POINTED FINITE TENSOR CATEGORIES OVER ABELIAN GROUPS

IVÁN ANGIONO AND CÉSAR GALINDO

Abstract. We characterize the finite pointed tensor categories equivalent to de-equivariantizations of finite dimensional pointed Hopf algebras over abelian groups.

1. Introduction

In this paper $\Bbbk$ will denote an algebraically closed field of characteristic zero. By tensor category we mean a $\Bbbk$-linear abelian category with finite dimensional Hom spaces and objects of finite length, endowed with a rigid $\Bbbk$-bilinear monoidal structure and such the unit object is simple. A tensor category is called finite if it is $\Bbbk$-linearly equivalent to the category of finite dimensional comodules over a finite dimensional $\Bbbk$-coalgebra.

Let H be a coquasi-Hopf algebra over $\Bbbk$. The category ${}^H\mathcal{M}$ of its finite dimensional corepresentations is a tensor category. Tensor categories of this form are characterized, via tannakian reconstruction arguments, as those possessing a quasi-fiber functor with values in the category of finite dimensional vector spaces over $\Bbbk$.

Comment 1 (by Cesar): el siguiente parrafo lo copié de la subsección 3.3, debemos reescribir esa parte, me parece que queda bien acá para explicar un poco más.

A tensor category is called pointed if every simple object is invertible. Example of such a categories are the category of finite dimensional comodules over a pointed coquasi-Hopf algebra. In fact, any finite pointed tensor category is equivalent to the category of comodules over a finite dimensional pointed coquasi-Hopf algebra.

In [3], we studied de-equivariantization of Hopf algebras, applying Tannakian techniques. We explicitly constructed a coquasi-bialgebra such that its tensor category of comodules realizes the de-equivariantization of a Hopf algebra, [3, Theorem 2.8]. As application, we defined a big family of pointed coquasi-Hopf algebras $A(H, G, \Phi)$ attached to a coradically graded pointed Hopf algebra H and some extra group-theoretical data, see [3, Proposition 3.3] and [3, Definition 3.5].

2010 *Mathematics Subject Classification.* 16W30, 18D10, 19D23.
I. Angiono was partially supported by CONICET, FONCyT-ANPCyT and Secyt (UNC), and C. Galindo by FondodeInvestigacionesdela Facultad de Ciencias y la Vicerrectoría de Investigaciones de la Universidad delos Andes.

The purpose of work is to characterize **Comment 2 (by Cesar)**: se te ocurre algo mejor? pointed finite tensor categories over abelian group constructed as de-equivariantization of the tensor category of comodules over finite dimensional pointed Hopf algebras. For Hopf algebras, the de-equivariantization process strictly generalize the theory of central extensions of Hopf algebra. However, now the central quotient is a coquasi-Hopf algebra.

We said that a tensor category $\mathcal{C}$ is coradically graded if $\mathcal{C}$ is equivalent to the category of comodules over a coradically graded coalgebra, see [11, Section 1.13] for a more categorical definition.

In [4] was proved that every finite-dimensional pointed Hopf algebra H with abelian group of group-like elements Γ is a cocycle deformation of $\mathcal{B}(V)\#\Bbbk\Gamma$, where $V \in^{\Bbbk\Gamma}_{\Bbbk\Gamma} \mathcal{YD}$ denotes the infinitesimal braiding of H. In particular, it implies that ${}^{H}\mathcal{M}$ and ${}^{\mathcal{B}(V)\#\Bbbk\Gamma}\mathcal{M}$ are tensor equivalent. Hence, the pointed tensor categoies obtained from H or ${}^{\mathcal{B}(V)\#\Bbbk\Gamma}\mathcal{M}$ are the same. That is the reason why are interested only in coradically graded coquasi-Hopf algebras.

For a tensor category $\mathcal{C}$ we will denote by $G(\mathcal{C})$ the group of isomorphism classes of invertible objects and by $\omega(\mathcal{C}) \in H^3(G(\mathcal{C}), \Bbbk^\times)$ the cohomology class defined by the associator of the full tensor subcategory of $\mathcal{C}$ of invertible objects.

Breen [7, Proposition 4.1] defined for every abelian group Λ a group homomorphism

$$\psi_\Lambda : H^3(\Lambda, \Bbbk^\times) \to \mathrm{Hom}(\wedge^3\Lambda, \Bbbk^\times),$$

that measure if the category of Yetter-Drinfeld modules ${}^{\Bbbk^\omega\Gamma}_{\Bbbk^\omega\Gamma}\mathcal{YD}$ is pointed, see Theorem 3.7.

Our main result can be summarized as:

Theorem 1.1. *A finite tensor category $\mathcal{C}$ is tensor equivalent to a de-equivariantization of a pointed Hopf algebra over an abelian group if and only if $\mathcal{C}$ is coradically graded, $G(\mathcal{C})$ is abelian and $\psi_{G(\mathcal{C})}(\omega_{\mathcal{C}}) \equiv 1$.*

Theorem 1.1 is proved in Section 4, where a pointed Hopf algebra of the form $\mathcal{B}(V)\#\Bbbk\Gamma$ is explicitly constructed. As a corollary, we obtain that every coradically graded pointed finite braided tensor category is tensor equivalent to a de-equivariantization of a coradically graded pointed Hopf algebra over an abelian group.

The organization of the paper is as follows. Section 2 is devoted to preliminaries. In Section 3 we define the map $\psi_\Lambda : H^3(\Lambda, \Bbbk^\times) \to \mathrm{Hom}(\wedge^3\Lambda, \Bbbk^\times)$ and characterizations of the condition $\psi_\Lambda(\omega) = 1$, which are used in the sequel. In Section 4 we proved generation in degree one for coradically graded coquasi-Hopf algebras A with associator, $\psi_{G(A)}(\omega) = 0$, where $\omega \in H^3(G(A), \Bbbk^\times)$ defined by the associator. We prove Theorem 1.1. We finish the section with an example of a coradically graded coquasi-Hopf algebra over $\Lambda = \mathbb{Z}/2\mathbb{Z}^{\oplus 3}$ with associator $\omega \in Z^3(\Lambda, \Bbbk^\times)$, such that $\psi_\Lambda(\omega) \neq 1$.

Comment 3 (by Cesar): Agregar en paper?: el diagrama de dinking esta bien definido

Comment 4 (by Cesar): Todo: 1) revisar notacion de unidad en cat mon. 2) agregar definition de product tensorial en ejem 2.3 (yetter-drinfeld)

2. Preliminaries

In this section we recall some definitions and results about coquasi-Hopf algebras and tensor categories. Throughout the paper we work over an algebraically closed arbitrary field of characteristic zero $\Bbbk$. Algebras and coalgebras are always defined over $\Bbbk$. For coalgebra (C, Δ, ε) we will use Sweedler's notation omitting the sum symbol, that is $\Delta(c) = c_1 \otimes c_2$ for all $c \in C$.

Given a group Γ, $\widehat{\Gamma}$ denotes the group of linear characters of Γ over $\Bbbk$, and $\langle \cdot, \cdot \rangle : \widehat{\Gamma} \times \Gamma \to \Bbbk^\times$ is the evaluation map.

Given $\theta \in \mathbb{N}_0$, then we denote $\mathbb{I}_\theta = \{n \in \mathbb{N} : n \leq \theta\}$, or simply $\mathbb{I}$ if θ is clear from the context. Also, $\delta_{x,y}$ is the Kronecker delta.

2.1. Coquasi-bialgebras.

A *coquasi-bialgebra* $(H, m, u, \omega, \Delta, \varepsilon)$ is a coalgebra (H, Δ, ε) together with coalgebra morphisms:

- the multiplication $m : H \otimes H \longrightarrow H$ (denoted $m(g \otimes h) = gh$),
- the unit $u : \Bbbk \longrightarrow H$ (where we call $u(1) = 1_H$),

and a convolution invertible element $\Omega \in (H \otimes H \otimes H)^*$ such that

$$h_1(g_1k_1)\Omega(h_2, g_2, k_2) = \Omega(h_1, g_1, k_1)(h_2g_2)k_2, \tag{1}$$

$$1_H h = h1_H = h, \tag{2}$$

$$\Omega(h_1g_1, k_1, l_1)\Omega(h_2, g_2, k_2l_2) = \Omega(h_1, g_1, k_1) \times\Omega(h_2, g_2k_2, l_1)\Omega(g_3, k_3, l_2), \tag{3}$$

$$\Omega(h, 1_H, g) = \varepsilon(h)\varepsilon(g), \tag{4}$$

for all $h, g, k, l \in H$. Note that

$$\Omega(1_H, h, g) = \Omega(h, g, 1_H) = \varepsilon(h)\varepsilon(g) \qquad \text{for all } g, h \in H.$$

A coquasi-bialgebra H is a *coquasi-Hopf algebra* if there is a coalgebra map $\mathcal{S} : H \longrightarrow H^{\mathrm{op}}$ (the *antipode*) and elements α, $\beta \in H^*$ such that

$$\alpha(h)1_H = \mathcal{S}(h_1)\alpha(h_2)h_3, \tag{5}$$

$$\beta(h)1_H = h_1\beta(h_2)\mathcal{S}(h_3), \tag{6}$$

$$\begin{aligned}\varepsilon(h) &= \omega(h_1\beta(h_2), \mathcal{S}(h_3), \alpha(h_4)h_5) \\ &= \omega^{-1}(\mathcal{S}(h_1), \alpha(h_2)h_3\beta(h_4), \mathcal{S}(h_5)), \qquad \text{for all } h \in H.\end{aligned} \tag{7}$$

Example 2.1. Let G be a discrete group. Recall that a (normalized) 3-cocycle $\omega \in Z^3(G, \Bbbk^\times)$ is a map $\omega : G \times G \to G \to \Bbbk^\times$ such that

$$\omega(ab, c, d)\omega(a, b, cd) = \omega(a, b, c)\omega(a, bc, d)\omega(b, c, d), \qquad \omega(a, 1, b) = 1$$

for all $a, b, c, d \in G$.

Given $\omega \in Z^3(G, \Bbbk^\times)$, we define coquasi-Hopf algebra $\Bbbk^\omega G$, with structure $(\Bbbk[G], \Omega_\omega, S, \alpha, \beta)$, where $\Bbbk[G]$ is the group algebra with the usual comultiplication $\Delta(g) = g \otimes g$ for all $g \in G$, and $\Omega_\omega(g, h, k) = \omega(g, h, k)$ for all $g, h, k \in G$. The antipode structure is given by

$$S(g) = g^{-1}, \qquad \alpha(g) = 1, \qquad \beta(g) = \omega(g, g^{-1}, g)^{-1}, \qquad \text{for all } g \in G.$$

2.2. **Braided tensor categories and the center construction.** By a tensor category we mean a k-linear abelian rigid tensor category $\mathcal{C}$ whose unit object $\mathbf{1}$ is simple, see [11].

Let H be a coquasi-Hopf algebra. The category of left H-comodules ${}^H\mathcal{M}$ is rigid and monoidal, where the tensor product is $\otimes_\Bbbk$, the comodule structure of the tensor product is the codiagonal one and the associator is

$$\begin{aligned} \phi_{U,V,W} \quad &: \quad (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W) \\ \phi_{U,V,W}((u \otimes v) \otimes w) \quad &= \quad \Omega(u_{-1}, v_{-1}, w_{-1}) u_0 \otimes (v_0 \otimes w_0) \end{aligned}$$

for $u \in U$, $v \in V$, $w \in W$ and $U, V, W \in {}^H\mathcal{M}$. The dual coactions are given by $\mathcal{S}$ and $\mathcal{S}^{-1}$, as in the case of Hopf algebras.

Example 2.2. Let G be a discrete group and $\omega \in Z^3(G, \Bbbk^\times)$. The tensor category ${}^{\Bbbk^\omega[G]}\mathcal{M}$ is Vec_G^ω, the category of G-graded vector spaces with associator induced by ω.

A braided tensor category is a tensor category $\mathcal{C}$ endowed with a braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$, see [15].

The main example of a braided tensor category in this paper will be the center $\mathcal{Z}(\mathcal{C})$ of a tensor category $\mathcal{C}$. The center construction produces a braided tensor category $\mathcal{Z}(\mathcal{C})$ from any tensor category $\mathcal{C}$. The objects of $\mathcal{Z}(\mathcal{C})$ are pairs $(Z, c_{-,Z})$, where $Z \in \mathcal{C}$ and $c_{X,Z} : X \otimes Z \to Z \otimes X$ are isomorphisms natural in X satisfying

$$c_{X \otimes Y, Z} = (c_{XZ} \otimes \mathrm{id}_Y)(\mathrm{id}_X \otimes c_{Y,Z}) \tag{8}$$

and $c_{I,Z} = \mathrm{id}_Z$, for all $X, Y \in \mathcal{C}$. The braided monoidal structure is given in the following way:

- the tensor product is $(Y, c_{-,Y}) \otimes (Z, c_{-,Z}) = (Y \otimes Z, c_{-,Y \otimes Z})$, where

$$c_{X,Y \otimes Z} = (\mathrm{id}_Y \otimes c_{X,Z})(c_{X,Y} \otimes \mathrm{id}_Z) : X \otimes Y \otimes Z \to Y \otimes Z \otimes X, \quad X \in \mathcal{C},$$

- the identity element is $(I, c_{-,I})$, $c_{Z,I} = \mathrm{id}_Z$
- the braiding is the morphism $c_{X,Y}$.

Example 2.3. *The Drinfeld center of* $\mathrm{Vec}_\Lambda^\omega$. Let Λ be a discrete group, and $\omega \in Z^3(\Lambda, \Bbbk^\times)$. The Drinfeld center of $\mathrm{Vec}_\Lambda^\omega$ is equivalent to ${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$, the category of Yetter-Drinfeld modules over $\Bbbk^\omega\Lambda$. The objects of ${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$ are Λ-graded vector spaces $V = \bigoplus_{g \in \Lambda} V_g$ with a linear map $\triangleright : \Bbbk^\omega\Lambda \otimes V \to V$ such that $1 \triangleright v = v$ for all $v \in V$,

$$(gh) \triangleright v = \frac{\omega(g, hkh^{-1}, h)}{\omega(g, h, k)\omega(ghkh^{-1}g^{-1}, g, h)} (g \triangleright (h \triangleright v)), \quad g, h, k \in \Lambda, \quad v \in V_k,$$

satisfying the following compatibility condition:

$$g \triangleright V_h \subseteq V_{ghg^{-1}} \qquad\qquad g, h \in \Lambda.$$

Morphisms in ${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$ are Λ-linear Λ-homogeneous maps. The category is tensor braided, with braiding $c_{V,W} : V \otimes W \to W \otimes V$, $V, W \in {}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$,

$$c_{V,W}(v \otimes w) = g \triangleright w \otimes v, \qquad g \in \Lambda, \qquad v \in V_g, \qquad w \in W.$$

2.3. **Bosonization for coquasi-Hopf algebras.** Now we recall the notation and results from [5] but restricted to pointed coquasi-Hopf algebras.

Given a Hopf algebra R in ${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$ with multiplication $\cdot : R \otimes R \to R$ and comultiplication $\mathit{\Delta} : R \to R \otimes R$, $\mathit{\Delta}(r) = r^{(1)} \otimes r^{(2)}$, the *bosonization* of R by $\Bbbk^\omega\Lambda$ [5, Definition 5.4] is the coquasi-Hopf algebra $R\#\Bbbk^\omega\Lambda$ with underlying vector space $R \otimes \Bbbk\Lambda$ and the following structure maps:

$$\begin{aligned}
(r\#g)(s\#h) &= \frac{\omega(g,l,h)\omega(k,l,gh)}{\omega(k,g,lh)\omega(l,g,h)} r \cdot (g \triangleright s)\#gh, \\
\Delta(r\#g) &= \frac{1}{\omega(kj^{-1},j,g)} r^{(1)}\#lg \ \otimes \ r^{(2)}\#g, \\
\Omega(r\#g, s\#h, t\#k) &= \varepsilon(r)\varepsilon(s)\varepsilon(t)\omega(g,h,k),
\end{aligned}$$

for all $g, h, k, l \in \Lambda$, $r \in R_k$, $s \in R_l$, $t \in R$, where $r^{(1)} \otimes r^{(2)} \in \oplus_j R_{kj^{-1}} \otimes R_j$.

We have two canonical coquasi-Hopf algebra maps

$$\pi : R\#\Bbbk^\omega\Lambda \to \Bbbk^\omega\Lambda,\ \pi(r\#g) = \varepsilon(r)g, \quad \iota : \Bbbk^\omega \to R\#\Bbbk^\omega\Lambda,\ \iota(g) = 1\#g,$$

such that $\pi \circ \iota = \mathrm{id}_{\Bbbk^\omega\Lambda}$.

Reciprocally, let H be a coquasi-Hopf algebra and assume that there exist coquasi-Hopf algebra maps $\pi : H \to \Bbbk^\omega\Lambda$, $\iota : \Bbbk^\omega\Lambda \to H$ such that $\pi \circ \iota = \mathrm{id}_{\Bbbk^\omega\Lambda}$. Then $H \simeq R\#\Bbbk^\omega\Lambda$, where $R = H^{\mathrm{co}\,\pi}$ admits a structure of Hopf algebra in ${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$ [5, Theorem 5.8].

In particular this applies for $H = \oplus_{n\geq 0} H_n$ coradically graded such that $H_0 = \Bbbk^\omega\Lambda$ [5, 6.1]. Here, R is a graded Hopf algebra in ${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$:

$$R = \oplus_{n\geq 0} R_n, \qquad \text{with } R_n = R \cap H_n,\ n \geq 0, \qquad \text{so } R_0 = \Bbbk 1.$$

2.4. **Nichols algebras.** Nichols algebras can be defined over any abelian braided tensor category see [19]. In particular we may consider Nichols algebra over $\mathcal{C} = \mathcal{Z}({}^H\mathcal{M})$ or $\mathcal{C} = {}^H_H\mathcal{YD}$, where H is a coquasi-bialgebra, see [1] for the definition when H is a Hopf algebra and [14] for $H = \Bbbk^\omega\Lambda$.

Given an object $V \in \mathcal{C}$ and $n \geq 3$, $V^{\otimes n}$ denotes $(\cdots((V \otimes V) \otimes \cdots) \otimes V)$, n copies of V. We consider the following (graded) Hopf algebras in $\mathcal{C}$:

- the tensor algebra $T(V) = \oplus_{n\geq 0} V^{\otimes n}$, with product given by the canonical isomorphism $V^{\otimes m} \otimes V^{\otimes n} \simeq V^{\otimes(m+n)}$; the coproduct $\Delta : T(V) \to T(V) \otimes T(V)$ is the unique graded algebra map such that $\Delta_{0,1} : V \to \Bbbk \otimes V$ and $\Delta_{1,0} : V \to V \otimes \Bbbk$ are the canonical isomorphisms.

- the tensor coalgebra $C(V) = \oplus_{n\geq 0} V^{\otimes n}$, with coproduct

  $$\Delta = \oplus_{m,n\geq 0} : C(V) \to C(V) \otimes C(V), \quad \Delta_{m,n} : V^{\otimes(m+n)} \xrightarrow{\simeq} V^{\otimes m} \otimes V^{\otimes n};$$

  the product $\Delta : T(V) \to T(V) \otimes T(V)$ is the unique graded coalgebra map induced by the canonical isomorphisms $\Bbbk \otimes V \simeq V \simeq V \otimes \Bbbk$.

There exists a unique graded Hopf algebra map $T(V) \to C(V)$ in $\mathcal{C}$, which is the identity on V. The *Nichols algebra* $\mathcal{B}(V)$ of V is the image of this map: it is a graded Hopf algebra in $\mathcal{C}$.

We may identify $\mathcal{B}(V)$ as a quotient $\mathcal{B}(V) = T(V)/\mathcal{J}(V)$ with the following universal property: $\mathcal{J}(V)$ is the largest coideal of $T(V)$ spanned by elements of $\mathbb{N}$-degree ≥ 2. There are other characterizations of $\mathcal{B}(V)$ [19].

A pre-Nichols algebra of V is any graded braided Hopf algebra in $V \in \mathcal{Z}({}^H\mathcal{M})$ intermediate between $T(V)$ and $\mathcal{B}(V)$, that is any braided Hopf algebra of the form $T(V)/I$ where $I \subseteq \mathcal{J}(V)$ is a homogeneous Hopf ideal.

Comment 5 (by Ivan): Completo mas tarde con post Nichols y referencia

3. Trivializations of elements in $H^3(\Lambda, \Bbbk^\times)$

Definition 3.1. Let $\omega \in H^n(\Lambda, \Bbbk^\times)$. We say that ω is *trivializable* if there exist a finite abelian group Γ and a group epimorphism $p : \Gamma \to \Lambda$ such that the pullback $p^*\omega \in H^n(\Gamma, \Bbbk^\times)$ is trivial. In this case we say that ω is *p-trivial*.

Example 3.2. Let C_n be the cyclic group of order n generated by σ. Then

$$\cdots \xrightarrow{N} \mathbb{Z}C_n \xrightarrow{\sigma-1} \mathbb{Z}C_n \xrightarrow{N} \mathbb{Z}C_n \xrightarrow{\sigma-1} \mathbb{Z}C_n \longrightarrow \mathbb{Z}$$

where $N = 1 + \sigma + \sigma^2 + \cdots + \sigma^{n-1}$ is a free resolution of $\mathbb{Z}$. Thus,

$$H^3(C_n, \Bbbk^\times) = \mathbb{G}_m(n) := \{a \in \Bbbk^\times : a^n = 1\}.$$

Let $m, n \in \mathbb{N}$ such that $n|m$ and $\pi : C_m \to C_n$ be the canonical group epimorphism. The induced map is

$$\pi^* : H^3(C_n, \Bbbk^\times) \to H^3(C_m, \Bbbk^\times), \qquad q \mapsto q^{\frac{m}{n}}.$$

Hence, if $q \in H^3(C_n, \Bbbk^\times)$ has order s, the canonical epimorphism $\pi : C_{sn} \to C_n$ trivializes q. Thus $\pi : C_{n^2} \to C_n$ trivializes all elements in $H^3(C_n, \Bbbk^\times)$.

Let Λ be a finite abelian group. We denote by $\wedge^n\Lambda$ the n-th exterior power of Λ, viewed as a $\mathbb{Z}$-module.

For each $\omega \in Z^3(\Lambda, \Bbbk^\times)$, Breen [7, Proposition 4.1] defined an alternating trilinear map

$$\psi_\Lambda(\omega)(l_1, l_2, l_3) = \prod_{\sigma \in \mathbb{S}_3} \omega(l_{\sigma(1)}, l_{\sigma(2)}, l_{\sigma(3)})^{\operatorname{sng}(\sigma)}, \qquad l_1, l_2, l_3 \in \Lambda.$$

The group homomorphism $\psi_\Lambda : Z^3(\Lambda, \Bbbk^\times) \to \operatorname{Hom}(\wedge^3\Lambda, \Bbbk^\times)$ induces a group homomorphism

$$\psi_\Lambda : H^3(\Lambda, \Bbbk^\times) \to \operatorname{Hom}(\wedge^3\Lambda, \Bbbk^\times).$$

Note that $\mathrm{Hom}(\Lambda^{\otimes 3}, \Bbbk^\times) \subset Z^3(\Lambda, \Bbbk^\times)$. Hence, if Λ is finite the restriction of ψ_Λ to $\mathrm{Hom}(\Lambda^{\otimes 3}, \Bbbk^\times)$ is surjective. Thus ψ_Λ is surjective.

Proposition 3.3. *If $\omega \in H^3(\Lambda, \Bbbk^\times)$ is trivializable then $\psi_\Lambda(\omega) = 0$.*

Proof. Let $p : \Gamma \to \Lambda$ be an epimorphism of finite abelian groups. By [6, §7.2, Proposition 3], the map

$$\wedge^n(p) : \wedge^n\Gamma \to \wedge^n\Lambda, \qquad g_1 \wedge \cdots \wedge g_n \mapsto p(g_1) \wedge \cdots \wedge p(g_n),$$

is surjective. Since Λ is finite, the group homomorphism

$$\begin{aligned}\wedge^n(p)^* : \mathrm{Hom}(\wedge^n\Lambda, \Bbbk^\times) &\to \mathrm{Hom}(\wedge^n\Gamma, \Bbbk^\times)\\ f &\mapsto [g_1 \wedge \cdots \wedge g_n \mapsto f(p(g_1) \wedge \cdots \wedge p(g_n))].\end{aligned}$$

is injective for all n.

Let $\omega \in H^3(\Lambda, \Bbbk^\times)$ such that $p^*(\omega) = 0$. Then $\wedge^{(3)}(p)^* \circ \psi_\Lambda(\omega) = 0$, since the diagram

$$\begin{array}{ccc} H^3(\Lambda, \Bbbk^\times) & \xrightarrow{p^*} & H^3(\Gamma, \Bbbk^\times) \\ \big\downarrow{\psi_\Lambda} & & \big\downarrow{\psi_\Gamma} \\ \mathrm{Hom}(\wedge^3\Lambda, \Bbbk^\times) & \xrightarrow{\wedge^{(3)}(p)^*} & \mathrm{Hom}(\wedge^3\Gamma, \Bbbk^\times) \end{array} \tag{9}$$

is commutative. By the injectivity of $\wedge^3(p)^*$, we have that $\psi_\Lambda(\omega) = 0$. □

Example 3.4. Let $\Lambda = (\mathbb{Z}/n\mathbb{Z})^{\oplus 3}$ and $\omega \in Z^3(\Lambda, \Bbbk^\times)$, defined by

$$\omega(\vec{x}, \vec{y}, \vec{z}) = \zeta^{x_1 y_2 z_3},$$

where ζ is a n-th root of unity and $\vec{x}, \vec{y}, \vec{z} \in \Lambda$. Then,

$$\psi_\Lambda(\omega)(\vec{x}, \vec{y}, \vec{z}) = \zeta^{\det([\vec{x},\vec{y},\vec{z}])}.$$

Thus, $\psi(\omega) \neq 0$ and $\langle\psi(\omega)\rangle = \mathrm{Hom}(\wedge^3\Lambda, \Bbbk^\times)$. It follows by Proposition 3.3 that ω is not trivializable.

Let $\omega \in Z^3(\Gamma, \Bbbk^\times)$. An abelian structure on ω is a map $c : \Gamma \times \Gamma \to \Bbbk^\times$ such that

$$\frac{c(a, bc)}{c(a, b)c(a, c)} = \frac{\omega(a, b, c)\omega(b, c, a)}{\omega(b, a, c)}$$
$$\frac{c(ab, c)}{c(a, c)c(b, c)} = \frac{\omega(a, c, b)}{\omega(a, b, c)\omega(c, a, b)},$$

for all $a, b, c \in \Gamma$. Following [9, 10] we denote by $Z^3_{ab}(\Gamma, \Bbbk^\times)$ the abelian group of all abelian 3-cocycles (ω, c).

Proposition 3.5. *Let $(\omega, c) \in Z^3_{ab}(\Lambda, \Bbbk^\times)$ be an abelian 3-cocycle. Then $\psi_\Lambda(\omega) = 0$.*

Proof. If $(\omega, c) \in Z^3_{ab}(\Lambda, \Bbbk^\times)$ the map

$$q : \Lambda \to \Bbbk^\times, \qquad g \mapsto c(g, g)$$

is a quadratic form on Λ, that is, $q(-a) = q(a)$ and the map

$$b_q(a, b) = \frac{q(ab)}{q(a)q(b)}, \qquad a, b \in \Lambda,$$

is a bicharacter. The quadratic form q determines completely the cohomology class of the pair (w, c), see [10, Theorem 26.1]. Using the map q, Quinn [18] defined an explicit 3-cocycle abelian 3-cocycle (h, c) with $c(a, a,) = q(a)$ for all $a \in \Lambda$. Assume that $\Lambda = \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_m\mathbb{Z}$. For each $i \in \{1, \dots, m\}$ let $q_i := q(\vec{e}_i)$ and $h_i \in Z^3(\mathbb{Z}/n_i\mathbb{Z}, \Bbbk^\times)$ defined by

$$h_i(a, b, c) = \begin{cases} 1, & \text{if } b + c < n_i, \\ q_i^{n_i a}. & \text{if } b + c \geq n_i, \end{cases}$$

where $0 \leq a, b, c < n_i$. Then by [18] and [10, Theorem 26.1], $h \in Z^3(\Lambda, \Bbbk^\times)$ given by

$$h(\vec{x}, \vec{y}, \vec{z}) = h(x_1, y_1, z_1)h(x_2, y_2, z_2) \cdots h(x_m, y_m, z_m),$$

is a 3-cocycle cohomologous to ω. By Example 3.2, the epimorphism

$$\pi : \mathbb{Z}/n_1^2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_m^2\mathbb{Z} \to \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_m\mathbb{Z},$$

trivializes h and then also trivializes ω. □

Remark 3.6. Let Λ be a cyclic group of odd order and $\omega \in H^3(\Lambda, \Bbbk^\times)$ a non-zero element. Then there is not $c \in C^2(\Lambda, \Bbbk^\times)$ such that $(\omega, c) \in Z^3_{ab}(\Lambda, \Bbbk^\times)$, however by Example 3.2 $\psi_\Lambda(\omega) = 0$.

Theorem 3.7. *Let $\omega \in H^3(\Lambda, \Bbbk^\times)$. The following statements are equivalent:*
(a) $\psi_\Lambda(\omega) \equiv 1$.
(b) *The braided fusion category ${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$ is pointed.*
(c) ω *is trivializable.*

Proof. For each $a \in \Lambda$, the map

$$\beta_a : \Lambda \times \Lambda \to \Bbbk^\times, \qquad \beta_a(g, h) = \frac{\omega(g, a, h)}{\omega(g, h, a)\omega(a, g, h)}$$

is a 2-cocycle, that is, satisfies the equation

$$\beta_a(g, h)\beta_a(gh, l) = \beta_a(g, hl)\beta_a(h, l),$$

for all $g, h, l \in \Lambda$. By [13, Example 6.3] we have an exact sequence of groups

$$0 \to \widehat{\Lambda} \to \operatorname{Inv}({}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}) \to G_\omega \to 0,$$

where $G_\omega = \{a \in \Lambda : 0 = [\beta_a] \in H^2(\Lambda, \Bbbk^\times)\}$. Then ${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$ is pointed if and only if $0 = [\beta_a]$ for all $a \in \Lambda$.

Since $\Bbbk^\times$ is divisible, β_a has trivial cohomology class if and only if β_a is symmetric. In conclusion, ${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$ is pointed if and only if $\beta_a(g,h) = \beta_a(h,g)$ for all $a, g, h \in \Lambda$. Since

$$\frac{\beta_a(b,c)}{\beta_a(c,b)} = \psi_\Lambda(a,b,c),$$

${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$ is pointed if and only if $\psi_\Lambda(\omega) = 1$. Hence, (a) $\iff$ (b).

Now (c) $\implies$ (a) by Proposition 3.3. Assume that (b) holds. Then there is a finite abelian group Γ and an abelian 3-cocycle $(\alpha, c) \in Z^3_{ab}(\Gamma, \Bbbk^\times)$ such that ${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD} \cong \operatorname{Vec}_\Gamma^{(\alpha,c)}$ as braided fusion categories. The forgetful functor ${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD} \to \operatorname{Vec}_\Lambda^\omega$ defines a group epimorphism $\pi_1 : \Gamma \to \Lambda$ such that $\pi_1^*([\omega]) = [\alpha]$. By Proposition 3.5, there exists an abelian group Γ_2 and an epimorphism $\pi_2 : \Gamma_2 \to \Gamma_1$ such that $\pi_1^*([\alpha]) = 0$, hence $\pi_2 \circ \pi_1 : \Gamma_2 \to \Lambda$ trivializes ω. □

4. Pointed coradically graded coquasi-Hopf algebras

Let Γ and Λ be abelian groups and $p : \Gamma \to \Lambda$ a group epimorphism. We fix a section $\iota : \Lambda \to \Gamma$ of p. Given $\omega \in Z^3(\Lambda, \Bbbk^\times)$, we will denote by $p^*\omega \in Z^3(\Gamma, \Bbbk^\times)$, the 3-cocycle defined by

$$p^*\omega(g,h,k) = \omega(p(g), p(h), p(k)), \qquad g,h,k \in \Gamma.$$

We assume that there is $\alpha : \Gamma \times \Gamma \to \Bbbk^\times$, such that $\delta(\alpha) = p^*\omega$; that is,

$$p^*\omega(g,h,k) = \frac{\alpha(g,h)\alpha(gh,k)}{\alpha(g,hk)\alpha(h,k)}, \qquad g,h,k \in \Gamma,$$

4.1. **Trivializing the non-associativity of Nichols algebras.** We consider the functor ${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD} \to {}^{\Bbbk^{p^*\omega}\Gamma}_{\Bbbk^{p^*\omega}\Gamma}\mathcal{YD}$ given on the objects by

$$V \mapsto \widehat{V}, \qquad \text{with } \Gamma\text{-grading} \qquad \widehat{V}_g = \begin{cases} V_k & g = \iota(k), \\ 0 & g \notin \iota(\Lambda), \end{cases}$$

and Γ-action via p; on the morphisms, it is just the identity.

Then there is a braided tensor equivalence

$$(F_\alpha, \overline{\alpha}) : {}^{\Bbbk^{p^*\omega}\Gamma}_{\Bbbk^{p^*\omega}\Gamma}\mathcal{YD} \to {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD},$$

where $F_\alpha(V) = V$ as Γ-graded vector spaces, with Γ-action

$$g \cdot v = \frac{\alpha(h,g)}{\alpha(g,h)} g \triangleright v, \qquad g,h \in \Gamma, \qquad v \in V_g;$$

the functor is the identity for morphisms; the isomorphism constraints are

$$\begin{aligned} \overline{\alpha}_{V,V'} : F_\alpha(V \otimes V') &\to F_\alpha(V) \otimes F_\alpha(V) \\ v \otimes v' &\mapsto \alpha(g,h)\ v \otimes v', \qquad g,h \in \Gamma,\ v \in V_g,\ v' \in V_h. \end{aligned}$$

4.2. **Generation in degree one.** Fix $V \in {}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$. Hence $W := F_\alpha(\widehat{V}) \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ is a braided vector space of diagonal type: there exists a basis $(x_i)_{i\in\mathbb{I}}$, elements $g_i \in \Gamma$, $\chi_i \in \widehat{\Gamma}$ such that $x_i \in W_{g_i}^{\chi_i}$, so the braiing is

$$c(x_i \otimes x_j) = g_i \cdot x_j \otimes x_i = q_{ij}\ x_j \otimes x_i, \qquad q_{ij} := \chi_j(g_i), \qquad i, j \in \mathbb{I}.$$

Coming back to V, let $\ell_i = p(g_i) \in \Lambda$, $i \in \mathbb{I}$. As $V = W$ as vector spaces and the Γ-grading on W is induced by ι, we have that $g_i = \iota(\ell_i)$ and $x_i \in V_{\ell_i}$ for all $i \in \mathbb{I}$. The *quasi*-braiding in ${}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$ is given by

$$c_V(x_i \otimes x_j) = g_i \triangleright x_j \otimes x_i = q_{ij}\frac{\alpha(\ell_i, \ell_j)}{\alpha(\ell_j, \ell_i)}\ x_j \otimes x_i, \qquad i, j \in \mathbb{I}.$$

We recall now some results about the FRT construction. Let $H(W)$ the bialgebra corresponding to (W, c) [16, VIII.6]: it is the algebra presented by generators T^i_j, $i, j \in \mathbb{I}$ and relations

$$q_{ij}T^n_jT^m_i - q_{nm}T^m_iT^n_j, \qquad i, j, m, n \in \mathbb{I}.$$

Hence $H(W)$ is a quantum linear space, so in particular it is $\mathbb{Z}^I$-graded, with $\deg T^j_i = \alpha_i$, $i, j \in \mathbb{I}$. The coproduct satisfies

$$\Delta(T^j_i) = \sum_{k\in\mathbb{I}} T^k_i \otimes T^j_k, \qquad i, j \in \mathbb{I},$$

while the R-matrix $\mathtt{r} : H(W) \otimes H(W) \to \Bbbk$ is determined by

$$\mathtt{r}(T^m_i \otimes T^n_j) = q_{ji}\delta_{i,m}\delta_{j,n}, \qquad i, j, m, n \in \mathbb{I}.$$

Hence W is a $H(W)$-comodule with coaction

$$\rho : W \to H(W) \otimes W, \qquad \rho(x_i) = \sum_{j\in\mathbb{I}} T^j_i \otimes x_j, \qquad i \in \mathbb{I}.$$

and c is also the braiding in the category of $H(W)$-comodules.

Theorem 4.1. *Let $R = \oplus_{n\geq 0} R_n \in {}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$ be a post-Nichols (respectively, pre-Nichols) algebra of $V = R_1$ such that $\dim R < \infty$. Then $R = \mathcal{B}(V)$.*

Proof. By abuse of notation, let $\alpha : H(W) \otimes H(W) \to \Bbbk$,

$$\alpha(T^{m_1}_{i_1} \dots T^{m_s}_{i_s}, T^{n_1}_{j_1} \dots T^{n_t}_{j_t}) = \delta_{i_1,m_1} \dots \delta_{i_s,m_s}\delta_{j_1,n_1} \dots \delta_{j_t,n_t}$$
$$\alpha(g_{i_1} \dots g_{i_s}, g_{j_1} \dots g_{j_t}), \qquad s, t \in \mathbb{N},\ i_k, m_k, j_l, n_l \in \mathbb{I}.$$

As $H(W)$ is $\mathbb{Z}^I$-graded, the map is well-defined, and $\alpha(1, x) = \alpha(x, 1) = \varepsilon(x)$ for all $x \in H(W)$. Hence we may consider the coquasi-bialgebra $H(W)^\alpha$ obtained by a 2-cocycle deformation by α.

Notice that (V, c_V) is the image of (W, c) under the braided equivalence ${}^{H(W)}\mathcal{M} \to {}^{H(W)^\alpha}\mathcal{M}$ induced by the 2-cocycle α, and this equivalence takes pre- and post-Nichols algebras of (W, c) to pre- and post-Nichols algebras of (V, c_V). Hence R is the image of a post-Nichols (respectively, pre-Nichols) algebra R' of (W, c), which is of diagonal type. By [2], $R' = \mathcal{B}(W)$, so $R = \mathcal{B}(V)$. $\square$

4.3. **Pointed coquasi-Hopf algebras and de-equivariantization.** Let H be a coquasi-Hopf algebra and G be an affine group scheme over $\Bbbk$. A central inclusion of G in H is a full braided embedding $\iota : \operatorname{Rep}(G) \to \mathcal{Z}({}^H\mathcal{M})$ such that the composition $\iota \circ U : \operatorname{Rep}(G) \to {}^H\mathcal{M}$ is full, where $U : \mathcal{Z}({}^H\mathcal{M}) \to {}^H\mathcal{M}$ is the forgetful functor.

Let $\mathcal{O}(G)$ be the algebra of regular function over G. The algebra $\mathcal{O}(G)$ is a commutative algebra in the symmetric category $\operatorname{Rep}(G)$, and thus a commutative algebra in the braided tensor category $\mathcal{Z}({}^H\mathcal{M})$. Following [12], we define the de-equivariantization ${}^H\mathcal{M}(G)$ of ${}^H\mathcal{M}$ by G, as the monoidal category of of left $\mathcal{O}(G)$-modules in ${}^H\mathcal{M}$, with the tensor product $M \otimes_{\mathcal{O}(G)} N$.

In [3], we studied de-equivariantization of Hopf algebras, applying Tannakian techniques. In [3, Theorem 2.8] we constructed a coquasi-bialgebra such that its tensor category of comodules realizes the de-equivariantization of a Hopf algebra. As application, we explicitly describe a big family of pointed coquasi-Hopf algebras $A(H, G, \Phi)$ attached to a coradically graded pointed Hopf algebra H and some extra data where G is a central subgroup of a finite group Γ and Φ is a group morphism between G and $\operatorname{Hom}(\Gamma, k^\times)$, satisfying some conditions, see [3, Proposition 3.3] and [3, Definition 3.5].

Theorem 4.2. *Let A be a finite-dimensional coradically graded coquasi-Hopf algebra such that $A_0 \simeq \Bbbk^\omega\Lambda$, where ω is trivializable. Then ${}^A\mathcal{M}$ is a de-equivariantization of a coradically graded pointed Hopf algebra over an abelian group.*

Proof. By [5] there exists a post-Nichols (respectively, pre-Nichols) algebra $R = \oplus_{n\geq 0} R_n \in {}^{\Bbbk^\omega\Lambda}_{\Bbbk^\omega\Lambda}\mathcal{YD}$ of $V = R_1$ such that $A \simeq R\#\Bbbk^\omega\Lambda$; hence $\dim R < \infty$, and by Theorem 4.1, $R = \mathcal{B}(V)$.

We consider $\widehat{V} \in {}^{\Bbbk^{p^*\omega}\Gamma}_{\Bbbk^{p^*\omega}\Gamma}\mathcal{YD}$: as the braiding is the same, $\mathcal{B}(V) \simeq \mathcal{B}(\widehat{V})$ as braided Hopf algebras, and

$$\pi := (\operatorname{id} \otimes p) : B := \mathcal{B}(\widehat{V})\#\Bbbk^{p^*\omega}\Gamma \to A = \mathcal{B}(V)\#\Bbbk^\omega\Lambda \tag{10}$$

is a projection of coquasi-Hopf algebras.

Given an epimorphism of finite dimensional coquasi-Hopf algebra $f : H \to Q$, it follows by [8, Proposition 5.1] that

$$H^{\operatorname{co} f} := \{b \in B : \operatorname{id} \otimes \pi\Delta(b) = b \otimes 1\},$$

admits a structure of commutative algebra in $\mathcal{Z}({}^H\mathcal{M})$ such that the tensor category of left $H^{\operatorname{co} f}$-modules in ${}^H\mathcal{M}$ is tensor equivalent to ${}^Q\mathcal{M}$.

We will see that there is a central inclusion $\iota : \operatorname{Rep}(\widehat{\ker(p)}) \to \mathcal{Z}({}^B\mathcal{M})$, such that the central algebra $\mathcal{O}(\widehat{\ker(p)}) = k[\ker(p)]$ is the central algebra associated to the epimorphism (10).

The inclusion $k[\ker(p)] \hookrightarrow B, a \mapsto 1\#a$, is an injective coquasi-Hopf algebra morphism, that induces a full tensor embedding

$$\operatorname{Rep}(\widehat{\ker(p)}) = \operatorname{Vec}_{\ker(p)} \hookrightarrow^B \mathcal{M}.$$

Let $V_a = \mathrm{Span}_k(v) \in {}^{k[\ker(p)]}\mathcal{M}$ be a one-dimensional comodule with $\Delta(v) = a \otimes v$. Since $1\#a$ is a central group-like of B, for any $M \in {}^{B}\mathcal{M}$, the flip map

$$\begin{aligned} c_{M,V_a} : M \otimes V_a &\to V_a \otimes M \\ m \otimes v &\mapsto v \otimes m, \end{aligned}$$

is an isomorphism of B-comodules. Equation (8), follows from the fact that

$$p_*(\omega)(a, g, h) = p_*(\omega)(g, a, h) = p_*(\omega)(g, h, a) = 1,$$

for all $g, h \in \Gamma, a \in \ker(p)$.

Since $k[\ker(p)] = B^{\mathrm{co}\,\pi} := \{b \in B : \mathrm{id} \otimes \pi\Delta(b) = b \otimes 1\}$, the central algebra associate to the surjective tensor functor $\pi_* : {}^{B}\mathcal{M} \to {}^{A}\mathcal{M}$ is exactly $k[\ker(p)]$. Then by [8, Proposition 5.1], ${}^{A}\mathcal{M}$ is a de-equivariantization of ${}^{B}\mathcal{M}$ by $\widehat{\ker(p)}$.

Since there is $\alpha : \Gamma \times \Gamma \to \Bbbk^{\times} \times$ such that $\delta(\alpha) = p^*(\omega)$, we can extend α linearly to a map $\alpha : B \otimes B \to \Bbbk$ such that $\alpha(B_n \otimes B) = \alpha(B \otimes B_n) = 0$ if $n > 0$, so α is a twist. Hence $H := B^{\alpha}$ is a Hopf algebra; as a coalgebra, $H = B$ is coradically graded, with $H_0 = \Bbbk\Gamma$. Hence $H \simeq R' \# \Bbbk\Gamma$ for some graded Hopf algebra $R' \in^{\Bbbk\Gamma}_{\Bbbk\Gamma} \mathcal{YD}$, where $R'_1 = F_{\alpha}(\widehat{V})$, so $H \simeq \mathcal{B}(W) \# \Bbbk\Gamma$. Since ${}^{H}\mathcal{M}$ is tensor equivalent to ${}^{B}\mathcal{M}$, we have that ${}^{A}\mathcal{M}$ is a de-equivariantization of the Hopf algebra H by the group $\widehat{\ker(p)}$.

□

4.4. **Example of a pointed coquasi-Hopf algebra over an abelian groups with non-trivializable associator.** Let A and B be finite abelian groups and $\alpha \in Z^2(A, \widehat{B})$ a 2-cocycle, that is, a map $\alpha : A \times A \to \widehat{B}$ such that

$$\alpha(x, y)\alpha(xy, z) = \alpha(x, yz)\alpha(y, z), \qquad x, y, z \in A.$$

We denote by $\widehat{B} \rtimes_{\alpha} A$ the central extension of A by $\widehat{B}$ associated to α. Explicitly, $\widehat{B} \rtimes_{\alpha} A = \widehat{B} \times A$ as a set, and product given by

$$(x_1, a_1)(x_2, a_2) = (x_1 x_2, \alpha(a_1, a_2), a_1 a_2).$$

The function

$$\omega_{\alpha}((a_1, g_1), (a_2, g_2), (a_3, g_3)) = \alpha(a_1, a_2)(g_3),$$

is a 3-cocycle $\omega_{\alpha} \in Z^3(A \oplus B, \Bbbk^{\times})$. It is easy to see that $\psi_{A \oplus B}(\omega_{\alpha}) = 0$ if and only if $\alpha(a_1, a_2) = \alpha(a_2, a_1)$ for all $a_1, a_2 \in A$.

By [20, Theorem 3.6], the braided fusion categories ${}^{\Bbbk^{\omega}_{\alpha} A \oplus B}_{\Bbbk^{\omega}_{\alpha} A \oplus B}\mathcal{YD}$ and ${}^{\Bbbk\widehat{B} \rtimes_{\alpha} A}_{\Bbbk\widehat{B} \rtimes_{\alpha} A}\mathcal{YD}$ are equivalent.

Let $A = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $B = \mathbb{Z}/2\mathbb{Z}$. We define

$$\alpha : A \times A \to \widehat{B}, \qquad \alpha((m_1, m_2), (n_1, n_2)) = \chi^{m_1 n_2},$$

where $\chi : \mathbb{Z}/2\mathbb{Z} \to \Bbbk^{\times}$ is the non-trivial character.

Since α is non-symmetric, $\psi_{A\oplus B}(\omega_\alpha) \neq 1$ and by Theorem 3.7 ω_α is non-trivializable. The group $\widehat{B} \rtimes_\alpha A$ is a non-abelian group of order eight with two elements of order four. Hence $\widehat{B} \rtimes_\alpha A$ is a isomorphic to D_4, the dihedral group of order 8.

In [17, Example 6.5], Milinski and Schneider constructed a Nichols algebra $\mathcal{B}(V)$ of dimension 64 over D_4. Since the braided categories ${}^{\Bbbk^\omega_\alpha A\oplus B}_{\Bbbk^\omega_\alpha A\oplus B}\mathcal{YD}$ and ${}^{\Bbbk\widehat{B}\rtimes_\alpha A}_{\Bbbk\widehat{B}\rtimes_\alpha A}\mathcal{YD}$ are equivalent, in ${}^{\Bbbk^\omega_\alpha A\oplus B}_{\Bbbk^\omega_\alpha A\oplus B}\mathcal{YD}$ there is a Nichols algebra $\mathcal{B}(V')$ of dimension 64. Hence, the bosonization

$$A := \mathcal{B}(V')\#\Bbbk_{\omega_\alpha}[\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}],$$

is a coradically graded coquasi-Hopf algebra with non-trivializable associator.

References

[1] N. Andruskiewitsch and H.-J. Schneider. Pointed hopf algebras. In *New directions in Hopf algebras*, MSRI series Cambridge Univ. Press, pages 1–68. 2002.

[2] I. Angiono. On Nichols algebras of diagonal type. *J. Reine Angew. Math.*, 683:189–251, 2013.

[3] I. Angiono, C. Galindo, and M. Pereira. De-equivariantization of Hopf algebras. *Algebr. Represent. Theory*, 17(1):161–180, 2014.

[4] I. Angiono and A. G. Iglesias. Liftings of nichols algebras of diagonal type ii. all liftings are cocycle deformations. *arXiv preprint arXiv:1605.03113*, 2016.

[5] A. Ardizzoni and A. Pavarin. Bosonization for dual quasi-bialgebras and preantipode. *J. Algebra*, 390:126–159, 2013.

[6] N. Bourbaki. *Algebra I. Chapters 1–3*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1998. Translated from the French, Reprint of the 1989 English translation [MR0979982 (90d:00002)].

[7] L. Breen. Monoidal categories and multiextensions. *Compositio Math.*, 117(3):295–335, 1999.

[8] A. Bruguières and S. Natale. Exact sequences of tensor categories. *Int. Math. Res. Not. IMRN*, (24):5644–5705, 2011.

[9] S. Eilenberg and S. Mac Lane. On the groups of $H(\Pi, n)$. I. *Ann. of Math. (2)*, 58:55–106, 1953.

[10] S. Eilenberg and S. Mac Lane. On the groups $H(\Pi, n)$. II. Methods of computation. *Ann. of Math. (2)*, 60:49–139, 1954.

[11] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[12] P. Etingof, D. Nikshych, and V. Ostrik. Weakly group-theoretical and solvable fusion categories. *Adv. Math.*, 226(1):176–205, 2011.

[13] C. Galindo and J. Y. Plavnik. Tensor functors between Morita duals of fusion categories. *Lett. Math. Phys.*, 107(3):553–590, 2017.

[14] H.-L. Huang, G. Liu, Y. Yang, and Y. Ye. Finite quasi-quantum groups of diagonal type. *arXiv preprint arXiv:1611.04096*, 2016.

[15] A. Joyal and R. Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993.

[16] C. Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[17] A. Milinski and H.-J. Schneider. Pointed indecomposable Hopf algebras over Coxeter groups. In *New trends in Hopf algebra theory (La Falda, 1999)*, volume 267 of *Contemp. Math.*, pages 215–236. Amer. Math. Soc., Providence, RI, 2000.

[18] F. Quinn. Group categories and their field theories. In *Proceedings of the Kirbyfest (Berkeley, CA, 1998)*, volume 2 of *Geom. Topol. Monogr.*, pages 407–453 (electronic). Geom. Topol. Publ., Coventry, 1999.

[19] P. Schauenburg. A characterization of the Borel-like subalgebras of quantum enveloping algebras. *Comm. Algebra*, 24(9):2811–2823, 1996.

[20] B. Uribe. On the classification of pointed fusion categories up to weak morita equivalence. *arXiv preprint arXiv:1511.05522*, 2015.

I. A.: FaMAF-CIEM (CONICET), Universidad Nacional de Córdoba, Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba, República Argentina.

E-mail address: angiono@famaf.unc.edu.ar

C. G.: Departamento de matemáticas, Universidad de los Andes, Carrera 1 N. 18A - 10, Bogotá, Colombia.

E-mail address: cn.galindo1116@uniandes.edu.co