x > -n \\ -n & \text{if } x = -n, \end{cases}$$

and

$$x^* = \begin{cases} m & \text{if } x = 0 \\ x & \text{if } x < 0 \\ p((p(x))^*) & \text{if } x > 0. \end{cases}$$

For convenience, we write $p(p(x)^*)$ for $p((p(x))^*)$. (Notice that the function $*$ is defined recursively for $x \geq 0$.)
Define the algebra $[-\mathbf{n}, \mathbf{m}]$ as follows:

$[-\mathbf{n}, \mathbf{m}] := \langle [-n, m]; \Rightarrow, 0 \rangle$, where $0 \in [-n, m]$ is the constant and $\Rightarrow$ is defined by

$$x \Rightarrow y = \begin{cases} max(x^*, y) & \text{if } x, y \geq 0 \\ min(x, y) & \text{otherwise.} \end{cases}$$

We set $x' := x \Rightarrow 0$.

We shall now illustrate the method described in the above definition by applying it to construct a 6-element $\mathbf{I_{2,0}}$-chain.

Let $k = 6$, and consider the interval $A = [-2, 3] = \{-2, -1, 0, 1, 2, 3\}$. Since $0 \Rightarrow 0 = max(0^*, 0) = max(3, 0) = 3$ and $a \Rightarrow b = min(a, b)$ if $a < 0$ or $b < 0$, we arrive at tha following partial table for $\Rightarrow$:

⇒	-2	-1	0	1	2	3
-2	-2	-2	-2	-2	-2	-2
-1	-2	-1	-1	-1	-1	-1
0	-2	-1	3	?	?	?
1	-2	-1	?	?	?	?
2	-2	-1	?	?	?	?
3	-2	-1	?	?	?	?

Next, we determine the operations p and $*$:

x	x^*
0	3
1	$p(p(1)^*) = p(0^*) = p(3) = 2$
2	$p(p(2)^*) = p(1^*) = p(2) = 1$
3	$p(p(3)^*) = p(2^*) = p(1) = 0$

x	$x \Rightarrow 0$
1	$max(1^*, 0) = max(2, 0) = 2$
2	$max(2^*, 0) = max(1, 0) = 1$
3	$max(3^*, 0) = max(0, 0) = 0$

Hence the table for $\Rightarrow$ becomes:

$\Rightarrow$	-2	-1	0	1	2	3
-2	-2	-2	-2	-2	-2	-2
-1	-2	-1	-1	-1	-1	-1
0	-2	-1	3	?	?	?
1	-2	-1	2	?	?	?
2	-2	-1	1	?	?	?
3	-2	-1	0	?	?	?

Observe that $0 \Rightarrow 1 = max(0^*, 1) = max(3, 1) = 3$, $1 \Rightarrow 1 = max(1^*, 1) = max(2, 1) = 2$, $2 \Rightarrow 1 = max(2^*, 1) = max(1, 1) = 1$ and $3 \Rightarrow 1 = max(3^*, 1) = max(0, 1) = 1$. Then we get

$\Rightarrow$	-2	-1	0	1	2	3
-2	-2	-2	-2	-2	-2	-2
-1	-2	-1	-1	-1	-1	-1
0	-2	-1	3	3	?	?
1	-2	-1	2	2	?	?
2	-2	-1	1	1	?	?
3	-2	-1	0	1	?	?

Iterating this process we obtain the following complete table for $\Rightarrow$:

$\Rightarrow$	-2	-1	0	1	2	3
-2	-2	-2	-2	-2	-2	-2
-1	-2	-1	-1	-1	-1	-1
0	-2	-1	3	3	3	3
1	-2	-1	2	2	2	3
2	-2	-1	1	1	2	3
3	-2	-1	0	1	2	3

Thus we have constructed the algebra $[-\mathbf{n}, \mathbf{m}]$. Observe that $-2 \sqsubset -1 \sqsubset 0 \sqsubset 1 \sqsubset 2 \sqsubset 3$ and $x'' = x^{**} = x$. Also, it is routine to verify $[-\mathbf{n}, \mathbf{m}] \in \mathbf{I_{2,0}}$. Hence it is an $\mathbf{I}_{2,0}$-chain.

Returning to the general method, we now aim to prove that $[-\mathbf{n}; \mathbf{m}]$ is an $\mathbf{I}_{2,0}$-chain. To prove this, we will need the following lemmas.

Lemma 4.3 *If $x \in [-\mathbf{n}, \mathbf{m}]$ and $0 \le x \le m$ then $x^* = m - x$ and, consequently, $x^* \in [0, m]$.*

Proof We prove this lemma by induction on the element x. Assume that $x = 0$. Then $0^* = m = m - 0$.

Next, suppose $x > 0$. Since $-n \le 0 < x$, we have $p(x) = x - 1$. Hence, by inductive hypothesis, we have

(4.1) $p(x)^* = m - p(x) = m - (x - 1) = m - x + 1.$

From $x > 0$, we can conclude that $m - x + 1 \leq m$. Also, since $x \leq m$, we obtain $0 \leq m - x$, thus $-n-1 < 0 \leq m-x$, implying $m-x+1 > -n$. Then we get $p(m-x+1) = m-x+1-1$. By (4.1), $x^* = p((p(x))^*) = p(m-x+1) = m-x$, completing the induction. It is clear that $x^* \in [0,m]$. $\square$

Corollary 4.4 *If $x \in [-\mathbf{n}, \mathbf{m}]$ then $x' = x^*$.*

Proof If $x < 0$ we have that $x' = x \Rightarrow 0 = min(x,0) = x = x^*$. If $x > 0$, then by Lemma 4.3, $x^* \geq 0$, and hence $x' = x \Rightarrow 0 = max(x^*, 0) = x^*$. $\square$

Lemma 4.5 *If $x \in [-\mathbf{n}, \mathbf{m}]$ then $x'' = x$.*

Proof We consider the following cases:

- If $x < 0$, then $x^* = x$, and hence $x^{**} = x$.
- If $x \geq 0$,

$$\begin{aligned} x^{**} &= (m-x)^* && \text{by Lemma 4.3 since } 0 < x \leq m \\ &= m-(m-x) && \text{by Lemma 4.3 since } 0 \leq m-x \leq m \\ &= x. \end{aligned}$$

Consequently, by Corollary 4.4, $x'' = x$. $\square$

Lemma 4.6 *If $x, y \in [-\mathbf{n}, \mathbf{m}]$ and $0 \leq x \leq y$ then $x^* \geq y^*$.*

Proof We prove this lemma by induction on the element x. If $x = 0$, $x^* = 0^* = m \geq y^*$ by Lemma 4.3.

Now assume that $x > 0$. Since $0 < x \leq y$, we have that $x^* = p(p(x)^*)$ and $y^* = p(p(y)^*)$. Note that $0 \leq p(x) \leq p(y)$. Then, by induction hypothesis, we get $p(y)^* \leq p(x)^*$. Hence $x^* = p(p(x)^*) \geq p(p(y)^*) = y^*$. $\square$

Lemma 4.7 *Let $k \in \mathbb{N}$. Let $m, n \in \omega$ be such that the interval $[-n, m] \subseteq \mathbb{Z}$ with $|[-n,m]| = k$ and $0 \leq n, m \leq k-1$. Then, $[-\mathbf{n}, \mathbf{m}] \in \mathbf{I}_{2,0}$.*

Proof The proof that $\langle [-n;m]; \Rightarrow, 0\rangle$ satisfies the identity (I) is long and computational, but routine. Hence we leave the verification to the reader with the recommendation that the following cases be considered, where $i, j, k \in [-n;m]$:

(1) $i,j,k \geq 0$, $i^* \geq j$, $i \geq k$

(2) $i,j,k \geq 0$, $i^* \geq j$, $i < k$

(3) $i,j,k \geq 0$, $i^* < j$, $k \geq i$

(4) $i,j,k \geq 0$, $i^* < j$, $k < i$, $j^* \leq k$

(5) $i,j,k \geq 0$, $i^* < j$, $k < i$, $j^* > k$

(6) $i,j \geq 0$ and $k < 0$

(7) $i \geq 0$, $j < 0$ and $k \geq 0$

(8) $i \geq 0$, $j < 0$ and $k < 0$

(9) $i < 0$, $j \geq 0$ and $k \geq 0$

(10) $i < 0$, $j \geq 0$ and $k < 0$

(11) $i < 0$, $j < 0$ and $k \geq 0$

(12) $i,j,k < 0$.

Observe that, if $x \in [-n,m]$, then, from Corollary 4.4, we have $x' = x^*$, and from Lemma 4.5 we have that $x'' = x$; and in particular $0'' = 0$. Thus, we conclude that $\langle [-n,m]; \Rightarrow, 0\rangle \in \mathbf{I}_{2,0}$. $\square$

In view of the above lemma and Theorem 3.8, the relation defined by

$$x \sqsubseteq y \quad \text{if and only if} \quad (x \Rightarrow y')' = x$$

is a partial order on $[-\mathbf{n}, \mathbf{m}]$. We now wish to show that $\sqsubseteq$ is indeed a total order.

Lemma 4.8 *Let* $[-\mathbf{n}, \mathbf{m}]$ *be the algebra, as defined in Definition 4.2. Then*

$$\langle [-n, m]; \sqsubseteq \rangle \cong \langle [-n, m]; \leq \rangle.$$

Proof Let $x, y \in [-n, m]$. It is enough to prove that $x \leq y$ if and only if $x \sqsubseteq y$.

Assume that $x \leq y$. We will consider the following cases:

- **Case 1**: $x < 0$. Then

  (4.2) $(x \Rightarrow y')' = (x \Rightarrow y^*)^* = [min(x, y^*)]^*$.

  We consider further the following subcases:

  - **Case 1.1**: $y < 0$.

    $$\begin{aligned} (x \Rightarrow y')' &= [min(x, y^*)]^* && \text{by (4.2)} \\ &= [min(x, y)]^* && \text{since } y < 0 \\ &= x^* && \text{since } x \leq y \\ &= x. && \text{since } x < 0 \end{aligned}$$

  - **Case 1.2**: $y \geq 0$.

    $$\begin{aligned} (x \Rightarrow y')' &= [min(x, y^*)]^* && \text{by (4.2)} \\ &= x^* && \text{since } y^* \geq 0 \text{ by Lemma 4.3, and } x < 0 \\ &= x. \end{aligned}$$

- **Case 2**: $x \geq 0$. Therefore $y \geq 0$. In this case

  $$\begin{aligned} (x \Rightarrow y')' &= (x \Rightarrow y^*)^* \\ &= [max(x^*, y^*)]^* \\ &= x^{**} && \text{by Lemma 4.6} \\ &= x \end{aligned}$$

Thus, in all these cases, $x \sqsubseteq y$.

For the converse, suppose $x \sqsubseteq y$.

- **Case 1**: $x < 0$. If $y \geq 0$ then $x < y$. So, we can assume $y < 0$. Then

  $$\begin{aligned} x &= x' && \text{since } x < 0 \\ &= (x \Rightarrow y')'' && \text{by hypothesis} \\ &= x \Rightarrow y' && \text{by Lemma 4.5} \\ &= x \Rightarrow y \\ &= min(x, y). \end{aligned}$$

  Hence $x \leq y$.

- **Case 2**: $x \geq 0$. Suppose $y < 0$. Then

$$\begin{array}{rcll} x & = & (x \Rightarrow y')' & \text{by hypothesis} \\ & = & (x \Rightarrow y)' & \\ & = & min(x,y)' & \\ & = & y' & \\ & = & y, & \end{array}$$

  a contradiction. Hence $y \geq 0$. Consequently,

$$\begin{array}{rcll} x' & = & (x \Rightarrow y')'' & \\ & = & x \Rightarrow y' & \text{by Lemma 4.5} \\ & = & max(x', y'), & \end{array}$$

  so, $x' \geq y'$. Then, by Lemma 4.5 and Lemma 4.6, $x = x'' \leq y'' = y$.

□

In view of Lemma 4.7 and Lemma 4.8, we have proved the following

Theorem 4.9 $[-\mathbf{n}, \mathbf{m}]$ *is an* $\mathbf{I_{2,0}}$*-chain, where*

$$-n \sqsubset -n+1 \sqsubset \ldots \sqsubset -1 \sqsubset 0 \sqsubset 1 \sqsubset 2 \sqsubset \ldots \sqsubset m.$$

5 Characterization of finite $\mathbf{I}_{2,0}$-chains

In this section we are going to prove our second main result. The following lemmas will be useful later in this section.

Lemma 5.1 *Let* $\mathbf{A} \in \mathbf{I}_{2,0}$. *Then* $0'$ *is the greatest element in* A, *relative to* $\sqsubseteq$.

Proof Let $a \in A$. Since $(a \to (0 \to 0)')' = (a \to 0'')' = (a \to 0)' = a'' = a$, we have $a \sqsubseteq 0'$. □

Lemma 5.2 *Let* $\mathbf{A} \in \mathbf{I}_{2,0}$ *and let* $a, b \in A$ *with* $0 \sqsubseteq a \sqsubseteq b$. *Then* $b' \sqsubseteq a'$.

Proof

$$\begin{array}{rcll} (b' \to a'')' & = & (b' \to a)' & \\ & = & (b' \to (a \to b')')' & \text{by hypothesis} \\ & = & ((a \to 0') \to b'')' & \text{by Lemma 2.6 (26)} \\ & = & ((a \to 0') \to b)' & \\ & = & ((0 \to a') \to b)' & \text{by Lemma 2.5 (a)} \\ & = & ((0 \to a')'' \to b)' & \\ & = & (0' \to b)' & \text{by hypothesis} \\ & = & b' & \text{by Lemma 2.4 (a).} \end{array}$$

□

Lemma 5.3 *Let* $\mathbf{A} \in \mathbf{I}_{2,0}$ *and let* $a \in A$. *If* $0 \sqsubseteq a$ *then* $0 \to a = 0'$.

Proof First notice that, since $0 \sqsubseteq a$, $0' = (0 \to a')'' = 0 \to a'$. Consequently,

(5.1) $0' = 0 \to a'$.

Then

$$\begin{array}{rcll} 0' & = & 0' \to 0' & \text{by Lemma 2.4 (a)} \\ & = & (0 \to a') \to 0' & \text{by (5.1)} \\ & = & (0' \to a') \to 0' & \text{by Lemma 2.6 (5)} \\ & = & a' \to 0' & \text{by Lemma 2.4 (a)} \\ & = & 0 \to a. & \text{by Lemma 2.5 (a)} \end{array}$$

□

Lemma 5.4 *Let* $\mathbf{A} \in \mathbf{I}_{2,0}$ *and let* $a, b \in A$. *If* $0 \sqsubseteq a$ *and* $0 \sqsubseteq b$ *then* $0 \sqsubseteq a \to b$.

Proof

$$\begin{array}{rcll} [0 \to (a \to b)']' & = & [(a \to b) \to 0']' & \text{by Lemma 2.5 (a)} \\ & = & (0 \to a) \to (b \to 0')' & \text{by (I)} \\ & = & (0 \to a) \to (0 \to b')' & \text{by Lemma 2.5 (a)} \\ & = & (0 \to a) \to 0 & \text{since } 0 \sqsubseteq b \\ & = & 0' \to 0 & \text{by Lemma 5.3 since } 0 \sqsubseteq a \\ & = & 0. & \text{by Lemma 2.4 (a)} \end{array}$$

□

Corollary 5.5 *Let* $\mathbf{A} \in \mathbf{I}_{2,0}$ *and* $a \in A$. *If* $a \sqsupseteq 0$ *then* $a' \sqsupseteq 0$.

Lemma 5.6 *Let* $\mathbf{A}$ *be an* $\mathbf{I}_{2,0}$*-chain and let* $a, b \in A$. *Then* $a' \to b' = b \to a$.

Proof Since $\mathbf{A}$ is a chain, we can assume that $b' \sqsubseteq a$ or $a \sqsubseteq b'$.

If $b' \sqsubseteq a$, $(b' \to a')' = b'$, then $b' \to a' = b$. Hence $b \to a = (b' \to a') \to a = [(a' \to b') \to (a' \to a)']'$, using (I). By Lemma 2.4 (d), $[(a' \to b') \to (a' \to a)']' = [(a' \to b') \to a']' = [[(a \to a') \to (b' \to a')']']' = (a \to a') \to (b' \to a')' = (a'' \to a') \to (b' \to a')' = a' \to b'$.

If $a \sqsubseteq b'$ then we have $a' = (a \to b'')'' = a \to b$, and the rest of the argument is similar to the previous case. □

Lemma 5.7 *Let* $\mathbf{A}$ *be a* $\mathbf{I}_{2,0}$*-chain with* $|A| \geq 2$ *and let* $a \in A$ *such that* $a \sqsubset 0$. *Then*

(a) $0 \to a' = a'$

(b) $0 \to a = a$

(c) $(a \to a) \to a = a \to a$

(d) $a \to a = a'$

(e) $a \to a = a$

(f) $a = a'$.

Proof

(a) Since $a \sqsubseteq 0$, we have that $a = (a \to 0')'$. Therefore, $a' = (a \to 0')'' = a \to 0' = 0 \to a'$ by Lemma 2.5 (b).

(b) Since $a \sqsubseteq 0$, we have

(5.2) $a = (a \to 0')'$.

Then we get

$$\begin{array}{rcll} (0 \to a) \to 0' & = & [(0 \to 0) \to (a \to 0')']' & \text{by (I)} \\ & = & [(0 \to 0) \to a]' & \text{by (5.2)} \\ & = & [0' \to a]' & \\ & = & a' & \text{by lemma 2.4 (a)} \end{array}$$

Using Lemma 2.5 (b), we obtain

(5.3) $a' = 0 \to (0 \to a)'$.

Since $\mathbf{A}$ is a chain, $0 \sqsubseteq 0 \to a$ or $0 \to a \sqsubseteq 0$. Suppose that $0 \sqsubseteq 0 \to a$. Then $(0 \to (0 \to a)')' = 0$ Therefore, by (5.3), $a = a'' = (0 \to (0 \to a)')' = 0$, a contradiction, since $a \neq 0$. Consequently, $0 \to a \sqsubseteq 0$. Hence, we have

$$\begin{array}{rcll} 0 \to a & = & ((0 \to a) \to 0')' & \text{since } 0 \to a \sqsubseteq 0 \\ & = & (0 \to (0 \to a)')' & \text{by lemma 2.5 (b)} \\ & = & a'' & \text{by (5.3)} \\ & = & a. & \end{array}$$

(c)

$$\begin{array}{rcll} a \to a & = & (0 \to a) \to a & \text{by item (b)} \\ & = & (a' \to 0') \to a & \text{by lemma 2.5 (a)} \\ & = & [(a' \to a') \to (0' \to a)']' & \text{by (I)} \\ & = & [(a \to a) \to (0' \to a)']' & \text{by Lemma 5.6} \\ & = & [(a \to a) \to a']' & \text{by lemma 2.5 (a)} \\ & = & [[(a'' \to a) \to (a \to a')']']' & \text{by (I)} \\ & = & (a \to a) \to (a \to a')' & \\ & = & (a \to a) \to (a'' \to a')' & \\ & = & (a \to a) \to a'' & \text{by lemma 2.5 (d)} \\ & = & (a \to a) \to a. & \end{array}$$

(d) Since $\mathbf{A}$ is a chain, $0 \to a' \sqsubseteq a$ or $a \sqsubseteq 0 \to a'$.

First, we assume that $0 \to a' \sqsubseteq a$. Then

$$\begin{array}{rcll} a \to a & = & (a \to a) \to a & \text{by (c)} \\ & = & (a' \to a') \to a & \text{by Lemma 5.6} \\ & = & a' \to (a' \to a')' & \text{by Lemma 5.6} \\ & = & (a \to 0) \to (a' \to a')' & \\ & = & [(a \to 0) \to (a' \to a')']'' & \\ & = & [(0 \to a') \to a']' & \text{using (I)} \\ & = & 0 \to a' & \text{since } 0 \to a' \sqsubseteq a \\ & = & a' & \text{using (a).} \end{array}$$

Next, we assume $a \sqsubseteq 0 \to a'$, i.e., $(a \to (0 \to a')')' = a$. Then, from (a), we have $a \to a = (a \to a'')'' = [(a \to (0 \to a')')']' = a'$.

(e) Using the items (c), (d) and Lemma 2.5 (d), we have $a \to a = (a \to a) \to a = a' \to a = a$.

(f) This follows immediately from the two preceding items.

$\square$

Lemma 5.8 *Let* $\mathbf{A}$ *be an* $\mathbf{I}_{2,0}$*-chain with* $|A| \geq 2$*, and let* $a, b \in A$*. If* $0 \sqsubseteq a$ *and* $b \sqsubset 0$ *then* $b \to a = b$ *and* $a \to b = b$*.*

Proof Since $0 \sqsubseteq a$ and $b \sqsubset 0$, we have that $(0 \to a')' = 0$ and $(b \to 0')' = b$. Therefore, using Lemma 5.6, $b = b'' = b' \to 0 = (b \to 0')'' \to (0 \to a')' = (b \to 0') \to (0 \to a')' = (0 \to b') \to (a \to 0')' = [(b' \to a) \to 0']'$. Hence,

(5.4) $b = [(b' \to a) \to 0']'$.

From the hypothesis and Lemma 5.7 (f), we have

(5.5) $b' = b$.

Suppose that $0 \sqsubseteq b' \to a$. Then $0 = [0 \to (b' \to a)']' = [(b' \to a) \to 0']'$ by Lemma 5.6, implying $0 = b$, which is a contradiction in view of (5.4). Consequently, $b' \to a \sqsubseteq 0$, since $\mathbf{A}$ is a chain. Hence,

(5.6) $b' \to a = [(b' \to a) \to 0']'$.

From (5.4), (5.5) and (5.6) we conclude $b = b \to a$, proving the first half of the conclusion of the lemma. From

$$\begin{aligned} b &= (b \to a')' && \text{since } b \sqsubseteq a, \text{ as } 0 \sqsubseteq a \text{ and } b \sqsubset 0 \\ &= (a'' \to b')' && \text{by Lemma 5.6} \\ &= (a \to b')' \\ &= (a \to b)' && \text{by (5.5)} \end{aligned}$$

we conclude that $a \to b = b' = b$ in view of (5.5), completing the second half. $\square$

Definition 5.9 *Let* $\mathbf{A} = \langle A; \to, 0\rangle$ *be a finite* $\mathbf{I}_{2,0}$*-chain. We let* $A^+ := \{a \in A : a \sqsupset 0\}$ *and* $A^- := \{a \in A : a \sqsubset 0\}$*. Observe that* $A = A^+ \cup \{0\} \cup A^-$*. Henceforth, without loss of generality, we will represent* $A = [-n, m]$ *with* $0 \leq n, m \leq |A| - 1$*, such that*

$$-n \sqsubset -n + 1 \sqsubset \ldots \sqsubset -1 \sqsubset 0 \sqsubset 1 \sqsubset 2 \sqsubset \ldots \sqsubset m.$$

Remark 5.10 *In view of the above definition, we can use the functions* $*$ *and* p *of Definition 4.2 as functions on the domain* $[-n, m]$ *of* $\mathbf{A}$ *as well.*

Now, we wish to prove that $\langle A; \to, 0\rangle = \langle [-n; m]; \Rightarrow, 0\rangle$. To achieve this, we need the following lemmas.

Lemma 5.11 *Let* $\mathbf{A} = \langle A; \to, 0\rangle$ *be a finite* $\mathbf{I}_{2,0}$*-chain with* $|A| \geq 2$*. If* $a \sqsupset 0$ *then* $a' = p(p(a)')$*.*

Proof By hypothesis we have that $a \sqsupset 0$. Then $p(a) \sqsupseteq 0$. Hence $0 \sqsubseteq p(a) \sqsubset a$. Then, by Lemma 5.2,

(5.7) $a' \sqsubseteq p(a)'$.

Since $a \sqsupset 0$, by Corollary 5.5, $a' \sqsupseteq 0$. Therefore, by (5.7),

(5.8) $0 \sqsubseteq p(a)'$.

If $a' = p(a)'$ then $a = p(a)$ and, consequently, $a = -n$, a contradiction, so $a' \sqsubset p(a)'$, and hence, $0 \sqsubseteq a' \sqsubseteq p(p(a)') \sqsubset p(a)'$. By lemma 5.2, $a \sqsupseteq [p(p(a)')]' \sqsupseteq p(a)$. Thus

(5.9) $[p(p(a)')]' \in \{a, p(a)\}$.

If $[p(p(a)')]' = p(a)$, we have that $p(p(a)') = [p(p(a)')]'' = p(a)'$, a contradiction, since $p(a)' \sqsupseteq 0$ by (5.8). Therefore $[p(p(a)')]' = a$ and therefore, $p(p(a)') = a'$. □

Lemma 5.12 *Let* $\mathbf{A} = \langle A; \rightarrow, 0\rangle$ *be a finite* $\mathbf{I}_{2,0}$*-chain. If* $a \in A$ *then* $a^* = a'$.

Proof The statement $0' = m = 0^*$ follows from Lemma 5.1. If $a \sqsubset 0$ then $a' = a$ by Lemma 5.7 (f), and $a = a^*$ by definition, implying $a = a^*$.

Now assume that $a \sqsupset 0$. We will verify that $a' = a^*$ by induction on a. If $a = 1$, then, as $0' = 0^*$, we have, by Lemma 5.11, that $1' = p(p(1)') = p(0') = p(0^*) = p(p(1)^*) = 1^*$. The inductive hypothesis is that $p(a)' = p(a)^*$. Hence, we have, by Lemma 5.11, $a' = p(p(a)') = p(p(a)^*) = a^*$. □

The following theorem shows that the general method described in Definition 4.2 essentially gives all finite $\mathbf{I}_{2,0}$-chains.

Theorem 5.13 *Let* $\mathbf{A}$ *be a finite* $\mathbf{I}_{2,0}$*-chain. Then* $\mathbf{A} \cong \langle [-n, m]; \Rightarrow, 0\rangle$ *for some* $0 \leq n, m \leq |A| - 1$.

Proof We will use the notation of Definition 5.9. Let $i, j \in A$. From Lemma 5.12, $i' = i^*$ and $j' = j^*$. It suffices to verify that

$$i \rightarrow j = \begin{cases} max(i', j) & \text{if } i, j \sqsupseteq 0 \\ min(i, j) & \text{otherwise} \end{cases}$$

with $0' = m$. We consider the following cases:

- **Case 1**: $j > 0$.
  We need the following subcases:
  - **Case 1.1**: $i > 0$.
    We make the following further subcases:
    * **Case 1.1.1**: $i' \geq j$.
      Since $i' \sqsupseteq j$, we observe that

      (5.10) $(j \rightarrow i'')' = j$.

      Hence

$$\begin{aligned} i \rightarrow j &= i \rightarrow (j \rightarrow i'')' && \text{by (5.10)} \\ &= i \rightarrow (j \rightarrow i)' \\ &= [(i \rightarrow j) \rightarrow i]' && \text{by Lemma 2.6 (4)} \\ &= [(0 \rightarrow j) \rightarrow i]' && \text{by Lemma 2.6 (5)} \\ &= [0' \rightarrow i]' && \text{by Lemma 5.3 since } j \sqsupseteq 0 \\ &= i' && \text{by Lemma 2.4 (a)} \\ &= max(i', j) && \text{since } i' \sqsupseteq j \end{aligned}$$

* **Case 1.1.2**: $i' < j$.
  Since $i' \sqsubseteq j$, we have

  (5.11) $(i' \to j')' = i'$.

  Therefore,

$$\begin{array}{rcll} i \to j & = & i'' \to j & \\ & = & (i' \to j')'' \to j & \text{by (5.11)} \\ & = & (i' \to j') \to j & \\ & = & (i' \to 0') \to j & \text{by Lemma 2.6 (1)} \\ & = & (0 \to i) \to j & \text{by Lemma 2.5 (a)} \\ & = & 0' \to j & \text{by Lemma 5.3 since } i \sqsupseteq 0 \\ & = & j & \text{by Lemma 2.4 (a)} \\ & = & max(i', j) & \text{since } i' \sqsubseteq j \end{array}$$

- **Case 1.2**: $i = 0$.
  Using Lemma 5.3 and Lemma 5.1, $0 \to j = 0' = max(0', j)$.
- **Case 1.3**: $i < 0$.

$$\begin{array}{rcll} i \to j & = & (0 \to i) \to j & \text{by Lemma 5.7 (b)} \\ & = & (i' \to 0') \to j & \\ & = & (i \to 0') \to j & \text{by Lemma 5.7 (f)} \\ & = & [(j' \to i) \to (0' \to j)']' & \text{by (I)} \\ & = & [(j' \to i) \to j']' & \\ & = & [(0 \to i) \to j']' & \text{by Lemma 2.6 (5)} \\ & = & (i \to j')' & \text{by Lemma 5.7 (b)} \\ & = & i & \text{since } i \sqsubset j \\ & = & min(i, j) & \end{array}$$

- **Case 2**: $j < 0$.
  It is useful to consider the following subcases:
  - **Case 2.1**: $i > 0$

$$\begin{array}{rcll} i \to j & = & i \to j' & \text{by Lemma 5.7 (f)} \\ & = & i \to (j \to i')'' & \text{since } j \sqsubset i \\ & = & i \to (j \to i') & \\ & = & j \to i' & \text{by Lemma 2.6 (17)} \\ & = & (j \to i')'' & \\ & = & j' & \text{since } j \sqsubset i \\ & = & j & \text{by Lemma 5.7 (f)} \\ & = & min(i, j) & \end{array}$$

  - **Case 2.2**: $i = 0$.

$$\begin{array}{rcll} i \to j & = & 0 \to j & \\ & = & j & \text{by Lemma 5.7 (b)} \\ & = & min(i, j) & \end{array}$$

  - **Case 2.3**: $i < 0$.

* **Case 2.3.1**: $i \leq j$.
  As $i \sqsubseteq j$, we have

  (5.12) $(i \to j')' = i$.

  Observe

$$\begin{array}{rcll} i \to j & = & i \to j' & \text{by Lemma 5.7 (f)} \\ & = & (i \to j')'' & \\ & = & i' & \text{by (5.12)} \\ & = & i & \text{by Lemma 5.7 (f)} \\ & = & min(i, j). & \end{array}$$

* **Case 2.3.2**: $i > j$. We have

  (5.13) $(j \to i')' = j$.

  as $j \sqsubseteq i$. Hence

$$\begin{array}{rcll} i \to j & = & j' \to i' & \text{by Lemma 5.6} \\ & = & j \to i' & \text{by Lemma 5.7 (f)} \\ & = & (j \to i')'' & \\ & = & j' & \text{by (5.13)} \\ & = & j & \text{by Lemma 5.7 (f)} \\ & = & min(j, i) & \end{array}$$

- **Case 3**: $j = 0$.
  - **Case 3.1**: $i \geq 0$.
    By Corollary 5.5, as $i \sqsupseteq 0$, we have that $i' = i \to 0 \sqsupseteq 0$. Hence $i \to 0 = i' = max(i', 0)$.
  - **Case 3.2**: $i < 0$. We have that

$$\begin{array}{rcll} i \to j & = & i \to 0 & \\ & = & i' & \\ & = & i & \text{by Lemma 5.7 (f)} \\ & = & min(i, j) & \end{array}$$

Hence $\mathbf{A} \cong \langle [-n; m]; \Rightarrow, 0 \rangle$. □

The following theorem, our second main result, is now immediate from the preceding results.

Theorem 5.14 *There are n non-isomorphic $I_{2,0}$-chains of size n, for $n \in \mathbb{N}$.*

A Appendix: Proofs

We would like to mention here that the identity: $x'' \approx x$ is used in these proofs frequently without explicit mention.

Proof of Lemma 2.6: Items (1) to (17) are proved in [3]. The proofs of (18) to (26) are given in [5]. Let $a, b, c, d \in A$.

(27)

$$\begin{aligned} (b \to c) \to a &= [(b \to c) \to a]' \to [(b \to c) \to a] && \text{by Lemma 2.4 (d)} \\ &= [(a' \to b) \to (c \to a)']'' \to [(b \to c) \to a] && \text{from (I)} \\ &= [(a' \to b) \to (c \to a)'] \to [(b \to c) \to a] \end{aligned}$$

(28)

$$\begin{aligned} [[0 \to (a \to b)'] \to (0 \to b')']' &= [\{0 \to (a \to b)'\} \to (b \to 0')']' && \text{by Lemma 2.5 (a)} \\ &= [(a \to b)' \to b] \to 0' && \text{by (I)} \\ &= 0 \to [(a \to b)' \to b]' && \text{by Lemma 2.5 (a)} \\ &= (a \to b) \to (0 \to b') && \text{by (10)} \\ &= 0 \to [(a \to b) \to b'] && \text{by (13)} \\ &= 0 \to [(a \to 0') \to b'] && \text{by (1)} \\ &= (a \to 0') \to (0 \to b') && \text{by (13)} \\ &= (0 \to a') \to (0 \to b') && \text{by Lemma 2.5 (a)} \\ &= a' \to (0 \to b') && \text{by (20)} \\ &= [(0 \to a) \to (0 \to b')']' && \text{by (8)} \\ &= [(0 \to a) \to (b \to 0')']' && \text{by Lemma 2.5 (a)} \\ &= (a \to b) \to 0' && \text{by (I)} \\ &= 0 \to (a \to b)' && \text{by Lemma 2.5 (a)} \end{aligned}$$

(29)

$$\begin{aligned} 0 \to [(a \to b) \to c] &= 0 \to [(c' \to a) \to (b \to c)']' && \text{by (I)} \\ &\sqsubseteq 0 \to (b \to c)'' && \text{by (28)} \\ &= 0 \to (b \to c) \end{aligned}$$

(30)

$$\begin{aligned} a' \to (b \to 0')' &= (a \to 0) \to (b \to 0')' \\ &= [\{(b \to 0') \to a\} \to \{0 \to (b \to 0')'\}']' && \text{by (I)} \\ &= [\{(b \to 0') \to a\} \to \{0 \to (0 \to b')'\}']' && \text{by Lemma 2.5 (a)} \\ &= [\{(b \to 0') \to a\} \to (0 \to b)']' && \text{by (9)} \\ &= [\{(0 \to b') \to a\} \to (0 \to b)']' && \text{by Lemma 2.5 (a)} \\ &= [[\{0 \to (0 \to b)'\} \to a] \to (0 \to b)']' && \text{by (9)} \\ &= [a \to (0 \to b)']' && \text{by (18)} \end{aligned}$$

(31)

$$\begin{aligned} [(0 \to a) \to b]' &= [(b \to a) \to b]' && \text{by (5)} \\ &= [b \to (a \to b)']'' && \text{by (4)} \\ &= b \to (a \to b)' \end{aligned}$$

(32)

$$\begin{aligned} [a \to (b \to 0')']' &= [a \to (0 \to b')']' && \text{by Lemma 2.5 (a)} \\ &= a' \to (b' \to 0')' && \text{by (30)} \\ &= a' \to (0 \to b)' && \text{by Lemma 2.5 (a)} \end{aligned}$$

(33)

$$\begin{aligned} b' \to a' &= (b \to 0) \to a' \\ &= [(a \to b) \to (0 \to a')']' && \text{by (I)} \\ &= [(a \to b) \to (a \to 0')']' && \text{by Lemma 2.5 (a)} \\ &= (a \to b)' \to (0 \to a)' && \text{by (32) with } x = a \to b, y = a \end{aligned}$$

(34)

$$\begin{aligned}
(0 \to a)' \to (0 \to b)' &= [(0 \to a) \to 0] \to (0 \to b)' \\
&= [\{(0 \to b) \to (0 \to a)\} \to \{0 \to (0 \to b)'\}']' \\
&\qquad \text{by (I)} \\
&= [\{0 \to (0 \to b)'\} \to (0 \to b)] \to [(0 \to a) \to \{0 \to (0 \to b)'\}']' \\
&\qquad \text{by (I)} \\
&= [\{(0 \to b) \to (0 \to b)'\} \to (0 \to b)] \to [(0 \to a) \to \{0 \to (0 \to b)'\}']' \\
&\qquad \text{by (5)} \\
&= [(0 \to b)' \to (0 \to b)] \to [(0 \to a) \to \{0 \to (0 \to b)'\}']' \\
&\qquad \text{by Lemma 2.4 (d)} \\
&= (0 \to b) \to [(0 \to a) \to \{0 \to (0 \to b)'\}']' \\
&\qquad \text{by Lemma 2.4 (d)} \\
&= (0 \to b) \to [(0 \to a) \to (0 \to b')']' \\
&\qquad \text{by (9)} \\
&= (0 \to b) \to [(0 \to a) \to (b \to 0')']' \\
&\qquad \text{by Lemma 2.5 (a)} \\
&= (0 \to b) \to [(a \to b) \to 0'] \\
&\qquad \text{by (I)} \\
&= (0 \to b) \to [0 \to (a \to b)'] \\
&\qquad \text{by Lemma 2.5 (a)} \\
&= 0 \to [(0 \to b) \to (a \to b)'] \\
&\qquad \text{by (13)} \\
&= 0 \to (a \to b)' \\
&\qquad \text{by (3)} \\
&= (a \to b) \to 0' \\
&\qquad \text{by Lemma 2.5 (a)} \\
&= [(0 \to a) \to (b \to 0')']' \\
&\qquad \text{by (I)} \\
&= [(0 \to a) \to (0 \to b')']' \\
&\qquad \text{by Lemma 2.5 (a)} \\
&= [a' \to (0 \to b')]'' \\
&\qquad \text{by (8)} \\
&= a' \to (0 \to b') \\
&= 0 \to (a' \to b') \qquad \text{by (13).}
\end{aligned}$$

(35)

$$\begin{aligned}
[(a \to b)' \to \{b \to (a \to b)'\}']' &= [(a \to b)' \to b] \to (a \to b)' && \text{by (4)} \\
&= (0 \to b) \to (a \to b)' && \text{by (5)} \\
&= (a \to b)' && \text{by (3)}
\end{aligned}$$

(36)

$$\begin{aligned}
(0 \to a) \to b &= (b \to a) \to b && \text{by (5)} \\
&= [b \to (a \to b)']' && \text{by (4)} \\
&\sqsubseteq (a \to b)' \to [b \to (a \to b)']' && \text{by (35) with } x = b, y = (a \to b)' \\
&= [\{(a \to b)' \to b\} \to (a \to b)']' && \text{by (4)} \\
&= [(0 \to b) \to (a \to b)']' && \text{by (5)} \\
&= (a \to b)'' && \text{by (3)} \\
&= a \to b && \text{since } x'' \approx x
\end{aligned}$$

(37)

$$\begin{array}{rcll}
[\{a \to (b \to a)'\} \to a'']' & = & [\{a \to (b \to a)'\} \to a]' & \\
& = & [\{0 \to (b \to a)'\} \to a]' & \text{by (5)} \\
& = & [\{(b \to a) \to 0'\} \to a]' & \text{by Lemma 2.5 (a)} \\
& = & [\{(b \to a) \to a'\} \to a]' & \text{by (1)} \\
& = & [\{(b \to 0') \to a'\} \to a]' & \text{by (1)} \\
& = & [\{(b \to 0') \to 0'\} \to a]' & \text{by (1)} \\
& = & [\{(b \to 0'') \to 0'\} \to a]' & \text{by (1)} \\
& = & [\{(b \to 0) \to 0'\} \to a]' & \\
& = & [(b' \to 0') \to a]' & \\
& = & [(0 \to b) \to a]' & \text{by Lemma 2.5 (a)} \\
& = & [(a \to b) \to a]' & \text{by (5)} \\
& = & a \to (b \to a)' & \text{by (4)}
\end{array}$$

□

Proof of Lemma 3.3

(1) Observe that by Lemma 2.5 (a), Lemma 2.6 (1) and the hypothesis we have that $(0 \to a') \to b = (a \to 0') \to b = (a \to b') \to b = (a \to b')'' \to b = a' \to b$.

(2)

$$\begin{array}{rcll}
b \to a' & = & [(0 \to a') \to b] \to a' & \text{by Lemma 2.6 (18)} \\
& = & (a' \to b) \to a' & \text{from (1)} \\
& = & (0 \to b) \to a' & \text{by Lemma 2.6 (5).}
\end{array}$$

(3)

$$\begin{array}{rcll}
b \to a' & = & (0 \to b) \to a' & \text{from (2)} \\
& = & (0 \to b) \to (a \to b')'' & \text{by hypothesis} \\
& = & (0 \to b) \to (a \to b') & \\
& = & (0 \to b'') \to (a \to b') & \\
& = & a \to b' & \text{by Lemma 2.6 (2)} \\
& = & (a \to b')'' & \\
& = & a' & \text{by hypothesis}
\end{array}$$

(4)

$$\begin{array}{rcll}
0 \to (a' \to b) & = & a' \to (0 \to b) & \text{by Lemma 2.6 (13)} \\
& = & 0 \to (a \to b')' & \text{by Lemma 2.6 (10)} \\
& = & 0 \to a & \text{by hypothesis}
\end{array}$$

(5) By hypothesis and (I) we have that $(d \to a) \to b' = [(b \to d) \to (a \to b')']' = [(b \to d) \to a]'$.

(6)

$$\begin{array}{rcll}
[\{d \to (0 \to b')\} \to a]' & = & (a' \to d) \to [(0 \to b') \to a]' & \text{by (I)} \\
& = & (a' \to d) \to [(a \to b') \to a]' & \text{by Lemma 2.6 (5)} \\
& = & (a' \to d) \to [(a \to b')'' \to a]' & \\
& = & (a' \to d) \to (a' \to a)' & \text{by hypothesis} \\
& = & (a' \to d) \to a' & \text{by Lemma 2.4 (d)} \\
& = & (a' \to d) \to (0' \to a)' & \text{by Lemma 2.4 (a)} \\
& = & [(d \to 0') \to a]' & \text{by (I)} \\
& = & (0 \to d) \to a' & \text{by Lemma 2.6 (11)}
\end{array}$$

(7)

$$\begin{aligned}
a \to [(a' \to d) \to \{(0 \to a) \to b'\}] &= a \to [(a' \to d) \to \{(b \to 0) \to (a \to b')'\}'] \\
&\quad \text{by (I)} \\
&= a \to [(a' \to d) \to \{(b \to 0) \to a\}'] \\
&\quad \text{by hypothesis} \\
&= a \to [\{d \to (b \to 0)\} \to a]' \\
&\quad \text{by (I)} \\
&= [[d \to (0 \to (b \to 0))] \to a]' \\
&\quad \text{by Lemma 2.6 (22) with } x = d, y = b \to 0, z = a \\
&= [[d \to (0 \to b')] \to a]' \\
&= (0 \to d) \to a' \\
&\quad \text{by (6)}
\end{aligned}$$

(8)

$$\begin{aligned}
a \to ((d \to a) \to b') &= a \to [(b \to d) \to a]' && \text{by (5)} \\
&= a'' \to [(b \to d) \to a]' \\
&= (a' \to 0) \to [(b \to d) \to a]' \\
&= [\{0 \to (b \to d)\} \to a]' && \text{by (I)} \\
&= [\{(b \to d)' \to 0'\} \to a]' \\
&= [\{((b \to d) \to 0) \to 0'\} \to a]' \\
&= [\{((b \to d) \to 0') \to 0'\} \to a]' && \text{by Lemma 2.6 (1)} \\
&= [\{((b \to d) \to 0') \to a'\} \to a]' && \text{by Lemma 2.6 (1)} \\
&= [\{((b \to d) \to a) \to a'\} \to a]' && \text{by Lemma 2.6 (1)} \\
&= [\{((b \to d) \to a) \to 0'\} \to a]' && \text{by Lemma 2.6 (1)} \\
&= [\{0 \to ((b \to d) \to a)'\} \to a]' \\
&= [\{0 \to ((d \to a) \to b')\} \to a]' && \text{by (5)} \\
&= [\{a \to ((d \to a) \to b')\} \to a]' && \text{by Lemma 2.6 (5)} \\
&= (a' \to a) \to [\{(d \to a) \to b'\} \to a]' && \text{by (I)} \\
&= a \to [\{(d \to a) \to b'\} \to a]' && \text{by Lemma 2.4 (d)} \\
&= a \to [\{a' \to (d \to a)\} \to (b' \to a)'] && \text{by (I)} \\
&= a \to [\{a' \to (d \to a)\} \to \{(b \to 0) \to a\}'] \\
&= a \to [\{a' \to (d \to a)\} \to \{(b \to 0) \to (a \to b')'\}'] && \text{by hypothesis} \\
&= a \to [\{a' \to (d \to a)\} \to \{(0 \to a) \to b'\}] && \text{by (I)} \\
&= [0 \to (d \to a)] \to a' && \text{by (7) with } d := d \to a \\
&= a \to (d \to a)' && \text{by Lemma 2.6 (23)}
\end{aligned}$$

(9)

$$\begin{aligned}
[0 \to (b \to d)] \to a &= [(a' \to 0) \to ((b \to d) \to a)']' && \text{by (I)} \\
&= [a \to ((b \to d) \to a)']' \\
&= [a \to ((d \to a) \to b')]' && \text{by (5)} \\
&= [a \to (d \to a)']' && \text{by (8)} \\
&= (a \to d) \to a && \text{by Lemma 2.6 (4)} \\
&= (0 \to d) \to a && \text{by Lemma 2.6 (5)}
\end{aligned}$$

(10)

$$\begin{aligned}(b \to (a \to d)) \to a &= [(a' \to b) \to \{(a \to d) \to a\}']' && \text{by (I)}\\ &= [(a' \to b) \to \{(0 \to d) \to a\}']' && \text{by Lemma 2.6 (5)}\\ &= [b \to (0 \to d)] \to a && \text{by (I)}\\ &= [0 \to (b \to d)] \to a && \text{by Lemma 2.6 (13)}\\ &= (0 \to d) \to a && \text{by (9)}\end{aligned}$$

(11)

$$\begin{aligned}b \to (0 \to a') &= (0 \to b) \to (0 \to a') && \text{by Lemma 2.6 (20)}\\ &= 0 \to [(0 \to b) \to a'] && \text{by Lemma 2.6 (13)}\\ &= 0 \to [(a \to 0) \to (b \to a')']' && \text{by (I)}\\ &= 0 \to [a' \to (b \to a')']' && \\ &= 0 \to (a' \to a'')' && \text{by (3)}\\ &= 0 \to (a' \to a)' && \\ &= 0 \to a' && \text{by Lemma 2.4 (d)}\end{aligned}$$

(12) From (I) and by hypothesis we have that $[(d \to a) \to b']' = (b \to d) \to (a \to b')' = (b \to d) \to a$.

(13)

$$\begin{aligned}a' \to b &= (a \to b') \to b && \text{by hypothesis}\\ &= [(b' \to a) \to (b' \to b)']' && \text{using (I)}\\ &= [(b' \to a) \to b']' && \text{by Lemma 2.4 (d)}\\ &= b' \to (a \to b')' && \text{by Lemma 2.6 (4)}\\ &= b' \to a && \text{by hypothesis}\end{aligned}$$

(14)

$$
\begin{aligned}
(d \to 0') \to (a' \to b) &= (a' \to b)' \to [(d \to 0') \to (a' \to b)] \\
&\qquad \text{by Lemma 2.6 (17)} \\
&= (a' \to b)' \to [(d \to 0') \to (b' \to a)] \\
&\qquad \text{by (13)} \\
&= (a' \to b)' \to [(d \to 0') \to \{(0 \to a) \to b'\}'] \\
&\qquad \text{by (12) with } d = 0 \\
&= (a' \to b)' \to [\{(0 \to a) \to b'\} \to \{d \to ((0 \to a) \to b')\}'] \\
&\qquad \text{by Lemma 2.6 (26) with} \\
&\qquad x = (0 \to a) \to b', y = d \\
&= (a' \to b)' \to [(b' \to a)' \to \{d \to ((0 \to a) \to b')\}'] \\
&\qquad \text{by (12) with } d = 0 \\
&= (a' \to b)' \to [(a' \to b)' \to \{d \to ((0 \to a) \to b')\}'] \\
&\qquad \text{by (13)} \\
&= (a' \to b)' \to [d \to \{(0 \to a) \to b'\}]' \\
&\qquad \text{by Lemma 2.6 (21)} \\
&= (b' \to a)' \to [d \to \{(0 \to a) \to b'\}]' \\
&\qquad \text{by (13)} \\
&= [(0 \to a) \to b'] \to [d \to \{(0 \to a) \to b'\}]' \\
&\qquad \text{by (12) with } d = 0 \\
&= [(0 \to d) \to \{(0 \to a) \to b'\}]' \\
&\qquad \text{by Lemma 2.6 (31) with} \\
&\qquad x = d, y = (0 \to a) \to b' \\
&= [(b' \to d) \to \{(0 \to a) \to b'\}]' \\
&\qquad \text{by Lemma 2.6 (15) with} \\
&\qquad x = b, y = d, z = a \\
&= [(b' \to d) \to (b' \to a)']' \\
&\qquad \text{by (12) with } d = 0 \\
&= [(b' \to d) \to (a' \to b)']' \\
&\qquad \text{by (13)} \\
&= (d \to a') \to b \qquad \text{by (I).}
\end{aligned}
$$

(15)

$$
\begin{aligned}
[(0 \to a') \to b]' &= [(a \to 0') \to b]' && \text{by Lemma 2.5 (a)} \\
&= (0 \to a) \to b' && \text{by Lemma 2.6 (11)}
\end{aligned}
$$

(16)

$$
\begin{aligned}
(a' \to b)' &= [(0 \to a') \to b]' && \text{by (1)} \\
&= (0 \to a) \to b' && \text{by (15)}
\end{aligned}
$$

(17)

$$
\begin{aligned}
[\{b' \to ((b \to d) \to a)\} \to (0 \to b)']' &= [\{b' \to ((d \to a) \to b')'\} \to (0 \to b)']' \\
&\qquad\qquad\qquad\qquad \text{by (12)} \\
&= [\{(d \to a) \to b'\}' \to 0] \to b \\
&\qquad\qquad\qquad\qquad \text{by (I)} \\
&= [(d \to a) \to b'] \to b \\
&= b' \to [(d \to a) \to b']' \\
&\qquad\qquad\qquad\qquad \text{by Lemma 2.6 (14) with} \\
&\qquad\qquad\qquad\qquad x = d \to a, y = b' \\
&= b' \to [(b \to d) \to a] \qquad \text{by (12).}
\end{aligned}
$$

□

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