Long range Casimir force induced by transverse electromagnetic modes

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We consider the interaction of two perfectly conducting plates of arbitrary shape that are inside a nonsimply connected cylinder with transverse section of the same shape. We show that the existence of transverse electromagnetic (TEM) modes produces a Casimir force that decays only as $1/a^2$, where a is the distance between plates. The TEM force does not depend on the area of the plates and dominates at large distances over the force produced by the transverse electric and transverse magnetic modes, providing in this way a physical realization of the $1 + 1$-dimensional Casimir effect. For the particular case of a coaxial circular cylindrical cavity, we compute the transverse electric, transverse magnetic and TEM contributions to the force, and find the critical distance for which the TEM modes dominate.

I. INTRODUCTION

In the last years, there has been an increasing interest in the Casimir effect [1]. The new generation of experiments [2] allowed a precise determination of the Casimir force, and stimulated theoretical calculations of the forces for different geometries, including finite temperature and conductivity corrections.

The analysis of the dependence of the Casimir force with the geometry is therefore of both theoretical and experimental relevance. In this paper, we will point out that, in nonsimply connected electromagnetic cavities, the presence of transverse electromagnetic (TEM) modes produces an additional contribution to the Casimir force, that it is independent of the section of the cavity and decays slower than the contributions of transverse electric (TE) and transverse magnetic (TM) modes. As far as we know, this is the first example in the literature that illustrates the relevance of the TEM modes in the *static* Casimir effect (for its relevance in the *dynamical* Casimir effect see [3,4]).

Concretely, we will consider the interaction between two identical perfectly conducting plates that are inside a very long cylinder of the same section, that is also perfectly conducting. These geometries are usually referred to as "Casimir pistons" [5], and have received considerable attention recently [6–8]. One of the reasons that triggered these investigations was the reconsideration of the Casimir energy for rectangular boxes, since repulsive forces have been predicted when considering only the zero-point energy of the internal modes of the rectangular cavities [9]. The validity of these results has been disputed [5] for at least two related reasons: the omission of the contribution of the external modes, and the ambiguity in the renormalization of the divergent quantities (however, there is no consensus in the literature on these issues; see for instance [10]). In any case, the advantage of the pistons is that, as long as the surfaces are perfectly conducting, one can compute the Casimir energy and forces unambiguously and without considering the external modes to the cavity.

The new aspect that we will introduce in this paper is the consideration of nonsimply connected cavities (see Fig. 1), allowing for the existence of TEM modes.

The paper is organized as follows. In Sec. II we will describe the different contributions to the zero-point energy in a nonsimply connected cavity. We will see that the Casimir energy for TEM modes is equivalent to that of a massless scalar field living in $1 + 1$ dimensions and satisfying Dirichlet boundary conditions on the plates. Moreover, the Casimir energy for TE and TM modes is equivalent to that of a set of massive, $1 + 1$-dimensional scalar fields, with the masses determined by the eigenfrequencies associated to the transverse section of the cavity. We will discuss the behavior of the force when the distance between the plates is much larger or much smaller than the transverse dimensions of the cavity, and conclude that the TEM force dominates above a critical distance. In Sec. III we will present a detailed analysis of the particular case of a coaxial cavity of circular section. We will evaluate the contributions of TE and TM modes to the Casimir energy for this geometry using a combination of the analytical result for a $1 + 1$ massive field and Cauchy's theorem to
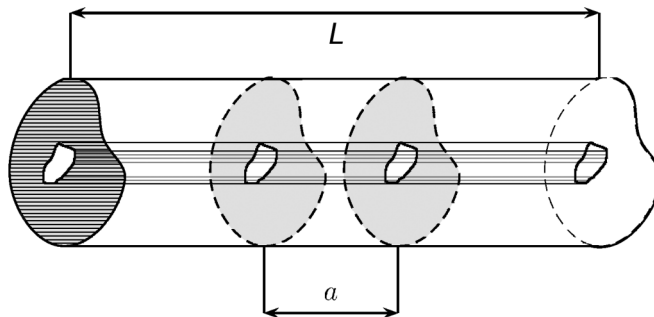

FIG. 1. Two pistons separated by a distance a inside a nonsimply connected cavity (the annular region between two cylinders of arbitrary section). The system is enclosed between another two plates separated by a distance $L \gg a$. All surfaces are perfect conductors.

perform the summation over the effective masses. We will also compare these contributions with that coming from TEM modes, and find the critical distance as a function of the radii of the inner and outer cylindrical shells. Section IV contains our final remarks.

II. CASIMIR ENERGY IN NONSIMPLY CONNECTED CAVITIES

Let us consider a very long electromagnetic cylindrical cavity, with an arbitrary section. We will assume that the cavity is nonsimply connected, i.e. that there is a second cylinder, also of arbitrary section, inside the larger one (see Fig. 1). The cavity is the annular region between the two cylinders and contains two plates (pistons) separated by a distance a (the pistons cover only the annular region between the cylinders). All surfaces are perfectly conducting. The $\mathbf{z}$ direction is the axis of the cavity, and we will denote by $\mathbf{x}_\perp$ the coordinates in the transverse sections.

At the classical level, the electromagnetic field admits a description in terms of independent TE, TM and TEM modes, which are defined with respect to the $\mathbf{z}$ direction. This is possible due to the particular geometries we are considering, that have an invariant section along the $\mathbf{z}$ axis. The TE and TM electromagnetic degrees of freedom can be written in terms of two different vector potentials $\mathbf{A}_{\text{TE}}$ and $\mathbf{A}_{\text{TM}}$ with null divergence and $\mathbf{z}$ component [11]. The TE electric and magnetic fields are given by

$$\mathbf{E}_{\text{TE}} = -\dot{\mathbf{A}}_{\text{TE}}; \qquad \mathbf{B}_{\text{TE}} = \nabla \times \mathbf{A}_{\text{TE}}, \qquad (1)$$

while the TM fields are given by the dual relations

$$\mathbf{B}_{\text{TM}} = \dot{\mathbf{A}}_{\text{TM}}; \qquad \mathbf{E}_{\text{TM}} = \nabla \times \mathbf{A}_{\text{TM}}. \qquad (2)$$

The vector potentials can be written in terms of the so-called (scalar) Hertz potentials [3,12] as

$$\mathbf{A}_{\text{TE}} = \hat{\mathbf{z}} \times \nabla \phi_{\text{TE}}, \qquad \mathbf{A}_{\text{TM}} = \hat{\mathbf{z}} \times \nabla \phi_{\text{TM}}. \qquad (3)$$

The Hertz potentials ϕ_{TE} and ϕ_{TM} satisfy the wave equation with Dirichlet and Neumann boundary conditions on the lateral surfaces, respectively. Both potentials satisfy Dirichlet boundary conditions on the pistons. The eigenfunctions can be chosen of the form

$$\phi_{\text{TE,TM}}(t, z, \mathbf{x}_\perp) = e^{-iwt} \sin\left(\frac{n\pi z}{a}\right) \varphi_{\text{TE,TM}}(\mathbf{x}_\perp), \qquad (4)$$

where n is a positive integer and $\varphi_{\text{TE,TM}}$ are eigenfunctions of the transverse Laplacian

$$\nabla_\perp^2 \varphi_{\text{TE,TM}} = -\lambda^2 \varphi_{\text{TE,TM}}. \qquad (5)$$

Therefore, the eigenfrequencies associated to the TE and TM modes are

$$w_{k,n}^{\text{TE}} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \lambda_{kD}^2}, \qquad w_{k,n}^{\text{TM}} = \sqrt{\left(\frac{n\pi}{a}\right)^2 + \lambda_{kN}^2}, \qquad (6)$$

where λ_{kD}^2 and λ_{kN}^2 are the eigenvalues of Eq. (5) when the eigenfunctions satisfy Dirichlet and Neumann boundary conditions, respectively.

When the cylindrical cavity is nonsimply connected, in addition to the TE and TM modes one should also consider the TEM modes, for which both the electric and magnetic fields have a vanishing z component. Working with the usual vector potential $\mathbf{A}$, the TEM solutions are of the form

$$\mathbf{A}(\mathbf{x}_\perp, z, t) = \mathbf{A}_\perp(\mathbf{x}_\perp) \phi_{\text{TEM}}(z, t),$$
$$\mathbf{E} = -(\partial_t \phi_{\text{TEM}}) \mathbf{A}_\perp, \qquad \mathbf{B} = (\partial_z \phi_{\text{TEM}}) \hat{\mathbf{z}} \times \mathbf{A}_\perp, \qquad (7)$$

where $\phi_{\text{TEM}}(z, t)$ is an additional scalar field. The transverse vector potential has vanishing rotor and divergence, and a zero tangential component on the transverse surfaces. Therefore, $\mathbf{A}_\perp$ is a solution of an *electrostatic* problem in the two transverse dimensions (in hollow cylindrical cavities the transverse potential vanishes and TEM modes do not exist). The scalar field ϕ_{TEM} satisfies Dirichlet boundary conditions on the longitudinal boundaries $z = 0$ and $z = a$, and the longitudinal wave equation $(\partial_t^2 - \partial_z^2)\phi_{\text{TEM}} = 0$. Thus, the eigenfrequencies of the TEM modes are

$$w_n^{\text{TEM}} = n\pi/a. \qquad (8)$$

In order to obtain the Casimir energy we introduce the regularized quantities

$$E_{\text{reg}}(a) = \frac{1}{2} \sum_n w_n^{\text{TEM}} e^{-\sigma w_n^{\text{TEM}}}$$
$$+ \frac{1}{2} \sum_{n,k} (w_{k,n}^{\text{TE}} e^{-\sigma w_{k,n}^{\text{TE}}} + w_{k,n}^{\text{TM}} e^{-\sigma w_{k,n}^{\text{TM}}})$$
$$\equiv E_{\text{reg}}^{\text{TEM}}(a) + E_{\text{reg}}^{\text{TE}}(a) + E_{\text{reg}}^{\text{TM}}(a), \qquad (9)$$

and two additional pistons separated at a very large distance L, enclosing the system. The physical Casimir energy is defined as the difference

$$E(a) = E_{\text{reg}}(a) + 2E_{\text{reg}}\left(\frac{L - a}{2}\right) - 3E_{\text{reg}}\left(\frac{L}{3}\right) \qquad (10)$$

in the limit when the cutoff σ tends to zero. Note that one can compute independently the TE, TM and TEM contributions to the energy. Note also that, as the pistons only cover the annular region between the cylinders, the internal modes of the smaller cylinder will be irrelevant for the interaction between plates.

To proceed, we note that the Casimir energy for this geometry is formally equivalent to that of a set of scalar fields living in $1 + 1$ dimensions and satisfying Dirichlet boundary conditions at $z = 0$ and $z = a$. Indeed, Eq. (8) implies that the TEM Casimir energy is equivalent to that of a massless scalar field, and the result is very well known [13]:

$$E^{\text{TEM}}(a) = -\frac{\pi}{12a}. \qquad (11)$$

From Eq. (6) we see that the TE and TM Casimir energies correspond to that of a set of massive scalar fields, with masses given by λ_{kD} and λ_{kN}. The Casimir energy E_m for a field of mass m in $1 + 1$ dimensions has been computed previously by many authors [14] using different methods of regularization (see, in particular, Ref. [8] for a calculation with an exponential cutoff). It reads [15]

$$E_m(a) = -\frac{1}{2\pi} \sum_{l=1}^{+\infty} \frac{mK_1(2lma)}{l}, \qquad (12)$$

where K_1 is the modified Bessel function of the second kind. Using this result and the analogy between the TE and TM eigenfrequencies [Eq. (6)] with the eigenfrequencies of massive scalar fields in $1 + 1$ dimensions, we can easily obtain the TE and TM contributions to the Casimir energy in the cylindrical cavity

$$E^{\text{TE}}(a) + E^{\text{TM}}(a) = -\frac{1}{2\pi} \sum_{l=1}^{+\infty} \left(\sum_{\lambda_{kD}} \frac{\lambda_{kD} K_1(2l\lambda_{kD}a)}{l} \right.$$
$$\left. + \sum_{\lambda_{kN}} \frac{\lambda_{kN} K_1(2l\lambda_{kN}a)}{l} \right). \qquad (13)$$

This equation has been previously obtained in Ref. [7] using a different method. We stress that the formula is valid for a cavity of arbitrary section. The energy can in principle be computed through a numerical evaluation of the eigenvalues of the transverse Laplacian. Alternatively, as we will describe in the next section, the summation over the eigenvalues can be performed using Cauchy's theorem. The force between pistons is easily obtained taking the derivative of the energy with respect to a.

Let us now discuss some generic properties of the different contributions to the Casimir energy. At small distances, when the separation between pistons is much smaller than the transverse dimensions of the cavity, one expects the proximity force approximation (PFA) to describe accurately the contributions of TE and TM modes. Indeed, using heat kernel techniques it can be shown [7] that, in this limit

$$E^{\text{TE}}(a) + E^{\text{TM}}(a) \approx -\frac{\pi^2}{720} \frac{A}{a^3}, \qquad (14)$$

where A is the area of the transverse sections. This is of course the well known result for parallel plates. It is worth stressing that the geometric properties of the transverse section, as the area, are contained in the eigenvalues λ_{kD} and λ_{kN}, which play the role of the masses of the fields in the $1 + 1$-dimensional analogy.

On the other hand, in the opposite limit we have $\lambda a \gg 1$. The TE and TM contributions to the Casimir energy are dominated by the lowest eigenvalue λ_{MIN}, and have the typical exponential suppression associated to massive fields, i.e.

$$E^{\text{TE}}(a) + E^{\text{TM}}(a) \approx -\kappa \sqrt{\frac{\lambda_{\text{MIN}}}{16\pi a}} e^{-2\lambda_{\text{MIN}}a}, \qquad (15)$$

where κ is the multiplicity of the eigenvalue.

From these results, we conclude that, at small distances, the total Casimir energy is dominated by the TE and TM contributions: it behaves as $1/a^3$ as for parallel plates, and it is proportional to the area of the pistons. For distances larger than a critical value $a > a_c$, the TEM is the leading contribution, and gives a long range Casimir energy that decays only as $1/a$. This contribution, typical of a massless scalar field in $1 + 1$ dimensions, is nonextensive, i.e. does not depend on the area of the pistons. The value of a_c depends of course on the particular form of the transverse section.

There are some additional properties which can be obtained using dimensional analysis. Let us denote by l_1 and l_2 the typical lengths associated to the sections of the internal and external cylinders of the cavity, respectively. On dimensional grounds we expect

$$E^{\text{TE}} + E^{\text{TM}} = \frac{1}{a} f\left(\frac{a}{l_2}, \frac{l_1}{l_2}\right). \qquad (16)$$

If the critical distance is defined by

$$f\left(\frac{a_c}{l_2}, \frac{l_1}{l_2}\right) = -\frac{\pi}{12}, \qquad (17)$$

then we have that

$$a_c = l_2 g(l_1/l_2). \qquad (18)$$

In the particular case $l_1 \ll a_c, l_2$, on physical grounds we expect the functions f and g defined above to have well defined limits:

$$E^{\text{TE}} + E^{\text{TM}} \approx \frac{1}{a} f\left(\frac{a}{l_2}, 0\right), \qquad a_c = l_2 g(0). \qquad (19)$$

Indeed, this limit can be achieved by inserting a thin wire inside a hollow cavity, so that the TE and TM modes of the hollow cavity are not disturbed, and so the TE and TM contributions to the zero-point energy are almost independent of the presence of the wire. In this situation the critical distance becomes a linear function of l_2. We will confirm this property in the particular example described in the next section.

The existence of the long range Casimir TEM force is of conceptual interest. One can wonder whether it is also relevant from an experimental perspective, i.e. if there is a chance of measuring this force in future experiments. There are two major limitations: on the one hand, being nonextensive in the area of the plates, the absolute value of the force is very small, and therefore it could only be measured at extremely short distances. Moreover, in this regime, the Casimir force would be dominated by the TE and TM contributions, unless the area of the plates is also small. We therefore address the question of whether the

TEM contribution to the force can be a significant fraction (α) of the total Casimir force in the PFA regime; i.e. we are interested in a configuration in which $F^{\text{TEM}} \geq \alpha(F^{\text{TE}} + F^{\text{TM}})$. For the sake of concreteness we study the case where the geometry of the plates is such that its area may be written as $b(l_2^2 - l_1^2)$, where b is some coefficient (this occurs, for instance, with a circle and any regular polygon). Within these assumptions, the conditions imposed to the forces become the following conditions in the parameters:

$$b \ll \frac{A}{a^2} \leq \frac{20}{\pi\alpha}, \quad (20)$$

where the first inequality follows from the validity of the PFA. This shows that, in principle, there is a region in the parameter space where the TEM force could be a significant fraction of the force, even in the proximity limit. However, we stress again that the smallness of the TEM force implies that its measurement is presently extremely difficult.

III. COAXIAL CYLINDRICAL CAVITY OF CIRCULAR SECTION

In this section we present and solve an explicit example of a nonsimply connected cavity. We compute the Casimir energy for a cavity formed by two concentric perfectly conducting circular cylinders closed by two (also nonsimply connected) plates in its extremes. The configuration is similar to that in Fig. 1 but with both inner and outer cylinders having circular sections of radii r_1 and r_2, respectively. We also study the Casimir force between the plates.

As discussed in the previous section, the vacuum energy of this nonsimply connected cavity will have contributions coming from the TEM modes, besides the usual TE and TM modes. The existence of TEM modes can be confirmed by obtaining explicitly the transverse vector potential $\mathbf{A}_\perp$ defined in Eq. (7), that for this particular geometry reads

$$\mathbf{A}_\perp = \frac{\hat{\mathbf{x}}_\perp}{|\mathbf{x}_\perp|}. \quad (21)$$

The TEM mode contribution to the energy is given by Eq. (11), and is independent of r_1 and r_2, as stated above.

On the other hand, the TE and TM mode contributions do not have such a simple expression and do depend on r_1 and r_2. To obtain this contribution, we may start from the already finite expression given in Eq. (13) and use Cauchy's theorem to convert the sum over λ_D's and λ_N's into a closed path integral in the complex plane of an appropriate function.

If $f(z)$ is a function with "1"-valued simple poles at $z = \lambda_{kD}$ and $z = \lambda_{kN}$ for all $\lambda_{kD,kN}$, then we may compute the sum in $\lambda = \lambda_{kD,kN}$ in Eq. (13) as a Cauchy integral,

$$\sum_\lambda \frac{\lambda K_1(2l\lambda a)}{l} = 2\pi i \int_\mathcal{C} z \frac{\lambda K_1(2lza)}{l} f(z), \quad (22)$$

if the contour $\mathcal{C}$ encloses all the poles of $f(z)$ in $z = \lambda_{kD,kN}$ and the function $K_1(2lza)$ is analytic in the interior of the curve. The explicit form of $f(z)$ comes out after observing that in the case of the cavity between two perfectly conducting concentric cylinders of radius r_1 and r_2, the Dirichlet and Neumann eigenfrequencies—which correspond to the TE and TM modes, respectively—are all the solutions of (see [16])

$$J_n(\lambda r_1)N_n(\lambda r_2) - J_n(\lambda r_2)N_n(\lambda r_1) = 0,$$
$$J_n'(\lambda r_1)N_n'(\lambda r_2) - J_n'(\lambda r_2)N_n'(\lambda r_1) = 0, \quad (23)$$

with n any integer number. (Notice that if λ is a solution, then $-\lambda$ is also a solution, but since both correspond to the same eigenfunction we may keep only the $\lambda > 0$ solutions to avoid double counting.) From here it is easy to see that

$$f(z) = \sum_n \frac{d}{dz} \ln[(J_n(zr_1)N_n(zr_2) - J_n(zr_2)N_n(zr_1))$$
$$\times (J_n'(zr_1)N_n'(zr_2) - J_n'(zr_2)N_n'(zr_1))] \quad (24)$$

satisfies the above requirements.

In order to choose the contour $\mathcal{C}$ we observe that $K_1(z)$ is singular at $z = 0$ but is analytic for $\text{Re}(z) > 0$, where it goes to zero as $z^{-1/2}e^{-\text{Re}(z)}$ for large $\text{Re}(z)$. Moreover, since the contour must enclose the real positive axis beginning in λ_{MIN} [the minimum of the solutions of Eq. (23) for all n] we choose the contour to be a *pizza slice* with its vertex at $z = \lambda_{\text{MIN}}/2$, angle $0 < \phi < \pi/2$, and centered in the real axis:

$$\mathcal{C} = \lim_{L\to\infty} \begin{cases} z = \frac{\lambda_{\text{MIN}}}{2} + \rho e^{-i\phi/2} & \rho \in (0, L), \\ z = \frac{\lambda_{\text{MIN}}}{2} + Le^{i\theta} & \theta \in (-\phi/2, +\phi/2), \\ z = \frac{\lambda_{\text{MIN}}}{2} + \rho e^{+i\phi/2} & \rho \in (L, 0). \end{cases}$$
$$(25)$$

Once $f(z)$ and $\mathcal{C}$ have been correctly chosen, we have an explicit expression for the TE + TM Casimir energy in terms of a double sum and a closed path integral:

$$E^{\text{TE}} + E^{\text{TM}} = -i \sum_{l=1,n=-\infty}^{\infty} \int_\mathcal{C} z \frac{\lambda K_1(2lza)}{l} f_n(z), \quad (26)$$

where $f_n(z)$ is each term in the sum in $f(z)$ [see Eq. (24)].

To compute explicitly Eq. (26) we need to truncate the l and n sum according to a given precision, and compute numerically the integral in the upper and lower segments of $\mathcal{C}$—which are essentially the same—since the contribution in the arc of radius L goes to zero. The criteria used to truncate the sum are best analyzed in Eq. (13), where the sum is exponentially damped by the Bessel function when its argument grows. In fact, we may divide in Eq. (13) the sum in λ as different sums for each n [see Eq. (23)], then we define $\lambda_{\text{min}}(n)$ as the minimum of the λ's for a given n, and then for each n we keep l's such that $2l\lambda_{\text{min}}(n)a < D$. Here D is chosen such that all the thrown away terms in the sum are damped by at least an e^{-D} factor. On the other hand, to truncate the sum in n, we set $l = 1$ and we choose

n such that $2\lambda_{\min}(n)a < D$. These criteria should give a precision of order e^{-D} to the final result in the sum. In our calculations we have taken $D = 8$ which is enough for our purposes.

It is worth noticing at this point that if we would have performed directly the sum in Eq. (13) over all relevant λ's instead of using the Cauchy integral approach, then we would have had to study the roots of Eq. (23) for each n and keep only those which satisfy $2l\lambda a < D$ for each l. Although more difficult, this would have also been a possible approach.

The TE + TM Casimir energy has been numerically computed using Eq. (26) for different cylinders radii (r_1 and r_2) and distance between the plates (a). We have also computed the Casimir force deriving Eq. (26) with respect to a and computing numerically the resulting expression:

$$F^{\text{TE}} + F^{\text{TM}} = -i \sum_{l=1,n=-\infty}^{\infty} \int_{\mathcal{C}} z^2 \lambda (K_0(2lza)$$
$$+ K_2(2lza))f_n(z). \quad (27)$$

As a check for the numerical TE + TM calculation we have corroborated that its behavior for small and large a corresponds to the expected proximity [Eq. (14)] and exponential [Eq. (15)] behaviors, respectively. In both cases we find, as expected, a convergence to unit in the ratio of the numerical energy and its expected asymptotic behavior. In Fig. 2 we show this convergence in the proximity limit, which is more complex from a numerical point of view, since it involves the summation of a large number of modes.

In order to explore the TE + TM to TEM transition, we have studied the ranges $r_2 = 1.1r_1$ to $r_2 = 40r_1$ and focused on the region of a's where the TE + TM energy (or force) equals the TEM energy (or force). As usual, the numerical calculation becomes harder when the surfaces
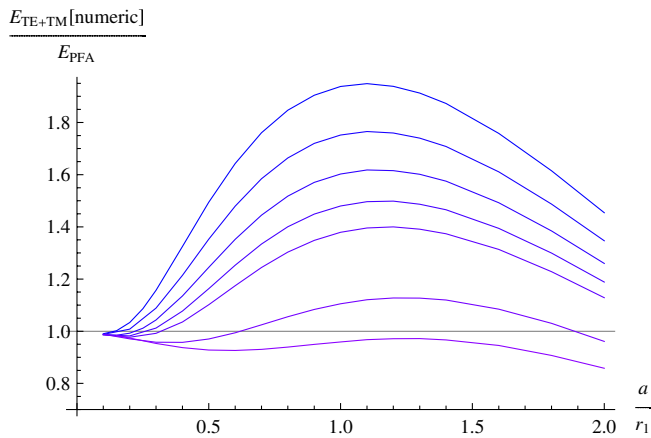
become closer, i.e. when r_2 approaches r_1. In Fig. 3 we have plotted the TE + TM Casimir energy as a function of a together with the TEM Casimir energy for three different r_2/r_1 configurations. As expected, the TE + TM energy dominates in the small-a region, but the TEM energy dominates when the distance between the plates is larger than the critical distance a_c. (Note that given r_1 and r_2 there are two different critical distances, one for the energy and another for the force. With no risk of confusion we use the notation a_c for both of them.)

We have plotted in Fig. 4 the Casimir energy and force critical distance a_c for several radii ratios r_2/r_1. As it can be seen in the plot, the dependence of a_c with r_2 becomes

FIG. 3 (color online). TEM (dotted line) and TE + TM (solid line) contributions to the Casimir energy for different r_2/r_1 configurations as a function of the distance a (in units of r_1) between the plates. From left to right the solid lines correspond to $r_2/r_1 = 1.2$, 2 and 4, and the corresponding critical distances are $a_c = 0.44r_1$, $0.68r_1$ and $1.41r_1$, respectively. Observe that the TEM contribution is independent of r_2/r_1.

FIG. 2 (color online). Ratio of the numerically computed TE + TM Casimir energy to the expected proximity behavior [Eq. (14)] in the small-a region. Our calculations reach $a/r_1 \approx 0.1$ which is enough for our purposes. The lines correspond, from lower to upper, to $r_2/r_1 = 2$, 1.8, 1.6, 1.55, 1.5, 1.45 and 1.4.
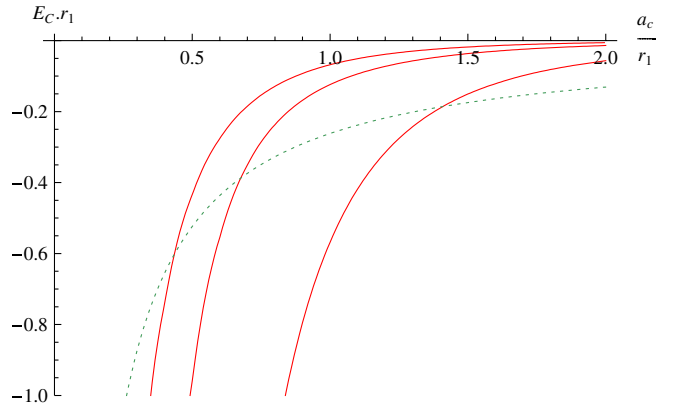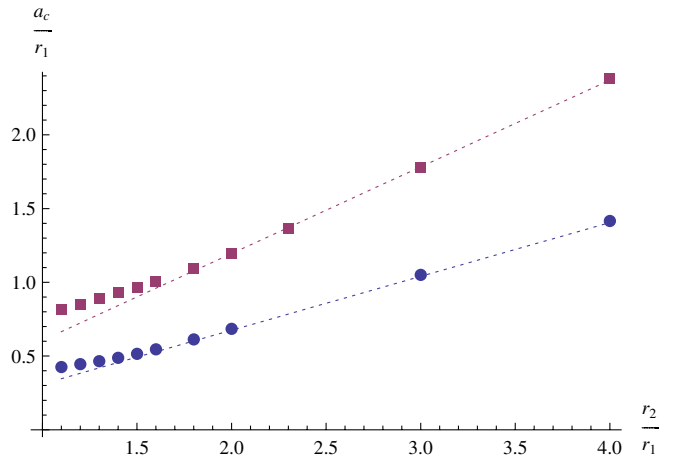

FIG. 4 (color online). Critical distance (in r_1 units) where the Casimir energy (lower plot) or force (upper) of the TE + TM modes equals that of the TEM modes for different r_2/r_1 configurations. As it can be seen, the dependence of a_c with r_2 becomes rapidly linear. The slopes of the asymptotes are 0.36 and 0.61.
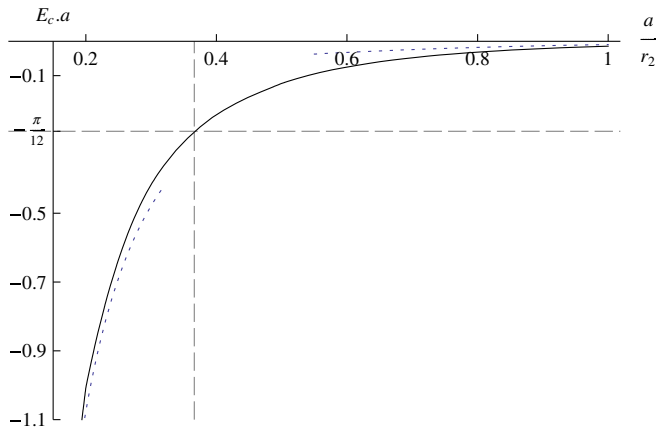
FIG. 5 (color online).   Casimir energy for two plates separated a distance a in a *hollow* circular cylinder of radius r_2. The expected PFA and exponential behavior for small and large a/r_2, respectively, is plotted in dotted lines. The dashed lines show that, as expected, this energy equals $E^{\mathrm{TEM}} = -\pi/12a$ at $a/r_2 \approx$ 0.36 (see text and Fig. 4).

rapidly linear. According to Eq. (18), this means that $a_c = r_2 g(r_1/r_2)$ approaches its limiting value $a_c \approx r_2 g(0)$ for $r_2 > 2r_1$. In this case we find $a_c \approx 0.36r_2$, for the critical distance associated to the energy.

As mentioned at the end of Sec. II, the linear relation between a_c and r_2 corresponds to the physical situation in which the TE + TM Casimir energy of the nonsimply connected cavity approaches that of a simply connected one of radius r_2. Therefore, one should be able to obtain the coefficient $g(0)$ by a comparison of the TEM Casimir energy with that of a hollow cylindrical cavity of radius r_2, which is given by Eq. (26) with

$$f_n(z) = \frac{d}{dz} \ln(J_n(zr_2) J'_n(zr_2)). \qquad (28)$$

We have checked that this is indeed the case. The Casimir energy for the hollow cavity, obtained again using Cauchy's theorem, is plotted in Fig. 5. As expected, this energy interpolates between the proximity result at short distances, and the exponential behavior at long distances. The number $g(0)$ is determined by the value of a/r_2 for which the energy of the hollow cavity equals $-\pi/12a$. In this way we obtain $g(0) \approx 0.36$, that coincides with the slope of the linear relation between the critical distance for the energy and r_2 presented in Fig. 4.

IV. CONCLUSIONS

In this paper we have described a geometry in which the existence of TEM modes induce a long range Casimir interaction. In particular, we have shown that the electromagnetic Casimir force between two parallel plates inside a nonsimply connected cylinder is essentially given by the sum of the Casimir forces for $1 + 1$ scalar fields with

different masses. For TE and TM modes, the masses are given by the eigenvalues of the Laplacian on a $z = \mathrm{const}$ section of the cylinder, with the appropriate boundary conditions, and are nonvanishing. The opposite happens to TEM modes, whose zero-point energy corresponds to a massless field, and this is the reason for the different qualitative behavior of their contribution to the force. On the one hand, the TEM force scales as $1/a^2$ at all distances and does not depend on the area of the plates. On the other hand, in the short distance limit TE and TM forces reproduce the parallel plates result proportional to A/a^4, and in the long distance limit they are exponentially suppressed due to the finite size of the plates (or, in the equivalent picture, to the nonvanishing effective masses). As a consequence, TE and TM modes dominate at short distances, while TEM modes do it at long distances. The critical distance where both contributions are balanced depends of course on the form of the section of the cylinder, and decreases with its area, as we explicitly showed in the particular example of a coaxial circular cylindrical cavity. The summation over the effective masses to compute the TE and TM contributions of the force for this particular case was performed using Cauchy's theorem, starting from the renormalized Casimir energy for a single massive field in $1 + 1$ dimensions.

Throughout the paper we considered the Casimir energy for perfect conductors at zero temperature. It would be interesting to generalize these results to take into account the combined effects of finite conductivity at nonzero temperature. In this context, it is worth remarking that the dominance of the TEM modes over the TE and TM contributions to the Casimir force is also valid at a nonzero temperature T, at least for perfect conductors. In fact, it has been shown [7] that at sufficiently long distances ($\lambda_{\mathrm{MIN}} a \gg 1$ and $aT \gg 1$), the Casimir force for TE and TM modes is proportional to $\lambda_{\mathrm{MIN}} T e^{-2\lambda_{\mathrm{MIN}} a}$, i.e. is exponentially suppressed. On the other hand, in the same situation the TEM force is proportional to T/a [17].

While the existence of the long range TEM Casimir force is of conceptual interest, it would be very difficult to measure it. Indeed, as discussed at the end of Sec. II, this contribution to the force does not depend on the area of the plates, its absolute value is extremely small, and therefore it would be measurable only at very short distances with the present technology. However, in this regime the TE and TM forces would be much larger than the TEM force, unless the area of the pistons is sufficiently small. Therefore, the measurement of the TEM Casimir force should involve thin rings at short distances, a rather difficult experiment, indeed.

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[15] This energy is usually computed for periodic boundary conditions in $1 + 1$ dimensions. Equation (12) contains an additional $1/2$ factor due to the fact that we are considering Dirichlet boundary conditions, and therefore the eigenfrequencies in Eq. (8) involve only positive integers.

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