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Isgur-Karl model revisited

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We show how to match the Isgur-Karl model to the spin-flavor quark operator expansion used in the $1/N_c$ studies of the nonstrange negative parity $L = 1$ excited baryons. Using the transformation properties of states and interactions under the permutation group S_3 we are able to express the operator coefficients as overlap integrals, without making any assumption on the spatial dependence of the quark wave functions. The general mass operator leads to parameter free mass relations and constraints on the mixing angles that are valid beyond the usual harmonic oscillator approximation. The Isgur-Karl model with harmonic oscillator wave functions provides a simple counterexample that demonstrates explicitly that the alternative operator basis for the $1/N_c$ expansion for excited baryons recently proposed by Matagne and Stancu is incomplete.

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I. INTRODUCTION

Excited baryons are the natural playground to test the spin-flavor structure of quark interactions in the low-energy regime and provide useful information about the nonperturbative aspects of quantum chromodynamics. A simple model used to study the masses and mixing angles of excited baryons is the Isgur-Karl (IK) model [1]. In this model the interaction Hamiltonian of two quarks contains two components: a contact spin-spin term and a tensor interaction. This is an approximation to the Breit interaction of two quarks mediated by one-gluon exchange [2] (the one-gluon exchange [OGE] model), obtained by neglecting the spin-orbit interaction. The physical motivation for neglecting the spin-orbit interaction is debatable; we will assume it from the start as defining the model considered here.

The predictions of the IK model have been obtained assuming a harmonic oscillator basis for the orbital wave functions [1]. With this assumption the model is very predictive: the entire mass spectrum of the $L = 1$ negative parity baryons is determined in terms of two free parameters, and the mixing angles are independent of the hadron masses.

In this paper we concentrate on these states and show how to rewrite the IK model predictions in an equivalent way, constructing its effective mass operator in terms of a spin-flavor quark operator expansion. This type of operator expansion is used in a systematic manner in the $1/N_c$ studies of excited baryons [3,4], where more general spin-flavor quark-quark interactions are allowed for.

The motivation for performing the matching of the IK model to the more general $1/N_c$ expansion is twofold: In the IK model the computation of the coefficients of the operator expansion is straightforward and illustrates the connection of a model calculation with the $1/N_c$ expansion explicitly. The second reason is that it provides a simple counterexample that shows the incompleteness of the alternative operator basis advocated recently by Matagne and Stancu in Ref. [5]. The usual basis with excited quark and core operators can reproduce the IK predictions, while a basis of symmetric operators as proposed in Ref. [5] cannot do it.

To compute the matching we use the method proposed in a recent paper [6], which considers the transformation properties of the states and operators under S_3, the permutation group of three objects acting on the spatial and spin-flavor degrees of freedom. Using these transformation properties under S_3 the coefficients of the operator expansion can be expressed as overlap integrals, without making any assumption on the spatial dependence of the quark wave functions. This allows one to obtain mass relations and constraints on the mixing angles that are valid beyond the harmonic oscillator approximation of the IK model.

Examining the transformation properties of states and operators under the permutation group S_3 also allows to count the number of unknown parameters (reduced matrix elements) that follow from a specific form of the quark-quark interaction, as was already discussed in Ref. [7]. In the IK model the spatial and spin-flavor components of the spin-spin and tensor interactions are both two-body symmetric interactions of dimension three that decompose as[1] $\mathrm{S} \oplus \mathrm{MS}$ under S_3. The spatial and spin-flavor part of the $L = 1$ excited baryons states we consider here transform both as MS. In the matrix elements only operators that transform as irreps contained in the decomposition of $\mathrm{MS} \otimes \mathrm{MS}$ can contribute. S and MS appear once in the decomposition of $\mathrm{MS} \otimes \mathrm{MS} = \mathrm{S} \oplus \mathrm{MS} \oplus \mathrm{A}$, which indi-

[1]In the following S, MS, and A are the symmetric, mixed symmetric and antisymmetric irreps of S_3 of dimensions one, two and one, respectively.

cates that there will be two unknown reduced matrix elements for each of the spin-spin and tensor interactions. The unit operator coming from the confinement potential is also present and transforms as S under S_3. This leads to five unknowns in the most general case. We will show in Sec. III that for a spin-spin contact interaction the two reduced matrix elements are related, and the most general mass operator depends then on four unknown coefficients, as shown in Eq. (24). In the particular case of the harmonic oscillator approximation taken in the original formulation of the IK model, all the reduced matrix elements that contribute to the splittings are related and can be parameterized by a single parameter, as will be shown in deriving Eq. (59).

The paper is organized as follows: In Sec. II, we present the excited baryon states, in Sec. III, we discuss the general form of the matrix elements using S_3, and in Sec. IV we give the general mass relations and constraints on the mixing angles. In Sec. V, we discuss the predictions of the IK model with harmonic oscillator wave functions. Finally, in Sec. VI we relate the mass operator of the IK model to the $1/N_c$ expansion and discuss the alternative basis proposed in [5]. In Sec. VII we give our conclusions.

II. THE STATES

The $L = 1$ quark model states for the excited baryons we will consider here, have both the spatial and the spin-flavor wave functions transforming in the mixed symmetric irreducible representation of S_3. A two-dimensional basis for the representation can be chosen as $\chi_i(\vec{r}_1, \vec{r}_2, \vec{r}_3)$, for the spatial wave functions, and ϕ_j for the spin-flavor wave functions, with $i, j = 2, 3$. The total wave function $|B\rangle$ is the tensor product of the spatial-spin-flavor wave functions which is completely symmetric (and antisymmetric in color).

A special choice of the MS basis wave functions was adopted in Ref. [6] (from here on referred to as I), motivated by computational ease in the arbitrary N_c case. This choice is defined by the transformation properties of the basis under permutations, given by Eqs. (6)–(8) in I. For $N_c = 3$ the defining properties of the basis states are

$$\begin{aligned} &P_{12}\chi_2 = -\chi_2, && P_{12}\chi_3 = \chi_3 - \chi_2, \\ &P_{13}\chi_2 = \chi_2 - \chi_3, && P_{13}\chi_3 = -\chi_3, \\ &P_{23}\chi_2 = \chi_3, && P_{23}\chi_3 = \chi_2. \end{aligned} \tag{1}$$

We will relate this basis to the ρ, λ basis commonly used in the IK model in Sec. V. The basis of spin-flavor wave functions ϕ_j can be chosen to have the same properties under permutations as χ_i. An explicit example for the ϕ_j basis can be found in Appendix B of Ref. I for the $N_{5/2}(1675)$ state. We will use the same basis here, which will allow us to use the results for matrix elements derived in I.

With the basis choice defined by Eq. (1), the complete baryon wave function is given by Eq. (10) of I

$$|B(J, m_J)\rangle = \frac{\sqrt{2}}{3}\sum_{i,j=2}^{3}\chi_i(L, m_L)\phi_j(S, m_S, I, I_3) \times \begin{pmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{pmatrix}_{ij} \langle J, m_J|L, S; m_L, m_S\rangle. \tag{2}$$

We made here explicit the angular momentum quantum numbers of the spatial χ_i and spin-flavor ϕ_j states, although for reasons of simplicity they will be omitted in the following. We also included a normalization factor that normalizes the states as $\langle B|B\rangle = 1$. These spatial (and similarly the spin-flavor) MS basis is normalized as $\langle\chi_i|\chi_j\rangle = 2$, if $i = j$, and $\langle\chi_i|\chi_j\rangle = 1$ if $i \neq j$. It is easy to verify using Eqs. (1) that the state $|B\rangle$ is indeed invariant under any permutation of two quarks.

The quark spin can be $S = 1/2, 3/2$, which is combined with the orbital angular momentum $L = 1$ to give the following N states: two states with $J = 1/2$ denoted $N_{1/2}$, $N'_{1/2}$, two states $J = 3/2$ denoted $N_{3/2}$, $N'_{3/2}$, and one state with $J = 5/2$ denoted $N_{5/2}$. In addition, there are also two Δ states, denoted as Δ_J with $J = 1/2, 3/2$.

States with the same quantum numbers mix, and we define the relevant mixing angles in the nonstrange sector as

$$N(1535) = \cos\theta_{N1}N_{1/2} + \sin\theta_{N1}N'_{1/2}, \tag{3}$$

$$N(1650) = -\sin\theta_{N1}N_{1/2} + \cos\theta_{N1}N'_{1/2}, \tag{4}$$

for the spin-$1/2$ nucleons, and

$$N(1520) = \cos\theta_{N3}N_{3/2} + \sin\theta_{N3}N'_{3/2}, \tag{5}$$

$$N(1700) = -\sin\theta_{N3}N_{3/2} + \cos\theta_{N3}N'_{3/2}, \tag{6}$$

for the spin-$3/2$ nucleons. The quark model basis states (N_J, N'_J) have quark spin $S = (1/2, 3/2)$, respectively. It is possible to bring the mixing angles into the range $(0°, 180°)$ by appropriate phase redefinitions of the physical states. We will use in the numerical analysis the hadronic masses in Table I, taken from Ref. [8].

III. THE MASS OPERATOR OF THE ISGUR-KARL MODEL

The Isgur-Karl model is defined by the quark Hamiltonian

$$\mathcal{H}_{\rm IK} = H_0 + \mathcal{H}_{\rm hyp}, \tag{7}$$

where H_0 contains the confining potential and kinetic terms of the quark fields, and is symmetric under spin and isospin. The hyperfine interaction $\mathcal{H}_{\rm hyp}$ is given by

$$\mathcal{H}_{\rm hyp} = A\sum_{i<j}\Big[\frac{8\pi}{3}\vec{s}_i\cdot\vec{s}_j\delta^{(3)}(\vec{r}_{ij}) + \frac{1}{r_{ij}^3}(3\vec{s}_i\cdot\hat{r}_{ij}\vec{s}_j\cdot\hat{r}_{ij} - \vec{s}_i\cdot\vec{s}_j)\Big], \tag{8}$$

where A determines the strength of the interaction, and $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j$ is the distance between quarks i, j. The first term is a local spin-spin interaction, and the second describes a tensor interaction between two dipoles. This interaction Hamiltonian is an approximation to the gluon-exchange interaction, neglecting the spin-orbit terms.[2]

In the original formulation of the IK model [1] the confining forces are harmonic and we will refer to this model as IK-h.o. (harmonic oscillator). We will derive in the following the form of the mass operator without making any assumption on the form of the confining quark forces. We refer to this version of the model as IK-V(r).

We obtain in the following the explicit form of the mass operator of this model in the system of the $L = 1$ negative parity baryons, following the method based on the permutation group S_3 presented in I. The interaction Hamiltonian Eq. (8) has the general form

$$\mathcal{H}_{\rm hyp} = \sum_{i<j}\mathcal{R}_{ij}\cdot\mathcal{O}_{ij}, \tag{9}$$

where $\mathcal{R}_{ij}$ are orbital operators acting on the coordinates of the quarks i, j, and $\mathcal{O}_{ij}$ are spin-flavor operators. Both can also carry spatial indices, which are contracted to form a scalar in $\mathcal{H}_{\rm hyp}$, as indicated by the dot product in Eq. (9).

The orbital and spin-flavor operators for the contact and tensor interactions are

$$R_{ij} = \frac{8\pi}{3}A\delta^{(3)}(\vec{r}_{ij}), \qquad O_{ij} = s_i\cdot s_j,$$
$$Q_{ij}^{ab} = \frac{A}{r_{ij}^3}(3\hat{r}_{ij}^a\hat{r}_{ij}^b - \delta^{ab}), \qquad O_{ij}^{ab} = \frac{1}{2}(s_i^a s_j^b + s_i^b s_j^a), \tag{10}$$

where a, b are spatial indices. All these operators are symmetric under the permutation of the two quark indices i, j, but belong to the reducible representation **3** under the permutation of the three quarks.

It has been shown in I that the hadronic matrix elements of the Hamiltonian $\mathcal{H}_{\rm hyp}$ can be expressed in terms of matrix elements of spin-flavor operators O_i that are related to the decomposition of $\mathcal{O}_{ij}$ into irreducible representations of S_3, the permutation group of three objects

$$\langle B|\mathcal{H}_{\rm hyp}|B\rangle = \sum_i c_i\langle\Phi(JSI)|O_i|\Phi(JSI)\rangle, \tag{11}$$

where the coefficients c_i contain the reduced matrix elements of the orbital operators $\mathcal{R}_{ij}$, and can be written in terms of overlap integrals of the quark model wave functions. The matrix elements of the spin-flavor operators in Eq. (11) are a convenient way to obtain the reduced matrix elements of the projections of $\mathcal{O}_{ij}$ onto irreducible representations of $S_3^{\rm sp\text{-}fl}$. They have been computed in I, and are taken between the states $|\Phi(JSI)\rangle$ constructed in Ref. [4] as the tensor product of the "excited" quark 1 with a core of unexcited quarks 2, 3, and projected onto the MS irrep of spin-flavor SU(4). The advantage of this representation is that the relevant matrix elements can be immediately read off from the tables in Ref. [4].

The general form of the matrix element of $\mathcal{H}_{\rm hyp}$ can be taken from Eq. (37) of I, which we repeat here for the convenience of the reader:

$$\langle B|\mathcal{H}_{\rm hyp}|B\rangle = \tfrac{1}{3}\langle\mathcal{R}^S\rangle\langle\mathcal{O}^S\rangle + \tfrac{1}{3}\langle\mathcal{R}^{\rm MS}\rangle\langle\mathcal{O}^{\rm MS}\rangle. \tag{12}$$

The reduced matrix elements $\langle\mathcal{O}^S\rangle$ and $\langle\mathcal{O}^{\rm MS}\rangle$ for the spin-spin and tensor interaction are written in terms of matrix elements of spin-flavor operators taken between the $|\Phi(JSI)\rangle$ states. The corresponding expressions for arbitrary N_c can be found in Eqs. (39), (42), (49), and (55) of I. Here, we present the $N_c = 3$ expression, where the spin-flavor operators are understood as their corresponding matrix elements

$$\begin{aligned}\langle B|\mathcal{H}_{\rm hyp}|B\rangle &= \tfrac{1}{3}\langle R_S\rangle(\tfrac{1}{2}\vec{S}^2 - \tfrac{9}{8}) \\ &+ \tfrac{1}{3}\langle R_{\rm MS}\rangle(-\vec{S}^2 + 3\vec{s}_1\cdot\vec{S}_c + \tfrac{9}{4}) \\ &+ \tfrac{1}{3}\langle Q_S\rangle(\tfrac{1}{4}L_2^{ab}\{S^a, S^b\}) \\ &+ \tfrac{1}{3}\langle Q_{\rm MS}\rangle(\tfrac{3}{2}L_2^{ab}\{s_1^a, S_c^b\} - \tfrac{1}{2}L_2^{ab}\{S^a, S^b\}).\end{aligned} \tag{13}$$

The first two lines correspond to the contact term, and the last two lines to the tensor term, with $L_2^{ab} = \frac{1}{2}\{L^a, L^b\} - \frac{1}{3}L(L+1)\delta^{ab}$. The reduced matrix elements of the orbital operators $\langle R_S\rangle$, $\langle R_{\rm MS}\rangle$, $\langle Q_S\rangle$, $\langle Q_{\rm MS}\rangle$ are given by (unknown) overlap integrals of the corresponding operators with the wave functions of the states of interest. This is shown below in Eq. (21) for the orbital operator R_{12} appearing in the definition of the spin-spin interaction, and in Eq. (22) for the orbital operator Q_{12}^{ab} appearing in the definition of the quadrupole interaction.

We now examine closer the structure of the orbital matrix elements. There are three orbital operators $\mathcal{R}_{ij}$, which transform as a combination of S and MS under S_3. The symmetric projection is

$$\mathcal{R}_S = \mathcal{R}_{12} + \mathcal{R}_{13} + \mathcal{R}_{23}, \tag{14}$$

and the MS operators are

$$\mathcal{R}_{\rm MS}^2 = \mathcal{R}_{13} - \mathcal{R}_{23}, \tag{15}$$

$$\mathcal{R}_{\rm MS}^3 = \mathcal{R}_{12} - \mathcal{R}_{23}. \tag{16}$$

Their matrix elements on a two-dimensional basis of MS

[2] In Ref. [1] A is taken as $A = \frac{2\alpha_S}{3m^2}$.

wave functions (χ_2, χ_3) with their reduced matrix elements defined by Eqs. (34)–(36) in I, are given by

$$\langle\chi_i|\mathcal{R}_S|\chi_j\rangle = \langle\mathcal{R}_S\rangle\begin{pmatrix} 2 & 1 \\ 1 & 2\end{pmatrix}_{ij}, \tag{17}$$

$$\langle\chi_i|\mathcal{R}^2_{\rm MS}|\chi_j\rangle = \langle\mathcal{R}_{\rm MS}\rangle\begin{pmatrix} 0 & 1 \\ 1 & 1\end{pmatrix}_{ij}, \tag{18}$$

$$\langle\chi_i|\mathcal{R}^3_{\rm MS}|\chi_j\rangle = \langle\mathcal{R}_{\rm MS}\rangle\begin{pmatrix} 1 & 1 \\ 1 & 0\end{pmatrix}_{ij}. \tag{19}$$

These equations can be solved for the matrix elements of $\mathcal{R}_{12}$, acting on quarks 1, 2, with the result

$$\langle\chi_i|\mathcal{R}_{12}|\chi_j\rangle = \frac{1}{3}\begin{pmatrix} 2(\langle\mathcal{R}_S\rangle + \langle\mathcal{R}_{\rm MS}\rangle) & \langle\mathcal{R}_S\rangle + \langle\mathcal{R}_{\rm MS}\rangle \\ \langle\mathcal{R}_S\rangle + \langle\mathcal{R}_{\rm MS}\rangle & 2\langle\mathcal{R}_S\rangle - \langle\mathcal{R}_{\rm MS}\rangle\end{pmatrix}_{ij}. \tag{20}$$

The spatial MS basis, as well as the operators, also carry angular momentum indices. Applying the Wigner-Eckart theorem for SU(2) one can factor the dependence on the magnetic quantum numbers m, m'. In the case of a scalar operator like the spin-spin interaction one obtains

$$\langle\chi_i(1m')|R_{12}|\chi_j(1m)\rangle = \frac{1}{3}\begin{pmatrix} 2(\langle R_S\rangle + \langle R_{\rm MS}\rangle) & \langle R_S\rangle + \langle R_{\rm MS}\rangle \\ \langle R_S\rangle + \langle R_{\rm MS}\rangle & 2\langle R_S\rangle - \langle R_{\rm MS}\rangle\end{pmatrix}_{ij}\delta_{mm'}. \tag{21}$$

In the case of a tensor operator one obtains

$$\langle\chi_i(1m')|Q^{ab}_{12}|\chi_j(1m)\rangle = \frac{1}{3}\begin{pmatrix} 2(\langle Q_S\rangle + \langle Q_{\rm MS}\rangle) & \langle Q_S\rangle + \langle Q_{\rm MS}\rangle \\ \langle Q_S\rangle + \langle Q_{\rm MS}\rangle & 2\langle Q_S\rangle - \langle Q_{\rm MS}\rangle\end{pmatrix}_{ij}\left(\frac{1}{2}\{L^a, L^b\} - \frac{2}{3}\delta^{ab}\right)_{m',m}. \tag{22}$$

The basis for the MS orbital wave functions in I is chosen such that χ_2 satisfies $P_{12}\chi_2 = -\chi_2$, and is thus odd under a permutation of the quarks 1, 2. This implies that $\chi_2(r_i)$ vanishes for $r_{12} = 0$, giving

$$\langle\chi_2|\delta^{(3)}(\vec{r}_{12})|\chi_2\rangle = 2(\langle R_S\rangle + \langle R_{\rm MS}\rangle) = 0, \tag{23}$$

which implies a relation among the R_S and $R_{\rm MS}$ reduced matrix elements, generally valid for any local interaction, $\langle R_{\rm MS}\rangle = -\langle R_S\rangle$.

Using this relation in Eq. (13), one finds that for $N_c = 3$ the most general mass operator in the IK model depends only on three unknown orbital overlap integrals, plus an additive constant c_0 related to the matrix element of H_0, and can be written as

$$\hat{M} = c_0 + aS_c^2 + bL_2^{ab}\{S_c^a, S_c^b\} + cL_2^{ab}\{s_1^a, S_c^b\}, \tag{24}$$

where the spin-flavor operators are understood to act on the state $|\Phi(JSI)\rangle$ constructed as a tensor product of the core of quarks 2, 3 and the "excited" quark 1. The coefficients are given by

$$a = \tfrac{1}{2}\langle R_S\rangle, \tag{25}$$

$$b = \tfrac{1}{12}\langle Q_S\rangle - \tfrac{1}{6}\langle Q_{\rm MS}\rangle, \tag{26}$$

$$c = \tfrac{1}{6}\langle Q_S\rangle + \tfrac{1}{6}\langle Q_{\rm MS}\rangle. \tag{27}$$

Evaluating the matrix elements using the tables in Ref. [4] we find the following explicit result for the mass matrix

$$M_{1/2} = \begin{pmatrix} c_0 + a & -\frac{5}{3}b + \frac{5}{6}c \\ -\frac{5}{3}b + \frac{5}{6}c & c_0 + 2a + \frac{5}{3}(b + c)\end{pmatrix}, \tag{28}$$

$$M_{3/2} = \begin{pmatrix} c_0 + a & \frac{\sqrt{10}}{6}b - \frac{\sqrt{10}}{12}c \\ \frac{\sqrt{10}}{6}b - \frac{\sqrt{10}}{12}c & c_0 + 2a - \frac{4}{3}(b + c)\end{pmatrix}, \tag{29}$$

$$M_{5/2} = c_0 + 2a + \frac{1}{3}(b + c), \tag{30}$$

$$\Delta_{1/2} = \Delta_{3/2} = c_0 + 2a. \tag{31}$$

In the next section we study the implications of these results.

IV. PREDICTIONS FROM THE IK-V(R) MODEL

The IK model makes several predictions that are independent of the values of the overlap integrals c_0, a, b, c and are valid beyond the harmonic oscillator approximation.

First, the masses of the $\Delta_{1/2}$ and $\Delta_{3/2}$ states are predicted to be equal. Experimentally, they are split by $\Delta_{3/2} - \Delta_{1/2} = 80 \pm 50$ MeV. This mass splitting is introduced by the spin-orbit coupling, which is neglected in the Isgur-Karl model.

Second, the splittings $\langle\Delta\rangle - N_{5/2}$ and $\langle N_{3/2}\rangle - \langle N_{1/2}\rangle$ are due to the tensor interaction and are predicted to be related as

$$\langle\Delta\rangle - N_{5/2} = \frac{2}{9}(\langle N_{3/2}\rangle - \langle N_{1/2}\rangle). \tag{32}$$

The angular brackets denote spin-weighted averaging over the corresponding doublets

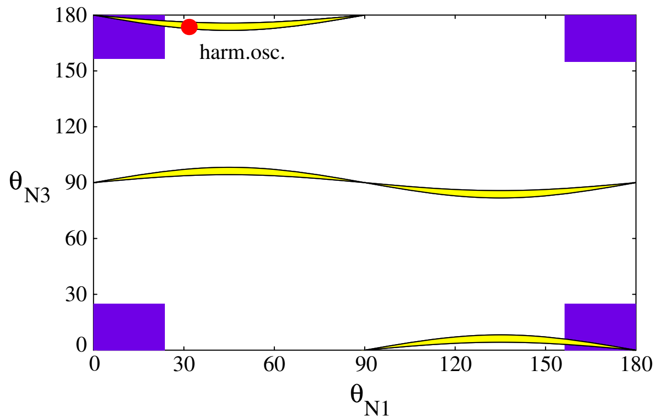

FIG. 1 (color online). Constraints on the mixing angles $(\theta_{N1}, \theta_{N3})$ in the general IK model, without any assumptions about the spatial wave functions. The four rectangles give the constraints from Eqs. (42) and (43), and the yellow bands represent the constraint Eq. (41). The red dot shows the mixing angles Eq. (64) obtained in the IK model with harmonic oscillator wave functions.

$$\langle\Delta\rangle = \tfrac{1}{3}\Delta_{1/2} + \tfrac{2}{3}\Delta_{3/2} = 1683 \pm 29\ \text{MeV}, \tag{33}$$

$$\langle N_{1/2}\rangle = \tfrac{1}{2}(N(1535) + N(1650)) = 1597 \pm 8\ \text{MeV}, \tag{34}$$

$$\langle N_{3/2}\rangle = \tfrac{1}{2}(N(1520) + N(1700)) = 1610 \pm 25\ \text{MeV}. \tag{35}$$

The experimental values of the two sides of Eq. (32) are (in MeV)

$$8 \pm 29 = 3 \pm 6, \tag{36}$$

which is well satisfied within errors.

Finally, there are also relations among hadronic parameters that do not involve the Δ states. These relations depend also on the splittings within the $J = 1/2, 3/2$ pair of states, defined as

$$\Delta N_{1/2} = N(1535) - N(1650) = -123 \pm 16\ \text{MeV}, \tag{37}$$

$$\Delta N_{3/2} = N(1520) - N(1700) = -180 \pm 50\ \text{MeV}. \tag{38}$$

There are three such relations:

$$\begin{aligned}\text{(I)}: &-\tfrac{5}{18}\Delta N_{1/2}\cos 2\theta_{N1} - \tfrac{2}{9}\Delta N_{3/2}\cos 2\theta_{N3} \\ &= N_{5/2} - \tfrac{5}{9}\langle N_{1/2}\rangle - \tfrac{4}{9}\langle N_{3/2}\rangle,\end{aligned} \tag{39}$$

$$\begin{aligned}\text{(II)}: &\tfrac{1}{2}\Delta N_{1/2}\cos 2\theta_{N1} - \tfrac{1}{2}\Delta N_{3/2}\cos 2\theta_{N3} \\ &= -\langle N_{1/2}\rangle + \langle N_{3/2}\rangle,\end{aligned} \tag{40}$$

$$\text{(III)}: \Delta N_{1/2}\sin 2\theta_{N1} + \sqrt{10}\Delta N_{3/2}\sin 2\theta_{N3} = 0. \tag{41}$$

Any two of these equations fix the mixing angles $(\theta_{N1}, \theta_{N3})$, with different results for the three ways of choosing two equations. In particular, the first two equations give

$$\Delta N_{1/2}\cos 2\theta_{N1} = \tfrac{2}{9}\langle N_{1/2}\rangle + \tfrac{16}{9}\langle N_{3/2}\rangle - 2N_{5/2}, \tag{42}$$

$$\Delta N_{3/2}\cos 2\theta_{N3} = \tfrac{20}{9}\langle N_{1/2}\rangle - \tfrac{2}{9}\langle N_{3/2}\rangle - 2N_{5/2}. \tag{43}$$

Using the experimental values for the masses, these equations give $\cos 2\theta_{N1} = 1.081 \pm 0.401$, $\cos 2\theta_{N3} = 0.889 \pm 0.246$, which leads to the allowed ranges for the mixing angles $\theta_{N1} = (0°, 23.6°)$, $(156.4°, 180°)$ and $\theta_{N3} = (0°, 25.0°), (155.0°, 180°)$. These ranges are shown in Fig. 1 as rectangles, along with the constraint from Eq. (41) [the yellow bands]. The three constraints intersect in the upper left and lower right corners of the figure.

The results for the mixing angles in the upper left region are close to the values determined from $N^* \to N\pi$ strong decays [9]. The analysis of the strong decays in Ref. [10] gave $(\theta_{N1}, \theta_{N3}) = (22.3°, 136.4°)$ and $(22.3°, 161.6°)$. The second point is favored by a $1/N_c$ analysis of the photoproduction amplitudes in Ref. [11].

In a recent paper [12] we presented the determination of the mixing angles in the OGE model, where we allow for a more general spatial dependence of the hyperfine interaction and also include the spin-orbit interaction. We comment on these results briefly, since the Isgur-Karl model considered here is a limiting case of the OGE model. Considering only the nonstrange states, the mixing angles of the OGE model are in agreement, within errors, with those extracted from strong decays; however, the predicted SU(3) splitting $\Lambda_{3/2}(1520) - \Lambda_{1/2}(1405)$ is in disagreement with the observed splitting. To correctly reproduce the splitting of these states one also needs flavor dependent

TABLE I. The experimental values for the masses of the nonstrange negative parity $L = 1$ baryons used as inputs (from Ref. [8]). The second and third lines give the predictions for these masses from fits to the IK model: IK-V(r) is the fit to the most general IK model, without assuming a specific form for the confining forces, and IK-h.o. is the fit to the IK model assuming a harmonic oscillator basis.

	$N_{1/2}(1535)$	$N_{1/2}(1650)$	$N_{3/2}(1520)$	$N_{3/2}(1700)$	$N_{5/2}(1675)$	$\Delta_{1/2}(1620)$	$\Delta_{3/2}(1700)$	χ^2
PDG(2008)	1535 ± 10	1658 ± 13	1520 ± 5	1700 ± 50	1675 ± 5	1630 ± 30	1710 ± 40	-
IK-V(r)	1523	1659	1523	1693	1674	1678	1678	5.0
IK-h.o.	1490	1657	1533	1749	1671	1686	1686	33.

operators [13] that partially cancel out the spin-orbit interaction coming from the one-gluon exchange interaction.

Finally, we quote briefly the best fit values for the coefficients c_0, a, b, c

$$c_0 = 1368 \pm 11 \text{ MeV}, \qquad a = 155 \pm 8 \text{ MeV}, \qquad b = -4^{+9}_{-10} \text{ MeV}, \qquad c = -8^{+11}_{-12} \text{ MeV}. \tag{44}$$

The resulting masses are listed in Table I as IK-V(r). The fit to the seven masses with four coefficients has 3 degrees of freedom. The resulting chi squared by degree of freedom is $\chi^2_{dof} = 1.7$. In this case we obtain $\theta_{N1} = (0°, 8.4°)$, $(171.6°, 180°)$, and $\theta_{N3} = (0°, 2.1°)$, $(177.9°, 180°)$ for the ranges of the mixing angles.

V. THE ISGUR-KARL MODEL WITH HARMONIC OSCILLATOR WAVE FUNCTIONS

In the usual treatment of the IK model [1] (denoted here as IK-h.o.), the leading order Hamiltonian H_0 describes three constituent quarks interacting by harmonic oscillator potentials

$$H_0 = \frac{1}{2m}\sum_i p_i^2 + \frac{K}{2}\sum_{i<j} r_{ij}^2. \tag{45}$$

This can be diagonalized exactly in terms of the reduced coordinates $\vec{\rho} = \frac{1}{\sqrt{2}}(\vec{r}_1 - \vec{r}_2)$, $\vec{\lambda} = \frac{1}{\sqrt{6}}(\vec{r}_1 + \vec{r}_2 - 2\vec{r}_3)$.

Expressed in terms of these coordinates, the Hamiltonian takes the form of two independent oscillators:

$$H = \frac{p_\rho^2}{2m} + \frac{p_\lambda^2}{2m} + \frac{3}{2}K\rho^2 + \frac{3}{2}K\lambda^2. \tag{46}$$

The eigenstates $\Psi^{\rho,\lambda}_{Lm}$ with $L = 1$, $m = 1$ are

$$\Psi^\rho_{11} = \rho_+ \frac{\alpha^4}{\pi^{3/2}} \exp\left(-\frac{1}{2}\alpha^2(\rho^2 + \lambda^2)\right), \tag{47}$$

$$\Psi^\lambda_{11} = \lambda_+ \frac{\alpha^4}{\pi^{3/2}} \exp\left(-\frac{1}{2}\alpha^2(\rho^2 + \lambda^2)\right), \tag{48}$$

where $\alpha = (3Km)^{1/4}$, $\rho_+ = -\rho_x - i\rho_y$, $\lambda_+ = -\lambda_x - i\lambda_y$ and the combination $\rho^2 + \lambda^2$ is invariant under permutations of the three quarks.

The relation to the χ basis in Eq. (1) is

$$\chi_2(1m) = \sqrt{2}\Psi^\rho_{1m}, \tag{49}$$

$$\chi_3(1m) = \frac{1}{\sqrt{2}}\Psi^\rho_{1m} + \sqrt{\frac{3}{2}}\Psi^\lambda_{1m}. \tag{50}$$

It is easy to check that these states transform under permutations as specified by the relations Eqs. (1), and are also normalized correctly.

The reduced matrix elements of the orbital operators $\langle R_S\rangle$, $\langle Q_S\rangle$, $\langle Q_{\rm MS}\rangle$ can be computed explicitly using the basis of wave functions Eqs. (47) and (48), where the expression for the 12 component of a general spatial operator, Eq. (20), takes the diagonal form

$$\langle \Psi^i_{11}|\mathcal{R}_{12}|\Psi^j_{11}\rangle = \frac{1}{3}\begin{pmatrix} \langle\mathcal{R}_S\rangle + \langle\mathcal{R}_{\rm MS}\rangle & 0 \\ 0 & \langle\mathcal{R}_S\rangle - \langle\mathcal{R}_{\rm MS}\rangle \end{pmatrix}_{ij}; \quad i, j = \rho, \lambda. \tag{51}$$

It is easy to understand that the off-diagonal matrix elements of $\mathcal{R}_{12}$ (which is symmetric under P_{12}) are zero because ρ and λ are antisymmetric and symmetric under P_{12}, respectively.

The reduced matrix element $\langle R_S\rangle$ of the spin-spin interaction can be extracted by considering the matrix element

$$\langle \Psi^\lambda_{11}|\delta^{(3)}(\vec{r}_{12})|\Psi^\lambda_{11}\rangle = \frac{\alpha^8}{2^{3/2}\pi^3}\int d^3\rho d^3\lambda\, \delta^{(3)}(\vec{\rho})(\lambda_x^2 + \lambda_y^2) \times e^{-\alpha^2(\rho^2+\lambda^2)} = \frac{\alpha^3}{(2\pi)^{3/2}} \tag{52}$$

which, using the definition of R_{12}, Eq. (10), gives

$$\langle R_S\rangle = A\frac{2\alpha^3}{\sqrt{2\pi}} \equiv \delta. \tag{53}$$

It is convenient to define the parameter δ as all the other reduced matrix elements can be written in terms of this single parameter.

The computation of the reduced matrix elements for the tensor interaction $\langle Q_S\rangle$, $\langle Q_{\rm MS}\rangle$ is more involved. The analog of Eq. (21) for the matrix element of the tensor interaction Q^{ab}_{12} acting on the quarks 1, 2 is given by Eq. (22).

The reduced matrix elements $\langle Q_S\rangle$ and $\langle Q_{\rm MS}\rangle$ can be determined from the matrix elements of Q^{ab}_{12} on the Ψ^λ, Ψ^ρ states. In this basis the matrix element of Q^{ab}_{12} is diagonal as in Eq. (51). The dependence on the angular momentum projections [shown in Eq. (22)] is easy to compute by choosing $a = b = 3$, which gives $(\frac{1}{2}\{L^3, L^3\} - \frac{2}{3})_{m'=1,m=1} = \frac{1}{3}$. The two matrix elements we need are

$$\langle \Psi^\lambda_{11}|Q^{33}_{12}|\Psi^\lambda_{11}\rangle = 0, \tag{54}$$

$$\langle \Psi^\rho_{11}|Q^{33}_{12}|\Psi^\rho_{11}\rangle = -A\frac{4\alpha^3}{15\sqrt{2\pi}} = -\frac{2}{15}\delta. \tag{55}$$

The first relation can be understood intuitively as following from the fact that the orbital angular momentum of the quarks 1, 2 in the Ψ^λ state vanishes, $L_\rho = 0$. The tensor operator Q^{ab}_{12} has $L_\rho = 2$ and thus its matrix element on these states vanishes. Explicitly, the matrix element is expressed as an integral over $\vec{\rho}$, $\vec{\lambda}$ as

$$\langle \Psi^\lambda_{11}|Q^{33}_{12}|\Psi^\lambda_{11}\rangle = A\frac{\alpha^8}{2^{3/2}\pi^3}\int d^3\rho d^3\lambda \frac{1}{\rho^5}(3\rho_z^2 - \rho^2) \times (\lambda_x^2 + \lambda_y^2)e^{-\alpha^2(\rho^2+\lambda^2)} = 0, \tag{56}$$

since the angular ρ integration vanishes $\int_{-1}^{1} d\cos\theta(3\cos^2\theta - 1) = 0$.

The matrix element in Eq. (55) can be computed straightforwardly with the result

$$\langle\Psi_{11}^{\rho}|Q_{12}^{33}|\Psi_{11}^{\rho}\rangle = A\frac{\alpha^8}{2^{3/2}\pi^3}\int d^3\rho d^3\lambda \frac{1}{\rho^5}(3\rho_z^2 - \rho^2) \times(\rho_x^2 + \rho_y^2)e^{-\alpha^2(\rho^2+\lambda^2)} = -A\frac{4\alpha^3}{15\sqrt{2\pi}}. \tag{57}$$

Comparing the results with Eq. (51), one finds that the reduced matrix elements in the IK model with harmonic oscillator wave functions are all related and can be expressed in terms of the single parameter δ as

$$\langle Q_{\rm MS}\rangle = \langle Q_S\rangle = -\frac{3}{5}\delta; \qquad \langle R_S\rangle = \delta. \tag{58}$$

This gives a relation among the coefficients a, b, c of the mass matrix Eq. (24)

$$a = \tfrac{1}{2}\delta, \qquad b = \tfrac{1}{20}\delta, \qquad c = -\tfrac{1}{5}\delta. \tag{59}$$

We recover the well-known result that in the harmonic oscillator model, the entire spectroscopy of the $L = 1$ baryons is fixed by one single constant δ, along with an overall additive constant c_0, and the model becomes very predictive. The explicit mass matrix is

$$M_{1/2} = \left(c_0 + \frac{3}{4}\delta\right)\mathbf{1} + \frac{1}{4}\delta\begin{pmatrix} -1 & -1 \\ -1 & 0 \end{pmatrix}, \tag{60}$$

$$M_{3/2} = \left(c_0 + \frac{3}{4}\delta\right)\mathbf{1} + \frac{1}{4}\delta\begin{pmatrix} -1 & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{9}{5} \end{pmatrix}, \tag{61}$$

$$M_{5/2} = \left(c_0 + \frac{3}{4}\delta\right) + \frac{1}{5}\delta, \tag{62}$$

$$\Delta_{1/2} = \Delta_{3/2} = \left(c_0 + \frac{3}{4}\delta\right) + \frac{1}{4}\delta. \tag{63}$$

This agrees with the mass matrix of Ref. [1]. Furthermore, the agreement on the signs of the mixing terms indicates that the phase convention of the states in Ref. [1] is the same as the phase convention of Ref. [4] used here.

The mixing angles are independent of the hadron masses, and are given by

$$\begin{aligned} \theta_{N1} &= \arctan\left(\frac{1}{2}(\sqrt{5} - 1)\right) = 31.7°, \\ \theta_{N3} &= \arctan\left(-\frac{\sqrt{10}}{14 + \sqrt{206}}\right) = 173.6°. \end{aligned} \tag{64}$$

The arguments of the previous section show that this prediction is specific to the harmonic oscillator model. However, the more general predictions of the IK-V(r) model for the mixing angles are close to this result, as can be seen from Fig. 1, where the point given in Eq. (64) is indicated as the red dot.

VI. RELATION TO THE $1/N_c$ EXPANSION

The predictions of the nonrelativistic quark model can be understood from QCD within the large N_c expansion. This method relies on a power counting scheme to organize the contributions of the different operators according to their order in $1/N_c$. At leading order in $1/N_c$ the spin-flavor contracted symmetry $SU(4)_c$ emerges in the baryon sector of QCD [14]. In the ground state baryon sector, the predictions of this symmetry reproduce the spin-flavor relations of the constituent quark model.

The situation is more complicated for the excited baryons, where the leading N_c predictions of the contracted symmetry do not generally agree with those of the quark model [3,15,16]. For example, at leading order in $1/N_c$ the masses of the nonstrange $L = 1$ negative parity baryons form three groups of degenerate states (towers), which differs from the quark model prediction of a degenerate **20** multiplet of SU(4) [15,16].

The mass operator of the IK model, Eq. (24), matches a subset of the operators that appear in the systematic $1/N_c$ expansion. The complete basis was given in Ref. [4], and it includes core and excited quark operators. The operators S_c^2 and $L_2^{ij}\{s_1^i, S_c^j\}$ contribute at order $O(1/N_c)$, and the operator $L_2^{ij}\{S_c^i, S_c^j\}$ appears only at order $O(1/N_c^2)$. Using the notation of Ref. [4] the predictions of the IK model encoded in Eq. (24) [supplemented by the relations Eq. (59) in the particular case of the IK-h.o. model], can be rewritten as

$$\begin{aligned} H^{\rm eff} &= c_1 O_1 + c_6 O_6 + c_8 O_8 + c_{17} O_{17} \\ &= c_1 N_c \mathbf{1} + c_6\left(\frac{1}{N_c}S_c^2\right) + c_8\left(\frac{1}{2N_c}L_2^{ab}\{s_1^a, S_c^b\}\right) \\ &\quad + c_{17}\left(\frac{1}{2N_c^2}L_2^{ab}\{S_c^a, S_c^b\}\right). \end{aligned} \tag{65}$$

These coefficients are related to the coefficients c_0, a, b, c used in Sec. III as

$$c_1 = \tfrac{1}{3}c_0 = \tfrac{1}{3}m_0 - \tfrac{1}{4}\delta = 462\ \text{MeV}, \tag{66}$$

$$c_6 = 3a = \tfrac{3}{2}\delta = 450\ \text{MeV}, \tag{67}$$

$$c_8 = 6c = -\tfrac{6}{5}\delta = -360\ \text{MeV}, \tag{68}$$

$$c_{17} = 18b = \tfrac{9}{10}\delta = 270\ \text{MeV}, \tag{69}$$

where $m_0 = 1610$ MeV and $\delta = 300$ MeV in the IK-h.o. model. In Table II the coefficients are compared with the result of the best fit made in Sec. IV. The success of the IK-h.o. basically lies in the correct prediction of the value of c_6 and the dominance of the operator O_6 in the general

TABLE II. The coefficients of the best fit in the IK-V(r) and the predicted values for the coefficients in the IK-h.o. model.

	c_1	c_6	c_8	c_{17}
IK-V(r)	456 ± 3.7	465 ± 23	-46^{+63}_{-74}	-69^{+165}_{-186}
IK-h.o.	462	450	-360	270

expansion. The predicted values for c_8 and c_{17} in the IK-h.o. model are too large and spoil the fit. In the best possible fit these two coefficients are compatible with zero within errors.

In the IK model with harmonic oscillator wave functions δ is also related to the splitting of the ground state baryons as $m_N = m_0' - \delta/2$, $m_\Delta = m_0' + \delta/2$. A simple calculation shows that the effective Hamiltonian for the ground state baryons that reproduces these IK predictions is

$$H_{gs}^{\mathrm{eff}} = g_1 N_c \mathbf{1} + g_3 \frac{1}{N_c} S^i S^i, \tag{70}$$

where

$$g_1 = \frac{1}{3} m_0' - \frac{1}{4}\delta = \frac{5M_N - M_\Delta}{12} \sim 287 \text{ MeV}, \tag{71}$$

$$g_3 = \delta = M_\Delta - M_N \sim 300 \text{ MeV}. \tag{72}$$

This explicit example is useful to discuss the alternative approach to the $1/N_c$ expansion for excited baryons presented in Ref. [5]. The authors of Ref. [5] propose the operator basis below, that differs from the one in Ref. [4] in that only a subset of the operators are allowed, arguing that the separation into excited quark and core quark operators is not necessary

$$Q_1 = N_c \mathbf{1}, \tag{73}$$

$$Q_2 = L^i s^i, \tag{74}$$

$$Q_3 = \frac{1}{N_c} S^i S^i, \tag{75}$$

$$Q_4 = \frac{1}{N_c} T^a T^a, \tag{76}$$

$$Q_5 = \frac{15}{N_c} L^{(2)ij} G^{ia} G^{ja}, \tag{77}$$

$$Q_6 = \frac{3}{N_c} L^i T^a G^{ia}, \tag{78}$$

$$Q_7 = \frac{3}{N_c^2} S^i T^a G^{ia}. \tag{79}$$

The first observation is that these seven operators are not independent. We find that the matrix elements of Q_7 for the states of interest can be rewritten in terms of those of Q_1, Q_3, Q_4 as $Q_7 = -\frac{3(4N_c - 9)}{16N_c^3} Q_1 + \frac{3(N_c - 1)}{8N_c}(Q_3 + Q_4)$. Furthermore, using the matrix elements from Table 3 in the first of Ref. [5] and equating $\sum_{i=1}^6 c_i Q_i$ to the matrix elements of the Isgur-Karl model, Eqs. (60)–(63) it is easy to see that it is not possible to find coefficients c_i that reproduce the predictions of the IK model. This is an explicit example that shows that the basis proposed in [5] is incomplete.

For the completely symmetric ground state baryons the $\{Q_i\}$ basis is correct, but overcomplete, as only Q_1, Q_3 are needed. The $\{Q_i\}$ basis constructed with symmetric operators is only correct for symmetric spin-flavor states like the [**56**, $L = 2$], see, for example, Ref. [17].

VII. CONCLUSIONS

We showed in this paper how to construct the effective mass operator of the Isgur-Karl model for the nonstrange negative parity $L = 1$ excited baryons. The effective mass operator is written as an operator expansion in Eq. (24), where the spatial dependence and spin-flavor dependence are factorized. This form of the mass operator is valid without making any assumptions about the spatial dependence of the quark wave functions and allows to explore the IK model beyond the harmonic oscillator approximation. The unknown spatial dependence is contained in the three coefficients Eqs. (25)–(27) of the expansion, which are written in terms of orbital overlap integrals in Eqs. (21) and (22). These explicit expressions for the coefficients are obtained exploiting the transformation properties of states and interactions under the permutation group S_3 acting on the spatial and spin-flavor degrees of freedom [6]. The spin-flavor structure of the model is manifest in the three nontrivial operators that appear in the expansion, whose matrix elements are calculable and can be conveniently read off from Tables II and III in Ref. [4].

The general operator form Eq. (24) leads to parameter free mass relations that also constrain the mixing angles and are well satisfied by data. The most noticeable disagreement is the prediction of the degeneracy of the two Δ states. The experimental data seems to point to the presence of a spin-orbit interaction. Smaller experimental errors on the masses of these two states would contribute to determine its strength.

In the particular case of harmonic oscillator wave functions the coefficients of the mass operator can be computed and written in terms of a single parameter as shown in Eq. (59). The mass operator Eq. (24) reproduces then exactly the predictions of the IK model as formulated in Ref. [1]. As is well known, in this approximation the mixing angles are fixed, independently of the hadronic parameters.

Recasting the predictions of the IK model in this way makes clear its relation to the $1/N_c$ studies of excited baryons, where the spin-flavor quark operator expansion is used in a systematic way. In Eq. (65)–(69) we present the result of the matching of the IK model to the operators of the $1/N_c$ expansion, using the notation of Ref. [4].

The matching of the IK model is also a simple example that shows that the alternative operator basis proposed in Ref. [5] can not reproduce the mass operator of the IK model with harmonic oscillator wave functions, and is thus incomplete.

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