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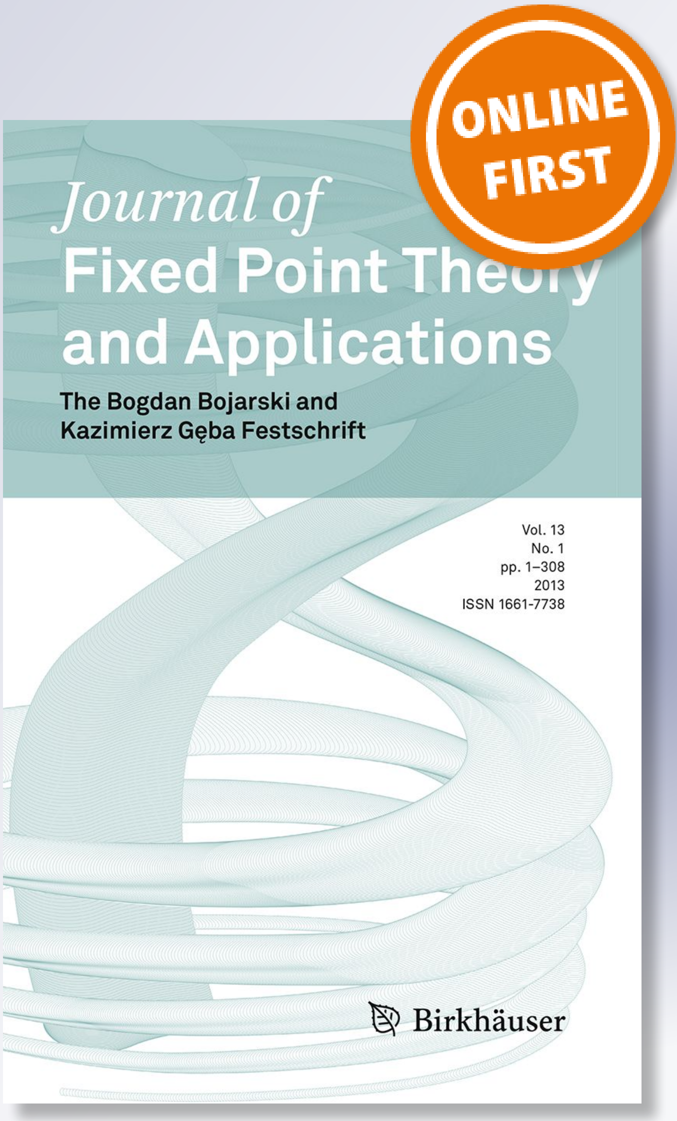

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Non-resonant Fredholm alternative and anti-maximum principle for the fractional p-Laplacian

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Dedicated to Paul H. Rabinowitz.

Abstract. In this paper we extend two nowadays classical results to a nonlinear Dirichlet problem to equations involving the fractional p-Laplacian. The first result is an existence in a non-resonant range more specific between the first and second eigenvalue of the fractional p-Laplacian. The second result is the anti-maximum principle for the fractional p-Laplacian.

Mathematics Subject Classification. 35R11, 47G20, 45G05.

Keywords. Fractional p-Laplacian, existence results, non-resonant, anti-maximum principle.

1. Introduction

This paper deals with existence and qualitative results for the following nonlinear Dirichlet problem with the fractional p-Laplacian

$$\begin{cases} (-\Delta_p)^s u = \lambda |u|^{p-2}u + f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c := \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.1}$$

Here and in the rest of this introduction, Ω is a smooth bounded open of $\mathbb{R}^N$, $s \in (0,1)$, and $p \in (1,\infty)$. The fractional p-Laplacian is a nonlocal version of the p-Laplacian and is an extension of the fractional Laplacian ($p = 2$). More precisely, the fractional p-Laplacian is defined as

$$(-\Delta_p)^s u(x) = 2\mathcal{K}\ \text{P.V.} \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\, \mathrm{d}y, \tag{1.2}$$

with

$$\mathcal{K} = p(1-s)\left(\int_{S^{N-1}} |\langle \omega, e\rangle|^p d\mathcal{H}^{N-1}(\omega)\right)^{-1}, \quad e \in S^{N-1}.$$

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where S^{N-1} denotes the unit sphere in $\mathbb{R}^N$ and $\mathcal{H}^{N-1}$ denotes the $N-1$-dimensional Hausdorff measure. For more details, see [14,17].

A pioneer work on existence of nonlinear one-dimensional integral equation (with L^2 kernels) under non-resonant case can be found in [27]. Besides that let us recall that the Fredholm alternative fails for p-Laplacian and the situation is much more complex than in the linear case. This can be found in a large number of results around Fredholm type alternative for the p-Laplacian, see for instance [9,23,28–31,43,50,51] and the references therein.

For the fractional Laplacian, the standard Fredholm alternative for compact operator can be applied. Observe that the spectrum for the fractional Laplacian is studied in [46,48].

Let us start by describing our existence results. Denote by $\lambda_1(s,p)$ and $\lambda_2(s,p)$ the first and second eigenvalues, respectively, for the fractional p-Laplacian with Dirichlet boundary condition. See Sect. 2 for the definition and basic properties of the eigenvalues of the fractional p-Laplacian.

First, by standard minimization argument, we show that if $\lambda < \lambda_1(s,p)$, then there is a unique weak solution of (1.1), see Sect. 3. Then, also in Section 3, we show the existence of solution to (1.2) for $\lambda \in (\lambda_1(s,p), \lambda_2(s,p))$ and $f \in W^{-s,p'}(\Omega)$. This existence part relies on an homotopy deformation of the degree as in [5], see also [4,6,13].

More precisely, we can prove the following Theorem:

Theorem 1.1. *Let $f \in W^{-s,p'}(\Omega)$. If $\lambda_1(s,p) < \lambda < \lambda_2(s,p)$ then there is a weak solution of* (1.1)*.*

Let us observe that Fredholm type alternative for fully non-linear operator can be found in [11, Section 5]. Notice that using the ideas of [11] and [21] a different homotopy (with respect to s) can be used to prove the above Theorem. Besides that let us also mention that from [21] other existence results can be proved using bifurcation from infinity for (1.1). These results can be found for the case of the p-Laplacian, for example in [9].

Our second aim was to show an anti-maximum principle for the fractional p-Laplacian. This principle has shown to be a powerful tool when analyzing nonlinear elliptic problems, see [8,12,18,36] and the references therein. For the p-Laplacian operator, the anti-maximum principle is proven in [32], see also [10,35]. On the other hand, the link between bifurcation theory and anti-maximum principle was observed for the first time in [8] (see for instance [8, Theorem 27] for an improvement of the anti-maximum principle for the p-Laplacian operator).

In Sect. 4, before proving our anti-maximum principle, we show the following maximum principle:

Theorem 1.2. *Let $f \in W^{-s,p'}(\Omega)$ be such that $f \not\equiv 0$.*

(1) *If $f \geq 0$ and $\lambda < \lambda_1(s,p)$, then $u > 0$ a.e. in Ω for any super-solution u of* (1.1)*.*
(2) *If $f \leq 0$, and $\lambda < \lambda_1(s,p)$, then $u < 0$ a.e. in Ω for any sub-solution u of* (1.1)*.*

Thus, we show the following anti-maximum principle:

Theorem 1.3. *Let $f \in W^{-s,p'}(\Omega)$ be such that $f \not\equiv 0$. Then there is $\delta = \delta(f) > 0$ such that*

(1) *if $f \geq 0$ and $\lambda \in (\lambda_1(s,p), \lambda_1(s,p) + \delta)$, then any weak solution u of* (1.1) *satisfies $u < 0$ a.e. in Ω.*
(2) *if $f \leq 0$ and $\lambda \in (\lambda_1(s,p), \lambda_1(s,p) + \delta)$, then any weak solution u of* (1.1) *satisfies $u > 0$ a.e. in Ω.*

Let us comment that, for the spectral fractional Laplacian (this is a different operator than $(-\Delta)^s$), the anti-maximum principle is only proved in the case $s = 1/2$, see [7]. In fact, we would like to mention that the proof in [7] can be easily extended to the case $s \in (0, 1)$. See also [34] where the anti-maximum principle is shown for non-singular kernel. So, as far we know, Theorem 1.3 is new even for the case $p = 2$. Therefore, we extend in particular the now classical anti-maximum principle of Clement and Peletier (see [18]) for all the range $s \in (0, 1)$ and $p \in (1, \infty)$.

We want to observe that our proof of the previous theorem is not a straightforward adaptation of the proof given in the local case due to that we do not have a suitable Hopf's lemma for the fractional p-Laplacian. To overcome this problem we will use Picone's identity (see Lemma 2.9) and show a lower bound for the measures of the negative (positive) sets of the weak super(sub)-solutions of (1.1) (see Lemma 4.5 and Remark 4.6 below).

In the linear case ($p = 2$), thanks to the regularity results up to the boundary and the Hopf lemma, we can prove a more general result improving Theorems 1.2 and 1.3:

Theorem 1.4. *Let Ω be a bounded domain with $C^{1,1}$ boundary, w_1 be a positive eigenfunction of $(-\Delta)^s$ associated with $\lambda_1(s,2)$. For any $f \in L^\infty(\Omega)$ with $\int_\Omega f(x)w_1 \mathrm{d}x \neq 0$, there is $\delta = \delta(f) > 0$ such that*

(1) *if $\int_\Omega f(x)w_1 \mathrm{d}x > 0$ then any weak solution u of* (1.1) *satisfies*
 (a) *$u < 0$ in Ω if $\lambda \in (\lambda_1(s,2), \lambda_1(s,2) + \delta)$;*
 (b) *$u > 0$ in Ω if $\lambda \in (\lambda_1(s,2) - \delta, \lambda_1(s,2))$;*
(2) *if $\int_\Omega f(x)w_1 \mathrm{d}x < 0$, then any weak solution u of* (1.1) *satisfies*
 (a) *$u > 0$ in Ω if $\lambda \in (\lambda_1(s,2), \lambda_1(s,2) + \delta)$;*
 (b) *$u < 0$ in Ω if $\lambda \in (\lambda_1(s,2) - \delta, \lambda_1(s,2))$.*

The paper is organized as follows: in Sect. 2 we review some preliminaries including the eigenvalue problems. In Sect. 3 we prove our existence results. Finally, in Sect. 4 we prove Theorems 1.2, 1.3 and 1.4.

2. Preliminaries

Let us start by introducing the notation and definitions that we will use in this work. We also gather some preliminary properties which will be useful in the forthcoming sections.

Here and hereafter, $s \in (0,1)$, $p \in (1,\infty)$ and we will denote by Ω an open set in $\mathbb{R}^N$. Given a subset A of $\mathbb{R}^N$ we set $A^c = \mathbb{R}^N \setminus A$, and $A^2 = A \times A$. For all function $u\colon \Omega \to \mathbb{R}$ we define

$$u_+(x) := \max\{u(x), 0\} \quad \text{and} \quad u_-(x) := \max\{-u(x), 0\},$$
$$\Omega_+ := \{x \in \Omega\colon u(x) > 0\} \quad \text{and} \quad \Omega_- := \{x \in \Omega\colon u(x) < 0\}.$$

2.1. Fractional Sobolev spaces

The fractional Sobolev spaces $W^{s,p}(\Omega)$ is defined to be the set of functions $u \in L^p(\Omega)$ such that

$$|u|^p_{W^{s,p}(\Omega)} := \int_{\Omega^2} \frac{|u(x) - u(y)|^p}{|x-y|^{N+sp}}\, \mathrm{d}x\mathrm{d}y < \infty.$$

The fractional Sobolev spaces admit the following norm:

$$\|u\|_{W^{s,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + |u|^p_{W^{s,p}(\Omega)}\right)^{\frac{1}{p}},$$

where

$$\|u\|^p_{L^p(\Omega)} := \int_\Omega |u(x)|^p \,\mathrm{d}x.$$

The space $W^{s,p}(\mathbb{R}^N)$ is defined similarly.

We will denote by $\widetilde{W}^{s,p}(\Omega)$ the space of all $u \in W^{s,p}(\Omega)$ such that $\tilde{u} \in W^{s,p}(\mathbb{R}^N)$, where $\tilde{u}$ is the extension by zero of u. The dual space of $\widetilde{W}^{s,p}(\Omega)$ is denoted by $W^{-s,p'}(\Omega)$ and the corresponding dual pairing is denoted by $\langle \cdot, \cdot \rangle$.

Remark 2.1. By [26, Lemma 6.1], if Ω is bounded, then there is a suitable constant $C = C(N,s,p) > 0$ such that for any $u \in \widetilde{W}^{s,p}(\Omega)$ we get

$$\begin{aligned} |u|^p_{W^{s,p}(\mathbb{R}^N)} &\geq \int_{\Omega\times\Omega^c} \frac{|u(x)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y = \int_\Omega |u(x)|^p \int_{\Omega^c} \frac{1}{|x-y|^{N+sp}}\mathrm{d}y\mathrm{d}x \\ &\geq \frac{C}{|\Omega|^{sp/N}} \|u\|^p_{L^p(\Omega)}, \end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Hence, the seminorm $|\cdot|_{W^{s,p}(\mathbb{R}^N)}$ is a norm in $\widetilde{W}^{s,p}(\Omega)$ equivalent to the standard norm.

If Ω is bounded, we set

$$\widehat{W}^{s,p}(\Omega) := \left\{u \in L^p_{loc}(\mathbb{R}^N)\colon \exists U \supset\supset \Omega \text{ s.t. } u \in W^{s,p}(U), [u]_{s,p} < \infty\right\},$$

where

$$[u]_{s,p} := \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+sp}}\mathrm{d}x.$$

Observe that $\widetilde{W}^{s,p}(\Omega) \subset \widehat{W}^{s,p}(\Omega)$.

We will denote by $p_s^\star$ the fractional critical Sobolev exponent, that is

$$p_s^\star := \begin{cases} \dfrac{Np}{N-sp} & \text{if } sp < N, \\ +\infty & \text{if } sp \geq N. \end{cases}$$

Remark 2.2. If $\mathcal{X} = W^{s,p}(\Omega)$ or $\widetilde{W}^{s,p}(\Omega)$ or $\widehat{W}^{s,p}(\Omega)$ and $u \in \mathcal{X}$, then $u_+, u_- \in \mathcal{X}$ owing to

$$|u_-(x) - u_-(y)| \leq |u(x) - u(y)| \quad \text{and} \quad |u_+(x) - u_+(y)| \leq |u(x) - u(y)|,$$

for all $x, y \in \Omega$.

Further information on fractional Sobolev spaces and many references may be found in [1,25,26,38,39].

2.2. Dirichlet problems

Let Ω be a bounded open set in $\mathbb{R}^N$, $s \in (0,1)$, and $f \in W^{-s,p'}(\Omega)$. We say that $f \geq (\leq) 0$ if for any $v \in \widetilde{W}^{s,p}(\Omega)$, $v \geq 0$ we have that $\langle f, v\rangle \geq (\leq) 0$.

We say that $u \in \widehat{W}^{s,p}(\Omega)$ is a weak super-solution of

$$\begin{cases} (-\Delta_p)^s u = f(x) & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{2.1}$$

if $u \geq 0$ a.e. in Ω^c and

$$\mathcal{K} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y \geq \langle f, v\rangle,$$

for each $v \in \widetilde{W}^{s,p}(\Omega), v \geq 0$.

A function $u \in \widehat{W}^{s,p}(\Omega)$ is a weak sub-solution of (2.1) if $-u$ is a weak super-solution. Finally, a function $u \in \widehat{W}^{s,p}(\Omega)$ is a weak solution of (2.1) if and only if it is both a weak super-solution and a weak sub-solution.

Our next result is a minimum principle.

Lemma 2.3. *Let $f \in W^{-s,p'}(\Omega)$ be such that $f \geq 0$, and u be a weak super-solution of (2.1). Then either $u > 0$ a.e. in Ω or $u = 0$ a.e. in Ω.*

Proof. Since u is a weak super-solution of (2.1), it follows from the comparison principle (see [39, Proposition 2.10]) that $u \geq 0$ in $\mathbb{R}^N$. Moreover, if Ω is connected, by [15, Theorem A.1], we get if $u \neq 0$ a.e. in Ω, then $u > 0$ a.e. in Ω.

Then, we only need to show that $u \not\equiv 0$ in Ω if and only if $u \not\equiv 0$ in all connected components of Ω. That is, we only need to show that if $u \not\equiv 0$ in Ω, then $u \not\equiv 0$ in all connected components of Ω.

Suppose, on the contrary, that is $u \not\equiv 0$ and there is a connected component U of Ω such that $u \equiv 0$ in U. Moreover, for any nonnegative function $v \in \widetilde{W}^{s,p}(U)$ we get

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y \\ &= -2 \int_U \int_{U^c} \frac{|u(x)|^{p-2} u(x) v(y)}{|x - y|^{N+sp}} \mathrm{d}x\mathrm{d}y \end{aligned}$$

due to $u \equiv 0$ in U. Then $u = 0$ a.e. in U^c, that is $u = 0$ a.e. in $\mathbb{R}^N$, which is a contradiction with the fact that $u \neq 0$ a.e. in Ω. ☐

To prove the Theorem 1.1, we will use the homotopy property of the Leray–Schauder degree. For this reason, we need to recall some properties of the Dirichlet problem for the fractional p-Laplace equations.

Let $f \in W^{-s,p'}(\Omega)$. If Ω is a smooth bounded domain, using the fractional Sobolev compact embedding theorem (see [1,25]), it is easily seen that (2.1) has a unique weak solution $u_f \in \widetilde{W}^{s,p}(\Omega)$. Moreover, the operator

$$\begin{aligned} \mathcal{R}_{s,p}\colon W^{-s,p'}(\Omega) &\to \widetilde{W}^{s,p}(\Omega) \\ f &\to u_f \end{aligned}$$

is continuous, see [21]

Now, let Ω be a smooth bounded domain, $f \in W^{-s,p'}(\Omega)$ and $t \in \mathbb{R}$; we define the operator $T_t\colon \widetilde{W}^{s,p}(\Omega) \to \widetilde{W}^{s,p}(\Omega)$ by

$$T_t(u) := \mathcal{R}_{s,p}(\lambda|u|^{p-2}u + tf).$$

Notice that by the fractional Sobolev compact embedding theorem and the continuity of $\mathcal{R}_{s,p}$, we have that T_t is a completely continuous operator.

2.3. Eigenvalue problems

Now we study the following eigenvalue problems:

$$\begin{cases} (-\Delta_p)^s u = \lambda|u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{2.2}$$

We say that λ is an eigenvalue of $(-\Delta_p)^s$ if there is a function $u \in \widetilde{W}^{s,p}(\Omega)\setminus\{0\}$ such that for any $v \in \widetilde{W}^{s,p}(\Omega)$

$$\mathcal{K}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y = \lambda\int_{\Omega}|u|^{p-2}uv\mathrm{d}x.$$

The function u is a corresponding eigenfunction of $(-\Delta_p)^s$ associated with λ.

Before showing the existence of a sequence of eigenvalues, we need to introduce some additional notation. Following [17], we define

$$\mathcal{S}^{s,p} := \left\{u \in \widetilde{W}^{s,p}(\Omega)\colon \|u\|_{L^p(\Omega)} = 1\right\},$$

and

$$\mathcal{W}^{s,p}_m := \{K \subset \mathcal{S}^{s,p}\colon K \text{ is symmetric and compact, } i(K) \geq m\}$$

for $m \in \mathbb{N}$. Here i denotes the Krasnosel'skiĭ genus.

For the proof of the following theorem, see [16,17,21,33,40] (for the local case, see [3,19,41,42,44]).

Theorem 2.4. *Let Ω a smooth bounded domain of $\mathbb{R}^N$. Then there is a sequence of eigenvalues of $(-\Delta_p)^s$*

$$\lambda_m(s,p) = \inf_{K\in\mathcal{W}^{s,p}_m}\max_{u\in K}\mathcal{K}|u|_{W^{s,p}(\mathbb{R}^N)}.$$

Moreover,

- *If u is an eigenfunction of $(-\Delta_p)^s$, then $u \in L^\infty(\Omega)$.*

- *$\lambda_1(s,p)$ is the first eigenvalue of $(-\Delta_p)^s$, that is*

$$\lambda_1(s,p) = \inf\left\{\mathcal{K}|u|^p_{W^{s,p}(\mathbb{R}^N)} : u \in \mathcal{S}^{s,p}\right\}.$$

- *$\lambda_1(s,p)$ is simple and isolated.*
- *Any eigenfunction of $(-\Delta_p)^s$ associated with $\lambda_1(s,p)$ has a constant sign.*
- *If u is an eigenfunction of $(-\Delta_p)^s$ associated with $\lambda > \lambda_1(s,p)$, then u must be sign-changing.*
- *$\lambda_2(s,p)$ is the second eigenvalue*

$$\begin{aligned}\lambda_2(s,p) &= \inf_{\gamma\in\Gamma(w_1,-w_1)} \max_{u\in \mathrm{Im}\gamma(0,1)} \mathcal{K}|u|^p_{W^{s,p}(\mathbb{R}^N)} \\ &= \inf\{\lambda\colon \lambda > \lambda_1(s,p) \text{ is an eigenvalue of } (-\Delta_p)^s\},\end{aligned}$$

where w_1 is an eigenfunction of $(-\Delta_p)^s$ associated with $\lambda_1(s,p)$ and $\Gamma(w_1,-w_1)$ is the set of continuous paths on $\mathcal{S}^{s,p}$ connecting to w_1 and $-w_1$.

Remark 2.5. It is not difficult to see that, if $u \in \widetilde{W}^{s,p}(\Omega)$ is such that

$$\lambda_1(s,p) = \frac{\mathcal{K}|u|^p_{W^{s,p}(\mathbb{R}^N)}}{\|u\|^p_{L^p(\Omega)}},$$

then u is eigenfunction of $(-\Delta_p)^s$ associated with $\lambda_1(s,p)$.

Let us finally observe that in [21], we also prove that $\lambda_1(\cdot,p)$ is continuous.

2.4. Regularity results

Here, we study the regularity up to the boundary of weak solutions of (1.1) when $f \in L^\infty(\Omega)$. For this, we need the following results:

Lemma 2.6. *Let $f \in L^\infty(\Omega)$ and $\lambda \in \mathbb{R}$. If u is a weak solution of (1.1) then $u \in L^\infty(\Omega)$.*

Proof. In this proof, we borrow ideas from [33,47].

If $ps > N$, then $u \in L^\infty(\Omega)$ due to the fractional Sobolev embedding theorem. For the rest of the proof, we assume $sp \le N$.

Let u be a weak solution of (1.1). Up to multiplying u by a small constant we may assume that

$$\|u\|_{L^p(\Omega)} = \sqrt{\delta},$$

where $\delta > 0$ will be selected below.

For any $k \in \mathbb{N}$, we define $v_k := (u - 1 + 2^{-k})_+$ and $U_k = \|v_k\|^p_{L^p(\Omega)}$. Observe that, for any $k \in \mathbb{N}$ we have that

$$\begin{aligned}&v_k \in \widetilde{W}^{s,p}(\Omega), \quad v_{k+1} \le v_k \text{ a.e. in } \mathbb{R}^N \quad \text{and} \\ &\{x \in \Omega\colon v_{k+1} > 0\} \subset \{x \in \Omega\colon v_k > 2^{-(k+1)}\}.\end{aligned} \tag{2.3}$$

Moreover, $U_k \to \|(u-1)_+\|_{L^p(\Omega)}$ as $k \to \infty$. Then, for any $k \in \mathbb{N}$

$$\begin{aligned}
\mathcal{K}|v_k|^p_{W^{s,p}(\Omega)} &= \mathcal{K}\int_{\Omega^2} \frac{|v_{k+1}(x)-v_{k+1}(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\
&\le \mathcal{K}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v_{k+1}(x)-v_{k+1}(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\
&= \lambda\int_\Omega v_{k+1}^p \mathrm{d}x + \int_\Omega f(x)v_{k+1}\mathrm{d}x \\
&\le |\lambda|U_k + \|f\|_{L^\infty(\Omega)}\int_\Omega v_{k+1}\mathrm{d}x \\
&\le |\lambda|U_k + \|f\|_{L^\infty(\Omega)}|\{x\in\Omega\colon v_{k+1}>0\}|^{1-1/p}U_k^{1/p}.
\end{aligned}$$

By (2.3), we get

$$U_k = \|v_k\|^p_{L^p(\Omega)} \ge 2^{-p(k+1)}|\{x\in\Omega\colon v_{k+1}>0\}|; \tag{2.4}$$

then

$$\mathcal{K}|v_k|^p_{W^{s,p}(\Omega)} \le \left(|\lambda| + \|f\|_{L^\infty(\Omega)}2^{(p-1)}\right)2^{(p-1)k}U_k. \tag{2.5}$$

Thus, given $q \in (p, p_s^\star)$, by the Holder inequality, the fractional Sobolev embedding theorem, (2.4) and (2.5), we have that

$$\begin{aligned}
U_{k+1} &\le \|v_{k+1}\|^p_{L^q(\Omega)}|\{x\in\Omega\colon v_{k+1}>0\}|^{1-p/q} \\
&\le C|v_k|^p_{W^{s,p}(\Omega)}\left(2^{p(k+1)}U_k\right)^{1-p/q} \\
&\le C\left(|\lambda| + \|f\|_{L^\infty(\Omega)}2^{(p-1)}\right)2^{(p-p^2/q)}2^{(2p-1-p^2/q)k}U_k^{2-q/p} \\
&\le \left\{\left[1 + C\left(|\lambda| + \|f\|_{L^\infty(\Omega)}2^{(p-1)}\right)2^{(p-p^2/q)}\right]2^{(2p-1-p^2/q)}\right\}^k U_k^{2-p/q} \\
&= C^k U_k^\rho,
\end{aligned}$$

where $C>1$ and $\rho = 2 - p/q > 1$.

Now, we choose the number $\delta > 0$ sufficiently small that

$$\delta^\rho < \frac{1}{C^{1/(\rho-1)}}$$

and proceeding as in the end of the proof of [49, Proposition 7], we can conclude that $u \le 1$ a.e. in Ω. By replacing u with $-u$ we obtain $\|u\|_{L^\infty(\Omega)} \le 1$. □

Then, by the previous lemma, [39, Theorem 1.1] and [45, Proposition 1.1 and Theorem 1.2], we have

Theorem 2.7. *Let Ω be a bounded domain with $C^{1,1}$ boundary, $f \in L^\infty(\Omega)$, $\lambda \in \mathbb{R}$, and $\delta(x) = dist(x, \partial\Omega)$. Then, there is $\alpha \in (0, s]$ and C, depending on Ω such that for all weak solution u of (1.1), $u \in C^\alpha(\overline{\Omega})$ and*

$$\|u\|_{C^\alpha(\overline{\Omega})} \le C\left(|\lambda|\|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\right).$$

In addition, if $p=2$ then $\alpha = s$ and

$$u/\delta^s \in C^\beta(\overline{\Omega}) \quad \text{and} \quad \|u/\delta^s\|_{C^\beta(\overline{\Omega})} \le D\left(|\lambda|\|u\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)},\right)$$

where $\beta \in (0, \min\{s, 1-s\})$. *The constants* β *and* D *depend only on* Ω *and* s.

Finally, in the linear case, as a consequence of the fractional Hopf lemma (see [22,37]), we have the next result:

Lemma 2.8. *Let* Ω *be a bounded domain with* $C^{1,1}$ *boundary,* $\delta(x) = dist(x, \partial\Omega)$, *and* w_1 *be an eigenfunction of* $(-\Delta)^s$. *If* $\{v_n\}_{n\in\mathbb{N}} \subset C^s(\overline{\Omega})$ *is such that* $v_n/\delta \in C(\overline{\Omega})$ *and*

$$v_n \to w_1 \quad \text{and} \quad \frac{v_n}{\delta^s} \to \frac{w_1}{\delta^s}$$

strongly in $\overline{\Omega}$, *then there is* $n_0 \in \mathbb{N}$ *such that* $v_n > 0$ *for all* $n \geq n_0$.

2.5. Picone inequality

For the proof of the following Picone inequality, see [2, Lemma 6.2]:

Lemma 2.9. *For every* $a_1, a_2 \geq 0$ *and* $b_1, b_2 > 0$

$$|a_1 - a_2|^p \geq |b_1 - b_2|^{p-2}(b_1 - b_2)\left(\frac{a_1^p}{b_1^{p-1}} - \frac{a_2^p}{b_2^{p-1}}\right).$$

The equality holds if and only if $(a_1, a_2) = k(b_1, b_2)$ *for some constant* k.

3. Non-resonant Fredholm alternative problem

Let us start this section proving the following existence results for Eq. (1.1) with $\lambda < \lambda_1(s,p)$. One of the principal results, that we will use through the rest of this work, is the fractional Sobolev compact embedding theorem. For this reason, throughout the rest of this work Ω is a smooth bounded domain of $\mathbb{R}^N$.

Theorem 3.1. *Let* $f \in W^{-s,p'}(\Omega)$. *If* $\lambda < \lambda_1(s,p)$ *then there is a weak solution of* (1.1).

Proof. The proof of this theorem is standard. First observe that weak solutions of (1.1) are critical points of the functional $J \colon \widetilde{W}^{s,p}(\Omega) \to \mathbb{R}$, where

$$J(u) := \frac{\mathcal{K}}{p}|u|^p_{W^{s,p}(\mathbb{R}^N)} - \frac{\lambda}{p}\|u\|^p_{L^p(\Omega)} - \langle f, u\rangle.$$

It follows from $\lambda < \lambda_1(s,p)$ that J is bounded below, coercive, strictly convex and sequentially weakly lower semi-continuous. Thus J has a unique critical point which is a global minimum. $\square$

Our next aim is to prove Theorem 1.1, to this end we will use the homotopy property of the Leray–Schauder degree. We first prove an a priori bound for the fixed points of T_t.

Lemma 3.2. *If* $\lambda_1(s,p) < \lambda < \lambda_2(s,p)$ *then there exists* $R > 0$ *such that for all* $t \in [0,1]$ *there is no solution of* $(I - T_t)u = 0$ *for* $|u|_{W^{s,p}(\mathbb{R}^N)} \geq R$

Proof. Suppose, to the contrary, that is for all $n \in \mathbb{N}$ there exist $t_n \in [0,1]$ and $u_n \in \widetilde{W}^{s,p}(\Omega)$ such that $(I - T_{t_n})u_n = 0$ and $|u_n|_{W^{s,p}(\mathbb{R}^N)} \to \infty$ as $n \to \infty$. Let us define

$$v_n = \frac{u_n}{|u_n|_{W^{s,p}(\mathbb{R}^N)}} \quad \forall n \in \mathbb{N}.$$

Then for all $n \in \mathbb{N}$, we have that v_n is a weak solution of

$$\begin{cases} (-\Delta_p)^s u = \lambda |u|^{p-2}u + \dfrac{t_n f(x)}{|u_n|^{p-1}_{W^{s,p}(\mathbb{R}^N)}} & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{3.1}$$

Using the fractional Sobolev compact embedding theorem, up to a subsequence (still denoted by v_n)

$$\begin{aligned} v_n &\rightharpoonup v \quad \text{weakly in } \widetilde{W}^{s,p}(\Omega), \\ v_n &\to v \quad \text{strongly in } L^p(\Omega). \end{aligned}$$

Since v_n is a weak solution of (3.1) we have

$$1 = |v_n|^p_{W^{s,p}(\mathbb{R}^N)} = \lambda \|v_n\|^p_{L^p(\Omega)} + \left\langle {}^{t_n f}\!/\!_{|u_n|^{p-1}_{W^{s,p}(\mathbb{R}^N)}}, v_n \right\rangle.$$

Then, using the fact that ${}^{t_n f}\!/\!_{|u_n|^{p-1}_{W^{s,p}(\mathbb{R}^N)}} \to 0$ strongly in $W^{-s,p'}(\Omega)$, together with the strong convergence of v_n in $L^p(\Omega)$, we find that $\|v\|^p_{L^p(\Omega)} = {}^1\!/\!_\lambda$. □

Now we are in position to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 3.2, the Leray–Schauder degree $d(I - T_t, B(0,R), 0)$ is well defined and constant for all in $t \in [0,1]$ by the invariance of the degree by homotopy. Thus $d(I - T_t, B(0,R), 0) = -1$ since $d(I - T_0, B(0,R), 0) = -1$ by Theorem 5.3 of [21], from here the existence result follows: □

Observe that, in the above proof, the fact $d(I - T_0, B(0,R), 0) \neq 0$ can be established without using the results of [21] as a consequence of Borsuk theorem (see for example [24, Theorem 8.3]).

4. Maximum and anti-maximum principle

In this section, we will denote by w_1 the positive eigenfunction of $(-\Delta_p)^s$ associated with $\lambda_1(s,p)$ whose L^p-norm is equal to 1. Since $w_1 \in L^\infty(\Omega)$, by [39], there is $\alpha \in (0,1)$ such that $w_1 \in C^\alpha(\overline{\Omega})$.

We start proving Theorem 1.2.

Proof of Theorem 1.2. We only prove the first statement; the another statement can be proved in an analogous way.

Since $u \geq 0$ a.e. in Ω^c we have that $u_- \in \widetilde{W}^{s,p}(\Omega)$. Then

$$\mathcal{K}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u_-(x)-u_-(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y$$
$$= -\lambda\int_\Omega |u_-|^p\mathrm{d}x + \langle f, u_-\rangle;$$

consequently,

$$\lambda\int_\Omega |u_-|^p\mathrm{d}x = -\mathcal{K}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u_-(x)-u_-(y))}{|x-y|^{N+sp}}$$
$$\mathrm{d}x\mathrm{d}y + \langle f, u_-\rangle$$
$$\geq \mathcal{K}\int_{\mathbb{R}^{2N}} \frac{|u_-(x)-u_-(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y.$$

Thus, if $u_- \not\equiv 0$ then

$$\lambda \geq \mathcal{K}\frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u_-(x)-u_-(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y}{\displaystyle\int_\Omega |u_-|^p\mathrm{d}x} \geq \lambda_1(s,p),$$

a contradiction. Therefore $u \geq 0$ in $\mathbb{R}^N$. Moreover, proceeding as in the proof of Lemma 2.3, we have that $u \not\equiv 0$ in all connected components of Ω. Finally, by [20, Theorem 2.9] , $u > 0$ a.e. in Ω. □

Before proving Theorem 1.3, we show some previous results:

Lemma 4.1. *Let $\lambda \geq \lambda_1(s,p)$, and $f \in W^{-s,p'}(\Omega)$ be such that $f \geq 0$ and $f \not\equiv 0$. Then the problem (1.1) has no non-negative weak super-solutions.*

Proof. Suppose, to the contrary, there is a non-negative weak super-solution u of (1.1). Then, by Lemma 2.3, $u > 0$ a.e. in Ω. By the definition of $\widehat{W}^{s,p}(\Omega)$, let $U \supset\supset \Omega$ be such that

$$\|u\|_{W^{s,p}(U)} + \int_{\mathbb{R}^N} \frac{|u|^{p-1}}{(1+|x|)^{N+sp}}\mathrm{d}x < \infty,$$

$n \in \mathbb{N}$ and $u_n := u + \dfrac{1}{n}$.

We begin by proving that $v_n := \dfrac{w_1^p}{u_n^{p-1}} \in \widetilde{W}^{s,p}(\Omega)$. It is immediate that $v_n > 0$ in Ω, $v_n = 0$ in Ω^c, and since $w_1 \in L^\infty(\Omega)$ we have that $v_n \in L^p(\Omega)$.

On the other hand,

$$\begin{aligned}|v_n(x)-v_n(y)| &= \left|\frac{w_1(x)^p - w_1(y)^p}{u_n(x)^{p-1}} + \frac{w_1(y)^p\left(u_n(y)^{p-1}-u_n(x)^{p-1}\right)}{u_n(y)^{p-1}u_n(x)^{p-1}}\right| \\ &\le n^{p-1}\left|w_1(x)^p - w_1(y)^p\right| + \|w_1\|^p_{L^\infty(\Omega)}\frac{\left|u_n(x)^{p-1}-u_n(y)^{p-1}\right|}{u_n(y)^{p-1}u_n(x)^{p-1}} \\ &\le n^{p-1}p(w_1(x)^{p-1}+w_1(y)^{p-1})|w_1(x)-w_1(y)| \\ &\quad +\|w_1\|^p_{L^\infty(\Omega)}(p-1)\frac{|u_n(x)^{p-2}+u_n(y)^{p-2}|}{u_n(y)^{p-1}u_n(x)^{p-1}}|u_n(x)-u_n(y)| \\ &\le 2\|w_1\|^{p-1}_{L^\infty(\Omega)}n^{p-1}p|w_1(x)-w_1(y)| \\ &\quad +n\|w_1\|^p_{L^\infty(\Omega)}(p-1)\left(\frac{1}{u_n(y)}+\frac{1}{u_n(x)}\right)|u(x)-u(y)| \\ &\le C(n,p,\|w_1\|_{L^\infty(\Omega)})\left(|w_1(x)-w_1(y)|+|u(x)-u(y)|\right),\end{aligned}$$

for all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$. Hence $v_n\in W^{s,p}(U)$ for all $m\in\mathbb{N}$ due to $w_1,u\in W^{s,p}(U)$. Then, since $v_n=0$ in Ω^c, and $v_n\in W^{s,p}(U)$ with $\Omega\subset\subset U$, we have

$$\begin{aligned}&\int_{\mathbb{R}^{2N}}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\ &\quad=\int_{U^2}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y + 2\int_{U\times U^c}\frac{|v_n(x)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\ &\quad=\int_{U^2}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y + 2\int_{\Omega\times U^c}\frac{|v_n(x)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\ &\quad=\int_{U^2}\frac{|v_n(x)-v_n(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y + 2n^p\|w_1\|_{L^\infty(\Omega)}\int_{\Omega\times U^c}\frac{\mathrm{d}x\mathrm{d}y}{|x-y|^{N+sp}} \\ &\quad<\infty,\end{aligned}$$

that is $v_n\in W^{s,p}(\mathbb{R}^N)$. Therefore, $v_n\in\widetilde{W}^{s,p}(\Omega)$.

Now, set

$$\begin{aligned}L(w_1,u_n) := |w_1(x)-w_1(y)|^p - |u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y)) \\ \times\left(\frac{w_1(x)^p}{u_n(x)^{p-1}}-\frac{w_1(x)^p}{u_n(y)^{p-1}}\right)\end{aligned}$$

By Lemma 2.9, we have

$$\begin{aligned}0 &\le \mathcal{K}\int_{\Omega^2}\frac{L(w_1,u_n)(x,y)}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \le \mathcal{K}\int_{\mathbb{R}^{2N}}\frac{L(w_1,u_n)(x,y)}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\ &\le \mathcal{K}\int_{\mathbb{R}^{2N}}\frac{|w_1(x)-w_1(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y \\ &\quad -\mathcal{K}\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\left(v_n(x)-v_n(y)\right)\mathrm{d}x\mathrm{d}y \\ &\le \lambda_1(s,p)\int_\Omega w_1(x)^p\,\mathrm{d}x - \lambda\int_\Omega u(x)^{p-1}v_n(x)\,\mathrm{d}x - \langle f,v_n\rangle\end{aligned}$$

$$\leq \lambda_1(s,p)\int_\Omega w_1(x)^p\,\mathrm{d}x - \lambda\int_\Omega u(x)^{p-1}\frac{w_1(x)^p}{u_n(x)^{p-1}}\,\mathrm{d}x - \left\langle f, \frac{w_1^p}{u_n^{p-1}}\right\rangle$$
$$\leq \lambda_1(s,p)\int_\Omega w_1(x)^p\,\mathrm{d}x - \lambda\int_\Omega u(x)^{p-1}\frac{w_1(x)^p}{u_n(x)^{p-1}}\,\mathrm{d}x,$$

due to w_1 is the positive eigenvalue associated with $\lambda_1(s,p)$, $u \in \widehat{W}^{s,p}(\Omega)$ is a weak super-solution of (1.1) and $f \geq 0$.

Since $\lambda_1(s,p) \leq \lambda$, by the Fatou's lemma and the dominated convergence theorem

$$\int_{\Omega^2}\frac{L(w_1,u)(x,y)}{|x-y|^{N+sp}}\,\mathrm{d}x\mathrm{d}y = 0.$$

Then, again by Lemma 2.9, $L(w_1,u)(x,y) = 0$ a.e. in Ω. and $u = kw_1$ a.e. in Ω for some constant $k > 0$. Then,

$$\begin{aligned}
&\lambda_1(s,p)\int_\Omega u(x)^{p-1}\varphi(x)\mathrm{d}x\\
&\quad= \mathcal{K}\int_{\mathbb{R}^2}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\\
&\quad\geq \lambda\int_\Omega u(x)^{p-1}\varphi(x)\mathrm{d}x + \langle f,\varphi\rangle,
\end{aligned}$$

for any $\varphi \in \widetilde{W}^{s,p}(\Omega)$, $\varphi \geq 0$. This is a contradiction since $\lambda \geq \lambda_1(s,p)$ and $f \geq 0$, $f \not\equiv 0$. $\square$

Remark 4.2. Observe that Lemma 4.1 implies that if $\lambda \geq \lambda_1(s,p)$, and $f \in W^{-s,p'}(\Omega)$ is such that $f \leq 0$ and $f \not\equiv 0$, then the problem (1.1) has no non-positive weak sub-solutions.

Corollary 4.3. *Let $f \in W^{-s,p'}(\Omega)$ be such that $f \geq 0$ and $f \not\equiv 0$. Then the problem (1.1) with $\lambda = \lambda_1(s,p)$ has no weak super-solutions.*

Proof. We argue by contradiction. If a weak super-solution u of (1.1) with $\lambda = \lambda_1(s,p)$ exists by Lemma 4.1 we have $u_- \not\equiv 0$ in Ω. Since $u_- \in \widetilde{W}^{s,p}(\Omega)$, we get, by the characterization of $\lambda_1(s,p)$ given in Theorem 2.4,

$$\begin{aligned}
&-\lambda_1(s,p)\int_\Omega u_-(x)^p\mathrm{d}x \leq \lambda_1(s,p)\int_\Omega |u(x)|^{p-2}u(x)u_-(x)\mathrm{d}x + \langle f, u_-\rangle\\
&\quad\leq \mathcal{K}\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^{p-2}(u(x)-u(y)(u_-(x)-u_-(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\\
&\quad\leq -\mathcal{K}\int_{\Omega_-^2}\frac{|u_-(x)-u_-(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\\
&\qquad- 2\mathcal{K}\int_{\Omega_-\times\Omega_-^c}\frac{(u_-(x)+u_+(y))^{p-1}u_-(x)}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\\
&\quad\leq -\mathcal{K}\int_{\mathbb{R}^{2N}}\frac{|u_-(x)-u_-(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Therefore,

$$\lambda_1(s,p) \geq \mathcal{K}\frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u_-(x)-u_-(y)|^p}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y}{\displaystyle\int_\Omega u_-(x)^p\mathrm{d}x},$$

that is u_- is a corresponding eigenfunction to $\lambda_1(s,p)$ (see Remark 2.5). Then there is $k>0$ such that $u_- = kw_1$, and, therefore, $u_- > 0$ in Ω, that is $u<0$ in Ω. Moreover,

$$\begin{aligned}\lambda_1(s,p)&\int_\Omega |u(x)|^{p-2}uv\mathrm{d}x\\ &= -\mathcal{K}\int_{\mathbb{R}^{2N}} \frac{|u_-(x)-u_-(y)|^{p-2}(u_-(x)-u_-(y))(v(x)-v(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\\ &\geq \mathcal{K}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\\ &\geq \lambda_1(s,p)\int_\Omega |u(x)|^{p-2}uv\mathrm{d}x + \langle f, v\rangle\end{aligned}$$

for any $v\in\widetilde{W}^{s,p}(\Omega)$, $v\geq 0$. This is a contradiction since $f\geq 0$, and $f\not\equiv 0$. □

Remark 4.4. Note that it follows straightforward from Corollary 4.3 that if $f\in W^{-s,p'}(\Omega)$ is such that $f\leq 0$ and $f\not\equiv 0$, then the problem (1.1) with $\lambda=\lambda_1(s,p)$ has no weak sub-solutions.

Lemma 4.5. *Let $\lambda\geq\lambda_1(s,p)$, and $f\in W^{-s,p'}(\Omega)$ be such that $f\geq 0$ and $f\not\equiv 0$. Then there exist $\alpha>1$ and a constant $C>0$ such that for all u is a weak super-solution of (1.1); we have that*

$$\left(\frac{C}{\lambda}\right)^\alpha \leq |\Omega_-|,$$

where $\Omega_- = \{x\in\Omega\colon u(x)<0\}$.

Proof. Let u be a weak super-solution of (1.1). By Lemma 4.1, $u_-\not\equiv 0$ in Ω. Taking u_- as test function, we have that

$$\begin{aligned}\mathcal{K}&\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u_-(x)-u_-(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\\ &\geq -\lambda\int_\Omega |u_-|^p\mathrm{d}x + \langle f, u_-\rangle.\end{aligned}$$

If $p<q<p_s^\star$, by fractional Sobolev embedding theorem, then there is a constant C such that

$$\begin{aligned}C\mathcal{K}\|u_-\|^p_{L^q(\Omega)} &\leq \mathcal{K}|u_-|^p_{W^{s,p}(\mathbb{R}^N)}\\ &\leq -\mathcal{K}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u_-(x)-u_-(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\end{aligned}$$

$$\leq -\mathcal{K}\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(u_-(x)-u_-(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y + \langle f, u_-\rangle$$
$$\leq \lambda \int_\Omega |u_-|^p \mathrm{d}x,$$

and using the Hölder inequality

$$C\mathcal{K}\|u_-\|^p_{L^q(\Omega)} \leq \lambda \|u_-\|^p_{L^q(\Omega)}|\Omega_-|^{1-p/q},$$

which, by using that $u_- \not\equiv 0$ in Ω, concludes the proof. □

Remark 4.6. As an immediate consequence of Lemma 4.5, we have that if $\lambda \geq \lambda_1(s,p)$, and $f \in W^{-s,p'}(\Omega)$ with $f \leq 0$ and $f \not\equiv 0$, then there exist $\alpha > 1$ and a constant $C > 0$ such that for all u weak sub-solution of (1.1) we have that

$$\left(\frac{C}{\lambda}\right)^\alpha \leq |\Omega_+|,$$

where $\Omega_+ = \{x \in \Omega \colon u(x) > 0\}$.

Next, we prove our first anti-maximum principle.

Proof of Theorem 1.3. Again, we only prove the first statement; as before the another statement can be proved in an analogous way.

Suppose, to the contrary, there are sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \searrow \lambda_1(s,p)$ and u_n is a weak solution of (1.1) with $\lambda = \lambda_n$ and $(u_n)_+ \not\equiv 0$ for all $n \in \mathbb{N}$.

We claim that

$$\|u_n\|_{L^q(\Omega)} \to \infty \tag{4.1}$$

for all $p \leq q < p_s^\star$.

Suppose not, so there exists $q \in (p, p_s^\star)$ such that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^q(\Omega)$. Then, using that u_n is a weak solution of (1.1) for all $n \in \mathbb{N}$, Hölder's inequality and $\lambda_n \searrow \lambda_1(s,p)$, we have that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\widetilde{W}^{s,p}(\Omega)$. Then, since T_1 is a completely continuous operator (see Sect. 2.2), up to a subsequence (still denoted by u_n)

$$u_n \to u \quad \text{strongly in } \widetilde{W}^{s,p}(\Omega),$$

where u is a weak solution of (1.1) with $\lambda = \lambda_1(s,p)$. By Corollary 4.3, this is a contradiction. We have to prove our claim.

Set $q \in (p, p_s^\star)$ and

$$v_n := \frac{u_n}{\|u_n\|_{L^q(\Omega)}} \quad \forall n \in \mathbb{N}.$$

Then for all $n \in \mathbb{N}$ v_n is a weak solution of

$$\begin{cases} (-\Delta_p)^s u = \lambda_n |u|^{p-2}u + \dfrac{f(x)}{\|u_n\|^{p-1}_{L^q(\Omega)}} & \text{in } \Omega,\\ u = 0 & \text{in } \Omega^c. \end{cases}$$

Now, using again that T_1 is a completely continuous operator and the fractional Sobolev compact embedding theorem, up to a subsequence (still denoted by v_n)

$$\begin{aligned} v_n &\to v \quad \text{strongly in } \widetilde{W}^{s,p}(\Omega),\\ v_n &\to v \quad \text{strongly in } L^q(\Omega). \end{aligned}$$

Thus, $v \not\equiv 0$ in Ω and, v is a weak solution of (2.2) since $\lambda_n \to \lambda_1(s,p)$ and $f/\|u_n\|^{p-1}_{L^q(\Omega)} \to 0$ strongly in $W^{-s,p'}(\Omega)$. That is, v is an eigenfunction of $(-\Delta_p)^s$ associated with $\lambda_1(s,p)$. Therefore, either $v > 0$ in Ω or $v < 0$ in Ω. The case $v > 0$ is a contradiction by Lemma 4.5. To complete the proof of the theorem it remains to consider the case when $v < 0$.

If $v < 0$ then $(v_n)_+ \to 0$ strongly in $L^q(\Omega)$. Therefore, using (4.1), it turns out that $\|(u_n)_+\|_{L^q(\Omega)} \to \infty$.

On the other hand, by the Sobolev embedding theorem, there is a constant C independent of n such that

$$\begin{aligned} C\mathcal{K}\|(u_n)_+\|^p_{L^q(\Omega)} &\le \mathcal{K}|(u_n)_+|^p_{W^{s,p}(\mathbb{R}^N)}\\ &\le \mathcal{K}\int_{\mathbb{R}^{2N}} \frac{|u_n(x)-u_n(y)|^{p-2}(u_n(x)-u_n(y))((u_n)_+(x)-(u_n)_+(y))}{|x-y|^{N+sp}}\mathrm{d}x\mathrm{d}y\\ &\le \lambda_n \int_\Omega (u_n)^p_+ \mathrm{d}x + \langle f(x), (u_n)_+\rangle\\ &\le \lambda_n \|(u_n)_+\|^p_{L^q(\Omega)} |\{x\in\Omega\colon u_n(x)>0\}|^{1-p/q}\\ &\quad + \|f\|_{W^{-s,p'}(\Omega)}|(u_n)_+|_{W^{s,p}(\mathbb{R}^N)} \end{aligned}$$

for all $n \in \mathbb{N}$. Then

$$C \le \lambda_n |\{x\in\Omega\colon u_n(x)>0\}|^{1-p/q} + \frac{\|f\|_{W^{-s,p'}(\Omega)}}{\|(u_n)_+\|^{p-1}_{L^q(\Omega)}}|v_n|_{W^{s,p}(\mathbb{R}^N)},$$

for all $n \in \mathbb{N}$. Therefore,

$$\frac{C}{\lambda_1(s,p)} \le \liminf_{n\to\infty} |\{x\in\Omega\colon u_n(x)>0\}|^{1-p/q},$$

which is a contradiction with the fact that $(v_n)_+ \to 0$ strongly in $L^q(\Omega)$.

Finally, We show our anti-maximum principle for the linear case.

Proof of Theorem 1.4. As before, we only prove the first statement; the other statements can be proved in an analogous way.

It suffices to prove that, for any two sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{u_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \searrow \lambda_1(s,2)$ and u_n is a weak solution of (1.1) with $\lambda = \lambda_n$, there is $n_0 \in \mathbb{N}$ such that $u_n < 0$ in Ω for all $n \ge n_0$. For such sequences, by Lemma 2.6, $u_n \in L^\infty(\Omega)$ for all $n \in \mathbb{N}$.

We claim that

$$\|u_n\|_{L^\infty(\Omega)} \to \infty.$$

If we assume by contradiction that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(\Omega)$. Then, using that u_n is a weak solution of (1.1) for all $n \in \mathbb{N}$, Hölder's inequality and $\lambda_n \searrow \lambda_1(s,p)$, we have that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $\widetilde{W}^{s,2}(\Omega)$. Then, since

T_1 is a completely continuous operator, up to a subsequence (still denoted by u_n)

$$u_n \to u \quad \text{strongly in } \widetilde{W}^{s,2}(\Omega),$$

where u is a weak solution of (1.1) with $\lambda = \lambda_1(s,2)$. Then

$$\begin{aligned}\lambda_1(s,2)\int_\Omega uw_1\mathrm{d}x &= \mathcal{K}\int_{\mathbb{R}^{2k}} \frac{(u(x)-u(y))(w_1(x)-w_1(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\\ &= \lambda_1(s,2)\int_\Omega uw_1\mathrm{d}x + \int_\Omega fw_1\mathrm{d}x.\end{aligned}$$

Therefore,

$$\int_\Omega fw_1\mathrm{d}x = 0,$$

and we have a contradiction. Thus our claim is proved.

Set

$$v_n := \frac{u_n}{\|u_n\|_{L^\infty(\Omega)}} \quad \forall n\in\mathbb{N}.$$

Then for all $n \in \mathbb{N}$ v_n is a weak solution of

$$\begin{cases}(-\Delta_p)^s u = \lambda_n |u|^{p-2}u + \dfrac{f(x)}{\|u_n\|_{L^\infty(\Omega)}} & \text{in } \Omega,\\ u = 0 & \text{in } \Omega^c.\end{cases}$$

Now, using again that T_1 is a completely continuous operator and the fractional Sobolev compact embedding theorem, up to a subsequence (still denoted by v_n)

$$v_n \to v \quad \text{strongly in } \widetilde{W}^{s,2}(\Omega).$$

Thus, $v \not\equiv 0$ in Ω, and v is a weak solution of (2.2) since $\lambda_n \to \lambda_1(s,2)$ and $f/\|u_n\|_{L^\infty(\Omega)} \to 0$ strongly in Ω. That is, v is an eigenfunction of $(-\Delta)^s$ associated with $\lambda_1(s,2)$. Therefore, either $v > 0$ in Ω or $v < 0$ in Ω.

On the other hand, for any $n \in \mathbb{N}$

$$(\lambda_1(s,2)-\lambda_n)\int_\Omega w_1 v_n\mathrm{d}x = \frac{1}{\|u\|_{L^\infty(\Omega)}}\int_\Omega f(x)w_1\mathrm{d}x > 0$$

then, since $\lambda_1(s,2) < \lambda_n$ for any $n\in\mathbb{N}$, we have that

$$\int_\Omega w_1 v_n\mathrm{d}x < 0 \quad \forall n \in \mathbb{N}$$

Therefore, $v < 0$ in Ω.

In addition, by Theorem 2.7 and the Arzela–Ascoli theorem, up to a subsequence (still denoted by v_n)

$$v_n \to w_1 \quad \text{and} \quad \frac{v_n}{\delta^s} \to \frac{w_1}{\delta^s}$$

strongly in $\overline{\Omega}$. Here $\delta(x) = \operatorname{dist}(x,\partial\Omega)$. Then, by Lemma 2.8, there is $n_0 \in \mathbb{N}$ such that $v_n < 0$ for all $n \in \mathbb{N}$. That is, there is $n_0\in\mathbb{N}$ such that $u_n < 0$ for all $n \geq n_0$. □

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