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The dynamics of thick curved beams constructed with functionally graded materials

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ABSTRACT

During the last two decades materials, that exhibit graded properties, left their spirit of conceptual laboratory specimens to become a technological reality with a well established background. However structural applications of these materials are not a fulfilled research. Models of straight and curved beams are normally reported in the scientific literature as the easiest way to understand some existing aspects in mechanics of structures. Most of these models are formulated appealing to numerical approaches such as the finite element method among others, without taking into account theoretical aspects that can be quite useful to reduce algebraic complexity.

In the present work a technical theory for dynamic analysis of thick curved beams is deduced within the context of functionally graded materials. The concept of material neutral-axis shifting is employed in the deduction procedure in order to reduce the algebraic handling and complexity of the motion equations. This leads to find analytical solutions of the governing differential system, even if it has variable coefficients. Parametric studies on the dynamics of curved beams are offered to show the versatility of the adopted formulation by means of solutions handled with the power series method.

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1. Introduction

In the last decade functionally graded materials (FGM) have been increasingly recognized as a feasible solution and potential answer to many challenging problems in a broad range of engineering applications, such as curved stiffeners of aerospace panels, among others. These materials have been reported in the middle eighties as a potential way to cope with the problem of failure and presence cracks in the interfaces of sandwich structures or laminated structures due to, for example, high thermo-mechanical stress gradients. Laminated composite structures and sandwich structures differ from structures constructed with FGM in what these last ones have mass, elastic and thermo-mechanical properties changing smoothly and continuously in prescribed directions. Structural models for FGM were introduced for different geometric configurations and scales covering a broad area from 3D solids through shells and plates and also beams or bars. The development of curved beams models has been the topic of interest of many researchers during the last 30 years. Those investigations were oriented to a wide range of engineering problems, such as instability and vibration analysis. Many curved beam models have been introduced to account for linear and non-linear behavior in structures made of both isotropic (Cortínez et al., 1999) and/or composite materials and arranged for thin-walled (Piovan and Cortínez, 2007) or solid cross-sections (Tufekci and Yasar Dogruer, 2006). A number of different methodologies such as Principle of virtual work, Hellinger–Reissner principle, Hu–Washizu principle among others were employed to develop a variety of models.

The classical theory of strength of materials, although considered old fashioned, has proved to be a conceptually easy way to derive generalized or more complex beam and bar models with curved or straight axis (Filipich, 1991; Filipich et al., 2003). It has to be noted that, despite its technological interest, very few studies on the dynamics of curved beams made of FGM have been performed in the past years according to the knowledge of the authors. In fact, Piovan et al. (2008a) developed a basic model for free vibration analysis of arcs under the presence of initial stresses. This model was derived employing the principle of Hellinger–Reissner and numerically implemented with the finite element method. Piovan and Sampaio (2009) presented a model to study transient vibrations of rotating curved beams made with FGM. Malekzadeh (2009) and Lim et al. (2009) carried out numerical approaches for in-plane vibrations of arches in the context of bi-dimensional formulations. Dryden (2007) introduced analytic solutions of an inhomogeneous curved beam in the context of bi-dimensional plane states. Bi-dimensional approaches allow analytical solutions only in a few simple cases. Three-dimensional models of these structures

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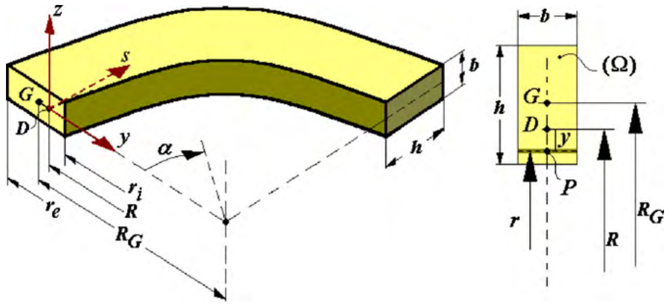

Fig. 1. Structural model: a circumferential thick arc.

should be analyzed with numerical formulations like finite element method among others. On the other hand, one-dimensional theories can reach an acceptable degree of approximation that could be nearly the same of 2D and 3D formulation but with a reasonable computational cost. Also one-dimensional models offer an easy conceptual understanding of the dynamic phenomena in the case of slender structures. Thus, in the present article a classical strength of materials theory for dynamic analysis of thick curved beams is derived in the context of functionally graded materials. The deduction process appeals to the concept of material neutral-axis shifting with the aim to reduce the algebraic handling. Then, equations of motion obtained under this conception are formally identical to the ones for isotropic materials (Filipich, 1991) or orthotropic materials (Piovan et al., 2008a), leading to the possibility to arrive at analytical solutions of the governing differential system. The solution of the free vibration problem of thick curved beams made of FGM is performed by means of the power series method. A recurrence scheme is employed in power series handling in order to reduce the number of unknown to only few unknown coefficients that can be selected according to the boundary equations. In fact, for this problem the maximum number of unknowns would be six if the curved beam has elastic supports. Some comparisons are performed with other beam approaches and 3D finite element approximations. An especial analysis of particular features in some boundary conditions is offered as well.

2. Model development

2.1. Hypotheses and definitions

In Fig. 1 a sketch of the structural element analyzed in this work is shown. As one can see, there are two relevant points in the cross-section: Point ***G*** corresponds to the centroid of the section, whereas point ***D*** is a point belonging to the neutral axis. In this model the following material properties are assumed as:

$$E = E_0\varphi_1(r), \qquad G = G_0\varphi_2(r), \qquad \rho = \rho_0\varphi_3(r), \tag{1}$$

where E, G and ρ are the Young modulus, Shear modulus and material density, respectively. On the other hand, the following variation is considered:

$$\varphi_j(r) = \left[k_j + \frac{(1-k_j)}{h^n}(r-r_i)^n\right], \quad (j = 1, 2, 3) \tag{2}$$

In Eq. (2), $k_1 = E_i/E_0$, $k_2 = G_i/G_0$ and $k_3 = \rho_i/\rho_0$, whereas E_i, G_i and ρ_i intend for the material properties at $r = r_i$ and E_0, G_0 and ρ_0 are the properties at $r = r_e$. The real exponent n rules the variation of the properties along the radial direction. The present study is confined within the context of strength of materials theory. The displacements of a generic point ***P*** of the cross-section can be represented in following form:

$$u_P(\alpha, t) = u(\alpha, t) + \theta(\alpha, t)y, \qquad w_P(\alpha, t) = w(\alpha, t), \tag{3}$$

where w is the transverse displacement, u is the tangential displacement and θ is the bending slope all of them measured with respect to point ***D***, α is the angular coordinate and t is the time, and $y = R - r$.

Taking into account the polar reference system, the representative strain components in terms of the displacements can be written (Filipich and Piovan, 2009) as:

$$\varepsilon_\alpha = \frac{1}{r}(u' - w + \theta' y), \qquad \gamma_{r\alpha} = \frac{1}{r}(u + w' + R\theta), \tag{4}$$

where primes mean derivation with respect to the angular variable α. According to classical beam theories, the remaining strain components are not considered. The strain components given in Eq. (4) are related to their corresponding stress components by means of a basic linear constitutive law, for instance see the work of Benatta et al. (2008):

$$\sigma_\alpha = E(r)\varepsilon_\alpha, \qquad \tau_{r\alpha} = G(r)\gamma_{r\alpha} \tag{5}$$

Notice that the second expression in Eq. (5) is not employed in the context of the strength of materials theory, needing a different approach which is analyzed in the following paragraphs.

2.2. Axial force and bending moment. Neutral axis

In this section the radial location of the neutral axis is obtained in order to simplify the deduction process, referring all the terminology to such axis. Thus, the axial force N and bending moment M_D are defined in terms of the normal stress by means of the following expressions:

$$N = \iint_\Omega \sigma_\alpha \, d\Omega, \qquad M_D = \iint_\Omega \sigma_\alpha y \, d\Omega. \tag{6}$$

Then employing the previous definitions and considering Eqs. (4) and (5) one obtains:

$$N = bE_0[\alpha_0(u' - w) + \alpha_1\theta'], \qquad M = bE_0[\alpha_1(u' - w) + \alpha_2\theta']. \tag{7}$$

where

$$\alpha_j = \int_{r_i}^{r_e} \frac{(R-r)^j}{r}\varphi_1(r)\,dr, \quad j = 0, 1, 2 \tag{8}$$

Thus, in order to obtain the expression for the neutral axis (for $N \neq 0$ and $M_D \neq 0$) one impose $\varepsilon_\alpha = \sigma_\alpha = 0$, or in other words if $y = a$ is the location of the neutral axis and taking into account the condition of null strain in Eq. (4) (i.e. $u' - w + \theta' a = 0$) one arrives to:

$$a = \frac{\alpha_2 N - \alpha_1 M}{\alpha_1 N - \alpha_0 M}. \tag{9}$$

Once the neutral axis has been generically defined, it is possible to rearrange the origin location (point ***D***) with the scope to substantially reduce the algebraic manipulation, thus one appeals to impose $a = 0$ when $N = 0$, which leads to $\alpha_1 = 0$. Consequently with $a = 0$ and taking into account Eq. (8) one obtains the expression of the neutral-axis radius as:

$$R = \frac{1}{\alpha_0}\int_{r_i}^{r_e} \varphi_1(r)\,dr \tag{10}$$

Now, with the value of R one deduces α_2 by simply employing Eq. (8). This procedure allows to reduce the algebra and to decouple the axial force and moment in terms of the displacements. Thus, Eq. (8) can be rewritten as:

$$N = \frac{E_0 A}{R}(u' - w), \qquad M_D = \frac{E_0 J}{R}\theta'. \tag{11}$$

where $A = bR\alpha_0$, $J = bR\alpha_2$. These entities are material area and material moment of inertia, respectively. In the case of a straight beam

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of homogeneous material, A and J are the area and centroid inertia moment of the cross-section, respectively.

2.3. Constitutive equations. Shear coefficient

As a previous step to deduce the expression of the shear force in terms of displacements, one needs to find the consistent form of the shear stress which has to satisfy the following internal equilibrium equation (where $\tau_{z\alpha}=0$ is assumed):

$$r\frac{\partial \tau_{r\alpha}}{\partial r}+2\tau_{r\alpha}-\sigma'_{\alpha}=0 \tag{12}$$

The gradient of the normal stress σ'_{α}, can be defined in terms of the shear force as (Filipich, 1991):

$$\sigma'_{\alpha}=\varphi_1(r)\left[\frac{R^2}{rJ}\left(R+\frac{\alpha_2}{R\alpha_0}\right)-\frac{R^2}{J}\right]Q \tag{13}$$

The solution of Eq. (12) leads to:

$$\tau_{r\alpha}=\Psi(r)Q \tag{14}$$

where

$$\Psi(r)=\frac{C}{r^2}+\frac{R^2}{J}\left(R+\frac{\alpha_2}{R\alpha_0}\right)f(r)-\frac{R^2}{J}g(r) \tag{15}$$

$$f(r)=\frac{(1-k_1)(r-r_i)^{n+1}}{h^n(n+1)r^2}+\frac{k_1}{r},$$
$$g(r)=\frac{(1-k_1)(r-r_i)^{n}}{h^n(n+2)}\left(1-\frac{r_i(nr+r_i)}{(n+1)r^2}\right)+\frac{k_1}{2} \tag{16}$$

In Eq. (15), C is a constant which can be indistinctly deduced from any of the following three conditions:

$$Q=\iint_{\Omega}\tau_{r\alpha}\,d\Omega \text{ or } \tau_{r\alpha}(r_e)=0 \text{ or } \tau_{r\alpha}(r_i)=0 \tag{17}$$

To deduce the formula of the shear coefficient and the expression of the shear force in terms of the displacements one should come across that the deformation energy due to shear effects can be written, according to elasticity theory and strength of materials, as:

$$2U_S=\iiint_V\left[\frac{\tau^2_{r\alpha}}{G_0\varphi_2(r)}\right]dV=\frac{mR}{G_0}\int_{\alpha}\left(\frac{Q^2}{A}\right)d\alpha$$
$$=\underline{\iiint_V\tau_{r\alpha}\gamma_{r\alpha}\,dV}=\int_{\alpha}Qr\gamma_{r\alpha}\,d\alpha \tag{18}$$

The underlined member is the classical definition of the work due to shear deformation. Now, comparing the second and fourth members of Eq. (23) and taking into account Eq. (4), one finally gets:

$$Q=\frac{G_0A}{mR}(u+w'+R\theta) \tag{19}$$

The shear coefficient m is obtained from the first and the second members of Eq. (18) and by means of Eq. (14), one obtains the following expression:

$$m=\frac{bA}{R}\int_{r_i}^{r_e}\frac{\Psi(r)^2r}{\varphi_2(r)}dr \tag{20}$$

It is interesting to note that the deduction of m is consistent with the present model. In other approaches and beam models of FGM that consider shear flexibility due bending this coefficient is normally employed without a careful thought, for example assuming $m=6/5$, which is obtained in the same way developed of the present article but for the case of "isotropic straight beam".

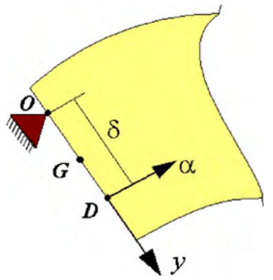

Fig. 2. Sketch of a generic end.

2.4. Equations of motion

In order to deduce the equations of motions the principle of Hamilton is employed. The kinetic energy K, the deformation energy U and the potential energy P due to external loads can be described in terms of the displacements in the following form:

$$2K=\rho_0b\int_{\alpha}[\gamma_0(\dot{u}^2+\dot{w}^2)+2\gamma_1\dot{\theta}\dot{u}+\gamma_2\dot{\theta}^2]\,d\alpha,$$
$$2U=bE_0\int_{\alpha}[\alpha_2\theta'^2+\alpha_0(u'-w)^2]\,d\alpha+\frac{bG_0\alpha_0}{m}\int_{\alpha}(u+w'+R\theta)^2\,d\alpha,$$
$$P=-R\int_{\alpha}[p_Rw+p_Tu+p_{\theta}\theta]\,d\alpha \tag{21}$$

where points indicate derivation with respect time t; p_R, p_T and p_{θ} are the distributed radial force on the neutral axis, the distributed tangential force on the neutral axis and the distributed moment applied on the same axis, respectively. The inertia coefficients γ_j are defined as:

$$\gamma_j=\int_{r_i}^{r_e}(R-r)^jr\varphi_3(r)\,dr,\quad j=0,1,2, \tag{22}$$

Finally performing the conventional variational steps in the Hamilton's principle one gets the following equations of motions in terms of displacements:

$$C_{11}u''-(C_{11}+C_{33})w'-C_{33}u-RC_{33}\theta-(D_{11}\ddot{u}+D_{22}\ddot{\theta})=-p_TR,$$
$$C_{33}w''+(C_{11}+C_{33})u'+RC_{33}\theta'-C_{11}w-D_{11}\ddot{w}=-p_RR,$$
$$C_{22}\theta''-RC_{33}(v'+u+R\theta)-(D_{22}\ddot{u}+D_{33}\ddot{\theta})=-p_{\theta}R, \tag{23}$$

where

$$C_{11}=bE_0\alpha_0,\quad C_{22}=bE_0\alpha_2,\quad C_{33}=\frac{bG\alpha_0}{m},\quad D_{11}=b\rho_0\gamma_0,$$
$$D_{22}=b\rho_0\gamma_1,\quad D_{33}=b\rho_0\gamma_2. \tag{24}$$

Eq. (23) is formally the same of the problem corresponding to the homogeneous case developed by Filipich (1991). Notice that the model is a completely coupled system, but if the structure is such that RG → ∞, the model is reduced to the case of a straight beam where the longitudinal motion is decoupled from the shear and bending motion. It is interesting to remark that Eq. (23) is a generalization of the Timoshenko straight beam approach, but as a particular case in the frame of functionally graded materials and curved beams.

In Fig. 2 one can see a sketch to describe a generalized simple support not necessarily located at the neutral axis. This boundary implies: $u_0=0$, $w_0=0$ and $M_0=0$, however appealing to Eq. (11) the expression of such general boundary condition can be written, in terms of displacements, as follows:

$$w=0,\quad u-\theta\delta=0,\quad J\theta'+Au'=0 \tag{25}$$

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On the other hand the boundary conditions corresponding to a clamped end are:

$$u = 0, \qquad w = 0, \qquad \theta = 0 \tag{26}$$

The boundary conditions related to a free end imply $N = 0$, $Q = 0$ and $M_D = 0$, but appealing to Eqs. (11) and (19) one can write them as:

$$u' - w = 0, \qquad \theta' = 0, \qquad u + w' + R\theta = 0 \tag{27}$$

3. Analysis and comparisons

3.1. Power series method for the natural frequencies problem

In order to study vibratory patterns of this type of structures, the motion equations are solved with a power series solution. The exact solution of the eigenvalue problem can be carried out by means of a generalization of the power series scheme developed originally by Filipich et al. (2003) and Rosales and Filipich (2006) for structural problems involving isotropic materials. It is convenient a previous non-dimensional re-definition of the differential equations, which implies that $x = R\alpha/L \in [0, 1]\ \forall\alpha \in [0, \Delta_\alpha]$, being L the circumferential length of the neutral axis of the curved beam, and Δ_α is the subtended angle of the curved beam.

The displacement variables have the common harmonic motion:

$$\{u, w, \theta\} = \left\{\bar{u}, \bar{w}, \bar{\theta}\right\} e^{i\omega t} \tag{28}$$

where ω is the circular frequency of the curved beam measured in rad/s and $i = \sqrt{-1}$, and $\left\{\bar{u}, \bar{w}, \bar{\theta}\right\}$ are the corresponding modal shapes. Now working without the presence of external loads and accepting Eq. (28) the differential equations system given previously can be re-arranged in the following form:

$$\begin{aligned}
&\frac{\partial^2\bar{u}}{\partial x^2} - \frac{L}{mR}(\Phi + m)\frac{\partial\bar{w}}{\partial x} - \frac{\Phi L^2}{mR^2}\left(\bar{u} + R\bar{\theta}\right) + \frac{\alpha_2}{L^2R^2\alpha_0^2}\lambda^2\left(\gamma_0\bar{u} + \gamma_1\bar{\theta}\right) = 0\\
&\frac{\partial^2\bar{w}}{\partial x^2} + \frac{L}{\Phi R}(\Phi + m)\frac{\partial\bar{u}}{\partial x} + L\frac{\partial\bar{\theta}}{\partial x} - \frac{mL^2}{\Phi R^2}\bar{w} + \frac{m\alpha_2\gamma_0}{L^2\Phi R^2\alpha_0^2}\lambda^2\bar{w} = 0\\
&\frac{\partial^2\bar{\theta}}{\partial x^2} - \frac{L\Phi\alpha_0}{m\alpha_2}\frac{\partial\bar{w}}{\partial x} - \frac{L^2\Phi\alpha_0}{m\alpha_2 R}\left(\bar{u} + R\bar{\theta}\right) + \frac{\lambda^2}{L^2R^2\alpha_0}\left(\gamma_1\bar{u} + \gamma_2\bar{\theta}\right) = 0
\end{aligned} \tag{29}$$

where

$$\lambda^2 = \frac{\rho_0 A}{E_0 J}\omega^2 L^4, \qquad \Phi = \frac{G_0}{E_0}, \tag{30}$$

The displacements are expanded with the following power series:

$$\left\{\bar{u}, \bar{v}, \bar{\theta}\right\} = \sum_{k=0}^{Z}\left\{U_k, W_k, \Theta_k\right\}x^k. \tag{31}$$

Theoretically $Z \to \infty$, however for practical purposes Z may be an arbitrary large integer. Applying the boundary conditions in non-dimensional form and appealing to a recurrence scheme of the power series (Piovan et al., 2008b) one can represent the solution system in the following form:

$$\begin{bmatrix} \varepsilon_{11}(\lambda) & \varepsilon_{12}(\lambda) & \varepsilon_{13}(\lambda)\\ \varepsilon_{21}(\lambda) & \varepsilon_{22}(\lambda) & \varepsilon_{23}(\lambda)\\ \varepsilon_{31}(\lambda) & \varepsilon_{32}(\lambda) & \varepsilon_{33}(\lambda) \end{bmatrix} \begin{Bmatrix} U^* \\ W^* \\ \Theta^* \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{32}$$

One should note that the differential system given in Eq. (29) can be re-arranged as a differential equation of sixth order; so one has six arbitrary integration constants. Three of them are imposed in one end and the remaining three – symbolically expressed as $\{U^*, W^*, \Theta^*\}$ in Eq. (32) – mean the three free coefficients after the substitution of the power series in the differential system (for further explanations see Piovan et al., 2008a,b and Rosales and Filipich, 2006). Thus, from Eq. (31) one can obtain the solution of the eigenvalue problem through the characteristic equation:

$$Det\left[\boldsymbol{\varepsilon}(\lambda)\right] = 0 \tag{33}$$

The aforementioned recurrence scheme allows to shrink the algebraic problem from $3(M+1)$ unknowns to only 3 unknown coefficients that can be selected according to the boundary equations. It should be stressed that the solution expressed in Eq. (33) lead to an arbitrary precision value for the frequency by selecting appropriately the limit Z of the power series.

3.2. Numerical analysis

In this section some numerical studies are performed in order to show certain features of the dynamics of curved beam associated with the imposition of the boundary conditions. In Table 1 the material properties of a ceramic and a metal are shown. All the following examples hold the same ceramic/metal distribution. In fact, the properties are graded according to Eq. (2) from a full metallic inner surface (at $r = r_i$) to a full ceramic outer surface (at $r = r_e$), with $n = 1$.

The first example corresponds to a comparison of the present model with other approaches. The present strength of materials approach for the curved beam is compared with the response of a flexible 3D general solver (called FlexPDE) of partial differential equations within the context of the finite element method. In this solver one can easily cope with the complex material laws to be included in the structural model as well as the model itself (see http://www.pdesolutions.com for further explanations and illustrative examples of the program). Also another one-dimensional model of a FGM curved beam (Piovan et al., 2008a) derived according to the Hellinger–Reissner principle (HR) is employed for comparison purposes. The boundary conditions of the curved beam can be clamped at both ends or clamped in one end and free to move

Table 1
Properties of metallic and ceramic materials.

Properties of materials	Steel	Alumina (Al_2O_3)
Young's modulus, E (GPa)	214.00	390.00
Shear modulus, G (GPa)	82.20	137.00
Material density, ρ (kg/m^3)	7800.00	3200.00

Table 2
Comparison of frequencies of different models and numerical approaches.

Boundary condition	Model [approach]	Frequencies (Hz)			
		First	Second	Third	Fourth
Clamped	1D present model [PSM]	2364.97	3388.57	6417.37	7657.81
	1D HR model [FEM]	2364.97	3388.57	6417.37	7657.81
	3D [FEM]	2366.59	3431.47	6497.00	7702.20
Clamped free	1D present model [PSM]	240.28	1237.25	3394.78	4397.86
	1D HR model [FEM]	240.28	1237.24	3394.78	4397.86
	3D [FEM]	241.39	1244.95	3418.79	4396.93

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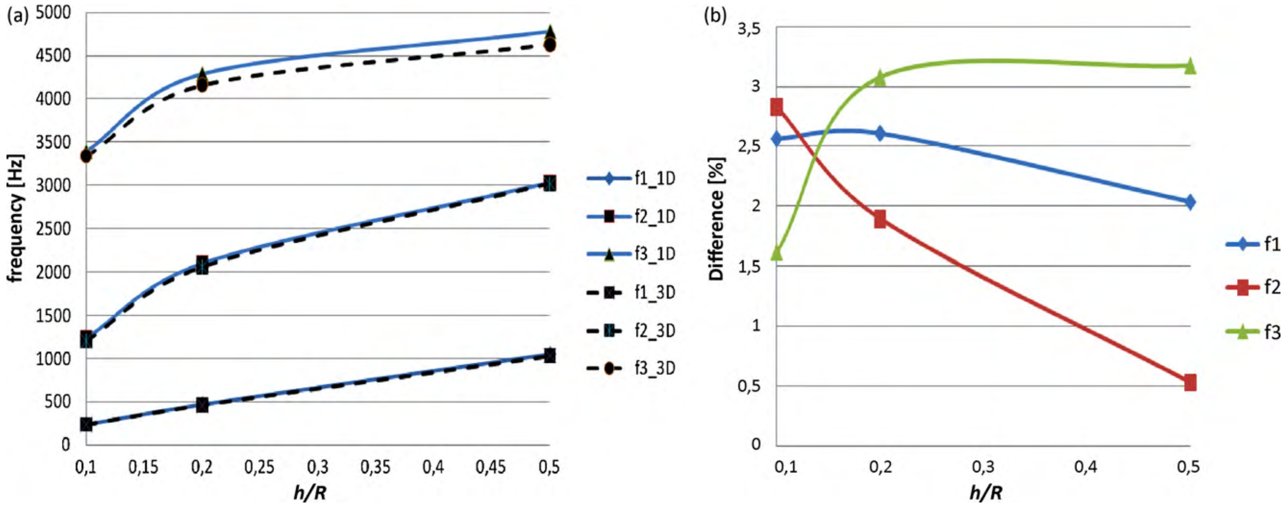

Fig. 3. Comparison of curved beam and 3D FEM approaches (a) variation with h/R and (b) percentage difference.

in the remaining. The geometric features of the curved beam are the following: $b = 20$ mm, $h = 50$ mm, $\Delta\alpha = 1$ rad and $R_G = 500$ mm, the shear coefficient is $m = 1.1619$ and the ratio $R/R_G = 1.004046$. Note that the neutral axis is greater than the R_G radius, conversely to the homogeneous classical approach. In Table 2 the comparison of the three models and numerical approaches is presented for the first four frequencies of the arch. PSM intend for power series method. The calculations were carried out with fifty terms in the PSM (or $Z = 50$), 10 quadric curved beam elements (for HR model) and nearly 850 tetrahedral elements in FlexPDE. As one can see in Table 2, differences between the approaches are negligible; however it should be mentioned that the 3D approach demanded more than 20 min to reach the desired precision on a 3.7 GHz Pentium IV computer (this is due to the quite fine mesh employed in order to cope with the non-homogeneity of the material). On the other hand both 1D numerical approaches demanded just a couple of seconds.

Fig. 3 shows a comparison of the present beam approach and the full 3D FEM approach. The first three frequencies are plotted with respect to the parameter h/R. This parameter gives a measure of the beam thickness. In this example, the geometric properties of the curved beam are $b = 20$ mm, $\Delta\alpha = \pi/3$ rad and $R_G = 500$ mm, the height h, varies according to parameter h/R. It is possible to see a very good agreement between both approaches, even in the case $h/R = 0.5$ which implies a very thick beam. Note also that frequencies calculated with the curved beam model have percentage difference less than 4% with respect to the 3D FEM approximation.

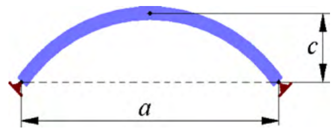

Fig. 4. Sketch of a shallow arc.

The third example reflects a study of the vibratory behavior for a beam with the generalized supports. Thus, Fig. 4 shows a curved beam defined by two parameters: the horizontal distance between the centroidal points of both ends 'a' and the arch

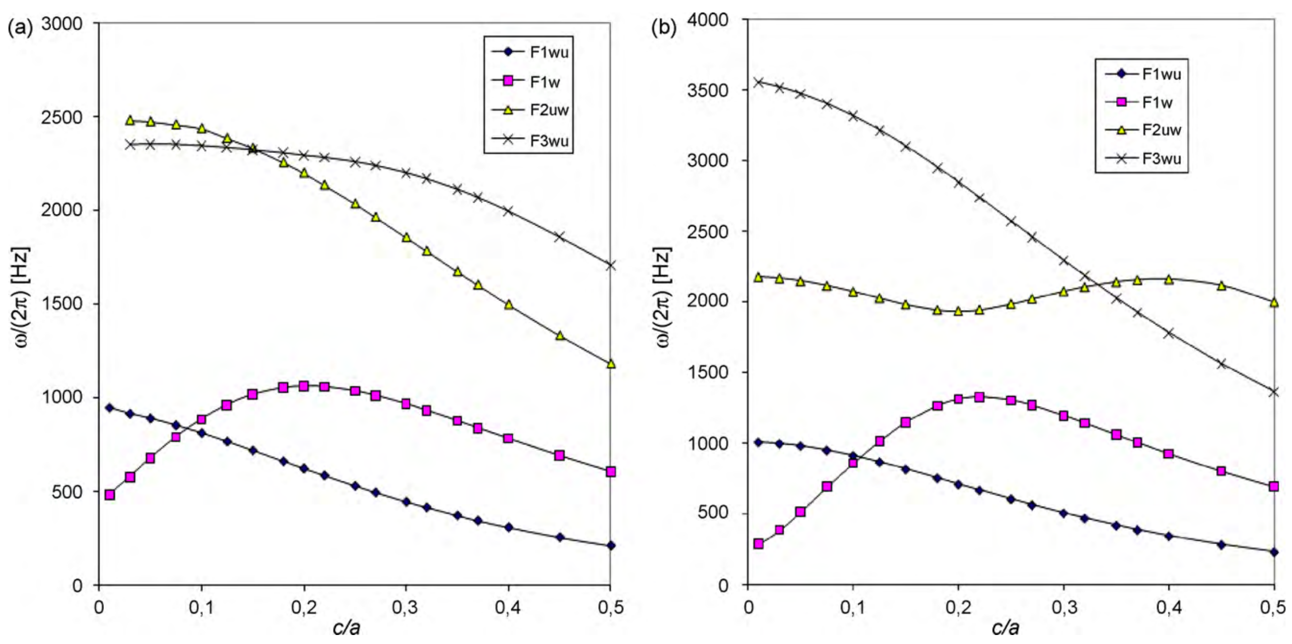

Fig. 5. Variation of the frequencies with parameter c/a for arches supported at (a) $r = r_i$ and (b) $r = r_D$.

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height parameter '*c*' which is measured between the horizontal level and the centroidal point (i.e. ***G***) of the mid cross-section. The influence of shallowness ratio c/a together with the position of the support at the ends is analyzed. The horizontal distance is fixed to $a = 1000$ mm. The cross-section is such that $b = 20$ mm and $h = 80$ mm. Fig. 5(a) shows the variations of the frequencies (related to a particular mode shape) with respect to the shallowness ratio c/a, for the case where the beam is supported at both ends in the point corresponding to the inner surface, i.e. r_i. Fig. 5(b) shows the frequency variation with c/a but for the case where the supports are located at the neutral axis (i.e. D).

In the previous two figures the nomenclature F1w intend for the first flexural dominant mode F1wu mean the first flexural dominant mode with membranal coupling; whereas F2uw, F3wu are second and third flexural–membranal coupled modes. In this nomenclature, the first letter after the number means the dominating motion, e.g. F3wu means a coupled mode with bending dominant motion. Although the modes F1w and F1wu have same qualitative behavior along the ratio 'c/a', it should be noted that the first axial-bending mode (F1uw) is not very sensible with the change of support position along the cross-section; conversely the flexural dominant mode (F1u) has a remarkable sensibility, especially in shallow arches.

4. Conclusions

In the present article a model for non-homogeneous thick curved beam according to the theory of strength of materials is developed. The structural non-homogeneity is confined in the frame of functionally graded materials. The derivation process consisted in the employment of the concept of neutral-axis shifting, which allows the possibility to reduce the algebraic manipulation. However, it is interesting to note that the inertial coupling between longitudinal displacement and bending slope is still observed; when the curvature radius tends to infinity (i.e. the case of straight beam), this coupling vanishes. This aspect eventually leads to find analytical solutions of the governing differential system, even if the differential system has variable coefficients. The motion equations are solved with a power series approach which gives arbitrary precision to the frequencies values. The values calculated with the present 1D model manifest a very good agreement with 3D finite element models. A special analysis featuring simply supported conditions was performed. This analysis has shown the strong dependence of the position along the end of the simply support restrictions in the first in-plane frequencies for the graded material, especially if the bending and axial modes are coupled. Also notice that for the same measure of relative error, the whole calculations of present 1D model demanded a thousandth of the time demanded by the 3D finite element code. This is due to the graded properties that need finer meshes. Clearly, the present strength of material approach has two advantages, i.e., shorter computation times allow the study a wide range of cases before facing deeper 3D analysis; even more one can cope also the cases of complex cross-section such as non-homogeneous, doubly connected, etc., without increasing the computing time. Evidently this aspect is quite important when one has to analyze features related to the optimization of properties gradation or the analysis of mechanical and gradation uncertainties (which are quite common in the fabrication of graded beams) needing important computational resources; this is matter of future research.

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