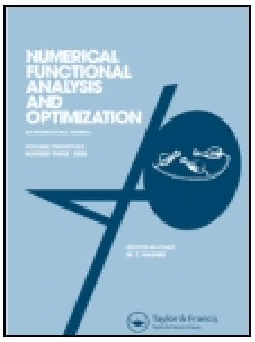

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ABSTRACT

In this article, we introduce the τ condition, which is weaker than the L^2 differentiability. If a function satisfies the τ condition on two points of $\mathbb{R}$, we prove the existence and characterization of the best local polynomial approximation on these points.

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1. Introduction

Let $x_1 \in \mathbb{R}, x_1 \neq 0, x_2 = -x_1$, and let $a > 0$ be such that $I_{a,i} := [x_i - a, x_i + a]$, $1 \leq i \leq 2$, are disjoint. Let $\mathcal{L}$ be the space of an equivalence class of Lebesgue measurable real functions defined on $I_a := I_{a,1} \cup I_{a,2}$. For each Lebesgue measurable set $A \subset I_a$, with $|A| > 0$, we consider the seminorm on $\mathcal{L}$,

$$\|f\|_A := \left(|A|^{-1} \int_A |f(x)|^2 dx \right)^{1/2},$$

where $|A|$ denotes the measure of the set A.

If $0 < \epsilon \leq a$, we denote $I_{\epsilon,i} = [x_i - \epsilon, x_i + \epsilon]$, $\|f\|_{\epsilon,i} = \|f\|_{I_{\epsilon,i}}$ and $\|f\|_\epsilon = \|f\|_{I_\epsilon}$ For a nonnegative integer s, let Π^s be the linear space of algebraic polynomials of degree at most s.

Henceforward, we consider $n, q, r \in \mathbb{N} \cup \{0\}$ such that $n + 1 = 2q + r, r < 2$.

If $f \in L^2(I_\epsilon)$, it is well known (see [1]) that there exists a unique best $\|.\|_\epsilon$ approximation of f from Π^n, say $P_\epsilon(f)$, satisfying

$$\|f - P_\epsilon(f)\|_\epsilon \leq \|f - P\|_\epsilon, \; P \in \Pi^n,$$

and it is characterized by the condition

$$\int_{I_\epsilon} (f - P_\epsilon(f))(x) P(x) dx = 0, \quad P \in \Pi^n. \tag{1.1}$$

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If f is an even (odd) function, it is easy to see that $P_\epsilon(f)$ is an even (odd) polynomial. If $\lim_{\epsilon\to 0} P_\epsilon(f)$ exists, say $P_0(f)$, it is called the *best local approximation of f on $\{x_1, x_2\}$ from Π^n*.

We recall that a function $f \in L^2(I_{a,i})$ is L^2 *differentiable of order s at x_i* and, according to Calderón and Zygmund in [2], $f \in t_s^2(x_i)$ if there exists $Q_i \in \Pi^s$ such that

$$\|f - Q_i\|_{\epsilon,i} = o(\epsilon^s), \quad \epsilon \to 0. \tag{1.2}$$

We also write $t_{-1}^2(x_i) = L^2(I_{a,i})$. It is well known that there exists at most one polynomial verifying (1.2) (see [3]).

The best local approximation at one point was formally introduced and studied in an article by Chui, Shisha and Smith [4]. In [5], this problem was considered for certain class of differentiable functions in the ordinary sense on two points. Later, in [3], [6], and [7], the authors extended it for L^2 differentiable and lateral L^2 differentiable functions on k points. In a recent article [8], the existence of the best local approximation for a class of functions satisfying C^p condition at one point was considered.

All of the exponents in this work will be nonnegative integers. We introduce the following definition.

Definition 1.1. A function $f \in L^2(I_{a,i})$ satisfies the τ condition of order s at x_i, if there exists $Q_i \in \Pi^s$ such that

$$\int_{x_i-\epsilon}^{x_i+\epsilon} (f - Q_i)(x)(x^2 - x_i^2)^j dx = o(\epsilon^{s+j+1}), \quad 0 \le j \le s, \ \epsilon \to 0. \tag{1.3}$$

If f verifies (1.3), we say that $f \in \tau_s(x_i)$.

Let $\tau_s(\pm x_i) := \tau_s(x_i) \cap \tau_s(-x_i)$ and $t_s^2(\pm x_i) := t_s^2(x_i) \cap t_s^2(-x_i)$.

We have the following uniqueness result.

Theorem 1.2. *Let $f \in L^2(I_{a,i})$. Then there exists at most a polynomial $Q_i \in \Pi^s$ satisfying (1.3).*

Proof. Assume that $Q_i, \overline{Q}_i \in \Pi^s$ and verify (1.3). It is easy to see that $T(x) = (Q_i - \overline{Q}_i)(x) := \sum_{m=0}^s a_m(x - x_i)^m$ satisfies

$$\sum_{m=0}^{s} a_m \int_{I_{\epsilon,i}} (x - x_i)^{m+j}(x + x_i)^j dx = o(\epsilon^{s+j+1}), \quad 0 \le j \le s, \ \epsilon \to 0. \tag{1.4}$$

We put $a_{-1} = 0$. If $a_m = 0$ for all m, $-1 \leq m \leq l < s$, then $a_{l+1} = 0$. In fact, considering $j = l + 1$ in (1.4), we get

$$a_{l+1} \int_{I_{\epsilon,i}} (x - x_i)^{2l+2}(x + x_i)^{l+1} dx + \sum_{m=l+2}^{s} a_m \int_{I_{\epsilon,i}} (x - x_i)^{m+l+1}(x + x_i)^{l+1} dx = o(\epsilon^{s+l+2}). \quad (1.5)$$

Since the summation in (1.5) is $O(\epsilon^{2l+4})$, we have

$$a_{l+1} \int_{I_{\epsilon,i}} (x - x_i)^{2l+2}(x + x_i)^{l+1} dx = o(\epsilon^{2l+3}). \quad (1.6)$$

Therefore, $a_{l+1} \int_{I_{\epsilon,i}} (x - x_i)^{2l+2} dx = o(\epsilon^{2l+3})$, i.e., $a_{l+1} = 0$. This proves the lemma. □

For $f \in \tau_s(x_i)(\tau_s(-x_i))$, we denote by $Q^s_{x_i}(f)(Q^s_{-x_i}(f))$ the unique polynomial of degree s verifying (1.3) at the points $x_i(-x_i)$.

The proof of the next theorem immediately follows.

Theorem 1.3. *Let s be a nonnegative integer number. Then $\tau_s(x_i)$ is a linear space and the operator $D_s : \tau_s(x_i) \to \Pi^s$ defined by $D_s(f)(x) = Q^s_{x_i}(f)(x)$, is linear. Moreover, $\tau_{s+1}(x_i) \subset \tau_s(x_i)$ and for $f \in \tau_{s+1}(x_i)$, we have $D_{s+1}(f)(x) = D_s(f)(x) + \alpha(x - x_i)^{s+1}$, $\alpha \in \mathbb{R}$.*

Now, if $f \in \tau_s(x_i)$ we can define the j-th τ derivative of f at x_i by

$$f^{(j)}(x_i) = (Q^s_{x_i}(f))^{(j)}(x_i), \quad 0 \leq j \leq s. \quad (1.7)$$

Remark 1.4. We have $t^2_s(x_i) \subset \tau_s(x_i)$. In fact, using the Hölder inequality we can see that the polynomial in Π^s that verifies $\|f - Q_i\|_{\epsilon,i} = o(\epsilon^s)$ also satisfies (1.3). In addition, the inclusion is strict as is shown in the following example. Let $f(x) = \sin(\frac{1}{x-x_i}), x \neq x_i$. It is easy to see that $f \in \tau_0(x_i)$, since f is odd. On other hand, if $f \in t^2_0(x_i)$, then there exists a constant $\alpha \in \mathbb{R}$ such that $\|f - \alpha\|_{\epsilon,i} = o(1)$. Since $\|f - \alpha\|_{\epsilon,i} = \|f + \alpha\|_{\epsilon,i}$, we get that $\alpha = 0$. However, $\|f\|_{\epsilon,i} \neq o(1)$, as we show below.

For $\epsilon_m = x_i + \frac{1}{m\pi}$ we have

$$\|f\|^2_{\epsilon_m,i} = m\pi \int_{x_i}^{x_i+\frac{1}{m\pi}} \sin^2\left(\frac{1}{x - x_i}\right) dx \geq m\pi \sin^2\left(m\pi + \frac{\pi}{4}\right) \sum_{l=m}^{\infty} \int_{x_i+\frac{1}{(l+1)\pi+\frac{\pi}{4}}}^{x_i+\frac{1}{(l+1)\pi-\frac{\pi}{4}}} dx \asymp m \sum_{l=m}^{\infty} \frac{1}{l^2} \nrightarrow 0.$$

In Section 2, we estimate the order of certain determinants depending on ϵ, and we prove some lemmas concerning to algebraic polynomials.

The main results of this article are in Section 3. We prove the existence of the best local approximation of a function f on $\{x_1, -x_1\}$ from Π^n, and we give a characterization of it under the following conditions: (a) n is even, $f \in \tau_q(\pm x_1)$, and the odd part of f belongs to $t^2_{q-1}(x_1)$ (Theorem 3.7). (b) n is odd, $f \in \tau_{q-1}(\pm x_1)$, and the odd part of f belongs to $t^2_{q-2}(x_1)$ (Theorem 3.11). Our theorems extend the mentioned results proved in [3] to a wider class of functions in L^2. We remark that the existence of best local approximation in Π^n is unknown for functions non L^2 differentiable even in two points.

2. Auxiliary results

We begin this section by estimating the order of the determinant of certain matrix depending on ϵ.

Lemma 2.1. *Let $u \in \mathbb{N}\cup\{0\}$ and let $A = (a_{jl})$ be a matrix of order $(u+1)\times(u+1)$, with $a_{jl} = a_{jl}(\epsilon) := \int_{1-\epsilon}^{1+\epsilon} (x^2-1)^{j+l} w(x)dx, 0 \le j, l \le u$, where w is a continuous function in a neighborhood of 1 such that $w(1) = 1$. Then the determinant of A, say $D(\epsilon)$, satisfies*

$$D(\epsilon) = (M + o(1))\epsilon^{(u+1)^2}, \tag{2.1}$$

where M is a non null constant.

Proof. Since A is a Gramian matrix of the set of linearly independent polynomials $\{(x^2 - 1)^j\}_{j=0}^u$ with the inner product $\langle\cdot,\cdot\rangle_{w,\epsilon}$ on $[1 - \epsilon, 1 + \epsilon]$, then $D(\epsilon) \neq 0$. For each pair j, l, the functions $(x + 1)^{j+l}w(x)$ is continuous and $(x - 1)^{j+l}$ is a integrable function with constant sign on the intervals $[1 - \epsilon, 1)$ and $(1, 1 + \epsilon]$, therefore by the First Value Mean theorem for integration there exist $\eta := \eta(\epsilon, j, l) \in [1, 1 + \epsilon]$ and $\eta' := \eta'(\epsilon, j, l) \in [1 - \epsilon, 1]$ such that

$$a_{jl} = w(\eta)(\eta + 1)^{j+l} b_{jl} + w(\eta')(\eta' + 1)^{j+l} b'_{jl}, \ 0 \le j, l \le u, \tag{2.2}$$

where $b_{jl} = b_{jl}(\epsilon) := \int_1^{1+\epsilon} (x - 1)^{j+l}dx$ and $b'_{jl} = b'_{jl}(\epsilon) := \int_{1-\epsilon}^1 (x - 1)^{j+l}dx$.

We observe that $a_{jl} = [2^{j+l} + o_{jl}(1)]b_{jl} + [2^{j+l} + o'_{jl}(1)]b'_{jl}$, where $o_{jl}(1), o'_{jl}(1)$ are functions of the variable ϵ which tend to zero as $\epsilon \to 0$.

It is well known that if p is an arbitrary permutation of the set $S = \{0, 1, \ldots, u\}$, then

$$\det(A) = \sum_p \mathrm{sg}(p) \prod_{j=0}^u a_{jp(j)}. \tag{2.3}$$

We consider the matrix $B = (b_{jl})$ and $B' = (b'_{jl})$. By Lemma 2.1 in [9] we obtain $\det(B) = \sum_p \mathrm{sg}(p) \prod_{j=0}^u b_{jp(j)} = C\epsilon^{(u+1)^2}$, where C is a constant non null.

In addition, it is easy to see that

$$\det(B') = \sum_p \text{sg}(p) \prod_{j=0}^{u} b'_{jp(j)} = \sum_p \text{sg}(p)(-1)^{n(n+1)} \prod_{j=0}^{u} b_{jp(j)} = \det(B). \quad (2.4)$$

On the other hand, expanding $\prod_{j=0}^{u} a_{jp(j)}$ in groups of terms containing only the factors b, only the factors b', and the mixed products, we have

$$\prod_{j=0}^{u} a_{jp(j)} = \prod_{j=0}^{u} [2^{j+p(j)} + o_{jp(j)}(1)] b_{jp(j)} + \prod_{j=0}^{u} [2^{j+p(j)} + o'_{jp(j)}(1)] b'_{jp(j)} + K\epsilon^{(u+1)^2}$$
$$+ o_p(\epsilon^{(u+1)^2}) = 2^{u(u+1)} \prod_{j=0}^{u} b_{jp(j)} + 2^{u(u+1)} \prod_{j=0}^{u} b'_{jp(j)} + K\epsilon^{(u+1)^2}$$
$$+ o'_p(\epsilon^{(u+1)^2}), \quad (2.5)$$

for some constant K. Then, from (2.3)–(2.5) we get $D(\epsilon) = (M + o(1))\epsilon^{(u+1)^2}$, where $M = 2^{u(u+1)+1}C + K$. □

Lemma 2.2. *Let $s, u \in \mathbb{N} \cup \{0\}, s \leq u$. Let $C = (c_{jl})$ be the matrix of order $(u+1) \times (u+1)$ defined by*

$$c_{jl} := c_{jl}(\epsilon) = \begin{cases} \langle (x^2-1)^j, (x^2-1)^l \rangle_{w,\epsilon} & 0 \leq j, l \leq u, \ l \neq s, \\ \epsilon^{j+u+1} O_j(1) & 0 \leq j \leq u, \ l = s, \end{cases} \quad (2.6)$$

where w is as in Lemma 2.1 and $O_j(1)$ is a function of the variable ϵ which is bounded for $\epsilon \to 0$. Then the determinant of C, say $N(s,\epsilon)$, satisfies $N(s,\epsilon) = O(\epsilon^{u-s+(u+1)^2})$.

If in (2.6) we replace $O_j(1)$ by $o_j(1)$, then $N(s,\epsilon) = o(\epsilon^{u-s+(u+1)^2}), 0 \leq s \leq u$.

Proof. Let C'_{jl} denote the sub matrix of C, where we have omitted the j-th file and the l-th column. Expanding the determinant of C'_{jl} by elements of the s-th column, we obtain

$$N(s,\epsilon) = \sum_{j=0}^{u} (-1)^{j+s} c_{js} \det(C'_{js}) = \sum_{j=0}^{u} \epsilon^{j+u+1} O_j(1) \det(C'_{js}). \quad (2.7)$$

Let $p := p_{js}$ be an arbitrary bijection of the set $\{0, \ldots, j-1, j+1, \ldots, u\}$ onto $\{0, \ldots, s-1, s+1, \ldots, u\}$. Then

$$\det(C'_{js}) = \sum_p \text{sg}(p) \prod_{k=0, k\neq j}^{u} a_{kp(k)}, \quad (2.8)$$

where the elements $a_{kp(k)}$ were given in Lemma 2.1.

Multiplying by ϵ^{j+s+1} and its inverse in (2.8), from (2.7) we get

$$N(s,\epsilon) = \sum_{j=0}^{u} \epsilon^{u-s} O_j(1) \left[\epsilon^{j+s+1} \sum_{p} \mathrm{sg}(p) \prod_{k=0,k\neq j}^{u} a_{kp(k)} \right]. \quad (2.9)$$

Since the expression in the bracket is $O'_j(\epsilon^{(u+1)^2})$, from (2.9) we obtain

$$N(s,\epsilon) = \sum_{j=0}^{u} \epsilon^{u-s} O''_j(1)\epsilon^{(u+1)^2} = O(\epsilon^{u-s})\epsilon^{(u+1)^2}. \quad (2.10)$$

Finally, the last assertion of the lemma analogously follows to the above proof. □

As a consequence of the previous lemmas, we get some results on the nets of even polynomials.

Lemma 2.3. *Let $T_\epsilon(x) = \sum_{l=0}^{q-1} b_l(\epsilon)x^2(x^2-1)^l$ be a net of polynomials in Π^{2q} such that*

$$\int_{1-\epsilon}^{1+\epsilon} T_\epsilon(x)(x^2-1)^j dx = o(\epsilon^{q+j}), \quad 0 \le j \le q-1. \quad (2.11)$$

Then

$$b_l(\epsilon) = o(\epsilon^{q-l-1}), \quad 0 \le l \le q-1. \quad (2.12)$$

In particular, the net $\{T_\epsilon\}_{\epsilon>0}$ converges to zero as $\epsilon \to 0$.

Proof. From (2.11), we have the following linear system,

$$\sum_{l=0}^{q-1} a_{jl}(\epsilon)b_l(\epsilon) = o(\epsilon^{q+j}), \quad 0 \le j \le q-1, \quad (2.13)$$

where $a_{jl}(\epsilon)$ was introduced in Lemma 2.1 with $w(x) = x^2$. Now, applying Lemma 2.1 and Lemma 2.2 with $u = q-1$, and later the Cramer rule we obtain (2.12). □

Lemma 2.4. *Let $T_\epsilon(x) = \sum_{l=0}^{q} b_l(\epsilon)(x^2-1)^l$ be a net of polynomials in Π^{2q}. If*

$$\int_{1-\epsilon}^{1+\epsilon} T_\epsilon(x)(x^2-1)^j dx = O(\epsilon^{q+j+1}), \quad 0 \le j \le q, \quad (2.14)$$

then for each $0 \le l \le q$,

$$b_l(\epsilon) = O(\epsilon^{q-l}) \quad \text{and} \quad T_\epsilon^{(l)}(\pm 1) = O(\epsilon^{q-l}). \quad (2.15)$$

In particular, the net $\{T_\epsilon\}_{\epsilon>0}$ is uniformly bounded on compact sets as $\epsilon \to 0$.

Proof. From (2.14) we have the following linear system,

$$\sum_{l=0}^{q} a_{jl}(\epsilon)b_l(\epsilon) = O(\epsilon^{q+j+1}), \quad 0 \le j \le q,$$

where $a_{jl}(\epsilon)$ was introduced in Lemma 2.1 with $w = 1$. Now, applying Lemma 2.1 and Lemma 2.2 with $u = q$, and later the Cramer rule we obtain

$$b_l(\epsilon) = O(\epsilon^{q-l}), \quad 0 \le l \le q. \tag{2.16}$$

The Leibnitz rule implies that

$$\begin{aligned} T_\epsilon^{(s)}(1) &= \sum_{l=0}^{q} b_l(\epsilon) \sum_{m=0}^{s} \binom{s}{m} [(x-1)^l]^{(m)}[(x+1)^l]^{(s-m)}|_{x=1} \\ &= \sum_{l=0}^{s} b_l(\epsilon) \binom{s}{l} l![(x+1)^l]^{(s-l)}|_{x=1} = O(\epsilon^{q-s}), 0 \le s \le q, \end{aligned}$$

where the last equality is a consequence of (2.16).

Since T_ϵ is even, then $T_\epsilon^{(s)}(-1) = (-1)^s T_\epsilon^{(s)}(1) = O(\epsilon^{q-s}), 0 \le s \le q.$ □

An analogous proof to the previous lemma with $u = q - 1$ gives the next lemma.

Lemma 2.5. *Let $T_\epsilon(x) = \sum_{l=0}^{q-1} b_l(\epsilon)(x^2-1)^l$ be a net of polynomials in Π^{2q-2}. If*

$$\int_{1-\epsilon}^{1+\epsilon} T_\epsilon(x)(x^2-1)^j dx = o(\epsilon^{q+j}), \quad 0 \le j \le q-1, \tag{2.17}$$

then $b_l(\epsilon) = o(\epsilon^{q-l-1}), 0 \le l \le q-1$. In particular, the net $\{T_\epsilon\}_{\epsilon>0}$ converges to 0 as $\epsilon \to 0$.

3. Existence of the best local approximation

In this section, we prove the existence of the best local approximation. Without loss of generality, we assume $x_1 = 1$. In fact, for $x_1 > 0$ we consider the function $\tilde{h}(t) = h(-x_1 t), t \in [-1-\epsilon/x_1, -1+\epsilon/x_1]$ and $\tilde{h}(t) = h(x_1 t), t \in [1-\epsilon/x_1, 1+\epsilon/x_1]$. It easy to see that if $h \in \tau_s(\pm x_1)$ then $\tilde{h} \in \tau_s(\pm 1)$ and if $h \in t_s^2(\pm x_1)$ then $\tilde{h} \in t_s^2(\pm 1)$. In addition, the best approximation of h on $[-1-\epsilon, -1+\epsilon] \cup [1-\epsilon, 1+\epsilon]$ from Π^n is the best approximation of $\tilde{h}$ on $[-1-\epsilon/x_1, -1+\epsilon/x_1] \cup [1-\epsilon/x_1, 1+\epsilon/x_1]$ from Π^n.

3.1. The n even case

In this subsection we assume n even, i.e., $r = 1$, and $f \in \tau_q(\pm 1)$. For $q \geq 1$, we define the following set

$$\mathcal{S}(f) = \{H \in \Pi^{2q} : H^{(j)}(\pm 1) = f^{(j)}(\pm 1), \quad 0 \leq j \leq q-1\}.$$

Let $S_0 \in \mathcal{S}(f)$ be a fixed polynomial. Then any polynomial in $\mathcal{S}(f)$ can be written as $S_0(x) + \lambda(x^2-1)^q, \lambda \in \mathbb{R}$. If $q = 0$ we put $\mathcal{S}(f) = \Pi^0$.

We consider the function $g = f - S_0$. According to (1.7) and Theorem 1.3, it is easy to see that

$$g \in \tau_q(\pm 1), \quad g^{(j)}(\pm 1) = 0, 0 \leq j \leq q-1, \quad \text{and} \quad \mathcal{S}(f) = S_0 + \mathcal{S}(g). \quad (3.1)$$

The proof of the following lemma is immediate.

Lemma 3.1. *It verifies that $P_\epsilon(f) = S_0 + P_\epsilon(g)$. In addition, $P_0(g)$ exists if and only if $P_0(f)$ exists, and $P_0(f) = S_0 + P_0(g)$.*

Now, our purpose is to prove the existence and characterization of $P_0(g)$. We consider the even and odd parts of g, i.e., $g^e(x) = \frac{g(x)+g(-x)}{2}$ and $g^o(x) = \frac{g(x)-g(-x)}{2}$, respectively. If there exist $P_0(g^e)$ and $P_0(g^o)$, clearly $P_0(g) = P_0(g^e) + P_0(g^o)$.

Lemma 3.2. *It verifies that $g^e, g^o \in \tau_q(\pm 1)$.*

Proof. By (3.1), $g \in \tau_q(\pm 1)$, and $Q^q_{\pm 1}(g)(x) = \alpha_{\pm 1}(x \mp 1)^q$ for some real numbers α_{-1}, α_{+1}, which verify

1. $\int_{1-\epsilon}^{1+\epsilon}(g(x) - \alpha_{+1}(x-1)^q)(x^2-1)^j dx = o(\epsilon^{q+j+1})$, $0 \leq j \leq q$.
2. $\int_{-1-\epsilon}^{-1+\epsilon}(g(x) - \alpha_{-1}(x+1)^q)(x^2-1)^j dx = o(\epsilon^{q+j+1})$, $0 \leq j \leq q$.

If in b) we make the change of variable $x = -t$, and then we add the equation a) member to member, we obtain

$$\int_{1-\epsilon}^{1+\epsilon}(g^e(x) - \gamma_{+1}(x-1)^q)(x^2-1)^j dx = o(\epsilon^{q+j+1}), \quad 0 \leq j \leq q, \quad (3.2)$$

where $\gamma_{+1} := \frac{\alpha_{+1}+(-1)^q\alpha_{-1}}{2}$. So, $Q^q_{+1}(g^e)(x) = \gamma_{+1}(x-1)^q$ and $g^e \in \tau_q(+1)$. An analogous proof with the polynomials

$$Q^q_{-1}(g^e)(x) = \gamma_{-1}(x+1)^q, \quad Q^q_{\pm 1}(g^o)(x) = \beta_{\pm 1}(x \mp 1)^q, \quad (3.3)$$

where $\gamma_{-1} := \frac{(-1)^q\alpha_{+1}+\alpha_{-1}}{2}$, $\beta_{-1} := \frac{\alpha_{-1}-(-1)^q\alpha_{+1}}{2}$, $\beta_{+1} := \frac{\alpha_{+1}-(-1)^q\alpha_{-1}}{2}$, yields $g^e \in \tau_q(-1)$ and $g^o \in \tau_q(\pm 1)$. □

Proposition 3.3. *If $g^o \in t^2_{q-1}(1)$, then $P_0(g^o) = 0$.*

Proof. According to (3.3) we have

$$\int_{1-\epsilon}^{1+\epsilon} (g^o(x) - \beta_{+1}(x-1)^q)(x^2-1)^j dx = o(\epsilon^{q+j+1}), \quad 0 \leq j \leq q. \tag{3.4}$$

Therefore,

$$\int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j dx = o(\epsilon^{q+j}), \quad 0 \leq j \leq q-1. \tag{3.5}$$

Thus, $Q_1^{q-1}(g^o) = 0$ and $g^o \in \tau_{q-1}(1)$. From hypothesis, there is $Q \in \Pi^{q-1}$ such that $\|g^o - Q\|_{\epsilon,1} = o(\epsilon^{q-1})$. By Remark 1.4 and Theorem 1.2, $Q = 0$ holds.

By the Hölder inequality, we obtain

$$\left| \int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j x dx \right| \leq K\|g^o\|_{\epsilon,1}\epsilon^{j+1} = o(\epsilon^{q+j}), \quad 0 \leq j \leq q-1. \tag{3.6}$$

On the other hand, from the characterization of $P_\epsilon(g^o)$

$$\int_{1-\epsilon}^{1+\epsilon} (g^o - P_\epsilon(g^o))(x)(x^2-1)^j x dx = 0, \quad 0 \leq j \leq q-1, \tag{3.7}$$

taking into account that the integrand is an even function. From (3.6) and (3.7), it follows that

$$\int_{1-\epsilon}^{1+\epsilon} x P_\epsilon(g^o)(x)(x^2-1)^j dx = o(\epsilon^{q+j}), \quad 0 \leq j \leq q-1. \tag{3.8}$$

Since $\{(x^2-1)^j x\}_{j=0}^{q-1}$ is a basis of the subspace of the odd polynomials in Π^{2q-1}, we can write

$$P_\epsilon(g^o)(x) = \sum_{l=0}^{q-1} b_l(\epsilon)(x^2-1)^l x.$$

Therefore, (3.8) and Lemma 2.3 imply that $P_\epsilon(g^o) \to 0$, as $\epsilon \to 0$. □

Proposition 3.4. *The net of polynomials $\{P_\epsilon(g^e)\}_{\epsilon>0}$ is uniformly bounded on compact sets, as $\epsilon \to 0$.*

Proof. From (3.2) we get

$$\int_{1-\epsilon}^{1+\epsilon} g^e(x)(x^2-1)^j dx = O(\epsilon^{q+j+1}), \quad 0 \leq j \leq q. \tag{3.9}$$

Now, (1.1) implies that

$$\int_{1-\epsilon}^{1+\epsilon} (g^e - P_\epsilon(g^e))(x)(x^2-1)^j dx = 0, \quad 0 \leq j \leq q, \tag{3.10}$$

because the integrand is an even function. From (3.9) and (3.10), it follows that

$$\int_{1-\epsilon}^{1+\epsilon} P_\epsilon(g^e)(x)(x^2-1)^j dx = O(\epsilon^{q+j+1}), \quad 0 \le j \le q. \tag{3.11}$$

Expanding $P_\epsilon(g^e)$ in terms of the basis $\{(x^2-1)^j\}_{j=0}^q$, from (3.11) and Lemma 2.4, it follows that $\{P_\epsilon(g^e)\}_{\epsilon>0}$ is uniformly bounded on compact sets, as $\epsilon \to 0$. □

The proof of the next lemma follows directly.

Lemma 3.5. *Let $P \in \Pi^{2q}$ be an even polynomial. Then, there exist two unique even polynomials, say $U \in \mathcal{S}(g^e)$ and $S \in \Pi^{2q-2}$, such that $P = U + S$.*

Now, given a polynomial $P \in \Pi^{2q}$, let $P^* \in \Pi^q$ be defined by

$$P^*(x) = \gamma_{+1}(x-1)^q - q!^{-1}U^{(q)}(1)(x-1)^q - \sum_{l=0}^{q-1} l!^{-1}P^{(l)}(1)(x-1)^l, \tag{3.12}$$

where U is the polynomial mentioned in Lemma 3.5 and γ_{+1} was introduced in (3.2). If $q = 0$, we omit the last term in (3.12).

We consider the linear functional $F : L^2([0,1]) \times \Pi^{2q} \to \mathbb{R}$ defined by

$$F(h,P) = \int_{1-\epsilon}^{1+\epsilon} (h-P)(x)\frac{(x^2-1)^q}{\epsilon^{2q+1}}dx. \tag{3.13}$$

Lemma 3.6. *Let $\{P_\epsilon(g^e) = U_\epsilon + S_\epsilon\}_{\epsilon>0} \subset \Pi^{2q}$ be a net of polynomials where U_ϵ and S_ϵ are as in Lemma 3.5. Then $S_\epsilon \to 0$ and $F(0, P_\epsilon(g^e)^*) = o(1)$, as $\epsilon \to 0$.*

Proof. Clearly $P_\epsilon(g^e)^{(j)}(\pm1) = S_\epsilon^{(j)}(\pm1), 0 \le j \le q-1$. $P_\epsilon(g^e)$ is an even polynomial satisfying (3.11), thus Lemma 2.4 implies that $S_\epsilon^{(j)}(\pm1) = O(\epsilon^{q-j})$, $0 \le j \le q-1$. Since $S_\epsilon \in \Pi^{2q-2}$, $S_\epsilon \to 0$.
On the other hand,

$$\begin{aligned} F(g^e, P_\epsilon(g^e) + P_\epsilon(g^e)^*) &= \int_{1-\epsilon}^{1+\epsilon} (g^e(x) - \gamma_{+1}(x-1)^q)\frac{(x^2-1)^q}{\epsilon^{2q+1}}dx \\ &\quad - \int_{1-\epsilon}^{1+\epsilon} q!^{-1}S_\epsilon^{(q)}(1)(x-1)^q\frac{(x^2-1)^q}{\epsilon^{2q+1}}dx \\ &\quad - \int_{1-\epsilon}^{1+\epsilon} \sum_{l=q+1}^{2q} l!^{-1}P_\epsilon(g^e)^{(l)}(1)(x-1)^l\frac{(x^2-1)^q}{\epsilon^{2q+1}}dx. \end{aligned} \tag{3.14}$$

From (3.2) and by making the change of variable $x = 1 + \epsilon t$ in (3.14), we get

$$F(g^e, P_\epsilon(g^e) + P_\epsilon(g^e)^*) = o(1) - \int_{-1}^{1} q!^{-1} S_\epsilon^{(q)}(1) t^{2q}(2+\epsilon t)^q dt - \sum_{l=q+1}^{2q} \epsilon^{l-q} l!^{-1} P_\epsilon(g^e)^{(l)}(1) \int_{-1}^{1} t^{2q}(2+\epsilon t)^q dt. \tag{3.15}$$

Proposition 3.4 implies that $P_\epsilon(g^e) = O(1)$ as $\epsilon \to 0$. Therefore, since $S_\epsilon \to 0$ we obtain $F(g^e, P_\epsilon(g^e)+P_\epsilon(g^e)^*) = o(1)$. In addition, from (3.10) it follows that $F(g^e, P_\epsilon(g^e)) = 0$. In consequence, we get $F(0, P_\epsilon(g^e)^*) = o(1)$. □

Now, we establish one of our main results.

Theorem 3.7. *Let $n = 2q$ and let $f \in \tau_q(\pm 1)$ be such that $f^o \in t^2_{q-1}(1)$. Then there exists the best local approximation of f on $\{-1, 1\}$ from Π^n. Moreover, if $S_0 \in \mathcal{S}(f)$ and $g = f - S_0$ then*

$$P_0(f)(x) = S_0(x) + \frac{g^{(q)}(1) + (-1)^q g^{(q)}(-1)}{q! 2^{q+1}} (x^2 - 1)^q. \tag{3.16}$$

Proof. Since $f^o \in t^2_{q-1}(1)$ then $g^o \in t^2_{q-1}(1)$. So, Proposition 3.3 implies that $P_0(g^o) = 0$. Therefore, it is sufficient to find $P_0(g^e)$. From Lemma 3.6, (3.11) and Lemma 2.4, we have

$$F(0, P_\epsilon(g^e)^*) = o(1), \quad \text{and} \quad \epsilon^{l-q} P_\epsilon(g^e)^{(l)}(1) = O(1), \quad 0 \le l \le q, \tag{3.17}$$

From Lemma 3.5, $P_\epsilon(g^e) = U_\epsilon + S_\epsilon$ with $U_\epsilon \in \mathcal{S}(g^e)$, and so $U_\epsilon(x) = \lambda_\epsilon (x^2 - 1)^q$, $\lambda_\epsilon \in \mathbb{R}$. In consequence, from (3.12), (3.13), (3.17) and by making the change of variable $x = 1 + \epsilon t$, we conclude

$$\int_{-1}^{1} (2^q \lambda_\epsilon - \gamma_{+1}) t^{2q} (2 + \epsilon t)^q dt = o(1). \tag{3.18}$$

Proposition 3.4 implies that $P_\epsilon(g^e) = O(1)$. Further, by Lemma 3.6 $S_\epsilon \to 0$, thus $\lambda_\epsilon = O(1)$. Now, if $\{\lambda_{\epsilon_m}\}$ is a sequence converging to λ_0, from (3.18) we get

$$\int_{-1}^{1} (2^q \lambda_0 - \gamma_{+1}) t^{2q} 2^q dt = 0, \tag{3.19}$$

i.e.,

$$\lambda_0 = \frac{\gamma_{+1}}{2^q} = \frac{g^{(q)}(1) + (-1)^q g^{(q)}(-1)}{q! 2^{q+1}}. \tag{3.20}$$

Therefore, the net $\{\lambda_\epsilon\}_{\epsilon > 0}$ converges to λ_0, i.e., $P_\epsilon(g^e)(x) \to \frac{\gamma_{+1}}{2^q}(x^2 - 1)^q = P_0(g^e)(x)$ by (3.17). Finally, Lemma 3.1 implies (3.16). □

3.2. The n odd case

In this subsection we assume n odd, i.e., $r = 0$, and $f \in \tau_{q-1}(\pm 1)$. Let $R_0 \in \Pi^{2q-1}$ be the polynomial determined by the conditions $R_0^{(j)}(\pm 1) = f^{(j)}(\pm 1)$, $0 \leq j \leq q-1$, and let $g = f - R_0$. According to (1.7) and Theorem 1.3, it is easy to see that

$$g \in \tau_{q-1}(\pm 1), \quad g^{(j)}(\pm 1) = 0, \quad 0 \leq j \leq q-1. \tag{3.21}$$

Remark 3.8. We observe that Lemma 3.1 holds with R_0 instead of S_0. Further by (3.21), $Q_{\pm 1}^{q-1}(g) = 0$.

Using Remark 3.8, with an analogous proof to Lemma 3.2 we get the next lemma.

Lemma 3.9. *It verifies that $g^e, g^o \in \tau_{q-1}(\pm 1)$ with $Q_{\pm 1}^{q-1}(g^e) = Q_{\pm 1}^{q-1}(g^o) = 0$.*

Proposition 3.10. *If $g^o \in t_{q-2}^2(1)$, then $P_0(g^e) = P_0(g^o) = 0$.*

Proof. From Lemma 3.9 we get

$$\int_{1-\epsilon}^{1+\epsilon} g^e(x)(x^2-1)^j dx = o(\epsilon^{q+j}), \quad 0 \leq j \leq q-1. \tag{3.22}$$

Now, (1.1) implies that

$$\int_{1-\epsilon}^{1+\epsilon} (g^e - P_\epsilon(g^e))(x)(x^2-1)^j dx = 0, \quad 0 \leq j \leq q-1, \tag{3.23}$$

because the integrand is an even function. From (3.22) and (3.23) it follows that

$$\int_{1-\epsilon}^{1+\epsilon} P_\epsilon(g^e)(x)(x^2-1)^j dx = o(\epsilon^{q+j}), \quad 0 \leq j \leq q-1. \tag{3.24}$$

Since n is odd, then $P_\epsilon(g^e)$ is an even polynomial in Π^{2q-2}. Expanding $P_\epsilon(g^e)$ in terms of the basis $\{(x^2-1)^j\}_{j=0}^{q-1}$, from (3.24) and Lemma 2.5 it follows that $P_0(g^e) = 0$.

Next, we prove that $P_0(g^o) = 0$. Since $g^o \in t_{q-2}^2(1)$, there exists $Q \in \Pi^{q-2}$ such that $\|g^o - Q\|_{\epsilon,1} = o(\epsilon^{q-2})$. By Remark 1.4, $t_{q-2}^2(1) \subset \tau_{q-2}(1)$. From Theorem 1.3 and Lemma 3.9 we get $Q = Q_1^{q-2}(g^o) = 0$.
By Hölder inequality we obtain

$$\left| \int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j(x-1)dx \right| \leq K\|g^o\|_{\epsilon,1}\epsilon^{j+2} = o(\epsilon^{q+j}), \quad 0 \leq j \leq q-1, \tag{3.25}$$

for some constant K. By Lemma 3.9,

$$\int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j dx = o(\epsilon^{q+j}), \quad 0 \le j \le q-1. \tag{3.26}$$

From (3.25) and (3.26) we have,

$$\int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j x dx = \int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j (x-1) dx + \int_{1-\epsilon}^{1+\epsilon} g^o(x)(x^2-1)^j dx = o(\epsilon^{q+j}). \tag{3.27}$$

On the other hand, (1.1) implies that

$$\int_{1-\epsilon}^{1+\epsilon} (g^o - P_\epsilon(g^o))(x)(x^2-1)^j x dx = 0, \quad 0 \le j \le q-1, \tag{3.28}$$

because the integrand is an even function. From (3.27) and (3.28) we get

$$\int_{1-\epsilon}^{1+\epsilon} P_\epsilon(g^o)(x)(x^2-1)^j x dx = o(\epsilon^{q+j}), \quad 0 \le j \le q-1. \tag{3.29}$$

Since $P_\epsilon(g^o)$ is an odd polynomial in Π^{2q-1}, we can expand $P_\epsilon(g^o)$ in terms of the basis $\{(x^2-1)^j x\}_{j=0}^{q-1}$. Therefore, from (3.29) and Lemma 2.3 it follows that $xP_\epsilon(g^o) \to 0$, as $\epsilon \to 0$, i.e., $P_0(g^o) = 0$. □

Now, we establish the second main result.

Theorem 3.11. *Let $n = 2q-1$ and let $f \in \tau_{q-1}(\pm 1)$ be such that $f^o \in t_{q-2}^2(1)$. Then there exists the best local approximation of f on $\{-1, 1\}$ from Π^n, and it is determined by the conditions*

$$P_0^{(j)}(f)(\pm 1) = f^{(j)}(\pm 1), \quad 0 \le j \le q-1.$$

Proof. Since $f^o \in t_{q-2}^2(1)$, then $g^o \in t_{q-2}^2(1)$. In consequence, by Proposition 3.10 we get $P_0(g) = 0$. Finally, the theorem follows from Remark 3.8. □

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References

1. E. W. Cheney (1966). *Introduction to Approximation Theory*. McGraw-Hill, New York.
2. A. P. Calderón and A. Zygmund (1961). Local properties of solution of elliptic partial differential equations. *Studia Math.* 20:171–225.

3. M. Marano (1986). Mejor Aproximación Local. Ph.D. Dissertation. Universidad Nacional de San Luis, Argentina.
4. C. K. Chui, O. Shisha, and P. W. Smith (1975). Best local approximation. *J. Approx. Theory* 15:371–381.
5. L. Y. Su (1979). *Best Local Approximation*. Ph.D. Dissertation. Texas University. College Station, Texas.
6. H. H. Cuenya and C. N. Rodriguez (2013). Differentiability and best local approximation. *Rev. Un. Mat. Argentina* 54(1):15–25.
7. H. H. Cuenya and D. E. Ferreyra (2013). Best local approximation and differentiability lateral. *Jaen J. Approx.* 5(1):81–99.
8. H. H. Cuenya and D. E. Ferreyra (2015). C^p condition and the best local approximation. *Anal. Theory Appl.* 31(1):58–67.
9. C. K. Chui, P. W. Smith, and J. D. Ward (1978). Best L_2 local approximation. *J. Approx. Theory* 22(3):254–261.