METASTABILITY FOR SMALL RANDOM PERTURBATIONS OF A PDE WITH BLOW-UP

PABLO GROISMAN, SANTIAGO SAGLIETTI AND NICOLAS SAINTIER

Abstract. We study small random perturbations by additive space-time white noise of a reaction-diffusion equation with a unique stable equilibrium and solutions which blow up in finite time. We show that for initial data in the domain of attraction of the stable equilibrium the perturbed system exhibits metastable behavior: its time averages remain stable around this equilibrium until an abrupt and unpredictable transition occurs which leads to explosion in a finite (but exponentially large) time. On the other hand, for initial data in the domain of explosion we show that the explosion time of the perturbed system converges to the explosion time of the deterministic solution.

1. Introduction

We consider, for $\varepsilon > 0$, the stochastic process $U^{u,\varepsilon}$ which formally satisfies the stochastic partial differential equation

$$\left\{\begin{array}{rll} \partial_t U^{u,\varepsilon} & = \partial^2_{xx} U^{u,\varepsilon} + g(U^{u,\varepsilon}) + \varepsilon \dot{W} & t > 0\,,\ 0 < x < 1 \\ U^{u,\varepsilon}(t,0) & = U^{u,\varepsilon}(t,1) = 0 & t > 0 \\ U^{u,\varepsilon}(0,x) & = u(x) & \end{array}\right. \tag{1.1}$$

where $g : \mathbb{R} \to \mathbb{R}$ is given by $g(u) := u|u|^{p-1}$ for fixed $p > 1$, $\dot{W}$ is space-time white noise and u is a continuous function satisfying $u(0) = u(1) = 0$.

This process can be thought of as a random perturbation of the dynamical system U^u given by the solution of (1.1) with $\varepsilon = 0$, i.e. U^u satisfies the partial differential equation

$$\left\{\begin{array}{rll} \partial_t U^u & = \partial^2_{xx} U^u + g(U^u) & t > 0\,,\ 0 < x < 1 \\ U^u(t,0) & = 0 & t > 0 \\ U^u(t,1) & = 0 & t > 0 \\ U^u(0,x) & = u(x) & 0 < x < 1. \end{array}\right. \tag{1.2}$$

Equation (1.2) is of reaction-diffusion type, a broad class of evolution equations which naturally arise in the study of phenomena as diverse as diffusion of a fluid through a porous material, transport in a semiconductor, coupled chemical reactions with spatial diffusion, population genetics, among others. In all these cases, the equation represents an approximate model of the phenomenon and thus it is of interest to understand how its description might change if subject to small random perturbations.

An important feature of (1.2) is that it admits solutions which are only local in time and blow up in a finite time. Indeed, the system has a unique stable equilibrium, the null function $\mathbf{0}$, and a countable family of unstable equilibriums, all of which are saddle points. The stable equilibrium possesses a domain of attraction $\mathcal{D}_{\mathbf{0}}$ such that if $u \in \mathcal{D}_{\mathbf{0}}$ then the solution U^u of (1.2) with initial datum u is globally defined and converges to $\mathbf{0}$ as time tends to infinity. Similarly, each unstable equilibrium has its own stable manifold,

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the union of which constitutes the boundary of $\mathcal{D}_{\mathbf{0}}$. Finally, for $u \in \mathcal{D}_e := \overline{\mathcal{D}_{\mathbf{0}}}^c$ the system blows up in finite time, i.e. there exists a time $0 < \tau^u < +\infty$ such that the solution U^u is defined for all $t \in [0, \tau^u)$ but satisfies

$$\lim_{t \nearrow \tau^u} \|U^u(t, \cdot)\|_\infty = +\infty.$$

The behavior of the system is, in some aspects, similar to the double-well potential model studied in [1, 12]. Indeed, (1.2) can be reformulated as

$$\partial_t U^u = -\frac{\partial S}{\partial \varphi}(U^u)$$

where S is the potential formally given by

$$S(v) = \int_0^1 \left[\frac{1}{2} \left(\frac{dv}{dx} \right)^2 + G(v) \right],$$

where we take $G(v) := -\frac{|v|^{p+1}}{p+1}$ as opposed to the term $G(v) = \frac{\lambda}{4} v^4 - \frac{\mu}{2} v^2$ appearing in the double-well potential model. In our system, instead of having two wells, each being the domain of attraction of the two stable equilibriums of the system, we have only one which corresponds to $\mathcal{D}_{\mathbf{0}}$. Since our potential tends to $-\infty$ along every direction, we can imagine the second well in our case as being infinity and thus there is no return from there once the system reaches its bottom. Moreover, since the potential behaves like $-\lambda^{p+1}$ in every direction, if the system falls into this "infinite well" it will reach its bottom (infinity) in a finite time (blow-up).

Upon adding a small noise to (1.2), one wonders if there are any qualitative differences in behavior between the deterministic system (1.2) and its stochastic perturbation (1.1). For short times both systems should behave similarly, since in this case the noise term will be typically of much smaller order than the remaining terms in the right hand side of (1.1). However, due to the independent and normally distributed increments of the perturbation, when given enough time the noise term will eventually reach sufficiently large values so as to induce a significant change of behavior in (1.1). We are interested in understanding what changes might occur in the blow-up phenomenon due to this situation and, more precisely, which are the asymptotic properties as $\varepsilon \to 0$ of the explosion time of (1.1) for the different initial data. Based on all of the considerations above, we expect the following scenario:

i. *Thermalization.* For initial data in $\mathcal{D}_{\mathbf{0}}$, the stochastic system is at first attracted towards this equilibrium. Once near it, the terms in the right hand side of (1.2) become negligible and so the process is then pushed away from the equilibrium by noise. Being away from $\mathbf{0}$, the noise becomes overpowered by the remaining terms in the right hand side of (1.1) and this allows for the previous pattern to repeat itself: a large number of attempts to escape from the equilibrium, followed by a strong attraction towards it.

ii. *Tunneling.* Eventually, after many frustrated attempts, the process succeeds in escaping $\mathcal{D}_{\mathbf{0}}$ and reaches the domain of explosion, the set of initial data for which (1.2) blows up in finite time. Since the probability of such an event is very small, we expect this escape time to be exponentially large. Furthermore, due to the large number of attempts that are necessary, we also expect this time to show little memory of the initial data.

iii. *Final excursion.* Once inside the domain of explosion, the stochastic system is forced to explode by the dominating source term g.

This type of phenomenon is known as *metastability*: the system behaves for a long time as if it were under equilibrium, but then performs an abrupt transition towards the real equilibrium (in our case, towards infinity). The former description was proved rigorously for the (infinite-dimensional) double-well potential model in [1, 12], inspired by the work in [10] for its finite-dimensional analogue. Their proofs rely heavily on large deviations estimates for $U^{u,\varepsilon}$ established in [8] for the infinite-dimensional system and in [9] for the finite-dimensional setting. In our case, we are only capable of proving the existence of local solutions of (1.1) and in fact, explosions *will* occur for $U^{u,\varepsilon}$. As a consequence, we will not be able to apply these same estimates directly, as the validity of these estimates relies on a proper control of the growth of solutions which does not hold in our setting. Localization techniques apply reasonably well to deal with the process until it escapes any fixed bounded domain but they cannot be used to say what happens from then onwards. Since we wish to focus specifically on trajectories that blow up in finite time, it is clear that a new approach is needed for this last part, one which involves a careful study of the blow-up phenomenon. Unfortunately, when dealing with perturbations of differential equations with blow-up, understanding how the behavior of the blow-up time is modified or even showing the persistence of the blow-up phenomenon itself is by no means an easy task in most cases. There are no general results addressing this matter, not even for nonrandom perturbations. This is why the usual approach to this kind of problems is to consider particular models such as ours.

The article is organized as follows. In Section 2 we give some preliminary definitions, introduce the local Freidlin-Wentzell estimates and then detail the results of this article. In Section 3 we give a detailed description of the deterministic system (1.2). Section 4 focuses on the explosion time of the stochastic system for initial data in the domain of explosion. The construction of an auxiliary domain G is performed in Section 5 and the study of the escape from G is carried out in Section 6. In Section 7 we establish metastable behavior for solutions with initial data in the domain of attraction of the stable equilibrium. Finally, we include at the end an appendix with some auxiliary results to be used throughout our analysis.

2. Definitions and results

2.1. **The deterministic PDE.** Our purpose in this section is to study equation (1.2). We assume that the source term $g : \mathbb{R} \to \mathbb{R}$ is given by $g(u) = u|u|^{p-1}$ for fixed $p > 1$ and also that u belongs to the space $C_D([0,1])$ of continuous functions on $[0,1]$ satisfying homogeneous Dirichlet boundary conditions, namely

$$C_D([0,1]) = \{v \in C([0,1]) : v(0) = v(1) = 0\}.$$

The space $C_D([0,1])$ is endowed with the supremum norm, i.e.

$$\|v\|_\infty = \sup_{x\in[0,1]} |v(x)|.$$

For any choice of $r > 0$ and $v \in C_D([0,1])$, we let $B_r(v)$ denote the closed ball in $C_D([0,1])$ of center v and radius r. Whenever the center is the null function $\mathbf{0}$, we simply write B_r. Equation (1.2) can be reformulated as

$$\partial_t U = -\frac{\partial S}{\partial \varphi}(U) \tag{2.1}$$

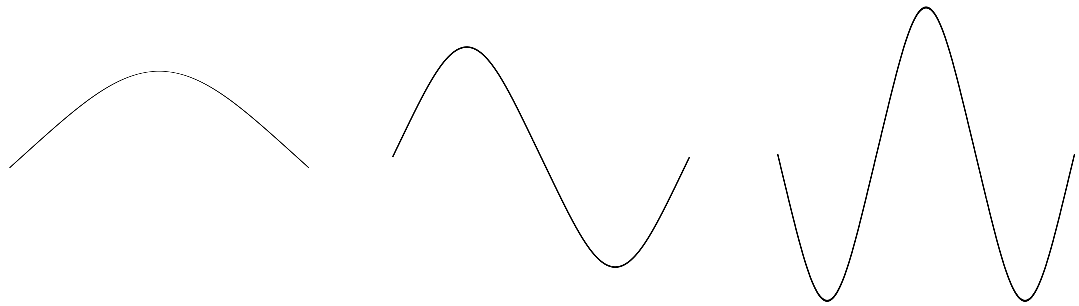

FIGURE 1. Examples of unstable equilibria: z, $z^{(2)}$ and $z^{(-3)}$.

where the *potential* S is the functional on $C_D([0,1])$ given by

$$S(v) = \begin{cases} \displaystyle\int_0^1 \left[\frac{1}{2}\left(\frac{dv}{dx}\right)^2 - \frac{|v|^{p+1}}{p+1} \right] & \text{if } v \in H_0^1((0,1)) \\ +\infty & \text{otherwise.} \end{cases}$$

Here $H_0^1((0,1))$ denotes the Sobolev space of square-integrable functions defined on $[0,1]$ with square-integrable weak derivative which vanish at the boundary $\{0,1\}$. Recall that $H_0^1((0,1))$ can be embedded into $C_D([0,1])$ so that the potential is indeed well-defined. We refer the reader to the appendix for a review of some of the main properties of S which shall be required throughout our work.

The formulation on (2.1) is interpreted as the validity of

$$\int_0^1 \partial_t U(t,x)\varphi(x)dx = \lim_{h\to 0} \frac{S(U+h\varphi)-S(U)}{h}$$

for any $\varphi \in C^1([0,1])$ with $\varphi(0) = \varphi(1) = 0$. It is known that for any $u \in C_D([0,1])$ there exists a unique solution U^u to equation (1.2) defined on some maximal time interval $[0,\tau^u)$ where $0 < \tau^u \leq +\infty$ is called the *explosion time* of U^u (see [17] for further details). In general, we will say that this solution belongs to the space

$$C_D([0,\tau^u) \times [0,1]) = \{v \in C([0,\tau^u) \times [0,1]) : v(\cdot,0) = v(\cdot,1) \equiv 0\}.$$

However, whenever we wish to make its initial datum u explicit we will do so by saying that the solution belongs to the space

$$C_{D_u}([0,\tau^u) \times [0,1]) = \{v \in C([0,\tau^u) \times [0,1]) : v(0,\cdot) = u \text{ and } v(\cdot,0) = v(\cdot,1) \equiv 0\}.$$

The origin $\mathbf{0} \in C_D([0,1])$ is the unique stable equilibrium of the system and is in fact asymptotically stable. It corresponds to the unique local minimum of the potential S. There is also a family of unstable equilibria of the system corresponding to the remaining critical points of the potential S, all of which are saddle points. Among these unstable equilibria there exists only one of them which is nonnegative, which we shall denote by z. It can be shown that this equilibrium z is in fact strictly positive for $x \in (0,1)$, symmetric with respect to the axis $x = \frac{1}{2}$ (i.e. $z(x) = z(1-x)$ for every $x \in [0,1]$) and that is both of minimal potential and minimal norm among all the unstable equilibria. The remaining equilibria are obtained by alternating scaled copies of both z and $-z$ as Figure 1 shows. We establish this fact rigurously in Section 3.

2.2. **Definition of solution for the SPDE.** In general, equations like (1.1) do not admit strong solutions in the usual sense as they may not be globally defined but instead defined *up to an explosion time*. In the following we formalize the idea of explosion and properly define the concept of solutions of (1.1).

First, we fix a probability space $(\Omega, \mathcal{F}, P)$ on which we have defined a Brownian sheet

$$W = \{W(t,x) : (t,x) \in \mathbb{R}^+ \times [0,1]\},$$

i.e. a stochastic process satisfying the following properties:

i. W has continuous paths, i.e. $(t,x) \mapsto W(t,x)(\omega)$ is continuous for every $\omega \in \Omega$.

ii. W is a centered Gaussian process with covariance structure given by

$$\mathrm{Cov}(W(t,x), W(s,y)) = (t \wedge s)(x \wedge y)$$

for every $(t,x), (s,y) \in \mathbb{R}^+ \times [0,1]$.

Then, for every $t \geq 0$ we define

$$\mathcal{G}_t = \sigma(W(s,x) : 0 \leq s \leq t, x \in [0,1])$$

and denote its augmentation by $\mathcal{F}_t$.[1] The family $(\mathcal{F}_t)_{t\geq 0}$ constitutes a filtration on $(\Omega, \mathcal{F})$. A *solution up to an explosion time* of the equation (1.1) on $(\Omega, \mathcal{F}, P)$ with respect to the Brownian sheet W and with initial datum $u \in C_D([0,1])$ is a stochastic process $U^{u,\varepsilon} = \{U^{u,\varepsilon}(t,x) : (t,x) \in \mathbb{R}^+ \times [0,1]\}$ satisfying the following properties:

i. $U^{u,\varepsilon}(0,\cdot) \equiv u$

ii. $U^{u,\varepsilon}$ has continuous paths taking values in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

iii. $U^{u,\varepsilon}$ is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e. for every $t \geq 0$ the mapping

$$(\omega, x) \mapsto U^{u,\varepsilon}(t,x)(\omega)$$

is $\mathcal{F}_t \otimes \mathcal{B}([0,1])$-measurable.

iv. If Φ denotes the fundamental solution of the heat equation on the interval $[0,1]$ with homogeneous Dirichlet boundary conditions, which is given by the formula

$$\Phi(t,x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n\in\mathbb{Z}} \left[\exp\left(-\frac{(2n+y-x)^2}{4t} \right) - \exp\left(-\frac{(2n+y+x)^2}{4t} \right) \right],$$

and for $n \in \mathbb{N}$ we define the stopping time $\tau_\varepsilon^{(n),u} := \inf\{t > 0 : \|U^{u,\varepsilon}(t,\cdot)\|_\infty \geq n\}$ then for every $n \in \mathbb{N}$ we have P-a.s.:

- $$\int_0^1 \int_0^{t\wedge\tau_\varepsilon^{(n),u}} |\Phi(t \wedge \tau_\varepsilon^{(n),u} - s, x, y) g(U^{u,\varepsilon}(s,y))| dsdy < +\infty \text{ for all } t \in \mathbb{R}^+$$

- $U^{u,\varepsilon}(t \wedge \tau_\varepsilon^{(n),u}, x) = I_H^{(n)}(t,x) + I_N^{(n)}(t,x)$ for all $(t,x) \in \mathbb{R}^+ \times [0,1]$, where

$$I_H^{(n)}(t,x) = \int_0^1 \Phi(t \wedge \tau_\varepsilon^{(n),u}, x, y) u(y) dy$$

and

$$I_N^{(n)}(t,x) = \int_0^{t\wedge\tau_\varepsilon^{(n),u}} \int_0^1 \Phi(t \wedge \tau_\varepsilon^{(n),u} - s, x, y) \left(g(U^{u,\varepsilon}(s,y)) dyds + \varepsilon dW(s,y) \right).$$

We call the random variable $\tau_\varepsilon^u := \lim_{n\to+\infty} \tau_\varepsilon^{(n),u}$ the *explosion time* of $U^{u,\varepsilon}$. Notice that the assumption of continuity of $U^{u,\varepsilon}$ over $\overline{\mathbb{R}}$ implies that:

[1]This means that $\mathcal{F}_t = \sigma(\mathcal{G}_t \cup \mathcal{N})$ where $\mathcal{N}$ denotes the class of all P-null sets of $\mathcal{G}_\infty = \sigma(\mathcal{G}_t : t \in \mathbb{R}^+)$.

- $\tau^u_\varepsilon = \inf\{t > 0 : \|U^{u,\varepsilon}(t,\cdot)\|_\infty = +\infty\}$
- $\|U^{u,\varepsilon}((\tau^u_\varepsilon)^-,\cdot)\|_\infty = \|U^{u,\varepsilon}(\tau^u_\varepsilon,\cdot)\|_\infty = +\infty$ on $\{\tau^u_\varepsilon < +\infty\}$.

We stipulate that $U^{u,\varepsilon}(t,\cdot) \equiv U^{u,\varepsilon}(\tau^u_\varepsilon,\cdot)$ for $t \geq \tau$ whenever $\tau^u_\varepsilon < +\infty$ but we do not assume that $\lim_{t\to+\infty} U^{u,\varepsilon}(t,\cdot)$ exists if $\tau^u_\varepsilon = +\infty$. Furthermore, since any initial datum $u \in C_D([0,1])$ is bounded, we always have $P(\tau^u_\varepsilon > 0) = 1$. It can be shown that there exists a (pathwise) unique solution $U^{u,\varepsilon}$ of (1.1) up to an explosion time and that it has the strong Markov property, i.e. if $\tilde\tau$ is a stopping time of $U^{u,\varepsilon}$ then, conditional on $\tilde\tau < \tau^u_\varepsilon$ and $U^{u,\varepsilon}(\tilde\tau,\cdot) = v$, the future $\{U^{u,\varepsilon}(t+\tilde\tau,\cdot) \colon 0 < t < \tau^u_\varepsilon - \tilde\tau\}$ is independent of the past $\{U^{u,\varepsilon}(s,\cdot) \colon 0 \leq s \leq \tilde\tau\}$ and identical in law to the solution of (1.1) with initial datum v. We refer to [13, 21] for details.

2.3. **Local Freidlin-Wentzell estimates.** One of the main tools we use in the study of solutions of (1.1) is the local large deviations principle we briefly describe next.

Given $u \in C_D([0,1])$ and $T > 0$, we consider the metric space of continuous functions

$$C_{D_u}([0,T]\times[0,1]) = \{v \in C([0,T]\times[0,1]) : v(0,\cdot) = u \text{ and } v(\cdot,0) = v(\cdot,1) \equiv 0\}$$

with the distance d_T induced by the supremum norm, i.e. for $v, w \in C_{D_u}([0,T]\times[0,1])$

$$d_T(v,w) := \sup_{(t,x)\in[0,T]\times[0,1]} |v(t,x) - w(t,x)|,$$

and define the rate function $I^u_T : C_{D_u}([0,T]\times[0,1]) \to [0,+\infty]$ by the formula

$$I^u_T(\varphi) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^T\int_0^1 |\partial_t\varphi - \partial_{xx}\varphi - g(\varphi)|^2 & \text{if } \varphi \in W^{1,2}_2([0,T]\times[0,1])\,,\ \varphi(0,\cdot) = u \\ +\infty & \text{otherwise.} \end{cases}$$

Here $W^{1,2}_2([0,T]\times[0,1])$ is the closure of $C^\infty([0,T]\times[0,1])$ with respect to the norm

$$\|\varphi\|_{W^{1,2}_2} = \left(\int_0^T\int_0^1 \left[|\varphi|^2 + |\partial_t\varphi|^2 + |\partial_x\varphi|^2 + |\partial_{xx}\varphi|^2\right]\right)^{\frac{1}{2}},$$

i.e. the Sobolev space of square-integrable functions defined on $[0,T]\times[0,1]$ with one square-integrable weak time derivative and two square-integrable weak space derivatives.

By following the lines of [1, 8, 20], it is possible to establish a large deviations principle for solutions of (1.1) with rate function I as given above whenever the source term g is globally Lipschitz. Unfortunately, this is not the case for us. Nonetheless, by employing localization arguments like the ones carried out in [11], one can obtain a weaker version of this principle which only holds locally, i.e. while the process remains inside any fixed bounded region. More precisely, we have the following result.

Theorem 2.1. *If for each $n \in \mathbb{N}$ and $u \in C_D([0,1])$ we define*

$$\tau^{(n),u} := \inf\{t > 0 : \|U^u(t,\cdot)\|_\infty \geq n\} \qquad \text{and} \qquad \mathcal{T}^{(n),u}_\varepsilon := \tau^{(n),u}_\varepsilon \wedge \tau^{(n),u}$$

where $\tau^{(n),u}_\varepsilon$ is defined as in Section 2.2, then the following estimates hold:

- **Lower bound**. *For any $\delta, h > 0$ and $n \in \mathbb{N}$, there exists ε_0 such that*

$$P\left(d_{T\wedge\mathcal{T}^{(n),u}_\varepsilon}(U^{u,\varepsilon},\varphi) < \delta\right) \geq e^{-\frac{I^u_T(\varphi)+h}{\varepsilon^2}} \tag{2.2}$$

for all $0 < \varepsilon < \varepsilon_0$, $u \in C_D([0,1])$ and $\varphi \in C_{D_u}([0,T]\times[0,1])$ with $\|\varphi\|_\infty \leq n$.

- **Upper bound**. *For any $\delta > 0$ there exist $\varepsilon_0 > 0$ and $C > 0$ such that*

$$\sup_{u \in C_D([0,1])} P\left(d_{T \wedge \mathcal{T}_\varepsilon^{(n),u}}(U^{u,\varepsilon}, U^u) > \delta\right) \leq e^{-\frac{C}{\varepsilon^2}}, \tag{2.3}$$

for all $0 < \varepsilon < \varepsilon_0$.

The usual large deviations estimates for these type of systems usually feature a more refined version of the upper bound than the one we give here (see [1], for example). However, the estimate in (2.3) is enough for our purposes and so we do not pursue any generalizations of it here. Also, notice that both estimates are somewhat uniform in the initial datum. This uniformity is obtained as in [1] by using the fact that g is Lipschitz when restricted to bounded sets. We refer to [1, 8] for further details.

2.4. **Main results.** Our purpose is to study the asymptotic behavior as $\varepsilon \to 0$ of $U^{u,\varepsilon}$, the solution of (1.1), for the different initial data $u \in C_D([0,1])$. We now state our results. In the following we write P_u to denote the distribution of $U^{u,\varepsilon}$. Whenever we choose to make the initial datum clear in this way, we will drop the superscript u from the notation for simplicity purposes.

Our first result is concerned with the continuity of the explosion time for initial data in the domain of explosion $\mathcal{D}_e$. In this case one expects the stochastic and deterministic systems to both exhibit a similar behavior for $\varepsilon > 0$ sufficiently small, since then the noise will not be able to grow fast enough so as to overpower the quickly exploding source term. We show this to be truly the case for $u \in \mathcal{D}_e$ such that U^u remains bounded from one side.

Theorem 2.2. *Let $\mathcal{D}_e^*$ be the set of initial data $u \in \mathcal{D}_e$ such that U^u explodes only through one side, i.e. U^u remains bounded either from below or above until its explosion time τ^u. Then given $\delta > 0$ and a bounded set $\mathcal{K} \subseteq \mathcal{D}_e^*$ at a positive distance from $\partial\mathcal{D}_e^*$ there exists a constant $C > 0$ such that*

$$\sup_{u \in \mathcal{K}} P_u(|\tau_\varepsilon - \tau| > \delta) \leq e^{-\frac{C}{\varepsilon^2}}.$$

The main differences in behavior between the stochastic and deterministic systems appear for initial data in $\mathcal{D}_{\mathbf{0}}$, where metastable behavior is observed. According to the characterization of metastability for stochastic processes in [3, 10], this behavior is given by two facts: the time averages of the process remain stable until an abrupt transition occurs and a different value is attained; furthermore, the time of this transition is unpredictable in the sense that, when suitably rescaled, it should have an exponential distribution. We manage to establish this description rigorously for our system whenever $1 < p < 5$. This rigorous description is contained in the remaining results.

Define the quantity $\Delta := 2(S(z) - S(\mathbf{0}))$. Our next result states that for any $u \in \mathcal{D}_{\mathbf{0}}$ the asymptotic magnitude of τ_ε^u is, up to logarithmic equivalence, of order $e^{\frac{\Delta}{\varepsilon^2}}$.

Theorem 2.3. *Given $\delta > 0$ and a bounded set $\mathcal{K} \subseteq \mathcal{D}_{\mathbf{0}}$ at a positive distance from $\partial\mathcal{D}_{\mathbf{0}}$, if $1 < p < 5$ then we have*

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in \mathcal{K}} \left| P_u\left(e^{\frac{\Delta - \delta}{\varepsilon^2}} < \tau_\varepsilon < e^{\frac{\Delta + \delta}{\varepsilon^2}}\right) - 1\right|\right] = 0.$$

Theorem 2.3 suggests that, for initial data $u \in \mathcal{D}_{\mathbf{0}}$, the typical route of $U^{u,\varepsilon}$ towards infinity involves passing through one of the unstable equilibria of minimal energy, $\pm z$. This seems reasonable since, as we will see in Section 5, for $1 < p < 5$ the barrier imposed by the potential S is the lowest there. The following result establishes this fact rigorously.

Theorem 2.4. *Given $\delta > 0$ and a bounded set $\mathcal{K} \subseteq \mathcal{D}_{\mathbf{0}}$ at a positive distance from $\partial\mathcal{D}_{\mathbf{0}}$, if $1 < p < 5$ then we have*

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in \mathcal{K}} |P_u \left(\tau_\varepsilon(\mathcal{D}^c_{\mathbf{0}}) < \tau_\varepsilon \, , \, U^\varepsilon(\tau_\varepsilon(\mathcal{D}^c_{\mathbf{0}}), \cdot) \in B_\delta(\pm z) \right) - 1| \right] = 0,$$

where $\tau^u_\varepsilon(\mathcal{D}^c_{\mathbf{0}}) := \inf\{t > 0 : U^{u,\varepsilon}(t, \cdot) \notin \mathcal{D}_{\mathbf{0}}\}$ and $B_\delta(\pm z) := B_\delta(z) \cup B_\delta(-z)$.

Our next result is concerned with the the asymptotic loss of memory of τ^u_ε. For $\varepsilon > 0$ define the scaling coefficient

$$\beta_\varepsilon = \inf\{t \geq 0 : P_{\mathbf{0}}(\tau_\varepsilon > t) \leq e^{-1}\} \tag{2.4}$$

Observe that Theorem 2.3 implies that the family $(\beta_\varepsilon)_{\varepsilon>0}$ satisfies $\lim_{\varepsilon\to 0} \varepsilon^2 \log \beta_\varepsilon = \Delta$. This next result states that for any $u \in \mathcal{D}_{\mathbf{0}}$ the normalized explosion time $\frac{\tau^u_\varepsilon}{\beta_\varepsilon}$ converges in distribution to an exponential random variable of mean one.

Theorem 2.5. *Given $\delta > 0$ and a bounded set $\mathcal{K} \subseteq \mathcal{D}_{\mathbf{0}}$ at a positive distance from $\partial\mathcal{D}_{\mathbf{0}}$, if $1 < p < 5$ then for any $t > 0$ we have*

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in \mathcal{K}} \left| P_u(\tau_\varepsilon > t\beta_\varepsilon) - e^{-t} \right| \right] = 0.$$

Finally, we show the stability of time averages of continuous functions evaluated along paths of the process starting in $\mathcal{D}_{\mathbf{0}}$, i.e. they remain close to the value of the function at $\mathbf{0}$. These time averages are taken along intervals of length going to infinity and times may be taken as being almost (in a suitable scale) the explosion time. This tells us that, up until the explosion time, the system spends most of its time in a small neighborhood of $\mathbf{0}$.

Theorem 2.6. *There exists a sequence $(R_\varepsilon)_{\varepsilon>0}$ with $\lim_{\varepsilon\to 0} R_\varepsilon = +\infty$ and $\lim_{\varepsilon\to 0} \frac{R_\varepsilon}{\beta_\varepsilon} = 0$ such that given $\delta > 0$ for any bounded set $\mathcal{K} \subseteq \mathcal{D}_{\mathbf{0}}$ at a positive distance from $\mathcal{W}$ we have*

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in \mathcal{K}} P_u \left(\sup_{0 \leq t \leq \tau_\varepsilon - 3R_\varepsilon} \left| \frac{1}{R_\varepsilon} \int_t^{t+R_\varepsilon} f(U^\varepsilon(s, \cdot))ds - f(\mathbf{0}) \right| > \delta \right) \right] = 0$$

for any bounded continuous function $f : C_D([0, 1]) \to \mathbb{R}$.

Theorem 2.2 is proved in Section 4, the remaining results are proved in Sections 6 and 7. Perhaps the proof of Theorem 2.2 is where one finds the most differences with other works in the literature dealing with similar problems, namely [10, 12]. This is due to the fact that for this part we cannot use large deviations estimates as these do. The remaining results were established in [1, 12] for the tunneling time in an infinite-dimensional double-well potential model, i.e. the time the system takes to go from one well to the bottom of the other one. Our proofs are similar to the ones found in these references, although we have the additional difficulty of dealing with solutions which are not globally defined.

3. Phase diagram of the deterministic system

In this section we review the behavior of solutions to (1.2) for the different initial data in $C_D([0, 1])$. We begin by characterizing the unstable equilibria of the system.

Proposition 3.1. A function $w \in C_D([0, 1])$ is an equilibrium of the system if and only if there exists $n \in \mathbb{Z}$ such that $w = z^{(n)}$, where for each $n \in \mathbb{N}$ we define $z^{(n)} \in C_D([0, 1])$ by the formula

$$z^{(n)}(x) = \begin{cases} n^{\frac{2}{p-1}} z(nx - [nx]) & \text{if } [nx] \text{ is even} \\ \\ -n^{\frac{2}{p-1}} z(nx - [nx]) & \text{if } [nx] \text{ is odd} \end{cases}$$

and also define $z^{(-n)} := -z^{(n)}$ and $z^{(0)} := \mathbf{0}$. Furthermore, for each $n \in \mathbb{Z}$ we have

$$\|z^{(n)}\|_\infty = |n|^{\frac{2}{p-1}} \|z\|_\infty \qquad \text{and} \qquad S(z^{(n)}) = |n|^{2\left(\frac{p+1}{p-1}\right)} S(z).$$

Proof. It is simple to verify that for each $n \in \mathbb{Z}$ the function $z^{(n)}$ is an equilibrium of the system and that each $z^{(n)}$ satisfies both $\|z^{(n)}\|_\infty = |n|^{\frac{2}{p-1}}\|z\|_\infty$ and $S(z^{(n)}) = |n|^{2\left(\frac{p+1}{p-1}\right)} S(z)$. Therefore, we must only check that for any equilibrium of the system $w \in C_D([0,1]) - \{\mathbf{0}\}$ there exists $n \in \mathbb{N}$ such that w coincides with either $z^{(n)}$ or $-z^{(n)}$.

Thus, let $w \in C_D([0,1]) - \{\mathbf{0}\}$ be an equilibrium of (1.2) and define the sets

$$G^+ = \{x \in (0,1) : w(x) > 0\} \qquad \text{and} \qquad G^- = \{x \in (0,1) : w(x) < 0\}.$$

Since $w \neq \mathbf{0}$ at least one of these sets must be nonempty. On the other hand, if only one of them is nonempty then, since z is the unique nonnegative equilibrium different from $\mathbf{0}$, we must have either $w = z$ or $w = -z$. Therefore, we may assume that both G^+ and G^- are nonempty. Notice that since G^+ and G^- are open sets we may write them as

$$G^+ = \bigcup_{k \in \mathbb{N}} I_k^+ \qquad \text{and} \qquad G^- = \bigcup_{k \in \mathbb{N}} I_k^-$$

where the unions are disjoint and each $I_k^\pm$ is a (possibly empty) open interval.

We first show that each union must be finite. Take $k \in \mathbb{N}$ and suppose we can write $I_k^+ = (a_k, b_k)$ for some $0 \leq a_k < b_k \leq 1$. It is easy to check that $\tilde{w}_k : [0,1] \to \mathbb{R}$ given by

$$\tilde{w}_k(x) := (b_k - a_k)^{\frac{2}{p-1}} w(a_k + (b_k - a_k)x)$$

is a nonnegative equilibrium of the system different from $\mathbf{0}$ and thus it must be $\tilde{w}_k = z$. This implies that $\|w\|_\infty \geq (b_k - a_k)^{-\frac{2}{p-1}} \|\tilde{w}_k\|_\infty = (b_k - a_k)^{-\frac{2}{p-1}} \|z\|_\infty$ from where we see that an infinite number of nonempty I_k^+ would contradict the fact that w is bounded. Thus, we see that G^+ is a finite union of open intervals and by symmetry so is G^-. The same argument also implies that for each interval $I_k^\pm = (a_k, b_k)$ the graph of $w|_{I_k^\pm}$ coincides with that of $\pm z$ but when scaled by the factor $(b_k - a_k)^{-\frac{2}{p-1}}$. More precisely, for all $x \in [0,1]$ we have

$$w(a_k + (b_k - a_k)x) = \pm (b_k - a_k)^{-\frac{2}{p-1}} z(x). \tag{3.1}$$

Now, Hopf's Lemma [7, p. 330] implies that $\partial_x z(0^+) > 0$ and $\partial_x z(1^-) < 0$. Furthermore, since z is symmetric with respect to $x = \frac{1}{2}$ we have in fact that $\partial_x z(0^+) = -\partial_x z(1^-) > 0$. In light of (3.1) and the fact that w is everywhere differentiable, the former tells us that plus and minus intervals must present themselves in alternating order, that their closures cover all of $[0,1]$ and also that their lengths are all the same. Combining this with (3.1) we conclude the proof. □

As a consequence of Proposition 3.1 we obtain the following important corollary.

Corollary 3.2. The functions $\pm z$ minimize the pontential S and the infinity norm among all the unstable equilibria of (1.2). In particular, we have that $\inf_{u \in \mathcal{W}} S(u) = S(\pm z)$.

Proof. The first statement is clear from Proposition 3.1 while the second one is deduced from the first since the mapping $t \mapsto S(U^u(t,\cdot))$ is monotone decreasing and continuous for any $u \in H_0^1((0,1))$ (see Proposition 8.8). □

Concerning the asymptotic behavior of solutions to (1.2), the following dichotomy was proved by Cortázar and Elgueta in [4].

Proposition 3.3. Let U^u denote the solution to (1.2) with initial datum $u \in C_D([0,1])$. Then one of these two possibilities must hold:

i. $\tau^u < +\infty$ and U^u blows up as $t \nearrow \tau^u$, i.e. $\lim_{t \nearrow \tau^u} \|U^u(t,\cdot)\|_\infty = +\infty$
ii. $\tau^u = +\infty$ and U^u converges to some stationary solution $z^{(n)}$ as $t \to +\infty$.

Proposition 3.3 allows us to split the space $C_D([0,1])$ of initial data into three parts

$$C_D([0,1]) = \mathcal{D}_{\mathbf{0}} \cup \mathcal{W} \cup \mathcal{D}_e \tag{3.2}$$

where $\mathcal{D}_{\mathbf{0}}$ denotes the domain of attraction of the origin $\mathbf{0}$, $\mathcal{W}$ is the union of all stable manifolds of the unstable equilibria and $\mathcal{D}_e$ is the domain of explosion of the system, i.e. the set of all initial data for which the system explodes in finite time. It can be seen that both $\mathcal{D}_{\mathbf{0}}$ and $\mathcal{D}_e$ are open sets and that $\mathcal{W}$ is the common boundary separating them. The following proposition gives a useful characterization of the domain of explosion $\mathcal{D}_e$.

Proposition 3.4 ([17, Theorem 17.6])**.** The domain of explosion $\mathcal{D}_e$ satisfies

$$\mathcal{D}_e = \{u \in C_D([0,1]) : S(U^u(t,\cdot)) < 0 \text{ for some } 0 \leq t < \tau^u\}.$$

Furthermore, we have $\lim_{t \nearrow \tau^u} S(U^u(t,\cdot)) = -\infty$.

From these results one can obtain a precise description of the domains $\mathcal{D}_{\mathbf{0}}$ and $\mathcal{D}_e$ in the region of nonnegative data. Cortázar and Elgueta proved the following result in [5].

Proposition 3.5.

i. Assume $u \in C_D([0,1])$ is nonnegative and such that U^u is globally defined and converges to z as $t \to +\infty$. Then for $v \in C_D([0,1])$ we have that:
   - $\mathbf{0} \lneq v \lneq u \Longrightarrow U^v$ is globally defined and converges to $\mathbf{0}$ as $t \to +\infty$.
   - $u \lneq v \Longrightarrow U^v$ explodes in finite time.

ii. For every nonnegative $u \in C_D([0,1])$ there exists $\lambda_c^u > 0$ such that for every $\lambda > 0$:
   - $0 < \lambda < \lambda_c^u \Longrightarrow U^{\lambda u}$ is globally defined and converges to $\mathbf{0}$ as $t \to +\infty$.
   - $\lambda = \lambda_c^u \Longrightarrow U^{\lambda u}$ is globally defined and converges to z as $t \to +\infty$.
   - $\lambda > \lambda_c^u \Longrightarrow U^{\lambda u}$ explodes in finite time.

This last result yields the existence of an unstable manifold of the saddle point z which is contained in the region of nonnegative initial data and which we shall denote by $\mathcal{W}_u^z$. It is 1-dimensional, has nonempty intersection with both $\mathcal{D}_{\mathbf{0}}$ and $\mathcal{D}_e$ and joins z with $\mathbf{0}$. By symmetry, a similar description also holds for the opposite unstable equilibrium $-z$. Figure 2 depicts the decomposition in (3.2) together with the unstable manifolds $\mathcal{W}_u^{\pm z}$. By exploiting the structure of the remaining unstable equilibria given by Proposition 3.1 one can verify for each of them the analogue of (ii) in Proposition 3.5. Details in [18].

4. Asymptotic behavior of τ_ε^u for $u \in \mathcal{D}_e$

In this section we investigate the continuity properties of the explosion time τ_ε^u for initial data in the domain of explosion $\mathcal{D}_e$. We show that, under suitable conditions on the initial datum $u \in \mathcal{D}_e$, the random explosion time τ_ε^u converges in probability to the deterministic explosion time τ^u as $\varepsilon \to 0$. To be more precise, let us consider the sets of initial data in $\mathcal{D}_e$ which explode only through $+\infty$ or $-\infty$, i.e.

$$\mathcal{D}_e^+ = \left\{ u \in \mathcal{D}_e : \inf_{(t,x) \in [0,\tau^u) \times [0,1]} U^u(t,x) > -\infty \right\}$$

and

$$\mathcal{D}_e^- = \left\{ u \in \mathcal{D}_e : \sup_{(t,x) \in [0,\tau^u) \times [0,1]} U^u(t,x) < +\infty \right\}.$$

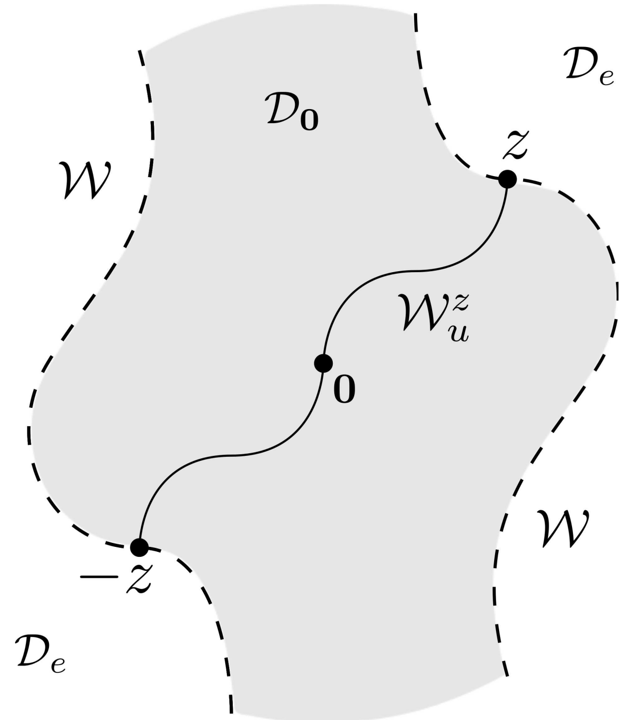

FIGURE 2. The phase diagram of equation (1.2).

Notice that $\mathcal{D}_e^+$ and $\mathcal{D}_e^-$ are disjoint and also that they satisfy the relation $\mathcal{D}_e^- = -\mathcal{D}_e^+$. Furthermore, we shall see below that $\mathcal{D}_e^+$ is an open set. Let us write $\mathcal{D}_e^* := \mathcal{D}_e^+ \cup \mathcal{D}_e^-$. The result we are to prove is the following.

Theorem 4.1. *For any bounded set $\mathcal{K} \subseteq \mathcal{D}_e^*$ at a positive distance from $\partial\mathcal{D}_e^*$ and $\delta > 0$ there exists a constant $C > 0$ such that*

$$\sup_{u \in \mathcal{K}} P_u(|\tau_\varepsilon - \tau| > \delta) \leq e^{-\frac{C}{\varepsilon^2}}.$$

We split the proof of Theorem 4.1 into two parts: proving first a lower bound and then an upper bound for τ_ε. The first one is a consequence of the continuity of solutions to (1.1) with respect to ε on intervals where the deterministic solution remains bounded. The precise estimate is contained in the following proposition.

Proposition 4.1. For any bounded set $\mathcal{K} \subseteq \mathcal{D}_e$ and $\delta > 0$ there exists a constant $C > 0$ such that

$$\sup_{u \in \mathcal{K}} P_u(\tau_\varepsilon < \tau - \delta) \leq e^{-\frac{C}{\varepsilon^2}}. \tag{4.1}$$

Proof. From the continuity of solutions respect to the initial datum (see Proposition 8.3) we have that $\inf_{u \in \mathcal{K}} \tau^u > 0$ so that we may assume $\tau^u > \delta$ for all $u \in \mathcal{K}$. For each $u \in \mathcal{D}_e$ we define the quantity

$$M_u := \sup_{0 \leq t \leq \max\{0, \tau^u - \delta\}} \|U^u(t, \cdot)\|_\infty.$$

By the continuity of solutions once again we obtain that the application $u \mapsto M_u$ is both upper semicontinuous and finite on $\mathcal{D}_e$ and hence, by Propositions 8.2 and 8.5, we conclude that $M := \sup_{u \in \mathcal{K}} M_u < +\infty$. Similarly, since $u \mapsto \tau^u$ is both continuous and finite on $\mathcal{D}_e$ (see Corollary 4.4 below for a proof of this) we also obtain that $\mathcal{T} := \sup_{u \in \mathcal{K}} \tau^u < +\infty$.

Hence, for $u \in \mathcal{K}$ we get

$$P_u(\tau_\varepsilon^u < \tau^u - \delta) \leq P_u\left(d_{\tau^u-\delta}\left(U^{M_u+1,\varepsilon}, U^{M_u+1}\right) > \frac{1}{2}\right) \leq P_u\left(d_{\mathcal{T}-\delta}\left(U^{M+1,\varepsilon}, U^{M+1}\right) > \frac{1}{2}\right).$$

By the estimate (2.3) we conclude (4.1). □

To establish the upper bound we consider for each $u \in \mathcal{D}_e^+$ the process

$$Z^{u,\varepsilon} := U^{u,\varepsilon} - V^{\mathbf{0},\varepsilon}$$

where $U^{u,\varepsilon}$ is the solution of (1.1) with initial datum u and $V^{\mathbf{0},\varepsilon}$ is the solution of (1.1) with source term $g \equiv 0$ and initial datum $\mathbf{0}$ constructed using the same Brownian sheet as $U^{u,\varepsilon}$. Let us observe that $Z^{u,\varepsilon}$ satisfies the random partial differential equation

$$\left\{\begin{array}{rll} \partial_t Z^{u,\varepsilon} &= \partial^2_{xx} Z^{u,\varepsilon} + g(Z^{u,\varepsilon} + V^{\mathbf{0},\varepsilon}) & t > 0\,,\, 0 < x < 1 \\ Z^{u,\varepsilon}(t,0) &= Z^{u,\varepsilon}(t,1) = 0 & t > 0 \\ Z^{u,\varepsilon}(0,x) &= u(x). & \end{array}\right. \tag{4.2}$$

Furthermore, since $V^{\mathbf{0},\varepsilon}$ is globally defined and remains bounded on finite time intervals, we have that $Z^{u,\varepsilon}$ and $U^{u,\varepsilon}$ share the same explosion time. Hence, to obtain the desired upper bound on τ_ε^u we may study the behavior of $Z^{u,\varepsilon}$. The advantage of this approach is that, in general, the behavior of $Z^{u,\varepsilon}$ will be easier to understand than that of $U^{u,\varepsilon}$. Indeed, each realization of $Z^{u,\varepsilon}$ is the solution of a partial differential equation which one can handle with PDE techniques.

Now, a straightforward calculation using the mean value theorem shows that whenever $\|V^{\mathbf{0},\varepsilon}\|_\infty < 1$ the process $Z^{u,\varepsilon}$ satisfies the inequality

$$\partial_t Z^{u,\varepsilon} \geq \partial^2_{xx} Z^{u,\varepsilon} + g(Z^{u,\varepsilon}) - h|Z^{u,\varepsilon}|^{p-1} - h \tag{4.3}$$

where $h := p2^{p-1}\|V^{\mathbf{0},\varepsilon}\|_\infty > 0$. Therefore, to establish the upper bound on τ_ε^u we first consider for $h > 0$ the solution $\underline{Z}^{(h),u}$ to the equation

$$\left\{\begin{array}{rll} \partial_t \underline{Z}^{(h),u} &= \partial^2_{xx} \underline{Z}^{(h),u} + g(\underline{Z}^{(h),u}) - h|\underline{Z}^{(h),u}|^{p-1} - h & t > 0\,,\, 0 < x < 1 \\ \underline{Z}^{(h),u}(t,0) &= \underline{Z}^{(h),u}(t,1) = 0 & t > 0 \\ \underline{Z}^{(h),u}(0,x) &= u(x). & \end{array}\right. \tag{4.4}$$

and obtain a convenient upper bound for the explosion time of this new process valid for every h sufficiently small. By showing then that for h suitably small the process $\underline{Z}^{(h),u}$ explodes through $+\infty$, the fact that $Z^{u,\varepsilon}$ is a supersolution to (4.4) will yield the desired upper bound on the explosion time of $Z^{u,\varepsilon}$, provided that $\|V^{\mathbf{0},\varepsilon}\|_\infty$ remains small enough. For this last part is where the assumption that $u \in \mathcal{D}_e^+$ is necessary. Lemma 4.3 below contains the proper estimate on $\underline{\tau}^{(h),u}$, the explosion time of $\underline{Z}^{(h),u}$.

Definition 4.2. For $h \geq 0$ we define the potential $\underline{S}^{(h)}$ on $C_D([0,1])$ associated to (4.4) by the formula

$$\underline{S}^{(h)}(v) = \left\{\begin{array}{ll} \displaystyle\int_0^1 \left[\frac{1}{2}\left(\frac{dv}{dx}\right)^2 - \frac{|v|^{p+1}}{p+1} + hg(v) + hv\right] & \text{if } v \in H_0^1((0,1)) \\ & \\ +\infty & \text{otherwise.} \end{array}\right.$$

Notice that $\underline{S}^{(0)}$ coincides with our original potential S. Moreover, it is easy to check that for all $h \geq 0$ the potential $\underline{S}^{(h)}$ satisfies all properties established for S in the appendix.

Lemma 4.3. Given $\delta > 0$ there exists $M > 0$ such that:

i. For every $0 \leq h < 1$, any $u \in C_D([0,1])$ with $\underline{S}^{(h)}(u) \leq -\frac{M}{2}$ verifies $\underline{\tau}^{(h),u} < \frac{\delta}{2}$.

ii. Given $K>0$ there exist constants $\rho_{M,K}, h_{M,K}>0$ depending only on M and K such that any $u \in C_D([0,1])$ satisfying $S(u) \leq -M$ and $\|u\|_\infty \leq K$ verifies

$$\sup_{v \in B_{\rho_{M,K}}(u)} \underline{\tau}^{(h),v} < \delta$$

for all $0 \leq h < h_{M,K}$.

Proof. Let us take $\delta > 0$ and show first that (i) holds for an appropriate choice of M. For fixed $M > 0$ and $0 \leq h < 1$, let $u \in C_D([0,1])$ be such that $\underline{S}^{(h)}(u) \leq -\frac{M}{2}$ and consider the application $\phi^{(h),u} : [0, \tau^{(h),u}) \to \mathbb{R}^+$ given by the formula

$$\phi^{(h),u}(t) = \int_0^1 \left(\underline{Z}^{(h),u}(t,x)\right)^2 dx.$$

It is simple to verify that $\phi^{(h),u}$ is continuous and that for any $t_0 \in (0, \underline{\tau}^{(h),u})$ it satisfies

$$\frac{d\phi^{(h),u}}{dt}(t_0) \geq -4\underline{S}^{(h)}(u^{(h)}_{t_0}) + 2\int_0^1 \left[\left(\frac{p-1}{p+1}\right)|u^{(h)}_{t_0}|^{p+1} - h\left(\frac{p+2}{p}\right)|u^{(h)}_{t_0}|^p - h|u^{(h)}_{t_0}|\right] \tag{4.5}$$

where we write $u^{(h)}_{t_0} := \underline{Z}^{(h),u}(t_0,\cdot)$ for convenience. Hölder's inequality reduces (4.5) to (4.6)

$$\frac{d\phi^{(h),u}}{dt}(t_0) \geq -4\underline{S}^{(h)}(u^{(h)}_{t_0}) + 2\left[\left(\frac{p-1}{p+1}\right)\|u^{(h)}_{t_0}\|^{p+1}_{L^{p+1}} - h\left(\frac{p+2}{p}\right)\|u^{(h)}_{t_0}\|^p_{L^{p+1}} - h\|u^{(h)}_{t_0}\|_{L^{p+1}}\right].$$

Observe that, by definition of $\underline{S}^{(h)}$ and the fact that the map $t \mapsto \underline{S}^{(h)}(u^{(h)}_t)$ is decreasing, we obtain the inequalities

$$\frac{M}{2} \leq -\underline{S}^{(h)}(u^{(h)}_{t_0}) \leq \frac{1}{p+1}\|u^{(h)}_{t_0}\|^{p+1}_{L^{p+1}} + h\|u^{(h)}_{t_0}\|^p_{L^{p+1}} + h\|u^{(h)}_{t_0}\|_{L^{p+1}}$$

from which we deduce that by taking M sufficiently large one can force $\|u^{(h)}_{t_0}\|_{L^{p+1}}$ to be large enough so as to guarantee that

$$\left(\frac{p-1}{p+1}\right)\|u^{(h)}_{t_0}\|^{p+1}_{L^{p+1}} - h\left(\frac{p+2}{p}\right)\|u^{(h)}_{t_0}\|^p_{L^{p+1}} - h\|u^{(h)}_{t_0}\|_{L^{p+1}} \geq \frac{1}{2}\left(\frac{p-1}{p+1}\right)\|u^{(h)}_{t_0}\|^{p+1}_{L^{p+1}}$$

is satisfied for any $0 \leq h < 1$. Therefore, we see that if M sufficiently large then for all $0 \leq h < 1$ the application $\phi^{(h),u}$ satisfies

$$\frac{d\phi^{(h),u}}{dt}(t_0) \geq 2M + \left(\frac{p-1}{p+1}\right)\left(\phi^{(h),u}(t_0)\right)^{\frac{p+1}{2}} \tag{4.7}$$

for every $t_0 \in (0, \tau^{(h),u})$, where to obtain (4.7) we have once again used Hölder's inequality and the fact that the map $t \mapsto \underline{S}^{(h)}(u^{(h)}_t)$ is decreasing. Now, it is straightforward to show that the solution y of the ordinary differential equation

$$\begin{cases} \dot{y} = 2M + \left(\frac{p-1}{p+1}\right) y^{\frac{p+1}{2}} \\ y(0) \geq 0 \end{cases}$$

explodes before time

$$T = \frac{\delta}{4} + \frac{2^{\frac{p+1}{2}}(p+1)}{(p-1)^2(M\delta)^{\frac{p-1}{2}}}.$$

Indeed, either y explodes before time $\frac{\delta}{4}$ or $\tilde{y} := y(\cdot + \frac{\delta}{4})$ satisfies

$$\begin{cases} \dot{\tilde{y}} \geq \left(\frac{p-1}{p+1}\right) \tilde{y}^{\frac{p+1}{2}} \\ \tilde{y}(0) \geq \frac{M\delta}{2} \end{cases}$$

which can be seen to explode before time

$$\tilde{T} = \frac{2^{\frac{p+1}{2}}(p+1)}{(p-1)^2(M\delta)^{\frac{p-1}{2}}}$$

by performing the standard integration method. If M is taken sufficiently large then T can be made strictly smaller than $\frac{\delta}{2}$ which, by (4.7), implies that $\tau^{(h),u} < \frac{\delta}{2}$ as desired.

Now let us show statement (ii). Given $K > 0$ let us take $M > 0$ as above and consider $u \in C_D([0,1])$ satisfying $S(u) \leq -M$ and $\|u\|_\infty \leq K$. Using Propositions 8.10 and 8.8 adapted to the system (4.4) we may find $\rho_{M,K} > 0$ sufficiently small so as to guarantee that for some small $0 < t_u < \frac{\delta}{2}$ any $v \in B_{\rho_{M,K}}(u)$ satisfies

$$\underline{S}^{(h)}(\underline{Z}^{(h),v}(t_u,\cdot)) \leq \underline{S}^{(h)}(u) + \frac{M}{4}$$

for all $0 \leq h < 1$. Notice that this is possible since the constants in Proposition 8.10 adapted to this context can be taken independent from h provided that h remains bounded. These constants still depend on $\|u\|_\infty$ though, so that the choice of $\rho_{M,K}$ will inevitably depend on both M and K. Next, let us take $0 < h_{M,K} < 1$ so as to guarantee that $\underline{S}^{(h)}(u) \leq -\frac{3M}{4}$ for every $0 \leq h < h_{M,K}$. Notice that, since $\underline{S}^{(h)}(u) \leq S(u) + h(K^p + K)$, it is possible to choose $h_{M,K}$ depending only on M and K. Thus, for any $v \in B_{\rho_{M,K}}(u)$ we obtain $\underline{S}^{(h)}(\underline{Z}^{(h),v}(t_u,\cdot)) \leq -\frac{M}{2}$ which, by the choice of M, implies that $\tau^{(h),v} < t_u + \frac{\delta}{2} < \delta$. This concludes the proof. □

Let us observe that the system $\overline{Z}^{(0),u}$ coincides with U^u for every $u \in C_D([0,1])$. Thus, by the previous lemma we obtain the following corollary.

Corollary 4.4. The application $u \mapsto \tau^u$ is continuous on $\mathcal{D}_e$.

Proof. Given $u \in \mathcal{D}_e$ and $\delta > 0$ we show that there exists $\rho > 0$ such that for all $v \in B_\rho(u)$ we have

$$-\delta + \tau^u < \tau^v < \tau^u + \delta.$$

To see this we first notice that by Proposition 8.3 there exists $\rho_1 > 0$ such that $-\delta + \tau^u < \tau^v$ for any $v \in B_{\rho_1}(u)$. Moreover, by (i) in Lemma 4.3 we may take $M, \tilde{\rho}_2 > 0$ such that $\tau^{\tilde{v}} < \delta$ for any $\tilde{v} \in B_{\tilde{\rho}_2}(\tilde{u})$ with $\tilde{u} \in C_D([0,1])$ verifying $S(\tilde{u}) \leq -M$. For any such M, by Proposition 3.4 we may find some $0 < t_M < t^u$ such that $S(U^u(t_M,\cdot)) \leq -M$ and using Proposition 8.3 we may take $\rho_2 > 0$ such that $U^v(t_M,\cdot) \in B_{\tilde{\rho}_2}(U^u(t_M,\cdot))$ for any $v \in B_{\rho_2}(u)$. This implies that $\tau^v < t_M + \delta < t^u + \delta$ for all $v \in B_{\rho_2}(u)$ and thus by taking $\rho = \min\{\rho_1, \rho_2\}$ we obtain the result. □

The following two lemmas provide the necessary tools to obtain the uniformity in the upper bound claimed in Theorem 4.1.

Lemma 4.5. Given $M > 0$ and $u \in \mathcal{D}_e$ let us define the quantities

$$\mathcal{T}^u_M = \inf\{t \in [0,\tau^u) : S(U^u(t,\cdot)) < -M\} \qquad \text{and} \qquad \mathcal{R}^u_M = \sup_{0 \leq t \leq \mathcal{T}^u_M} \|U^u(t,\cdot)\|_\infty.$$

Then the applications $u \mapsto \mathcal{T}^u_M$ and $u \mapsto \mathcal{R}^u_M$ are both upper semicontinuous on $\mathcal{D}_e$.

Proof. We must see that the sets $\{\mathcal{T}_M < \alpha\}$ and $\{\mathcal{R}_M < \alpha\}$ are open in $\mathcal{D}_e$ for all $\alpha > 0$. But the fact that $\{\mathcal{T}_M < \alpha\}$ is open follows at once from Proposition 8.10 and $\{\mathcal{R}_M < \alpha\}$ is open by Proposition 8.3. □

Lemma 4.6. For each $u \in \mathcal{D}_e^+$ let us define the quantity

$$\mathcal{I}^u := \inf_{(t,x)\in[0,\tau^u)\times[0,1]} U^u(t,x).$$

Then the application $u \mapsto I^u$ is lower semicontinuous on $\mathcal{D}_e^+$.

Proof. Notice that $\mathcal{I}^u \geq 0$ for any $u \in \mathcal{D}_e^+$ since $U^u(t,0) = U^u(t,1) = 0$ for all $t \in [0,\tau^u)$. Therefore, it will suffice to show that the sets $\{\alpha < \mathcal{I}\}$ are open in $\mathcal{D}_e^+$ for every $\alpha < 0$. With this purpose in mind, given $\alpha < 0$ and $u \in \mathcal{D}_e^+$ such that $\alpha < \mathcal{I}^u$, take $\beta_1, \beta_2 < 0$ such that $\alpha < \beta_1 < \beta_2 < \mathcal{I}^u$ and let y be the solution to the ordinary differential equation

$$\begin{cases} \dot{y} = -|y|^p \\ y(0) = \beta_2. \end{cases} \tag{4.8}$$

Define $t_\beta := \inf\{t \in [0, t^y_{max}) : y(t) < \beta_1\}$, where t^y_{max} denotes the explosion time of y. Notice that by the lower semicontinuity of S for any $M > 0$ we have $S(U^u(\mathcal{T}^u_M, \cdot)) \leq -M$ and thus, by Lemma 4.3, we may choose M such that

$$\sup_{v\in B_\rho(U^u(\mathcal{T}^u_M,\cdot))} \tau^v < t_\beta \tag{4.9}$$

for some small $\rho > 0$. Moreover, if $\rho < \mathcal{I}^u - \beta_2$ then every $v \in B_\rho(U^u(\mathcal{T}^u_M, \cdot))$ satisfies $\inf_{x\in[0,1]} v(x) \geq \beta_2$ so that U^v is in fact a supersolution to the equation (4.8). By (4.9) this implies that $v \in \mathcal{D}_e^+$ and $\mathcal{I}^v \geq \beta_1 > \alpha$. On the other hand, by Proposition 8.3 we may take $\delta > 0$ sufficiently small so that for every $w \in B_\delta(u)$ we have $\mathcal{T}^u_M < \tau^w$ and

$$\sup_{t\in[0,\mathcal{T}^u_M]} \|U^w(t,\cdot) - U^u(t,\cdot)\|_\infty < \rho.$$

Combined with the previous argument, this yields the inclusion $B_\delta(u) \subseteq \mathcal{D}_e^+ \cap \{\alpha < \mathcal{I}\}$. In particular, this shows that $\{\alpha < \mathcal{I}\}$ is open and thus concludes the proof. □

Remark 4.7. The preceding proof shows, in particular, that the set $\mathcal{D}_e^+$ is open.

The conclusion of the proof of Theorem 4.1 is contained in the next proposition.

Proposition 4.8. For any bounded set $\mathcal{K} \subseteq \mathcal{D}_e^*$ at a positive distance from $\partial\mathcal{D}_e^*$ and $\delta > 0$ there exists a constant $C > 0$ such that

$$\sup_{u\in\mathcal{K}} P_u(\tau_\varepsilon > \tau + \delta) \leq e^{-\frac{C}{\varepsilon^2}}. \tag{4.10}$$

Proof. Since $\mathcal{D}_e^- = -\mathcal{D}_e^+$ and $U^{-u} = -U^u$ for $u \in C_D([0,1])$, without loss of generality we may assume that $\mathcal{K}$ is contained in $\mathcal{D}_e^+$. Let us begin by noticing that for any $M > 0$

$$\mathcal{T}_M := \sup_{u\in\mathcal{K}} \mathcal{T}^u_M < +\infty \qquad \text{and} \qquad \mathcal{R}_M := \sup_{u\in\mathcal{K}} \mathcal{R}^u_M < +\infty.$$

Indeed, by Propositions 8.2 and 8.5 we may choose $t_0 > 0$ small so as to guarantee that the orbits $\{U^u(t,\cdot) : 0 \leq t \leq t_0, u \in \mathcal{K}\}$ remain uniformly bounded and the family $\{U^u(t_0,\cdot) : u \in \mathcal{K}\}$ is contained in a compact set $\mathcal{K}' \subseteq \mathcal{D}_e^+$ at a positive distance from $\partial\mathcal{D}_e^+$. But then we have

$$\mathcal{T}_M \leq t_0 + \sup_{u\in\mathcal{K}'} \mathcal{T}^u_M \qquad \text{and} \qquad \mathcal{R}_M \leq \sup_{0\leq t\leq t_0, u\in\mathcal{K}} \|U^u(t,\cdot)\|_\infty + \sup_{u\in\mathcal{K}'} \mathcal{R}^u_M$$

and both right hand sides are finite due to Lemma 4.5 and the fact that $\mathcal{T}^u_M$ and $\mathcal{R}_M$ are both finite for each $u \in \mathcal{D}_e$ by Proposition 3.4. Similarly, by Lemma 4.6 we also have

$$\mathcal{I}_\mathcal{K} := \inf_{u\in\mathcal{K}} \mathcal{I}^u > -\infty.$$

Now, for each $u \in \mathcal{K}$ and $\varepsilon > 0$ by the Markov property we have for any $\rho > 0$

$$P_u(\tau_\varepsilon > \tau + \delta) \leq P(d_{\mathcal{T}_M}(U^{(\mathcal{R}_M+1),u,\varepsilon}, U^{(\mathcal{R}_M+1),u}) > \rho) + \sup_{v \in B_\rho(U^u(\mathcal{T}^u_M,\cdot))} P_v(\tau_\varepsilon > \delta). \tag{4.11}$$

The first term on the right hand side is taken care of by (2.3) so that in order to show (4.10) it only remains to deal with the second term by choosing M and ρ appropriately. The argument given to deal with this term is similar to that of the proof of Lemma 4.6. Let y be the solution to the ordinary differential equation

$$\begin{cases} \dot{y} = -|y|^p - |y|^{p-1} - 1 \\ y(0) = \mathcal{I}_{\mathcal{K}} - \frac{1}{2}. \end{cases} \tag{4.12}$$

Define $t_{\mathcal{I}} := \inf\{t \in [0, t^y_{max}) : y(t) < \mathcal{I}_{\mathcal{K}} - 1\}$, where t^y_{max} denotes the explosion time of y. By Lemma 4.3, we may choose M such that

$$\sup_{v \in B_{\rho_M}(U^u(\mathcal{T}^u_M,\cdot))} \tau^{(h),v} < \min\{\delta, t_{\mathcal{I}}\} \tag{4.13}$$

for all $0 \leq h < h_M$, where $\rho_M > 0$ and $h_M > 0$ are suitable constants. The key observation here is that, since $\mathcal{R}_M < +\infty$, we may choose these constants so as not to depend on u but rather on M and $\mathcal{R}_M$ themselves. Moreover, if $\rho_M < \frac{1}{2}$ then every $v \in B_{\rho_M}(U^u(\mathcal{T}^u_M,\cdot))$ satisfies $\inf_{x\in[0,1]} v(x) \geq \mathcal{I}_{\mathcal{K}} - \frac{1}{2}$ so that $\underline{Z}^{(h),v}$ is in fact a supersolution to the equation (4.12) for all $0 \leq h < \min\{h_M, 1\}$. By (4.13) the former implies that $\underline{Z}^{(h),v}$ explodes through $+\infty$ and that it remains bounded from below by $\mathcal{I}_{\mathcal{K}} - 1$ until its explosion time which, by (4.13), is smaller than δ. In particular, we see that if $\|V^{\mathbf{0},\varepsilon}\|_\infty < \min\{1, \frac{h_M}{p2^{p-1}}\}$ then $Z^{v,\varepsilon}$ explodes before $\underline{Z}^{(h),v}$ does, so that we have that $\tau_\varepsilon < \delta$ under such conditions. Hence, we conclude that

$$\sup_{v \in B_{\rho_M}(U^u(\mathcal{T}^u_M,\cdot))} P_v(\tau_\varepsilon > \delta) \leq P\left(\sup_{t\in[0,\delta]} \|V^{\mathbf{0},\varepsilon}(t,\cdot)\|_\infty \leq \min\left\{1, \frac{h_M}{p2^{p-1}}\right\}\right)$$

which, by recalling the estimate (2.3), gives the desired control on the second term in the right hand side of (4.11). Thus, by taking ρ equal to ρ_M in (4.11), we obtain the result. □

This last proposition in fact shows that for $\delta > 0$ and a given bounded set $\mathcal{K} \subseteq \mathcal{D}^*_e$ at a positive distance from $\partial\mathcal{D}^*_e$ there exist constants $M, C > 0$ such that

$$\sup_{u\in\mathcal{K}} P_u(\tau_\varepsilon > \mathcal{T}^u_M + \delta) \leq e^{-\frac{C}{\varepsilon^2}}.$$

By exploiting the fact $\mathcal{T}_M < +\infty$ for every $M > 0$ we obtain the following useful corollary.

Corollary 4.9. For any bounded $\mathcal{K} \subseteq \mathcal{D}^*_e$ at a positive distance from $\partial\mathcal{D}^*_e$ there exist constants $\tau_K, C > 0$ such that

$$\sup_{u\in\mathcal{K}} P_u(\tau_\varepsilon > \tau_K) \leq e^{-\frac{C}{\varepsilon^2}}.$$

5. Construction of an auxiliary domain

To study the behavior of the explosion time for initial data in $\mathcal{D}_{\mathbf{0}}$ it is convenient to introduce an auxiliary bounded domain $G \subseteq C_D([0,1])$ containing a neighborhood B_c of the stable equilibrium and such that for any initial data $u \in B_c$ the escape time from this domain is asymptotically equivalent to the explosion time. By doing so we can then reduce our original problem to a simpler one: characterizing the escape from this domain. This becomes a simpler problem because, since the escape only depends on the behavior of the system while it remains inside a bounded region, local large deviation estimates can be successfully applied to its study. This approach is not new, it was originally proposed in [10] to study the finite-dimensional double-well potential model. However, in our present setting the construction of this auxiliary domain is much more involved and, as a matter of fact, a priori it is not even clear that such a domain exists for every value of $p > 1$. The aim of this section is to construct such a domain for $1 < p < 5$. The following lemma will play a key role in this.

Lemma 5.1 ([19])**.** If $1 < p < 5$ then the set $\{u \in \overline{\mathcal{D}_{\mathbf{0}}} : S(u) \leq a\}$ is bounded in $C([0,1])$ for any $a > 0$.

Proof. For $a > 0$ and $v \in \{u \in \overline{\mathcal{D}_{\mathbf{0}}} : S(u) \leq a\}$ consider $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ given by

$$\psi(t) := \int_0^1 (U^v(t,\cdot))^2.$$

A direct computation shows that for every $t_0 > 0$ the function ψ satisfies

$$\frac{d\psi(t_0)}{dt} = -4S(U^v(t,\cdot)) + 2\left(\frac{p-1}{p+1}\right)\int_0^1 |U^v(t,\cdot)|^{p+1}.$$

By Proposition 8.8 and Hölder's inequality we then obtain

$$\frac{d\psi(t_0)}{dt} \geq -4a + 2\left(\frac{p-1}{p+1}\right)(\psi(t_0))^{\frac{p+1}{2}}$$

which implies that $\psi(0) \leq B := \left[2a\left(\frac{p+1}{p-1}\right)\right]^{\frac{2}{p+1}}$ since otherwise ψ (and therefore U^v) would explode in finite time. Now, by the Gagliardo-Niremberg interpolation inequality (recall that v is absolutely continuous since $S(v) < +\infty$)

$$\|v\|_\infty^2 \leq C_{GN}\|v\|_{L^2}\|\partial_x v\|_{L^2},$$

we obtain

$$\int_0^1 |v|^{p+1} \leq \|v\|_{L^2}^2 \|v\|_\infty^{p-1} \leq C_{GN}^{\frac{p-1}{2}} B^{\frac{p+3}{4}} \|\partial_x v\|_{L^2}^{\frac{p-1}{2}} \leq C_{GN}^{\frac{p-1}{2}} B^{\frac{p+3}{4}} (2a + \int_0^1 |v|^{p+1})^{\frac{p-1}{4}}$$

which for $p < 5$ implies the bound

$$\int_0^1 |v|^{p+1} \leq B' := \max\left\{2a, \left[C_{GN}^{\frac{p-1}{2}} B^{\frac{p+3}{4}} 2^{\frac{p-1}{4}}\right]^{\frac{4}{5-p}}\right\}. \tag{5.1}$$

Since $S(v) \leq a$ we see that (5.1) implies the bound $\|\partial_x v\|_{L^2} \leq \sqrt{2B'}$. Thus, we conclude

$$\|v\|_\infty \leq \left(C_{GN}^{p-1} 2BB'\right)^{\frac{1}{4}}$$

which shows that $\{u \in \overline{\mathcal{D}_{\mathbf{0}}} : 0 \leq S(u) \leq a\}$ is bounded. □

Remark 5.2. The proof of Lemma 5.1 is the only instance throughout our work in which the assumption $p < 5$ is used. As a matter of fact, we only require the weaker condition that there exists $\alpha > 0$ such that the set $\{u \in \overline{\mathcal{D}_{\mathbf{0}}} : S(u) \leq S(z) + \alpha\}$ is bounded. However, determining the validity of this condition for arbitrary $p > 1$ does not seem simple.

Before we can carry on with the next proposition, we need to introduce some definitions.

Definition 5.3. Given $T > 0$ and $\varphi \in C_D([0,T] \times [0,1])$ we define the *rate* $I(\varphi)$ of φ by the formula

$$I(\varphi) := I_T^{\varphi(0,\cdot)}(\varphi),$$

where $I_T^{\varphi(0,\cdot)}$ is defined as in Section 2.3.

Definition 5.4. We say that a function $\varphi \in C_D([0,T]\times[0,1])$ is *regular* if both derivatives $\partial_t\varphi$ and $\partial^2_{xx}\varphi$ exist and belong to $C_D([0,T]\times[0,1])$.

Proposition 5.5. Given $T>0$, for any $\varphi \in C_D \cap W_2^{1,2}([0,T]\times[0,1])$ such that $\partial^2_{xx}\varphi(0,\cdot)$ exists and belongs to $C_D([0,1])$ we have that

$$I(\varphi) \geq 2\left[\sup_{0\leq T'\leq T}\left(S(\varphi(T',\cdot)) - S(\varphi(0,\cdot))\right)\right]. \tag{5.2}$$

Proof. Assume first that φ is regular. Using that $(x-y)^2 = (x+y)^2 - 4xy$ for $x,y\in\mathbb{R}$, for any $0\leq T'\leq T$ we obtain that

$$\begin{aligned}
I(\varphi) = \frac{1}{2}\int_0^T\int_0^1 |\partial_t\varphi - \partial^2_{xx}\varphi - g(\varphi)|^2 &\geq \frac{1}{2}\int_0^{T'}\int_0^1 |\partial_t\varphi - \partial^2_{xx}\varphi - g(\varphi)|^2\\
&= \frac{1}{2}\int_0^{T'}\int_0^1 \left[|\partial_t\varphi + \partial^2_{xx}\varphi + g(\varphi)|^2 - 4\left(\partial^2_{xx}\varphi + g(\varphi)\right)\partial_t\varphi\right]\\
&= \frac{1}{2}\int_0^{T'}\left[\left(\int_0^1 |\partial_t\varphi + \partial^2_{xx}\varphi + g(\varphi)|^2\right) + 4\frac{dS(\varphi(t,\cdot))}{dt}\right]\\
&\geq 2\left(S(\varphi(T',\cdot)) - S(\varphi(0,\cdot))\right).
\end{aligned}$$

Taking supremum on T' yields the result in this particular case. Now, if φ is not necessarily regular then by [8, Theorem 6.9] we may take a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of regular functions converging to φ on $C_{D_{\varphi(0,\cdot)}}([0,T]\times[0,1])$ and such that $\lim_{n\to+\infty} I(\varphi_n) = I(\varphi)$ is satisfied. The result in the general case then follows from the validity of (5.2) for regular functions and the lower semicontinuity of S. □

In order to properly interpret the content of Proposition 5.5 we need to introduce the concept of *quasipotential* for our system. We do so in the following definitions.

Definition 5.6. Given $u,v \in C_D([0,1])$ a *path from u to v* is a continuous function $\varphi \in C_D([0,T]\times[0,1])$ for some $T>0$ such that $\varphi(0,\cdot)=u$ and $\varphi(T,\cdot)=v$.

Definition 5.7. Given $u,v\in C_D([0,1])$ we define the *quasipotential* $V(u,v)$ from u to v by the formula

$$V(u,v) = \inf\{I(\varphi) : \varphi \text{ path from } u \text{ to } v\}.$$

Furthermore, given a subset $B\subseteq C_D([0,1])$ we define the quasipotential from u to B as

$$V(u,B) := \inf\{V(u,v) : v\in B\}.$$

We refer the reader to the appendix for a review of the properties of V we shall use.

In a limiting sense, made rigorous through the large deviations estimates in Section 2.3, the quasipotential $V(u,v)$ represents the energy cost for the stochastic system to travel from u to (an arbitrarily small neighborhood of) v. Notice that Lemma 5.1 implies that $\lim_{n\to+\infty} V(\mathbf{0}, \partial B_n \cap \mathcal{D}_{\mathbf{0}}) = +\infty$, which says that the energy cost for the stochastic system starting from $\mathbf{0}$ to explode in a finite time while remaining inside $\mathcal{D}_{\mathbf{0}}$ is infinite. Thus, should explosion occur, it would involve the system stepping outside $\mathcal{D}_{\mathbf{0}}$ and crossing $\mathcal{W}$. In view of Proposition 5.5, the crossing of $\mathcal{W}$ will typical take place through $\pm z$ since the energy cost for performing such a feat is the lowest there. Therefore, if we wish the escape from G to capture the essential characteristics of the explosion phenomenon in the stochastic system (at least when starting from $\mathbf{0}$) then it is important to guarantee that this escape involves passing through (an arbitrarily small neighborhood of) $\pm z$. Not only this, but we also require that once the system escapes this domain then it explodes with overwhelming probability in a quick fashion, i.e. before some time τ^* which does not depend on ε. More precisely, we wish to consider a bounded domain $G \subseteq C_D([0,1])$ verifying the following properties:

Conditions 5.8.

i. There exists $r_{\mathbf{0}} > 0$ such that $B_{2r_{\mathbf{0}}} \subseteq \mathcal{D}_{\mathbf{0}} \cap G$.
ii. There exists $c > 0$ such that $B_c \subseteq B_{r_{\mathbf{0}}}$ and for all $v \in B_c$ the solution U^v to (1.2) with initial datum v is globally defined and converges to $\mathbf{0}$ without escaping $B_{r_{\mathbf{0}}}$.
iii. There exists a closed subset $\partial^{\pm z}$ of the boundary ∂G which satisfies
   • $V(\mathbf{0}, \partial G - \partial^{\pm z}) > V(\mathbf{0}, \partial^{\pm z}) = V(\mathbf{0}, \pm z)$.
   •• $\partial^{\pm z}$ is contained in $\mathcal{D}^*_e$ and at a positive distance from its boundary.

In principle, we have seen that such a domain is useful to study the behavior of the explosion time whenever the initial datum of the stochastic system is (close to) the origin. Nevertheless, by the local estimate (2.3), when starting inside $\mathcal{D}_{\mathbf{0}}$ the system will typically visit a small neighborhood of the origin before crossing $\mathcal{W}$ and thus such a choice of G will also be suitable to study the explosion time for arbitrary initial data in $\mathcal{D}_{\mathbf{0}}$.

The construction of the domain G is done as follows. Since $\mathcal{D}_{\mathbf{0}}$ is open we may choose $r_{\mathbf{0}} > 0$ such that $B_{3r_{\mathbf{0}}}$ is contained in $\mathcal{D}_{\mathbf{0}}$. Moreover, by the asymptotic stability of $\mathbf{0}$ we may choose $c > 0$ verifying (ii) in Conditions 5.8. Now, given $\zeta_1 > 0$ by Lemma 5.1 we may take $n_0 \in \mathbb{N}$ such that $n_0 > 3r_{\mathbf{0}}$ and the set $\{u \in \overline{\mathcal{D}_{\mathbf{0}}} : S(u) \leq S(z) + \zeta_1\}$ is contained in the interior of the ball $B_{n_0 - 1}$. We then define the pre-domain $\tilde{G}$ as

$$\tilde{G} := B_{n_0} \cap \overline{\mathcal{D}_{\mathbf{0}}}. \tag{5.3}$$

Notice that since both B_{n_0} and $\overline{\mathcal{D}_{\mathbf{0}}}$ are closed sets we have that

$$\partial \tilde{G} = (\mathcal{W} \cap B_{n_0}) \cup (\partial B_{n_0} \cap \mathcal{D}_{\mathbf{0}})$$

which, by the particular choice of n_0 and Proposition 8.8, implies $\min_{u \in \partial \tilde{G}} S(u) = S(z)$. By Propositions 5.5 and 8.9 we thus obtain $V(\mathbf{0}, \partial \tilde{G}) \geq \Delta$. Next, if for $u \in C_D([0,1])$ we let u^- denote the negative part of u, i.e. $u^- = \max\{-u, 0\}$, then since $z^- = \mathbf{0}$ we may find $\tilde{r}_z > 0$ such that $u^- \in \mathcal{D}_{\mathbf{0}}$ for any $u \in B_{\tilde{r}_z}(z)$. Finally, if for $r > 0$ we write $B_r(\pm z) := B_r(z) \cup B_r(-z)$ and take $r_z > 0$ such that $r_z < \frac{\tilde{r}_z}{2}$, $B_{2r_z}(\pm z)$ is contained in the interior of B_{n_0} and z is the unique equilibrium point of the system lying inside $B_{r_z}(z)$, then we define our final domain G as

$$G = \tilde{G} \cup B_{r_z}(\pm z).$$

Let us now check that this domain satisfies all the required conditions. We begin by noticing that (i) and (ii) in Conditions 5.8 are immediately satisfied by the choice of n_0. Now, let us also observe that for any $r > 0$

$$\inf\{S(u) : u \in \partial \tilde{G} - B_r(\pm z)\} > S(z). \tag{5.4}$$

Indeed, if this is not the case then, since we have $S(z) = \inf_{u \in \mathcal{W}} S(u)$ by Corollary 3.2, there exists a sequence $(u_k)_{k \in \mathbb{N}} \subseteq [\mathcal{W} \cap B_{n_0} - B_r(\pm z)]$ such that $\lim_{k \to +\infty} S(u_k) = S(z)$. Then, by Proposition 8.2 we have that there exists $t_0 > 0$ sufficiently small satisfying

$$\sup_{k \in \mathbb{N}} \left[\sup_{t \in [0,t_0]} \|U^{u_k}(t,\cdot)\|_\infty \right] < +\infty \qquad \text{and} \qquad \inf_{k \in \mathbb{N}} \|U^{u_k}(t_0,\cdot) - (\pm z)\|_\infty > \frac{r}{2}$$

and therefore by Proposition 8.5 we may conclude that there exists a subsequence $(u_{k_j})_{j\in\mathbb{N}}$ such that $U^{u_{k_j}}(t_0,\cdot)$ converges to a limit $u_\infty \in C_D([0,1])$ as $j\to+\infty$. Since the potential is lower semicontinuous and $\mathcal{W}$ is both closed and invariant under the deterministic flow, by Proposition 8.8 we conclude that $u_\infty = \pm z$ which contradicts the fact that the sequence $(U^{u_{k_j}}(t_0,\cdot))_{j\in\mathbb{N}}$ is at a positive distance from these equilibriums. Hence, we obtain (5.4). In particular, this implies that $V(\mathbf{0},\partial\tilde{G}-B_r(\pm z)) > \Delta$ is satisfied for any choice of $r>0$. Let us then take $\zeta_2>0$ such that $\Delta+\zeta_2 < V(\mathbf{0},\partial\tilde{G}-B_{\frac{r_z}{2}}(\pm z))$ and define

$$\tilde{\partial}^z := \{u \in \partial B_{r_z}(z)\cap\overline{\mathcal{D}_e} : V(\mathbf{0},u)\le \Delta+\zeta_2\}.$$

Notice that $\tilde{\partial}^z$ is nonempty and satisfies $d(\tilde{\partial}^z,\mathcal{W})>0$. Indeed, it is possible to show that for each $\alpha>0$ there exists a path from $\mathbf{0}$ to $B_{r_z}(z)\cap\overline{\mathcal{D}_e}$ with rate function less than $\Delta+\alpha$, which immediately implies that $\tilde{\partial}^z$ is nonempty. This path is essentially obtained by going from $\mathbf{0}$ to z by describing the orbit given by the unstable manifold $\mathcal{W}_u^z$ in reverse order, then making a linear interpolation towards $(1+h)z$ for some $h\in(0,r_z)$ sufficiently small and ultimately following the deterministic flow until it reaches $B_{r_z}(z)$ (notice that this will eventually happen since $(1+h)z\in\mathcal{D}_e$ by Proposition 3.5 and $\mathcal{D}_e$ is invariant). We refer to [18, Lemma 4.3] for details on the construction. On the other hand, if $d(\tilde{\partial}^z,\mathcal{W})=0$ then we can construct sequences $(u_k)_{k\in\mathbb{N}}\subseteq\mathcal{W}$ and $(v_k)_{k\in\mathbb{N}}\subseteq\tilde{\partial}^z$ such that $\lim_{k\to+\infty} d(u_k,v_k)=0$. The growth estimates on the appendix imply that there exists $t_1>0$ sufficiently small such that

$$\lim_{k\to+\infty} d(U^{u_k}(t_1,\cdot),U^{v_k}(t_1,\cdot))=0 \;\text{ and }\; \frac{r_z}{2} < \inf_{k\in\mathbb{N}} d(U^{v_k}(t_1,\cdot),z) \le \sup_{k\in\mathbb{N}} d(U^{v_k}(t_1,\cdot),z) < 2r_z.$$

By Proposition 8.5 we obtain that for some appropriate subsequence we have

$$\lim_{j\to+\infty} U^{u_{k_j}}(t_1,\cdot) = \lim_{j\to+\infty} U^{v_{k_j}}(t_1,\cdot) = v_\infty.$$

Observe that $v_\infty\in\mathcal{W}\cap B_{n_0}-B_{\frac{r_z}{2}}(\pm z)$ and thus that $v_\infty\in\partial\tilde{G}-B_{\frac{r_z}{2}}(\pm z)$. Furthermore, by the lower semicontinuity of $V(\mathbf{0},\cdot)$ and the fact that the mapping $t\mapsto V(\mathbf{0},U^u(t,\cdot))$ is monotone decreasing for any $u\in C_D([0,1])$ (see the appendix for details), we obtain that $V(\mathbf{0},v_\infty)\le\Delta+\zeta_2$ which, together with the previous observation, implies the contradiction $\Delta+\zeta_2\ge V(\mathbf{0},\partial\tilde{G}-B_{\frac{r_z}{2}}(\pm z))$. Hence, we see that $d(\tilde{\partial}^z,\mathcal{W})>0$ and thus we may define

$$\partial^z = \left\{u\in\partial B_{r_z}(z)\cap\overline{\mathcal{D}_e} : d(u,\mathcal{W}) \ge \frac{d(\tilde{\partial}^z,\mathcal{W})}{2}\right\}$$

and set $\partial^{\pm z} := \partial^z\cup(-\partial^z)$. Since one can easily check that

$$\partial G = [\partial\tilde{G}-B_{r_z}(\pm z)]\cup[\partial B_{r_z}(\pm z)\cap\overline{D_e}]$$

we conclude that $V(\mathbf{0},\partial G-\partial^{\pm z})\ge\Delta+\zeta_2$. On the other hand, by using Proposition 5.5 together with the existence of paths as described above, which go from $\mathbf{0}$ to $\tilde{\partial}^z$ by passing through z and have a rate function which can be made arbitrarily close to Δ, we get that $V(\mathbf{0},\partial^z)=V(\mathbf{0},\tilde{\partial}^z)=V(\mathbf{0},\pm z)=\Delta$ from which one obtains

$$V(\mathbf{0},\partial G-\partial^{\pm z}) > V(\mathbf{0},\partial^z) = V(\mathbf{0},\pm z).$$

Furthermore, by the comparison principle and the choice of $\tilde{r}_z$ we have $B_{\tilde{r}_z}(\pm z)\cap\mathcal{D}_e\subseteq\mathcal{D}_e^*$. Therefore, since we clearly have $d(\partial^{\pm z},\mathcal{W})>0$ by definition of $\partial^{\pm z}$, upon recalling that $\partial^{\pm z}\subseteq\mathcal{D}_e$ and $r_z<\frac{\tilde{r}_z}{2}$ we see that $\partial^{\pm z}\subseteq\mathcal{D}_e^+$ and $d(\partial^{\pm z},\partial\mathcal{D}_e^*)\ge\min\{d(\partial^{\pm z},\mathcal{W}),\frac{\tilde{r}_z}{2}\}>0$, so that condition (iii) also holds. See Figure 3.

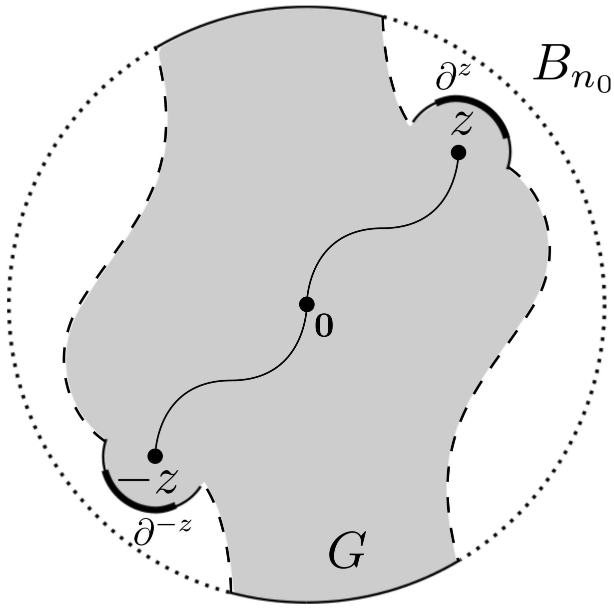

FIGURE 3. The auxiliary domain G

Remark 5.9. Let us notice that, by Corollary 4.9, (••) in Conditions 5.8 implies that there exist constants $\tau^*, C > 0$ such that

$$\sup_{u \in \partial^{\pm z}} P_u(\tau_\varepsilon > \tau^*) \leq e^{-\frac{C}{\varepsilon^2}}$$

for all $\varepsilon > 0$ sufficiently small. Since (•) guarantees that the escape from G will typically take place through $\partial^{\pm z}$, this tells us that both τ_ε and $\tau_\varepsilon(\partial G)$ are asymptotically equivalent, so that it will suffice to study the escape from G in order to establish each of our results.

6. The escape from G

The problem of escaping a bounded domain with similar characteristics to the ones detailed in Conditions 5.8 already appears in the literature. In [10, 15], the authors study the escape from a finite-dimensional domain containing a stable equilibrium and only one saddle point. Our domain G bears the additional difficulties of being infinite-dimensional and also of possibly containing other unstable equilibria besides $\pm z$. On the other hand, in [1] they do deal with an infinite-dimensional domain, but this domain has unstable equilibria only in its boundary and does not contain any of them in its interior as opposed to what happens in our current situation. Despite the fact that our domain does not quite fall into any of the cases studied before, all the results of interest in our present setting can still be obtained by combining the ideas from these previous works, eventually making some slight modifications along the way. We outline below the main results regarding the escape from the domain G and refer the reader to [18] for details on their proofs. Hereafter, $c > 0$ is taken as in Conditions 5.8.

The first result is concerned with the asymptotic order of magnitude of the exit time.

Theorem 6.1. *Given $\delta > 0$ we have*

$$\lim_{\varepsilon \to 0} \left[\sup_{u \in B_c} \left| P_u \left(e^{\frac{\Delta - \delta}{\varepsilon^2}} < \tau_\varepsilon(\partial G) < e^{\frac{\Delta + \delta}{\varepsilon^2}} \right) - 1 \right| \right] = 0,$$

where $\tau^u_\varepsilon(\partial G) := \inf\{t > 0 : U^{u,\varepsilon}(t, \cdot) \in \partial G\}$ denotes the exit time from G.

The second result gives information about the typical escape routes chosen by U^ε.

Theorem 6.2. *The stochastic system verifies*

$$\lim_{\varepsilon\to 0}\left[\sup_{u\in B_c} P_u\left(U^\varepsilon(\tau_\varepsilon(\partial G),\cdot)\notin \partial^{\pm z}\right)\right]=0.$$

Furthermore, if $\tilde{G}$ is the pre-domain constructed in Section 5, then for any $\delta>0$

$$\lim_{\varepsilon\to 0}\left[\sup_{u\in B_c} P_u\left(U^\varepsilon\left(\tau_\varepsilon(\partial\tilde{G}),\cdot\right)\notin B_\delta(\pm z)\right)\right]=0. \tag{6.1}$$

The asymptotic distribution of the exit time is established in this third result.

Theorem 6.3. *For each $\varepsilon>0$ define the normalization coefficient $\gamma_\varepsilon>0$ by the relation*

$$P_{\mathbf{0}}(\tau_\varepsilon(\partial G)>\gamma_\varepsilon)=e^{-1}.$$

Then there exists $\rho>0$ such that for every $t\geq 0$

$$\lim_{\varepsilon\to 0}\left[\sup_{u\in B_\rho}\left|P_u(\tau_\varepsilon(\partial G)>t\gamma_\varepsilon)-e^{-t}\right|\right]=0. \tag{6.2}$$

Finally, the stability of time averages is shown in the forth and last result.

Theorem 6.4. *There exists a sequence $(R_\varepsilon)_{\varepsilon>0}$ with $\lim_{\varepsilon\to 0}R_\varepsilon=+\infty$ and $\lim_{\varepsilon\to 0}\frac{R_\varepsilon}{\gamma_\varepsilon}=0$ such that given $\delta>0$ we have*

$$\lim_{\varepsilon\to 0}\left[\sup_{u\in B_c} P_u\left(\sup_{0\leq t\leq \tau_\varepsilon(\partial G)-3R_\varepsilon}\left|\frac{1}{R_\varepsilon}\int_t^{t+R_\varepsilon} f(U^\varepsilon(s,\cdot))ds - f(\mathbf{0})\right|>\delta\right)\right]=0$$

for any bounded continuous function $f: C_D([0,1])\to\mathbb{R}$.

Remark 6.1. We would like to point out that the main technical point in the proof of Theorem 6.3 is to show that for small $\rho>0$

$$\lim_{\varepsilon\to 0}\left[\sup_{u,v\in B_\rho}\left[\sup_{t>t_0}\left|P_u(\tau_\varepsilon(\partial G)>t\gamma_\varepsilon)-P_v(\tau_\varepsilon(\partial G)>t\gamma_\varepsilon)\right|\right]\right]=0. \tag{6.3}$$

We do this as in [2] with the help of the coupling of solutions with different initial data proposed in [14]. Some technical difficulties which are not present in [2] arise in the construction of the coupling due to the behavior of the source term g but, nonetheless, it is still possible to couple solutions with initial data sufficiently close to $\mathbf{0}$ so that (6.3) can be obtained. We refer to [18] for details.

7. Asymptotic behavior of τ^u_ε for $u\in\mathcal{D}_{\mathbf{0}}$

We now use the analysis from Section 6 to derive our main results with respect to the metastable behavior of $U^{u,\varepsilon}$ for initial data $u\in\mathcal{D}_{\mathbf{0}}$. We begin by showing that, uniformly over bounded sets at a positive distance from $\mathcal{W}$, for $u\in\mathcal{D}_{\mathbf{0}}$ the system $U^{u,\varepsilon}$ typically visits a small neighborhood of $\mathbf{0}$ before explosion without ever exiting $\mathcal{D}_{\mathbf{0}}$.

Lemma 7.1. For any bounded set $\mathcal{K}\subseteq\mathcal{D}_{\mathbf{0}}$ at a positive distance from $\partial\mathcal{D}_{\mathbf{0}}$ and $\rho>0$

$$\lim_{\varepsilon\to 0}\left[\sup_{u\in\mathcal{K}}\left|P_u\left(\tau_\varepsilon(B_\rho)<\min\{\tau_\varepsilon(\mathcal{D}^c_{\mathbf{0}}),\tau_\varepsilon\}\right)-1\right|\right]=0 \tag{7.1}$$

where $\tau^u_\varepsilon(B_\rho):=\inf\{t>0: U^{u,\varepsilon}(t,\cdot)\in B_\rho\}$.

Proof. Let us observe that for any $u \in \mathcal{D}_{\mathbf{0}}$ the system U^u reaches the set $B_{\frac{\varrho}{2}}$ in a finite time $\tau^u(B_{\frac{\varrho}{2}})$ while remaining at all times inside the ball B_{r^u}, where $r^u := \sup_{t\geq 0} \|U^u(t,\cdot)\|_\infty$, and at a certain positive distance $d^u := \inf_{t\geq 0} d(U^u(t,\cdot), \partial\mathcal{D}_{\mathbf{0}})$ from the boundary of $\mathcal{D}_{\mathbf{0}}$. Therefore, if $\mathcal{K}$ is any bounded set contained in $\mathcal{D}_{\mathbf{0}}$ at a positive distance from $\partial\mathcal{D}_{\mathbf{0}}$ then we have that the quantities $\tau_{\mathcal{K},\frac{\varrho}{2}} := \sup_{u\in\mathcal{K}} \tau^u(B_{\frac{\varrho}{2}})$ and $r_{\mathcal{K}} := \sup_{u\in\mathcal{K}} r^u$ are both finite while $d_{\mathcal{K}} := \inf_{u\in\mathcal{K}} d^u$ is strictly positive. Indeed, the finiteness of $\tau_{\mathcal{K},\frac{\varrho}{2}}$ follows at once from Proposition 8.7 whereas $r_{\mathcal{K}}$ is finite since by Proposition 8.2 one may find $t_0 > 0$ sufficiently small such that $\sup_{u\in\mathcal{K}} \left[\sup_{t\in[0,t_0]} \|U^u(t,\cdot)\|_\infty\right]$ is finite. That $\sup_{u\in\mathcal{K}} \left[\sup_{t\geq t_0} \|U^u(t,\cdot)\|_\infty\right]$ is finite then follows as in the proof of Proposition 8.7 due to the fact that the mapping $u \mapsto r^u$ is both upper semicontinuous and finite on $\mathcal{D}_{\mathbf{0}}$. Finally, the fact that the quantity $d_{\mathcal{K}}$ is strictly positive follows in a similar manner, using the fact that the mapping $u \mapsto d^u$ is both lower semicontinuous and positive on $\mathcal{D}_{\mathbf{0}}$. Now, if we write $\mathcal{T}_\varepsilon^{\mathcal{K}} := \tau_{\mathcal{K},\frac{\varrho}{2}} \wedge \tau_\varepsilon^{(r_{\mathcal{K}}+1)}$ then notice that for any $u \in \mathcal{K}$ we have the bound

$$P_u\left(\tau_\varepsilon \leq \tau_\varepsilon(B_\rho)\right) \leq P_u\left(\tau_\varepsilon(B_\rho) > \tau_{\mathcal{K},\frac{\varrho}{2}}\right) + P_u\left(\tau_\varepsilon \leq \tau_{\mathcal{K},\frac{\varrho}{2}}\right) + P_u\left(\tau_\varepsilon(\mathcal{D}_{\mathbf{0}}^c) \leq \tau_{\mathcal{K},\frac{\varrho}{2}}\right) \tag{7.2}$$

with

$$P_u\left(\tau_\varepsilon(B_\rho) > \tau_{\mathcal{K},\frac{\varrho}{2}}\right) \leq P_u\left(d_{\mathcal{T}_\varepsilon^{\mathcal{K}}}(U^\varepsilon, U) > \min\left\{\frac{\rho}{2}, \frac{1}{2}\right\}\right)$$

and

$$P_u\left(\tau_\varepsilon \leq \tau_{\mathcal{K},\frac{\varrho}{2}}\right) \leq P_u\left(d_{\mathcal{T}_\varepsilon^{\mathcal{K}}}(U^\varepsilon, U) > \frac{1}{2}\right).$$

As for the third term in the right hand side of (7.2), we have

$$\begin{aligned} P_u\left(\tau_\varepsilon(\mathcal{D}_{\mathbf{0}}^c) \leq \tau_{\mathcal{K},\frac{\varrho}{2}}\right) &\leq P_u\left(\tau_\varepsilon^{(r_{\mathcal{K}}+1)} \leq \tau_{\mathcal{K},\frac{\varrho}{2}}\right) + P_u\left(\tau_\varepsilon(\mathcal{D}_{\mathbf{0}}^c) \leq \tau_{\mathcal{K},\frac{\varrho}{2}} < \tau_\varepsilon^{(r_{\mathcal{K}}+1)}\right) \\ &\leq P_u\left(d_{\mathcal{T}_\varepsilon^{\mathcal{K}}}(U^\varepsilon, U) > \frac{1}{2}\right) + P_u\left(d_{\mathcal{T}_\varepsilon^{\mathcal{K}}}(U^\varepsilon, U) > \frac{d_{\mathcal{K}}}{2}\right) \end{aligned}$$

The uniform bounds given by (2.3) now allow us to conclude the result. □

The next step is to show that, for initial data in a small neighborhood of the origin, the explosion time and the exit time from G are asymptotically equivalent.

Lemma 7.2. If $\tau^* > 0$ is taken as in Remark 5.9 then

$$\lim_{\varepsilon\to 0}\left[\sup_{u\in B_c} P_u(\tau_\varepsilon > \tau_\varepsilon(\partial G) + \tau^*)\right] = 0. \tag{7.3}$$

Proof. For any $u \in B_c$ the strong Markov property implies that

$$P_u(\tau_\varepsilon > \tau_\varepsilon(\partial G) + \tau^*) \leq \sup_{v\in B_c} P_v\left(U^\varepsilon(\tau_\varepsilon(\partial G),\cdot) \notin \partial^{\pm z}\right) + \sup_{v\in\partial^{\pm z}} P_v(\tau_\varepsilon > \tau^*).$$

We may now conclude the result by using Theorem 6.2 and Remark 5.9. □

With these two lemmas at hand, we can now show the remaning results of Section 2.4. Indeed, Theorem 2.3 follows from Theorem 6.1 by using the strong Markov property together with Lemmas 7.1 and 7.2. Furthermore, Theorem 2.4 follows from Lemma 7.2 for $\rho = c$, where c is as in Conditions 5.8, together with (6.1) for $\delta > 0$ sufficiently small so as to guarantee that $B_\delta(\pm z)$ is contained in the interior of B_{n_0}, where n_0 is as in (5.3). Finally, Lemma 7.2 implies that $\lim_{\varepsilon\to 0} \frac{\beta_\varepsilon}{\gamma_\varepsilon} = 1$ from which, together with Lemma 7.2 and the strong Markov property, we get Theorems 2.5 and 2.6 by using Theorems 6.3 and 6.4. We leave the details to the reader, which are completely straightforward.

8. Appendix

8.1. Comparison principle.

Proposition 8.1. Let $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ be globally Lipschitz functions. For $u, v \in C([0,1])$ consider U^u and U^v the solutions of the equation

$$\partial_t U = \partial^2_{xx} U + f_1(U) + f_2(U)\dot{W}$$

with initial data u and v, respectively, and boundary conditions satisfying

$$P(U(t,\cdot)|_{\partial[0,1]} \geq V(t,\cdot)|_{\partial[0,1]} \text{ for all } t \geq 0) = 1.$$

Then, if $u \geq v$ we have that

$$P\left(U^u(t,x) \geq U^v(t,x) \text{ for all } t \geq 0, x \in [0,1]\right) = 1.$$

A proof of this result can be found on [6, p. 130]. Let us notice that by taking $f_2 \equiv 0$ one obtains a comparison principle for deterministic partial differential equations.

8.2. Growth and regularity estimates.

Proposition 8.2. Given a bounded set $B \subseteq C_D([0,1])$ there exists $t_B > 0$ such that

- $\tau^u > t_B$ for any $u \in B$
- There exists $b : [0, t_B] \to \mathbb{R}^+$ such that $\lim_{t \to 0^+} b(t) = 0$ and for any $t \in [0, t_B]$

$$\sup_{u \in B} \|U^u(t,\cdot) - u\|_\infty \leq b(t).$$

Proof. The first assertion follows at once from the comparison principle if one takes as t_B any time for which the solution to the equation

$$\begin{cases} \dot{y} = g(y) \\ y(0) = \sup_{u \in B} \|u\|_\infty \end{cases}$$

is defined. For the second assertion, take b as the solution to the equation

$$\begin{cases} \dot{y} = g(y + \sup_{u \in B} \|u\|_\infty) \\ y(0) = 0. \end{cases}$$

and then apply the comparison principle again to obtain the desired inequality. □

Proposition 8.3. The following local and pointwise growth estimates hold:

i. Given a bounded set $B \subseteq C_D([0,1])$ there exist $C_B, t_B > 0$ such that
   - $\tau^u > t_B$ for any $u \in B$
   - For any pair $u, v \in B$ and $t \in [0, t_B]$

$$\|U^u(t,\cdot) - U^v(t,\cdot)\|_\infty \leq e^{C_B t}\|u - v\|_\infty.$$

ii. Given $u \in C_D([0,1])$ and $t \in [0, \tau^u)$ there exist $C_{u,t}, \delta_{u,t} > 0$ such that
   - $\tau^v > t$ for any $v \in B_{\delta_{u,t}}(u)$
   - For any $v \in B_{\delta_{u,t}}(u)$ and $s \in [0,t]$

$$\|U^u(s,\cdot) - U^v(s,\cdot)\|_\infty \leq e^{C_{u,t} s}\|u - v\|_\infty.$$

Proof. These are standard continuity estimates with respect to the initial datum which can be found, for example, in [16]. □

Proposition 8.4. If $u \in C_D([0,1])$ then $\partial^2_{xx} U^u$ exists for any $t \in (0, \tau^u)$. Furthermore, for any bounded set $B \subseteq C_D([0,1])$ there exists a time $t_B > 0$ such that

- $\tau^u > t_B$ for any $u \in B$
- For any $t \in (0, t_B)$ we have $\sup_{u \in B} \left[\max\{\|\partial_x U^u(t,\cdot)\|_\infty, \|\partial^2_{xx} U^u(t,\cdot)\|_\infty\}\right] < +\infty$.

Proof. One can obtain this result by following the analysis in the proof of [1, Lemma A.1]. □

Proposition 8.5. For any bounded set $B \subseteq C_D([0,1])$ there exists $t_B > 0$ such that

- $\tau^u > t_B$ for any $u \in B$
- For any $t \in (0, t_B)$ there exist positive constants R_t, N_t such that for every $u \in B$ the function $U^u(t, \cdot)$ belongs to the compact set

$$\gamma_{R_t,N_t} = \{v \in C_D([0,1]) : \|v\|_\infty \le R_t\,,\ |v(x) - v(y)| \le N_t|x - y| \text{ for all } x, y \in [0,1]\}.$$

Proof. This is direct consequence of Propositions 8.2-8.4 and the mean value theorem. □

Proposition 8.6. The following local and pointwise growth estimates hold:

i. Given a bounded set $B \subseteq C_D([0,1])$ there exists $t_B > 0$ such that
   - $\tau^u > t_B$ for any $u \in B$
   - For any $t \in (0, t_B)$ there exists $C_{t,B} > 0$ such that for all $u, v \in B$
   $$\|\partial_x U^u(t,\cdot) - \partial_x U^v(t,\cdot)\|_\infty \le C_{t,B}\|u - v\|_\infty.$$

ii. Given $u \in C_D([0,1])$ and $t \in (0, \tau^u)$ there exist $C_{u,t}, \delta_{u,t} > 0$ such that
   - $\tau^v > t$ for any $v \in B_{\delta_{u,t}}(u)$
   - For any $v \in B_{\delta_{u,t}}(u)$
   $$\|\partial_x U^u(t,\cdot) - \partial_x U^v(t,\cdot)\|_\infty \le C_{u,t}\|u - v\|_\infty.$$

Proof. These estimates also follow from the analysis in the proof of [1, Lemma A.1]. □

Proposition 8.7. For any equilibrium point w of the deterministic system let us consider its stable manifold $\mathcal{W}^w$ defined as

$$\mathcal{W}^w := \{u \in C_D([0,1]) : U^u \text{ is globally defined and } U^u(t,\cdot) \underset{t\to+\infty}{\longrightarrow} w\}.$$

Notice that $\mathcal{W}^{\mathbf{0}} = \mathcal{D}_{\mathbf{0}}$. Then for any bounded set $B \subseteq \mathcal{W}^w$ there exists $t_B > 0$ such that for any $t_0 \in [0, t_B]$ and $r > 0$ we have

$$\sup_{u\in B}\,[\inf\{t \ge t_0 : d(U^u(t,\cdot), w) \le r\}] < +\infty$$

whenever one of the following conditions hold:

- $w \neq \mathbf{0}$
- $w = \mathbf{0}$ and B is at a positive distance from $\mathcal{W} := \bigcup_{n\in\mathbb{Z}-\{0\}} \mathcal{W}^{z^{(n)}}$.

Furthermore, if $B \subseteq \mathcal{D}_e$ is bounded and at a positive distance from $\mathcal{W}$ then for any $n \in \mathbb{N}$ we have

$$\sup_{u\in B} \tau^{(n),u} < +\infty.$$

Proof. Let us suppose first that $w \neq \mathbf{0}$. Then, since $\mathcal{W}^\omega$ is a closed set, by Proposition 8.5 we have that the family $\{U^u(t_B,\cdot) : u \in B\}$ is contained in a compact set $B' \subseteq \mathcal{W}^w$ for some suitably small $t_B > 0$. Hence, we obtain that

$$\sup_{u\in B}\,[\inf\{t \ge t_0 : d(U^u(t,\cdot), w) \le r\}] \le t_B + \sup_{v\in B'}\,[\inf\{t \ge 0 : d(U^v(t,\cdot), w) < r\}]$$

Since the application $v \mapsto \inf\{t \ge 0 : d(U^v(t,\cdot), w) < r\}$ is upper semicontinuous and finite on $\mathcal{W}^w$, we conclude that the right hand side is finite and thus the result follows in this case. Now, if $w = \mathbf{0}$ then once again by Proposition 8.5 we have that the family $\{U^u(t_B,\cdot) : u \in B\}$ is contained in a compact set $B' \subseteq \mathcal{D}_{\mathbf{0}}$. By Proposition 8.2 we may choose $t_B > 0$ sufficiently small so as to guarantee that B' is at a positive distance from

$\mathcal{W}$. From here we conclude the proof as in the previous case. Finally, the last statement of the proposition is proved in a completely analogous fashion. □

8.3. **Properties of the potential S.**

Proposition 8.8. The mapping $t \mapsto S(U^u(t,\cdot))$ is monotone decreasing and continuous for any $u \in H_0^1((0,1))$.

Proof. An easy computation shows that

$$\frac{d}{dt}S(U^u(t,\cdot)) = -\int_0^1 (\partial_t U^u(t,x))^2\,dx \leq 0$$

from which the result follows. Details can be found in [17, Lemma 17.5]. □

Proposition 8.9. The potential S is lower semicontinuous.

Proof. See [7]. □

Proposition 8.10. Given $u \in C_D([0,1])$ and $t \in (0,\tau^u)$ there exist constants $C_{u,t},\delta_{u,t} > 0$ such that

- $\tau^v > t$ for any $v \in B_{\delta_{u,t}}(u)$
- For any $v \in B_{\delta_{u,t}}(u)$ one has

$$\|S(U^u(t,\cdot)) - S(U^v(t,\cdot))\|_\infty \leq C_{u,t}\|u-v\|_\infty.$$

Proof. This is a direct consequence of Propositions 8.6 and 8.3. □

8.4. **Properties of the quasipotential V.**

Proposition 8.11. The mapping $u \mapsto V(\mathbf{0},u)$ is lower semicontinuous on $C_D([0,1])$.

Proof. Let $(u_k)_{k\in\mathbb{N}} \subseteq C_D([0,1])$ be a sequence converging to some limit $u_\infty \in C_D([0,1])$. We must check that

$$V(\mathbf{0},u_\infty) \leq \liminf_{k\to+\infty} V(\mathbf{0},v_k). \tag{8.1}$$

If $S(u_\infty) = +\infty$ then by Proposition 5.5 we see that $V(\mathbf{0},u_\infty) = +\infty$ and thus by the lower semicontinuity of S we conclude that $\lim_{v\to u} V(\mathbf{0},v) = +\infty$ which establishes (8.1) in this particular case. Now, if $S(u_\infty) < +\infty$ then, by the lower semicontinuity of S and the continuity in time of the solutions to (1.2), given $\delta > 0$ there exists $t_0 > 0$ sufficiently small such that $S(U^{u_\infty}(t_0,\cdot)) > S(u_\infty) - \frac{\delta}{2}$. Moreover, by Proposition 8.3 we may even assume that t_0 is such that

$$\|U^{u_k}(t_0,\cdot) - U^{u_\infty}(t_0,\cdot)\|_\infty \leq 2\|u_k - u_\infty\|_\infty$$

for any $k \in \mathbb{N}$ sufficiently large. Thus, given k sufficiently large and a path φ_k from $\mathbf{0}$ to u_k we construct a path $\varphi_{k,\infty}$ from $\mathbf{0}$ to u_∞ by the following steps:

i. We start from $\mathbf{0}$ and follow φ_k until we reach u_k.
ii. From u_k we follow the deterministic flow U^{u_k} until time t_0.
iii. We then join $U^{u_k}(t_0,\cdot)$ and $U^{u_\infty}(t_0,\cdot)$ by a linear interpolation of speed one.
iv. From $U^{u_\infty}(t_0,\cdot)$ we follow the reverse deterministic flow until we reach u_∞.

By the considerations made in the proof of [18, Lemma 4.3] it is not difficult to see that there exists $C > 0$ such that for any $k \in \mathbb{N}$ sufficiently large we have

$$I(\varphi_{k,\infty}) \leq I(\varphi_k) + C\|u_k - u_\infty\|_\infty + \delta$$

so that we ultimately obtain

$$V(\mathbf{0}, u_\infty) \leq \liminf_{k\to+\infty} V(\mathbf{0}, u_k) + \delta.$$

Since $\delta > 0$ can be taken arbitrarily small we conclude (8.1). $\square$

Proposition 8.12. For any $u, v \in C_D([0,1])$ the map $t \mapsto V\left(u, U^v(t,\cdot)\right)$ is decreasing.

Proof. Given $0 \leq s < t$ and a path φ from u to $U^u(s,\cdot)$ we may extend ϕ to a path $\tilde{\varphi}$ from u to $U^u(t,\cdot)$ simply by following the deterministic flow afterwards. It follows that

$$V\left(u, U^v(t,\cdot)\right) \leq I(\tilde{\varphi}) = I(\varphi)$$

which, by taking infimum over all paths from u to $U^u(s,\cdot)$, yields the desired monotonicity. $\square$

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Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Pabellón I, Ciudad Universitaria
C1428EGA Buenos Aires, Argentina.
E-mail address: pgroisma@dm.uba.ar, ssaglie@dm.uba.ar, nsaintie@dm.uba.ar
Web page: http://mate.dm.uba.ar/~pgroisma