HODGE ISO

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Abstract. We constructi many infinite sequences of pairs of Lens spaces that are p-isospectral for all p but are not strongly isospectral. Such pairs are associated to special $\| \|_1$-isospectral lattices on $\mathbb{Z}^n$.

Contents

1. Introduction

Two compact riemannian manifolds M, M' are said to be isospectral if the spectrum of M and M' with respect to the Laplace operator are the same. More generally, they are said to be p-isospectral if the spectrum on p-forms for M, M' are the same. Clearly, if $M, M?$ are isometric, they are p-isospectral for all p. The first examples of manifolds that are p-isospectral for every p and still they are not isometric are due to J.Milnor ([Mi]). Since then many more examples have been constructed showing connections between spectra and geometry of a riemannian manifold (see...). In [Su], Sunada gave a general procedure to construct isospectral pairs of manifolds by using orbit spaces of two finite groups F, F' of isometries acting on a riemannian manifold M, where F, F' are almost conjugate in a bigger finite group G, that is, there is a bijection between F and F' that preserves G-conjugacy classes. Sunada's method yields many new examples, but always provides manifolds that are strongly isospectral, that is they are isospectral for every strongly natural operator acting on sections of a vector bundle E over M, in particular they are p-isospectral for all p.

The first pair of manifolds that are isospectral on functions and not on 1-forms is due to Carolyn Gordon, who used left invariant metrics on Heisenberg nilmanifolds. Later, Ikeda studied the spectrum of spherical space forms and produced, for each p, lens spaces that are q-isospectral for every $q \leq p$ and are not $p+1$-isospectral. For a given p, Ikeda's examples have dimension of the order of about $n = 2p$. Several people produced, subsequently, new examples of strongly isospectral, non isometric, spherical space forms, by Sunada's method (J.A.Wolf, P.

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Gilkey, D. Schueth, A. Ikeda). However, in the class of *non strongly isospectral* spherical space forms, we are just left with Ikeda's examples ([Ik]) and those found by Gornet-Mc Gowan by computer methods. These examples cannot be Sunada isospectral, since two lens spaces that are Sunada isospectral are necessarily isometric.

In this paper we take up the construction of lens spaces that are p-isospectral for all p but still they are not strongly isospectral. This question has been open for some time and was raised by J.A. Wolf in [Wo, Introduction]. Our approach consists in using the representation theory of a compact Lie group and some invariant theory to derive a formula for the multiplicity of each eigenvalue of Δ_p, the p-Laplace operator, in terms of the multiplicities of the weights of representations of G and on the number of solutions of certain congruences, depending on the data of the lens space (see Theorem). One can associate to each lens space $L(q:p_1,\ldots,p_m)$ the integral lattice contaning $q\mathbb{Z}^r$ given by

$$\Lambda(q:p_1,\ldots,p_m)=\{(z_1,\ldots,z_m)\in\mathbb{Z}^r:\sum z_ip_i\equiv 0 \mod q\}.$$

We show that two lens spaces $L(q:p_1,\ldots,p_m)$ and $L(q:p_1',\ldots,p_m')$ are p-isospectral for every $1\leq p\leq n$, if and only if the lattices $\Lambda(q:p_1,\ldots,p_m)$ and $\Lambda(q:p_1',\ldots,p_m')$ are strongly $\|\,\|_1$-isospectral (see Definition).

In Section **??** we construct an infinite family of strongly $\|\,\|_1$-isospectral lattices. The proof of isospectrality is involved and is based on a subdivision of the lattice into layers and in showing that the number of elements in each layer, having a fixed $\|\,\|_1$ length, is the same for both lattices. This is attained by partitioning each layer into finitely many regions (6?) and defining, for each of these regions, an $\|\,\|_1$-isometry that gives a bijective correspondence between the elements of one lattice in that layer and those of the other lattice. As mentioned, the associated lens spaces are p-isospectral for every p, but they cannot be strongly isospectral since we give a representation τ of G such that the lattices are not τ-isospectral. We point out that the corresponding lens spaces are homotopically equivalent but not homeomorphic to each other.

Finally, in Section **??** we show (see Theorem **??**) that to check the $\|\,\|_1$-isospectrality of the lattices in

2. Preliminaries

it is sufficient to do this for all elements having $\|\,\|_1$ less than or equal to .., thus reducing the verification to a finite set of lengths. We conclude the section with an Appendix that gives a list, obtained by computer methods, of all pairs for $m=3$ and $m=4$, for $q\leq 500$. As we can see, there are many examples which are not included in the family constructed in the previous section.

-bundles

2.1. **Homogeneous vector bundles.** For each finite dimensional unitary representation (τ,W_τ) of K, we consider the *homogeneous vector bundle*

$$E_\tau=G\times_\tau W_\tau\longrightarrow X=G/K,$$

that is the quotient of $G\times W_\tau$ under the right action of K given as $(x,v)\cdot k=(xk,\tau(k^{-1})v)$ (see for instance [Wa73, Ch. 5] or [LMR13, §2.1]). The space $\Gamma^\infty(E_\tau)$ of smooth sections of E_τ is in one to one correspondence with the space $C^\infty(G/K;\tau)$ of smooth functions $f:G\to W_\tau$ such that $f(xk)=\tau(k^{-1})f(x)$.

We restrict to Γ the left actions of G on $X=G/K$, E_τ, $\Gamma^\infty(E_\tau)$ and $C^\infty(G/K;\tau)$. The space $\Gamma^\infty(\Gamma\backslash E_\tau)$ of Γ-invariant elements in $\Gamma^\infty(E_\tau)$ is a homogeneous vector bundle over the compact Riemannian manifolds $\Gamma\backslash X$, and it isomorphic to the space $C^\infty(\Gamma\backslash G/K;\tau)$ of Γ-invariant functions in $C^\infty(G/K;\tau)$. We denote by $L^2(\Gamma\backslash E_\tau)$ the closure of $C^\infty(\Gamma\backslash G/K;\tau)$ with respect to the inner product $(f_1,f_2)=\int_{\Gamma\backslash X}\langle f_1(x),f_2(x)\rangle\,\mathrm{d}x$.

The (semisimple) Lie algebra $\mathfrak{g}_0$ of G, its complexification $\mathfrak{g}$ and the universal enveloping algebra $U(\mathfrak{g})$ act on $C^\infty(G/K;\tau)$ in a natural way. We shall denote by $C=\sum X_i^2\in U(\mathfrak{g})$, where $X_1,\dots,X_n$ is any orthonormal basis of $\mathfrak{g}$, the *Casimir* element of $\mathfrak{g}$; C defines a second order elliptic *differential operator* Δ_τ on $C^\infty(G/K;\tau)\simeq\Gamma^\infty(E_\tau)$. This operator commutes with the left action of G, in particular with elements in Γ, thus Δ_τ induces a differential operator $\Delta_{\tau,\Gamma}$ acting on $\Gamma^\infty(\Gamma\backslash E_\tau)$.

ex2:S^n **Example 2.1.** In the next sections we shall fix the following symmetric pair $(\mathrm{O}(n+1),\mathrm{O}(n))$. Its associated symmetric space is the n-dimensional sphere S^n. For $0\le p\le n$, let τ_p denote the p-exterior representation $\bigwedge^p(\mathbb{C}^n)$ of $K=\mathrm{O}(n)$. Then Δ_{τ_p} (resp. $\Delta_{\tau_p,\Gamma}$) coincides with the Hodge-Laplace operator on complex valued differential forms of degree p on S^n (resp. $\Gamma\backslash S^n$).

We now recall some notions and facts from the Introduction. [1]

tau_iso **Definition 2.2.** Let (G,K) be a Riemannian symmetric pair, let $X=G/K$ and let $\tau\in\widehat{K}$. Let Γ_1 and Γ_2 be two discrete subgroups of G acting freely on X. The spaces $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are said to be τ*-isospectral* if the Laplace type operators Δ_{τ,Γ_1} and Δ_{τ,Γ_2} have the same spectrum.

f2:p_iso **Definition 2.3.** Two compact Riemannian spaces M_1 and M_2 are called p-isospectral if the Hodge-Laplace operators acting on p-forms have the same spectrum.

We note that if a discrete [2] subgroup $\Gamma\subset\mathrm{O}(n+1)$ acts freely on S^n, then it must necessarily be included in $\mathrm{SO}(n+1)$, thus $\Gamma\backslash S^n$ is an orientable manifold. In particular, this implies that $\Gamma_1\backslash S^n$ and $\Gamma_2\backslash S^n$ are p-isospectral if and only if they are $(n-p)$-isospectral.

-reg-rep 2.2. **Right regular representation.** We consider the *right regular representation* R_Γ of G on $L^2(G)$ given by $(R(g)\cdot f)(x)=f(xg)$. Since G is a compact group, by the Peter-Weyl theorem the set $\bigcup_{\pi\in\widehat{G}}\left\{C^\pi_{v_i,v_j}:1\le i,j\le\dim V_\pi\right\}$ is an orthonormal basis of $L^2(G)$, where $\{v_1,\dots,v_{d_\pi}\}$ ($d_\pi:=\dim V_\pi$) is any orthonormal basis of V_π and $C^\pi_{v,w}:G\to\mathbb{C}$ is given by $C^\pi_{v,w}(x)=d_\pi^{-1/2}\langle\pi(x)v,w\rangle$. Let $\pi\in\widehat{G}$. By fixing any $w\in V_\pi$, we have that the subspace $\{C^\pi_{v,w}:v\in V_\pi\}$ is invariant by the right regular representation and isomorphic to V_π, thus the representation π has multiplicity d_π in $L^2(G)$.[3] Hence

$$L^2(G)=\sum_{\pi\in\widehat{G}}d_\pi\,V_\pi.$$

We restrict the right regular representation R to the subspace

$$L^2(\Gamma\backslash G):=\left\{f\in L^2(G):f(\gamma x)=f(x)\quad\forall\gamma\in\Gamma\right\}.$$

We denote by R_Γ the action of G on this subspace. We then have

$$\begin{aligned}C^\pi_{v,w}\in L^2(\Gamma\backslash G)\quad&\iff\quad\langle\pi(\gamma x)v,w\rangle=\langle\pi(x)v,w\rangle\qquad\forall x\in G,\ \forall\gamma\in\Gamma\\&\iff\quad\langle\pi(x)v,\pi(\gamma^{-1})w-w\rangle=0\qquad\forall x\in G,\ \forall\gamma\in\Gamma\\&\iff\quad\pi(\gamma)w=w\qquad\forall\gamma\in\Gamma\\&\iff\quad w\in V_\pi^\Gamma,\end{aligned}$$

[1] Cambio.
[2] finite?
[3] peq cambios

where V_π^Γ is the subspace of V_π invariant by the action of Γ. Let $d_\pi^\Gamma = \dim V_\pi^\Gamma$. Hence we have the decomposition

`(GammaG)`
$$L^2(\Gamma\backslash G) = \sum_{\pi\in\widehat{G}} d_\pi^\Gamma\, V_\pi. \tag{2.1}$$

Remark 2.4. In the notation in [LMR13], (2.1) tells us that $n_\Gamma(\pi) = d_\pi^\Gamma$ for all $\pi \in \widehat{G}$ when G is a compact group.

For any $\tau \in \widehat{K}$, set $\widehat{G}_\tau = \{\pi \in \widehat{G} : \operatorname{Hom}_K(W_\tau, V_\pi) \neq 0\}$ and let $R_{\Gamma,\tau}$ denote the restriction of R_Γ on the subspace

`aG)_tau`
$$L^2(\Gamma\backslash G)_\tau = \sum_{\pi\in\widehat{G}_\tau} d_\pi^\Gamma\, V_\pi. \tag{2.2}$$

`au-equiv` **Definition 2.5.** Let $\tau \in \widehat{K}$. Two discrete subgroups Γ_1 and Γ_2 of G that act freely on $X = G/K$ are called *τ-equivalent* if the representations $R_{\Gamma_1,\tau}$ and $R_{\Gamma_2,\tau}$ are equivalent, or equivalently, if $d_\pi^{\Gamma_1} = d_\pi^{\Gamma_2}$ for every $\pi \in \widehat{G}_\tau$.

There is a strong relation between the notions of τ-equivalence groups (Definition 2.5) and τ-isospectral manifolds (Definition 2.2). The next result (see [LMR13]) explains one part of this relation.

`>tau-iso` **Proposition 2.6.** *Let (G, K) be a symmetric pair of compact type and let Γ, Γ_1 and Γ_2 be discrete cocompact subgroups of G that act freely on $X = G/K$. Let $\Delta_{\tau,\Gamma}$ be the Laplace operator acting on the sections of the homogeneous vector bundle $\Gamma\backslash E_\tau$ (see Subsection 2.1) of the manifold $\Gamma\backslash X$.*

(1) *If $\lambda \in \mathbb{R}$, the multiplicity $d_\lambda(\tau,\Gamma)$ of the eigenvalue λ of $\Delta_{\tau,\Gamma}$ is given by*

`:_lambda`
$$d_\lambda(\tau,\Gamma) = \sum_{\pi\in\widehat{G}:\, \lambda(C,\pi)=\lambda} d_\pi^\Gamma\, \dim\left(\operatorname{Hom}_K(W_\tau^*, V_\pi)\right). \tag{2.3}$$

(2) *If Γ_1 and Γ_2 are τ^*-equivalent then $\Gamma_1\backslash X$ and $\Gamma_2\backslash X$ are τ-isospectral.*

The next result gives a description of d_π^Γ when Γ is a finite abelian group inside a maximal torus T of G. Let $\mathfrak{h}_0$ denote the Lie subalgebra of T, and let $\mathfrak{h}$ be its complexification. We denote by ... consider the root system of $(\mathfrak{g},\mathfrak{h})$. Any finite dimensional representation (π, V_π) of G decomposes as $V_\pi = \oplus_\mu V_\pi(\mu)$, where μ runs over the weight lattice $P(G) \subset \mathfrak{h}^*$ of G and

$$V_\pi(\mu) = \{v \in V_\pi : \pi(h)v = h^\mu\, v \quad \forall h \in T\},$$

where $h^\mu := e^{\mu(H)} \in \mathbb{C}$ for any $H \in \mathfrak{h}_0$ such that $\exp(H) = h$. The nonzero elements in $V_\pi(\mu)$ are called weight vectors of weight μ. The multiplicity of μ in π is denoted by $m_\pi(\mu) = \dim V_\pi(\mu)$.

`ax-torus` **Proposition 2.7.** *Let Γ be a finite subgroup of G, $\Gamma \subset T$, a maximal torus of G and let (π, V_π) be any finite dimensional representation of G. Then*

$$d_\pi^\Gamma = \dim V_\pi^\Gamma = \sum_{\substack{\mu\in P(G):\\ \gamma^\mu=1\,\forall\gamma\in\Gamma}} m_\pi(\mu).$$

Proof. Since $\Gamma \subset T$, the action of any $\gamma \in \Gamma$ diagonalizes on the decomposition $\oplus_\mu V_\pi(\mu)$. Moreover, any weight vector v of weight μ is invariant by $\gamma \in \Gamma$ if and only if $\gamma^\mu = 1$. Hence

$$V_\pi(\mu)^\Gamma = \begin{cases} V_\pi(\mu) & \text{if } \gamma^\mu = 1 \quad \forall\gamma\in\Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

The remaining assertions follows easily. □

es-forms

2.3. **Spherical spaces forms.** In this subsection we restrict our attention to the symmetric pair $(G,K)=(\mathrm{O}(n+1),\mathrm{O}(n))$, thus $X=S^n$ the n-dimensional sphere in $\mathbb{R}^{n+1}$. A spherical space form is a manifold of the form $\Gamma\backslash S^n$ where $\Gamma\subset\mathrm{O}(n+1)$ acts freely on S^n. Since even dimensional spheres S^n cover only S^n and $\mathbb{R}P^n$, we will look only at odd dimensional spheres, thus we assume from now on that $G=\mathrm{O}(2m)$ and $K=\mathrm{O}(2m-1)$. We also note that if a discrete subgroup $\Gamma\subset G$ acts freely on S^{2m-1}, [4] then it must necessarily be included in $G^0=\mathrm{SO}(2m)$; thus, all odd-dimensional spherical space forms are orientable.

Here and subsequently, we denote by $\mathfrak{g}_0=\mathfrak{so}(2m)$ the real skew-symmetric matrices, which is the Lie algebra of $G^0=\mathrm{SO}(2m)$ and of $G=\mathrm{O}(2m)$ as well. Let $\mathfrak{g}=\mathfrak{g}_0\otimes_{\mathbb{R}}\mathbb{C}\simeq\mathfrak{so}(2m,\mathbb{C})$, the complex skew-symmetric matrices. We fix the maximal torus in $\mathrm{SO}(2m)$ the subgroup

ax_torus

$$T=\left\{\begin{pmatrix}R(\theta_1)&&\\&\ddots&\\&&R(\theta_m)\end{pmatrix}:\theta_1,\dots,\theta_m\in\mathbb{R}\right\},\tag{2.4}$$

where $R(\theta):=\left(\begin{smallmatrix}\cos(2\pi\theta)&\sin(2\pi\theta)\\-\sin(2\pi\theta)&\cos(2\pi\theta)\end{smallmatrix}\right)$. The Lie algebra of T is

eq2:h_0

$$\mathfrak{h}_0=\left\{\begin{pmatrix}\begin{smallmatrix}0&2\pi\theta_1\\-2\pi\theta_1&0\end{smallmatrix}&&\\&\ddots&\\&&\begin{smallmatrix}0&2\pi\theta_m\\-2\pi\theta_m&0\end{smallmatrix}\end{pmatrix}:\theta_1,\dots,\theta_m\in\mathbb{R}\right\}.\tag{2.5}$$

Note that $h=\exp(H)$ if $h\in T$ and $H\in\mathfrak{h}_0$ as above. Let

eq2:h

$$\mathfrak{h}=\left\{\begin{pmatrix}\begin{smallmatrix}0&ih_1\\-ih_1&0\end{smallmatrix}&&\\&\ddots&\\&&\begin{smallmatrix}0&ih_m\\-ih_m&0\end{smallmatrix}\end{pmatrix}:h_1,\dots,h_m\in\mathbb{C}\right\},\tag{2.6}$$

thus $\mathfrak{h}\cong\mathfrak{h}_0\otimes_{\mathbb{R}}\mathbb{C}$ is a *Cartan subalgebra* of $\mathfrak{g}$. For $H\in\mathfrak{h}$ as in (2.6), set $\varepsilon_j(H)=h_j$ for $1\le j\le m$, thus $\{\varepsilon_1,\dots,\varepsilon_m\}$ is a basis of $\mathfrak{h}^*$. Note that $\varepsilon_j(H)=-2\pi i\theta_j$ for $h\in\mathfrak{h}_0$ as in (2.5).

Let us denote by $P(G^0)=\bigoplus_{j=1}^m\mathbb{Z}\varepsilon_j$ the *weight lattice* of $G^0=\mathrm{SO}(2m)$ and by $\Delta(\mathfrak{g},\mathfrak{h})=\{\pm\varepsilon_i\pm\varepsilon_j:1\le i<j\le m\}$ the *roots* of $\mathfrak{h}$ on $\mathfrak{g}$. We fix the set $\Delta^+(\mathfrak{g},\mathfrak{h})=\{\varepsilon_i\pm\varepsilon_j:1\le i<j\le m\}$ as *positive* roots, thus the corresponding *simple* roots are $\Pi(\mathfrak{g},\mathfrak{h})=\{\varepsilon_j-\varepsilon_{j+1}:1\le j\le m-1\}\cup\{\varepsilon_{m-1}+\varepsilon_m\}$ and the *dominant weight* $P_+(G^0)$ of G^0 are the elements $\sum_{j=1}^m a_j\varepsilon_j\in P(G^0)$ such that $a_1\ge\cdots\ge a_{m-1}\ge|a_m|$.

Similarly, we fix the Cartan subalgebra of $\mathfrak{k}=\mathfrak{so}(2m-1,\mathbb{C})$, the complexification of the Lie algebra $\mathfrak{k}_0$ of $K^0=\mathrm{SO}(2m-1)$, as in (2.6) except that the last two rows and columns are replaced by one with all coefficients equal to zero. In this case, the dominant weight $P_+(K^0)$ of K^0 are the elements $\sum_{j=1}^{m-1}a_j\varepsilon_j$ such that $a_j\in\mathbb{Z}$ for all j and $a_1\ge\cdots\ge a_{m-1}\ge a_{m-1}\ge 0$.

The highest weight theorem gives a one-one correspondence between the irreducible representations of $\mathrm{SO}(n)$ and the elements in $P_+(\mathrm{SO}(n))$. For $\mu\in P_+(\mathrm{SO}(2m-1))$ (resp. $\Lambda\in P_+(\mathrm{SO}(2m))$), we write (τ_μ,W_μ) (resp. (π_Λ,V_Λ)) the irreducible representation of $\mathrm{SO}(2m-1)$ (resp. $\mathrm{SO}(2m)$) with highest weight μ (resp. Λ).

The irreducible representation of $\mathrm{O}(2m-1)$ are in a one-one correspondence with the elements μ in $P_+(\mathrm{SO}(2m-1))$ and a parameter $\kappa\in\{\pm1\}$. We denote by $\tau_{\mu,\kappa}$ the representation associated to the parameters (μ,κ). The underlying vector space of $\tau_{\mu,\kappa}$ is W_μ, $\tau_{\mu,\kappa}$ valued in $\mathrm{SO}(2m-1)$ coincides with τ_μ and $\tau_{\mu,\kappa}(-Id)=\kappa\, Id_{W_\mu}$. We have that

$$\tau_{\mu,\kappa}|_{\mathrm{SO}(2m-1)}\cong\tau_\mu\quad\text{and}\quad\tau_{\mu,\kappa}\cong\tau_{\mu,-\kappa}\otimes\det.$$

[4] cambios

The even case is a bit more complicated (see [LMR13, Subsection 2.2] for more details). For $\Lambda = \sum_{j=1}^m a_i\varepsilon_i \in P_+(\mathrm{SO}(2m))$ such that $a_m = 0$ and $\delta \in \{\pm 1\}$, one associates $\pi_{\Lambda,\delta} \in \widehat{\mathrm{O}(2m)}$. Again we have $\pi_{\Lambda,\delta}|_{\mathrm{SO}(2m)} \cong \pi_\Lambda$ and $V_{\pi_{\Lambda,\delta}} = V_{\pi_\Lambda}$. The parameter δ depends on certain intertwining operator T_Λ (see [Pe1, p. 372] and [LMR13, eq. (2.7)]). In the case $a_m \neq 0$, we have only one representation $\pi_{\Lambda,0} \in \widehat{\mathrm{O}(2m)}$, which satisfies $\pi_{\Lambda,0}|_{\mathrm{SO}(2m)} \cong \pi_\Lambda \oplus \pi_{\overline{\Lambda}}$ and $V_{\pi_{\Lambda,0}} = V_{\pi_\Lambda} \oplus V_{\pi_{\overline{\Lambda}}}$, where $\overline{\Lambda} = \sum_{j=1}^{m-1} a_i\varepsilon_i - a_m\varepsilon_m$. Since $\Gamma \subset \mathrm{SO}(2m)$ if Γ acts freely on S^{2m-1} and $V_{\pi_{\Lambda,\delta}}$ is equal to $V_{\pi_\Lambda} \oplus V_{\pi_{\overline{\Lambda}}}$ or V_{π_Λ} according δ is zero or not, it follows that

nV^Gamma
$$d^\Gamma_{\pi_{\Lambda,\delta}} = \dim V^\Gamma_{\pi_{\Lambda,\delta}} = \begin{cases} \dim V^\Gamma_{\pi_\Lambda} & \text{if } a_m = 0,\\ \dim V^\Gamma_{\pi_\Lambda} + \dim V^\Gamma_{\pi_{\overline{\Lambda}}} & \text{if } a_m \neq 0. \end{cases} \tag{2.7}$$

We end this section by describing the p-spectrum of any odd-dimensional spherical space form in terms of the coefficients d^Γ_π. This result can be proved by using Proposition 2.6 (1) and the branching law from $\mathrm{O}(2m)$ to $\mathrm{O}(2m-1)$ (see [LMR13, Thm. 1.1]). We first introduce more notation:

mbda_k^p
$$\Lambda_k^p = \begin{cases} k\varepsilon_1 & \text{if } p = 1,\\ k\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_p & \text{if } 2 \le p \le m,\\ \Lambda_{k,2m-p} & \text{if } m+1 \le p \le 2m-1, \end{cases} \tag{2.8}$$

mbda_k^p
$$\lambda_k^p = k^2 + k(2m-2) + (p-1)(2m-1-p), \tag{2.9}$$

eq2:E_p
$$\mathcal{E}_p = \begin{cases} \{\lambda_k^p : k \in \mathbb{N}\} & \text{if } 1 \le p \le 2m-1,\\ \{0\} & \text{if } p = 0 \text{ or } 2m. \end{cases} \tag{2.10}$$

Hence $\Lambda_k^p \in P_+(\mathrm{SO}(2m))$ for every $k \in \mathbb{N}$ and every $1 \le p \le 2m-1$, $\mathcal{E}_p = \mathcal{E}_{2m-p}$ for every $0 \le p \le 2m$ and the sets $\mathcal{E}_p$ and $\mathcal{E}_{p+1}$ are disjoint for every $0 \le p \le 2m-1$

2:p-spec
Theorem 2.8. *Let Γ be a finite subgroup of $\mathrm{O}(2m)$ acting freely on S^{2m-1} and let $0 \le p \le 2m-1$. If $\lambda \in \mathrm{Spec}_p(\Gamma\backslash S^{2m-1})$ then $\lambda \in \mathcal{E}_p \cup \mathcal{E}_{p+1}$. Furthermore, if $\lambda = \lambda_k^p \in \mathcal{E}_p$ then*

p,Gamma)
$$d_\lambda(p,\Gamma) = d_\lambda(p-1,\Gamma) = d^\Gamma_{\pi_{\Lambda_k^p,\delta}} = \begin{cases} \dim V^\Gamma_{\pi_{\Lambda_k^p}} & \text{if } p \neq m,\\ \dim V^\Gamma_{\pi_{\Lambda_k^p}} + \dim V^\Gamma_{\pi_{\overline{\Lambda}_k^p}} & \text{if } p = m, \end{cases} \tag{2.11}$$

In particular, if $\lambda \in \mathrm{Spec}_0(\Gamma\backslash S^n)$ then $\lambda = k(k+2m-2)$ for some $k \in \mathbb{N}_0$ and

0,Gamma)
$$d_\lambda(0,\Gamma) = \dim V^\Gamma_{\pi_{k\varepsilon_1}}. \tag{2.12}$$

3. Lens spaces

In this section...[5]

Let q be a positive integer and let $s = (s_1, \dots, s_m) \in \mathbb{Z}^m$ such that $\gcd(s_j, q) = 1$ for all j. Let

$$\gamma = \begin{pmatrix} R(s_1/q) & & \\ & \ddots & \\ & & R(s_m/q) \end{pmatrix}.$$

Recall that $R(\theta) = \left(\begin{smallmatrix} \cos(2\pi\theta) & \sin(2\pi\theta) \\ -\sin(2\pi\theta) & \cos(2\pi\theta) \end{smallmatrix}\right)$. The element γ generates the cyclic subgroup $\Gamma(q;s) := \{\gamma^j\}_{j=1}^q$ of order q of $\mathrm{SO}(2m)$, which acts freely on S^{2m-1}. The manifold $L(q;s) := \Gamma(q;s)\backslash S^{2m-1}$ is called a *Lens space*.[6] The following properties are known (see for instance [Ik88, Thm. 3.1]).

[5] escribir algo

[6] creo que en todo el paper debera ser lens space con minscula, salvo al comienzo de oracin

Proposition 3.1. *Let $L = L(q; s)$ and $L' = L(q; s')$ be two Lens spaces of degree q. Then the following assertions are equivalent.*

1. *L is isometric to L'.*
2. *L is diffeomorphic to L'.*
3. *L is homeomorphic to L'.*
4. *There are $\ell \in \mathbb{Z}$ and $\epsilon \in \{\pm 1\}^m$ such that $(s_1, \dots, s_m)$ is a permutation of $(\epsilon_1 \ell s'_1, \dots, \epsilon_m \ell s'_m)$ $\pmod q$.*

Remark 3.2. If two Lens spaces are strongly isospectral then they are isometric.[7]

We next compute the p-spectrum for a Lens space $L = \Gamma\backslash S^{2m-1}$ with $\Gamma = \Gamma(q; s)$. From Theorem 2.8, if $\lambda \in \operatorname{Spec}_p(L)$ then $\lambda \in \mathcal{E}_p \cup \mathcal{E}_{p+1}$ where $\mathcal{E}_p$ and $\mathcal{E}_{p+1}$ are disjoint sets given in (2.10). Moreover, if $\lambda \in \mathcal{E}_p$ then $\lambda = \lambda_k^p$ (see (2.9)) for some $k \in \mathbb{N}$ and $1 \le p \le 2m-1$ and the multiplicities $d_\lambda(p, L)$ and $d_\lambda(p-1, L)$ of λ of the Hodge-Laplace operator on p and $p-1$-forms of the Lens space L are given by

$$d_\lambda(p, L) = d_\lambda(p-1, L) = d^{\Gamma}_{\pi_{\Lambda_k^p}, \delta} = \begin{cases} \dim V^{\Gamma}_{\pi_{\Lambda_k^p}} & \text{if } p \neq m, \\ \dim V^{\Gamma}_{\pi_{\Lambda_k^p}} + \dim V^{\Gamma}_{\pi_{\overline{\Lambda}_k^p}} & \text{if } p = m, \end{cases}$$

where $\Lambda_k^p \in P_+(\mathrm{SO}(2m))$ is given in (2.8) and $\pi_{\Lambda_k^p}$ is the irreducible representation of $\mathrm{SO}(2m)$ with highest weight Λ_k^p.

We denote by π_k^p the irreducible representation of $\mathrm{SO}(2m)$ with highest weight Λ_k^p (i.e. $\pi_k^p = \pi_{\Lambda_k^p}$) if $p \neq m$ and $\pi_k^m = \pi_{\Lambda_k^m} \oplus \pi_{\overline{\Lambda}_k^m}$. Hence

oda(P,L) (3.1)
$$d_{\lambda_k^p}(p, L) = d^{\Gamma}_{\pi_k^p}.$$

We write $\mathbb{Z}^m$ instead $P(\mathrm{SO}(2m))$, by corresponding $(a_1, \dots, a_m)$ with $\sum_{j=1}^m a_j \varepsilon_j$. For $\mu = \sum_j a_j \varepsilon_j \in \mathbb{Z}^m$ we set $|\mu| = \sum_j |a_j|$ and $Z(\mu) = \#\{j : a_j = 0\}$ the *height* and *zeros* of μ respectively. For each $k \in \mathbb{N}_0$ and $0 \le z \le m$, we define

$$M_\Gamma(k) = \#\{\mu \in \mathbb{Z}^m : |\mu| = k,\ \gamma^\mu = 1\}, \tag{3.2}$$
$$N_\Gamma(k, z) = \#\{\mu \in \mathbb{Z}^m : |\mu| = k,\ Z(\mu) = z,\ \gamma^\mu = 1\}. \tag{3.3}$$

The next result gives a sufficient condition for a pair of lens spaces to be 0-isospectral (resp. p-isospectral for all p).

odge-iso **Theorem 3.3.** *Let $L = \Gamma\backslash S^n$ and $L' = \Gamma'\backslash S^n$ be Lens spaces. Then:*

1. *If $M_\Gamma(k) = M_{\Gamma'}(k)$ for all $k \ge 0$, then L and L' are 0-isospectral.*
2. *If $N_\Gamma(k, z) = N_{\Gamma'}(k, z)$ for all $k \ge 0$ and $0 \le z \le m$, then L and L' are p-isospectral for all p.*

We first state a useful lemma which will be proved in Section 5.

icities **Lemma 3.4.** *Let $\mu, \eta \in \mathbb{Z}^m$ and let $\pi_k^p \in \widehat{\mathrm{SO}(2m)}$ with highest weight Λ_k^p (see (2.8)) if $p \neq m$ and $\pi_k^m = \pi_{\Lambda_k^m} \oplus \pi_{\overline{\Lambda}_k^m}$. Then the following assertions hold.*

1. *If $m_{\pi_k^p}(\mu) > 0$ then $|\Lambda_k^p| - |\mu| \in 2\mathbb{N}_0$.*
2. *If $|\mu| = |\eta|$ then $m_{\pi_{k\varepsilon_1}}(\mu) = m_{\pi_{k\varepsilon_1}}(\eta)$ for every $k \in \mathbb{N}_0$.*
3. *If $|\mu| = |\eta|$ and $Z(\mu) = Z(\eta)$ then $m_{\pi_k^p}(\mu) = m_{\pi_k^p}(\eta)$ for every $k \in \mathbb{N}$ and $1 \le p \le 2m-1$.*

[7] ampliar, pero creo que este es un buen lugar para ponerlo

Proof of Theorem 3.3. We shall prove (2) since (1) is very similar and simpler. Since Lens spaces are orientable manifolds, we only need to show that they are p-isospectral for every $0 \leq p \leq m-1$. By Theorem 2.8 and (3.1), it is sufficient to show that $d^{\Gamma}_{\pi^p_k} = d^{\Gamma'}_{\pi^p_k}$ for every $k \in \mathbb{N}$ and every $1 \leq p \leq m$.

We fix $k \in \mathbb{N}$ and $1 \leq p \leq m$. Since Γ is cyclic generated by γ and it lies inside the maximal torus T given in (2.4), by Proposition 2.7 we have that

$$d^{\Gamma}_{\pi^p_k} = \sum_{\mu \in \mathbb{Z}^m:\, \gamma^\mu = 1} m_{\pi^p_k}(\mu).$$

Lemma 3.4 (1) ensures that the weights μ of $V_{\pi^p_k}$ satisfy $|\mu| = |\Lambda| - 2r$ for some $r \in \mathbb{N}_0$, thus

t-ceros (3.4) $$d^{\Gamma}_{\pi^p_k} = \sum_{r=0}^{[|\Lambda|/2]} \sum_{z=0}^{m} \sum_{\substack{\mu \in \mathbb{Z}^m:\, \gamma^\mu = 1, \\ |\mu| = |\Lambda| - 2r,\, Z(\mu) = z}} m_{\pi^p_k}(\mu),$$

and similarly for $d^{\Gamma'}_{\pi^p_k}$. By Lemma 3.4 (3), the third sum in (3.4) is equal to $N_\Gamma(|\Lambda| - 2r, z)$ times $m_{\pi^p_k}(\mu_0)$, for any particular $\mu_0 \in \mathbb{Z}^m$ that satisfies the conditions. But $N_\Gamma(k, z) = N_{\Gamma'}(k, z)$ by hypothesis, thus the proof is complete. □

Remark 3.5. Let $\mu = \sum_{j=1}^m a_j \varepsilon_j \in \mathbb{Z}^m$ and let

$$H_\gamma = \operatorname{diag}\left(\begin{pmatrix} 0 & 2\pi s_1/q \\ -2\pi s_1/q & 0 \end{pmatrix} \dots \begin{pmatrix} 0 & 2\pi s_m/q \\ -2\pi s_m/q & 0 \end{pmatrix} \right).$$

We have that $\exp H_\gamma = \gamma$, thus

$$\gamma^\mu = e^{-2\pi i \left(\frac{a_1 s_1 + \dots + a_m s_m}{q} \right)}.$$

Hence

(3.5) $$\gamma^\mu = 1 \iff a_1 s_1 + \dots + a_m s_m \equiv 0 \pmod{q}.$$

This is the condition in (3.4).

Theorem 3.3 gives sufficient conditions for a pair of Lens spaces to be p-isospectral for all p. However, these conditions require infinitely many congruence equations to be satisfied [8]. The following theorem shows one can reduce the verification to only finitely many equations. We first introduce some new notions.

We call $\mu = \sum_{j=1}^m a_j \varepsilon_j \in \mathbb{Z}^m$ *q-primitive* if $|a_j| < q$ for every j. Now, for a Lens space $L = \Gamma \backslash S^n$ where $\Gamma = \langle \gamma \rangle$ has order q, we set

(3.6) $$N^*_\Gamma(k, z) = \#\{\mu \in \mathbb{Z}^m \ q\text{-primitive} : |\mu| = k,\ Z(\mu) = z,\ \gamma^\mu = 1\},$$

for any $k \in \mathbb{N}_0$ and any $0 \leq z \leq m$. Note that $N^*_\Gamma(k, z) = 0$ if $k > (m-1)q$ or if $z = m$ and $k > 0$.

dge-iso **Theorem 3.6.** *Let $L = \Gamma \backslash S^n$ and $L' = \Gamma' \backslash S^n$ be Lens spaces. If $N^*_\Gamma(k, z) = N^*_{\Gamma'}(k, z)$ for all $k \geq 0$ and $0 \leq z \leq m$, then L and L' are p-isospectral for all p.*

Proof. HACER (Juan Pablo tiene una fórmula.).[9] □

Remark 3.7. Theorems 3.3 and 3.6 give sufficient conditions for pairs of Lens spaces to be p-isospectral for all p. We do not known whether the converse is true. In other words, if $L = \Gamma \backslash S^{2m-1}$ and $L = \Gamma' \backslash S^{2m-1}$ are Lens space p-isospectral for all p, then $N_\Gamma(k, z) = N_{\Gamma'}(k, z)$ and/or $N^*_\Gamma(k, z) = N^*_{\Gamma'}(k, z)$ for every $k \in \mathbb{N}$ and $0 \leq z \leq m$.

[8] buscar mejor sinónimo

[9] hacer.

4. Explicit examples

ppendix

5. Appendix

In this section we prove Lemma 3.4. The first assertion follows immediately since any weight can be written as the highest weight minus a sum of simple roots, thus the height of $\Lambda - \mu$ is even and nonnegative since all simple roots of the root system associated to $\mathrm{SO}(2m)$ have height two. The rest shall be proved by using a realization of the irreducible representation of $\mathrm{SO}(2m)$ with highest weight Λ_p^k (see (2.8)) given by Ikeda and Taniguchi [IT78].

Let us denote by $\mathcal{P}_k$ the set of complex homogeneous polynomials of degree k in the variables $\bar{x} = (x_1, \dots, x_{2m})^t$. The group $\mathrm{SO}(2m)$ acts on $\mathcal{P}_k$ by $(g \cdot f)(\bar{x}) = f(g^{-1}\bar{x})$. It is well known that the subspace $\mathcal{H}_k$ of harmonic polynomials in $\mathcal{P}_k$ is invariant by the action of $\mathrm{SO}(2m)$ and is an irreducible representation of highest weight $k\varepsilon_1$. Moreover, $\mathcal{P}_k = \mathcal{H}_k \oplus \mathcal{P}_{k-2}$, where $\mathcal{P}_{k-2}$ is injected in $\mathcal{P}_k$ by $f(\bar{x}) \mapsto |\bar{x}|^2 f(\bar{x})$ where $|\bar{x}|^2 = x_1^2 + \cdots + x_{2m}^2$.

Again, we write $\mathbb{Z}^m$ instead $P(\mathrm{SO}(2m))$, by corresponding $(a_1, \dots, a_m)$ with $\sum_{j=1}^m a_j\varepsilon_j$, and write $|\mu| = \sum_{j=1}^m |a_j|$ and $Z(\mu) = \#\{1 \leq j \leq m : a_j = 0\}$. Furthermore, if (π, V) is any representation of $\mathrm{SO}(2m)$ and $\mu \in \mathbb{Z}^m$, we let

$$V(\mu) = \{v \in V : \pi(h) \cdot v = h^\mu v \quad \forall h \in T\},$$

where T is the maximal torus in (2.4) of $\mathrm{SO}(2m)$ and $h^\mu = e^{\mu(H)}$ for any $H \in \mathfrak{h}$ such that $\exp H = h$. The multiplicity of μ in V is $\dim V(\mu)$.

ult_P_k **Proposition 5.1.** *We denote by $\widetilde{m}_k(\mu)$ the multiplicity of $\mu \in \mathbb{Z}^m$ in $\mathcal{P}_k$. Then*

ult_P_k

$$\widetilde{m}_k(\mu) = \begin{cases} \dbinom{r+m-1}{m-1} & \text{if } r = \frac{1}{2}(k - |\mu|) \in \mathbb{N}_0, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Proof. For $1 \leq j \leq m$, let $\widetilde{x}_j = x_{2j-1} + ix_{2j}$ and $\widetilde{x}_{j+m} = x_{2j-1} - ix_{2j}$. We consider the basis in $\mathcal{P}_k$ given by

ase_P_k

$$\left\{ \widetilde{x}_1^{l_1} \dots \widetilde{x}_{2m}^{l_{2m}} : l_j \in \mathbb{N}_0\, \forall j,\ \textstyle\sum_{j=1}^{2m} l_j = k \right\}. \tag{5.2}$$

Given $h \in T$, one can check that $h \cdot \widetilde{x}_j = e^{-i\theta_j}\, \widetilde{x}_j$ and $h \cdot \widetilde{x}_{j+m} = e^{i\theta_j}\, \widetilde{x}_{j+m}$, then

asis_P_k

$$h \cdot \widetilde{x}_1^{l_1} \dots \widetilde{x}_{2m}^{l_{2m}} = e^{-i((l_1 - l_{m+1})\theta_1 + \cdots + (l_m - l_{2m})\theta_m)}\, \widetilde{x}_1^{l_1} \dots \widetilde{x}_{2m}^{l_{2m}} = e^{\mu(H)}\, \widetilde{x}_1^{l_1} \dots \widetilde{x}_{2m}^{l_{2m}}, \tag{5.3}$$

where $\mu = \sum_{j=1}^m (l_j - l_{j+m})\varepsilon_j$.

Equation (5.3) tells us that (5.2) is a basis of weight vectors of $\mathcal{P}_k$. It follows that an arbitrary element $\mu = \sum_j a_j\varepsilon_j \in \mathbb{Z}^m$ is a weight of $\mathcal{P}_k$ if and only if there are $l_1, \dots, l_{2m} \in \mathbb{N}_0$ such that $a_j = l_j - l_{j+m}$ and $\sum_j l_j = k$. The last condition is equivalent to $k - |\mu| \in 2\mathbb{N}_0$. In this case, if $k - |\mu| = 2r$ with $r \in \mathbb{N}_0$, then $\widetilde{m}_k(\mu)$ is equal to the different ways to write r as sum of m different nonnegative integer numbers, that is[10] $\binom{r+m-1}{m-1}$. □

We obtain the following Corollary, which proves the second assertion of Lemma 3.4 since $\pi_k^1 \in \widehat{\mathrm{SO}(2m)}$ has highest weight $k\varepsilon_1$.

Corollary 5.2. *Let π_k^1 be the irreducible representation of $\mathrm{SO}(2m)$ with highest weight $k\varepsilon_1$. Then*

$$m_{\pi_k^1}(\mu) = \begin{cases} \dbinom{r+m-2}{m-2} & \text{if } r = \frac{1}{2}(k - |\mu|) \in \mathbb{N}_0, \\ 0 & \text{otherwise.} \end{cases}$$

[10] REFERIR A ALGÚN LADO. Juan Pablo dice que es algo muy conocido

Proof. Since $\mathcal{P}_k = \mathcal{H}_k \oplus \mathcal{P}_{k-2}$, we have that $m_{k\varepsilon_1}(\mu) = \dim \mathcal{H}_k(\mu) = \widetilde{m}_k(\mu) - \widetilde{m}_{k-2}(\mu)$. The Corollary follows by applying Proposition 5.1. □

We now study the p-exterior representation $\bigwedge^p(\mathbb{C}^{2m})$ of $\mathrm{SO}(2m)$ for $0 \leq p \leq 2m$. The action of $\mathrm{SO}(2m)$ on $\bigwedge^p(\mathbb{C}^{2m})$ is given by $g \cdot (w_1 \wedge \cdots \wedge w_p) = (g.w_1) \wedge \cdots \wedge (g.w_p)$. If $p = 0$ or $2m$, this is the trivial representation. When $0 < p < m$ or $m < p < 2m$, the representation is irreducible with highest weight $\Lambda_1^p = \varepsilon_1 + \cdots + \varepsilon_p$, thus it is equivalent to π_1^p. If $p = m$ the representation splits as $\bigwedge_+^m(\mathbb{C}^{2m}) \oplus \bigwedge_-^m(\mathbb{C}^{2m})$ with each of those are irreducible with highest weight $\varepsilon_1 + \cdots + \varepsilon_{m-1} \pm \varepsilon_m$, thus $\bigwedge^m(\mathbb{C}^{2m})$ is equivalent π_1^m. Hence the p-exterior representation of $\mathrm{SO}(2m)$ is equivalent to π_1^p for every $1 \leq p \leq 2m-1$.

Proposition 5.3. *We denote by $\widetilde{m}^p(\mu)$ the multiplicity of $\mu \in \mathbb{Z}^m$ in $\bigwedge^p(\mathbb{C}^{2m})$. We write $\widetilde{p} = \min(p, 2m-p)$. Then*

$$\widetilde{m}^p(\mu) = \begin{cases} \binom{m - \widetilde{p} + 2r}{r} & \text{if } r = \frac{1}{2}(\widetilde{p} - |\mu|) \in \mathbb{N}_0 \text{ and } |a_j| \leq 1 \text{ for all } j, \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

Proof. We assume that $1 \leq p \leq m$ since the other cases are similar, thus $\widetilde{p} = p$. Let $\{e_1, \dots, e_{2m}\}$ be the canonical basis of $\mathbb{C}^{2m}$. For $1 \leq j \leq m$, we set $v_j = e_{2j-1} - ie_{2j}$ and $v_{j+m} = e_{2j-1} + ie_{2j}$. Hence $\{v_1, \dots, v_{2m}\}$ is also a basis of $\mathbb{C}^{2m}$ and

$$\{v_{j_1} \wedge \cdots \wedge v_{j_p} : j_1 < j_2 < \cdots < j_p\} \tag{5.5}$$

is a basis of $\bigwedge^p(\mathbb{C}^{2m})$. For $J = \{j_1 < j_2 < \cdots < j_p\}$ we write $\omega_J = v_{j_1} \wedge \cdots \wedge v_{j_p}$.

Let h be in the maximal torus T, then $h \cdot v_j = e^{i\theta_j} v_j$ and $h \cdot v_{j+m} = e^{-i\theta_j} v_{j+m}$. For $J = \{j_1 < \cdots < j_p\}$ we have that

$$h \cdot \omega_J = e^{\mu(H)}\, \omega_J, \tag{5.6}$$

where $\mu = \sum_{j=1}^m a_j \varepsilon_j$ with

$$a_j = \begin{cases} 1 & \text{if } j \in J \text{ and } j+m \notin J, \\ -1 & \text{if } j \notin J \text{ and } j+m \in J, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that an arbitrary element $\mu = \sum_j a_j \varepsilon_j \in \mathbb{Z}^m$ is a weight of $\bigwedge^p(\mathbb{C}^{2m})$ if and only if $p - |\mu| \in 2\mathbb{N}_0$ and $|a_j| \leq 1$ for all j.

Let $\mu = \sum_{j=1}^m a_j \varepsilon_j \in \mathbb{Z}^m$ such that $|a_j| \leq 1$ for all j and $r = \frac{1}{2}(p - |\mu|) \in \mathbb{N}_0$. Let $I = \{1 \leq j \leq m : a_j = 1\} \cup \{m+1 \leq j \leq 2m : a_{j-m} = -1\}$. Thus I has $p - 2r$ elements. It is a simple matter to check that ω_J is a weight vector with weight μ if and only if J has p elements, $I \subset J$ and satisfies that $j \in J \smallsetminus I \iff j+m \in J \smallsetminus I$. Clearly, there are $\binom{m-p+2r}{r}$ choices for J, hence $\widetilde{m}^p(\mu) = \binom{m-p+2r}{r}$. □

Following [IT78], we consider the subspace $\mathcal{P}_k^p$ of the complexified cotangent vector bundle $\bigwedge^p \mathbb{R}^{2m}$ of the manifolds $\mathbb{R}^{2m}$ generated by elements of the form

$$f(\bar{x})\, dx_{j_1} \wedge \cdots \wedge dx_{j_p},$$

where $f(\bar{x})$ is a homogeneous complex polynomial in the variables $\bar{x} = (x_1, \dots, x_{2m})^t$. Note that $\mathcal{P}_k^0 \cong \mathcal{P}_k$ and $\mathcal{P}_0^p \cong \bigwedge^p(\mathbb{C}^{2m})$. One can check that

$$\mathcal{P}_k^p \cong \mathcal{P}_k \otimes_{\mathbb{C}} \textstyle\bigwedge^p(\mathbb{C}^{2m}). \tag{5.7}$$

Proposition 5.4. *We denote by $\widetilde{m}_k^p(\mu)$ the multiplicity of $\mu \in \mathbb{Z}^m$ in $\mathcal{P}_k^p$. If $|\mu| = |\eta|$ and $Z(\mu) = Z(\eta)$, then $\widetilde{m}_k^p(\mu) = \widetilde{m}_k^p(\eta)$.*

Proof. We have to prove that $\widetilde{m}_k^p(\mu)$ depends only on $|\mu|$ and $Z(\mu)$. From (5.7) we have that

$$\widetilde{m}_k^p(\mu) = \sum_{\mu_1+\mu_2=\mu} \widetilde{m}_k(\mu_1)\, \widetilde{m}^p(\mu_2) = \sum_{\eta} \widetilde{m}_k(\mu-\eta)\, \widetilde{m}^p(\eta).$$

The sum is already over the weights η of $\bigwedge^p(\mathbb{C}^{2m})$. From Proposition 5.3, these weights are of the form $\eta = \sum_{j=1}^m a_j\varepsilon_j \in \mathbb{Z}^m$ such that $|a_j| \leq 1$ and $\widetilde{p} - |\eta| = 2r$ with $r \in \mathbb{N}_0$. We denote by Q the set of these weights.

According to the above remarks, we have that

de_m_k^p

$$\widetilde{m}_k^p(\mu) = \sum_{r=0}^{[\frac{\widetilde{p}}{2}]} \sum_{\eta\in Q:|\eta|=\widetilde{p}-2r} \widetilde{m}_k(\mu-\eta)\binom{m-\widetilde{p}+2r}{r} = \sum_{r=0}^{[\frac{\widetilde{p}}{2}]} \binom{m-\widetilde{p}+2r}{r} \sum_{\substack{\eta\in Q:\eta>0,\\ |\eta|=\widetilde{p}-2r}} \sum_{\sigma\in\mathbb{Z}_2^J} \widetilde{m}_k(\mu-\sigma(\eta)), \tag{5.8}$$

where $\eta = \sum_{j=1}^m a_j\varepsilon_j > 0$ means that $a_1 \geq \cdots \geq a_m \geq 0$, $J := \{1 \leq j \leq m : a_j \neq 0\}$, and $\mathbb{Z}_2^J$ denotes the group given by multiplying by ± 1 on the coordinates in J.

Claim: *For a fixed $\eta \in Q$ such that $\eta > 0$ and $|\eta| = \widetilde{p} - 2r$, the factor $\sum_{\sigma\in\mathbb{Z}_2^J} \widetilde{m}_k(\mu - \sigma(\eta))$ depends only on $|\mu|$ and the number of zero coordinates of μ for which η has a nonzero coordinate.*

We fix $\eta = \sum_{j=1}^m a_j\varepsilon_j \in Q$ such that $\eta > 0$ and $|\eta| = \widetilde{p} - 2r$. Recall that $a_j = \pm 1$ for every $j \in J$. Write $\mu = \sum_{j=1}^m b_j\varepsilon_j$.

We first consider the simplest case: if $b_j = 0$ for every $j \in J$, then $\mu - \sigma(\eta)$ has the same height $|\mu| + \#J = |\mu| + \widetilde{p} - 2r$ for every $\sigma \in \mathbb{Z}_2^J$, thus Proposition 5.1 implies that $\sum_{\sigma\in\mathbb{Z}_2^J} \widetilde{m}_k(\mu-\sigma(\eta))$ depends only on $|\mu|$ as we claimed. Actually,

$$\sum_{\sigma\in\mathbb{Z}_2^J} \widetilde{m}_k(\mu-\sigma(\eta)) = \begin{cases} 2^{\widetilde{p}-2r}\binom{\widetilde{r}+m-1}{m-1} & \text{if } \widetilde{r} = \frac{1}{2}(k-|\mu|-\widetilde{p}+2r) \in \mathbb{N}_0, \\ 0 & \text{otherwise.} \end{cases}$$

For the general case we set $J = J_0 \cup J_1$, $\ell_0 = \#J_0$ and $\ell_1 = \#J_1$, where $J_0 = \{j \in J : b_j = 0\}$ and $J_1 = \{j \in J : b_j \neq 0\}$. We fix $\sigma \in \mathbb{Z}_2^J$. We write $\sigma(\eta) = \sum_{j=1}^m \sigma_j a_j \varepsilon_j$ where $\sigma_j = \pm 1$ if $j \in J$ and $\sigma_j = 0$ otherwise. We set $I_\pm(\sigma) = \{j \in J_1 : \pm b_j\sigma_j a_j > 0\}$, thus $J_1 = I_+(\sigma) \cup I_-(\sigma)$ (disjoint union). Clearly, the height of $\mu - \sigma(\eta)$ is $|\mu|$ plus one for each coordinate in J_0, minus one for each coordinate in $I_+(\sigma)$, plus one for each coordinate in $I_-(\sigma)$. By setting $\alpha = \#I_+(\sigma)$, thus $\#I_-(\sigma) = \ell_1 - \alpha$, we have that $|\mu - \sigma(\eta)| = |\mu| + \ell_0 - \alpha + (\ell_1 - \alpha) = |\mu| + \ell_0 + \ell_1 - 2\alpha$.

By summing over the variable $t = \ell_1 - 2\alpha$, we conclude that

$$\sum_{\sigma\in\mathbb{Z}_2^J} \widetilde{m}_k(\mu-\sigma(\eta)) = \sum_t \binom{\ell_1}{\frac{\ell_1-|t|}{2}} A_t$$

where t runs over the integer numbers such that $-\ell_1 \leq t \leq \ell_1$ and $t \equiv \ell_1 \pmod 2$, and

$$A_t = \begin{cases} \binom{\widetilde{r}_t+m-1}{m-1} & \text{if } \widetilde{r}_t = \frac{1}{2}(k-|\mu|-\ell_0-t) \in \mathbb{N}_0, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, the mentioned factor depend only on $|\mu|$. ■

The Claim tells us that the factor $\sum_{\substack{\eta\in Q:\eta>0,\\ |\eta|=\widetilde{p}-2r}} \sum_{\sigma\in\mathbb{Z}_2^J} \widetilde{m}_k(\mu-\sigma(\eta))$ in (5.8) depends only on $|\mu|$ and $Z(\mu)$, which is the desired conclusion. ☐

Finally, we can prove the third assertion of Lemma 3.4.

Proof of Lemma 3.4 (3). By [Ik88, Prop. 1.12], we have that

$$V_{\pi_k^p} \cong \sum_{t=0}^{p} (-1)^t \left(\mathcal{P}_{k+t}^{p-t} - \mathcal{P}_{k-t-2}^{p-t}\right),$$

hence

$$m_{\pi_k^p}(\mu) = \sum_{t=0}^{p} (-1)^t \left(\widetilde{m}_{k+t}^{p-t}(\mu) - \widetilde{m}_{k-t-2}^{p-t}(\mu)\right),$$

which completes the proof by Proposition 5.4. ☐

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