A FINITE-DIMENSIONAL LIE ALGEBRA ARISING FROM A NICHOLS ALGEBRA OF DIAGONAL TYPE (RANK 2)

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Abstract. Let $\mathcal{B}_{\mathfrak{q}}$ be a finite-dimensional Nichols algebra of diagonal type corresponding to a matrix $\mathfrak{q} \in \mathbf{k}^{\theta\times\theta}$. Let $\mathcal{L}_{\mathfrak{q}}$ be the Lusztig algebra associated to $\mathcal{B}_{\mathfrak{q}}$ [AAR]. We present $\mathcal{L}_{\mathfrak{q}}$ as an extension (as braided Hopf algebras) of $\mathcal{B}_{\mathfrak{q}}$ by $\mathfrak{Z}_{\mathfrak{q}}$ where $\mathfrak{Z}_{\mathfrak{q}}$ is isomorphic to the universal enveloping algebra of a Lie algebra $\mathfrak{n}_{\mathfrak{q}}$. We compute the Lie algebra $\mathfrak{n}_{\mathfrak{q}}$ when $\theta = 2$.

1. Introduction

1.1. Let $\mathbf{k}$ be a field, algebraically closed and of characteristic zero. Let $\theta \in \mathbb{N}$, $\mathbb{I} = \mathbb{I}_\theta := \{1, 2, ..., \theta\}$. Let $\mathfrak{q} = (q_{ij})_{i,j\in\mathbb{I}}$ be a matrix with entries in $\mathbf{k}^\times$, V a vector space with a basis $(x_i)_{i\in\mathbb{I}}$ and $c^{\mathfrak{q}} \in GL(V \otimes V)$ be given by

$$c^{\mathfrak{q}}(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \qquad\qquad i, j \in \mathbb{I}.$$

Then $(c^{\mathfrak{q}} \otimes \mathrm{id})(\mathrm{id} \otimes c^{\mathfrak{q}})(c^{\mathfrak{q}} \otimes \mathrm{id}) = (\mathrm{id} \otimes c^{\mathfrak{q}})(c^{\mathfrak{q}} \otimes \mathrm{id})(\mathrm{id} \otimes c^{\mathfrak{q}})$, i.e. $(V, c^{\mathfrak{q}})$ is a braided vector space and the corresponding Nichols algebra $\mathcal{B}_{\mathfrak{q}} := \mathcal{B}(V)$ is called of diagonal type. Recall that $\mathcal{B}_{\mathfrak{q}}$ is the image of the unique map of braided Hopf algebras $\Omega : T(V) \to T^c(V)$ from the free associative algebra of V to the free associative coalgebra of V, such that $\Omega_{|V} = \mathrm{id}_V$. For unexplained terminology and notation, we refer to [AS].

Remarkably, the explicit classification of all $\mathfrak{q}$ such that $\dim \mathcal{B}_{\mathfrak{q}} < \infty$ is known [H2] (we recall the list when $\theta = 2$ in Table 1). Also, for every $\mathfrak{q}$ in the list of [H2], the defining relations are described in [A2, A3].

1.2. Assume that $\dim \mathcal{B}_{\mathfrak{q}} < \infty$. Two infinite dimensional graded braided Hopf algebras $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ and $\mathcal{L}_{\mathfrak{q}}$ (the Lusztig algebra of V) were introduced and studied in [A3, A5], respectively [AAR]. Indeed, $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ is a pre-Nichols, and $\mathcal{L}_{\mathfrak{q}}$ a post-Nichols, algebra of V, meaning that $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ is intermediate between $T(V)$ and $\mathcal{B}_{\mathfrak{q}}$, while $\mathcal{L}_{\mathfrak{q}}$ is intermediate between $\mathcal{B}_{\mathfrak{q}}$ and $T^c(V)$. This is summarized

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in the following commutative diagram:

$$
\begin{tikzcd}
T(V) \arrow[rr] \arrow[dr] \arrow[rrrr, bend left, "\Omega"] & & \mathcal{B}_{\mathfrak{q}} \arrow[rr] \arrow[dr] & & T^c(V) \\
& \widetilde{\mathcal{B}}_{\mathfrak{q}} \arrow[ur, "\pi"'] & & \mathcal{L}_{\mathfrak{q}} \arrow[ur] &
\end{tikzcd}
$$

The algebras $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ and $\mathcal{L}_{\mathfrak{q}}$ are generalizations of the positive parts of the De Concini-Kac-Procesi quantum group, respectively the Lusztig quantum divided powers algebra. The distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ is defined discarding some of the relations in [A3], while $\mathcal{L}_{\mathfrak{q}}$ is the graded dual of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$.

1.3. The following notions are discussed in Section 2. Let $\Delta_+^{\mathfrak{q}}$ be the generalized positive root system of $\mathcal{B}_{\mathfrak{q}}$ and let $\mathfrak{O}_{\mathfrak{q}} \subset \Delta_+^{\mathfrak{q}}$ be the set of Cartan roots of $\mathfrak{q}$. Let x_β be the root vector associated to $\beta \in \Delta_+^{\mathfrak{q}}$, let $N_\beta = \operatorname{ord} q_{\beta\beta}$ and let $Z_{\mathfrak{q}}$ be the subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ generated by $x_\beta^{N_\beta}$, $\beta \in \mathfrak{O}_{\mathfrak{q}}$. By [A5, Theorems 4.10, 4.13], $Z_{\mathfrak{q}}$ is a braided normal Hopf subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ and $Z_{\mathfrak{q}} = {}^{\operatorname{co}\pi}\widetilde{\mathcal{B}}_{\mathfrak{q}}$. Actually, $Z_{\mathfrak{q}}$ is a true commutative Hopf algebra provided that

$$
q_{\alpha\beta}^{N_\beta} = 1, \qquad \forall \alpha, \beta \in \mathfrak{O}_{\mathfrak{q}}. \tag{1}
$$

Let $\mathfrak{Z}_{\mathfrak{q}}$ be the graded dual of $Z_{\mathfrak{q}}$; under the assumption (1) $\mathfrak{Z}_{\mathfrak{q}}$ is a cocommutative Hopf algebra, hence it is isomorphic to the enveloping algebra $\mathcal{U}(\mathfrak{n}_{\mathfrak{q}})$ of the Lie algebra $\mathfrak{n}_{\mathfrak{q}} := \mathcal{P}(\mathfrak{Z}_{\mathfrak{q}})$. We show in Section 3 that $\mathcal{L}_{\mathfrak{q}}$ is an extension (as braided Hopf algebras) of $\mathcal{B}_{\mathfrak{q}}$ by $\mathfrak{Z}_{\mathfrak{q}}$:

$$
\mathcal{B}_{\mathfrak{q}} \overset{\pi^*}{\hookrightarrow} \mathcal{L}_{\mathfrak{q}} \overset{\iota^*}{\twoheadrightarrow} \mathfrak{Z}_{\mathfrak{q}}. \tag{2}
$$

The main result of this paper is the determination of the Lie algebra $\mathfrak{n}_{\mathfrak{q}}$ when $\theta = 2$ and the generalized Dynkin diagram of $\mathfrak{q}$ is connected.

Theorem 1.1. *Assume that* $\dim \mathcal{B}_{\mathfrak{q}} < \infty$ *and* $\theta = 2$*. Then* $\mathfrak{n}_{\mathfrak{q}}$ *is either 0 or isomorphic to* $\mathfrak{g}^+$*, where* $\mathfrak{g}$ *is a finite-dimensional semisimple Lie algebra listed in the last column of Table 1.*

Assume that there exists a Cartan matrix $\mathbf{a} = (a_{ij})$ of finite type, that becomes symmetric after multiplying with a diagonal (d_i), and a root of unit q of odd order (and relatively prime to 3 if $\mathbf{a}$ is of type G_2) such that $q_{ij} = q^{d_i a_{ij}}$ for all $i, j \in \mathbb{I}$. Then (2) encodes the quantum Frobenius homomorphism defined by Lusztig and Theorem 1.1 is a result from [L].

The penultimate column of Table 1 indicates the type of $\mathfrak{q}$ as established in [AA]. Thus, we associate Lie algebras in characteristic zero to some contragredient Lie (super)algebras in positive characteristic. In a forthcoming paper we shall compute the Lie algebra $\mathfrak{n}_{\mathfrak{q}}$ for $\theta > 2$.

Row	Generalized Dynkin diagrams	parameters	Type of $\mathcal{B}_{\mathfrak{q}}$	$\mathfrak{n}_{\mathfrak{q}} \simeq \mathfrak{g}^+$
1	$\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ}$	$q \neq 1$	Cartan A	A_2
2	$\overset{q}{\circ}\overset{q^{-1}}{\text{——}}\overset{-1}{\circ} \quad \overset{-1}{\circ}\overset{q}{\text{——}}\overset{-1}{\circ}$	$q \neq \pm 1$	Super A	A_1
3	$\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{q^2}{\circ}$	$q \neq \pm 1$	Cartan B	B_2
4	$\overset{q}{\circ}\overset{q^{-2}}{\text{——}}\overset{-1}{\circ} \quad \overset{-q^{-1}}{\circ}\overset{q^2}{\text{——}}\overset{-1}{\circ}$	$q \notin \mathbb{G}_4$	Super B	$A_1 \oplus A_1$
5	$\overset{\zeta}{\circ}\overset{q^{-1}}{\text{——}}\overset{q}{\circ} \quad \overset{\zeta}{\circ}\overset{\zeta^{-1}q}{\text{——}}\overset{\zeta q^{-1}}{\circ}$	$\zeta \in \mathbb{G}_3 \not\ni q$	$\mathfrak{br}(2, a)$	$A_1 \oplus A_1$
6	$\overset{\zeta}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ} \quad \overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}'_3$	Standard B	0
7	$\overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-\zeta^2}{\circ} \quad \overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-1}}{\text{——}}\overset{-1}{\circ} \quad \overset{-\zeta^{2}}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta^{3}}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ} \quad \overset{-\zeta^{3}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}'_{12}$	$\mathfrak{ufo}(7)$	0
8	$\overset{-\zeta^{2}}{\circ}\overset{\zeta}{\text{——}}\overset{-\zeta^2}{\circ} \quad \overset{-\zeta^{2}}{\circ}\overset{\zeta^{3}}{\text{——}}\overset{-1}{\circ} \quad \overset{-\zeta^{-1}}{\circ}\overset{-\zeta^3}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}'_{12}$	$\mathfrak{ufo}(8)$	A_1
9	$\overset{-\zeta}{\circ}\overset{\zeta^{-2}}{\text{——}}\overset{\zeta^3}{\circ} \quad \overset{\zeta^{3}}{\circ}\overset{\zeta^{-1}}{\text{——}}\overset{-1}{\circ} \quad \overset{-\zeta^{2}}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}'_9$	$\mathfrak{brj}(2; 3)$	$A_1 \oplus A_1$
10	$\overset{q}{\circ}\overset{q^{-3}}{\text{——}}\overset{q^3}{\circ}$	$q \notin \mathbb{G}_2 \cup \mathbb{G}_3$	Cartan G_2	G_2
11	$\overset{\zeta^{2}}{\circ}\overset{\zeta}{\text{——}}\overset{\zeta^{-1}}{\circ} \quad \overset{\zeta^{2}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-1}{\circ} \quad \overset{\zeta}{\circ}\overset{-\zeta}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}'_8$	Standard G_2	$A_1 \oplus A_1$
12	$\overset{\zeta^{6}}{\circ}\overset{-\zeta^{-1}}{\text{——}}\overset{-\zeta^{-4}}{\circ} \quad \overset{\zeta^{6}}{\circ}\overset{\zeta}{\text{——}}\overset{\zeta^{-1}}{\circ}$ $\overset{-\zeta^{-4}}{\circ}\overset{\zeta^{5}}{\text{——}}\overset{-1}{\circ} \quad \overset{\zeta}{\circ}\overset{\zeta^{-5}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}'_{24}$	$\mathfrak{ufo}(9)$	$A_1 \oplus A_1$
13	$\overset{\zeta}{\circ}\overset{\zeta^{2}}{\text{——}}\overset{-1}{\circ} \quad \overset{-\zeta^{-2}}{\circ}\overset{\zeta^{-2}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}'_5$	$\mathfrak{brj}(2; 5)$	B_2
14	$\overset{\zeta}{\circ}\overset{\zeta^{-3}}{\text{——}}\overset{-1}{\circ} \quad \overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{-1}{\circ}$ $\overset{-\zeta^{-2}}{\circ}\overset{\zeta^{3}}{\text{——}}\overset{-1}{\circ} \quad \overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}'_{20}$	$\mathfrak{ufo}(10)$	$A_1 \oplus A_1$
15	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{\zeta^5}{\circ} \quad \overset{\zeta^{3}}{\circ}\overset{-\zeta^{4}}{\text{——}}\overset{-\zeta^{-4}}{\circ}$ $\overset{\zeta^{5}}{\circ}\overset{-\zeta^{-2}}{\text{——}}\overset{-1}{\circ} \quad \overset{\zeta^{3}}{\circ}\overset{-\zeta^{2}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}'_{15}$	$\mathfrak{ufo}(11)$	$A_1 \oplus A_1$
16	$\overset{-\zeta}{\circ}\overset{-\zeta^{-3}}{\text{——}}\overset{-1}{\circ} \quad \overset{-\zeta^{-2}}{\circ}\overset{-\zeta^{3}}{\text{——}}\overset{-1}{\circ}$	$\zeta \in \mathbb{G}'_7$	$\mathfrak{ufo}(12)$	G_2

TABLE 1. Lie algebras arising from Dynkin diagrams of rank 2.

1.4. The paper is organized as follows. We collect the needed preliminary material in Section 2. Section 3 is devoted to the exactness of (2). The computations of the various $\mathfrak{n}_{\mathfrak{q}}$ is the matter of Section 4. We denote by $\mathbb{G}_N$ the group of N-th roots of 1, and by $\mathbb{G}'_N$ its subset of primitive roots.

2. Preliminaries

2.1. **The Nichols algebra, the distinguished-pre-Nichols algebra and the Lusztig algebra.** Let $\mathfrak{q}$ be as in the Introduction and let $(V, c^{\mathfrak{q}})$ be the corresponding braided vector space of diagonal type. We assume from now on that $\mathcal{B}_{\mathfrak{q}}$ is finite-dimensional. Let $(\alpha_j)_{j\in\mathbb{I}}$ be the canonical basis of $\mathbb{Z}^\theta$. Let $\mathbf{q}: \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbf{k}^\times$ be the $\mathbb{Z}$-bilinear form associated to the matrix $\mathfrak{q}$, i.e. $\mathbf{q}(\alpha_j, \alpha_k) = q_{jk}$ for all $j, k \in \mathbb{I}$. If $\alpha, \beta \in \mathbb{Z}^\theta$, we set $q_{\alpha\beta} = \mathbf{q}(\alpha, \beta)$. Consider the matrix $(c_{ij}^{\mathfrak{q}})_{i,j\in\mathbb{I}}$, $c_{ij} \in \mathbb{Z}$ defined by $c_{ii}^{\mathfrak{q}} = 2$,

$$c_{ij}^{\mathfrak{q}} := -\min\left\{n \in \mathbb{N}_0 : (n+1)_{q_{ii}}(1 - q_{ii}^n q_{ij} q_{ji}) = 0\right\}, \qquad i \neq j. \tag{3}$$

This is well-defined by [R]. Let $i \in \mathbb{I}$. We recall the following definitions:

- ⋄ The reflection $s_i^{\mathfrak{q}} \in GL(\mathbb{Z}^\theta)$, given by $s_i^{\mathfrak{q}}(\alpha_j) = \alpha_j - c_{ij}^{\mathfrak{q}}\alpha_i$, $j \in \mathbb{I}$.
- ⋄ The matrix $\rho_i(\mathfrak{q})$, given by $\rho_i(\mathfrak{q})_{jk} = \mathbf{q}(s_i^{\mathfrak{q}}(\alpha_j), s_i^{\mathfrak{q}}(\alpha_k))$, $j, k \in \mathbb{I}$.
- ⋄ The braided vector space $\rho_i(V)$ of diagonal type with matrix $\rho_i(\mathfrak{q})$.

A basic result is that $\mathcal{B}_{\mathfrak{q}} \simeq \mathcal{B}_{\rho_i(\mathfrak{q})}$, at least as graded vector spaces.

The algebras $T(V)$ and $\mathcal{B}_{\mathfrak{q}}$ are $\mathbb{Z}^\theta$-graded by $\deg x_i = \alpha_i$, $i \in \mathbb{I}$. Let $\Delta_+^{\mathfrak{q}}$ be the set of $\mathbb{Z}^\theta$-degrees of the generators of a PBW-basis of $\mathcal{B}_{\mathfrak{q}}$, counted with multiplicities [H1]. The elements of $\Delta_+^{\mathfrak{q}}$ are called (positive) roots. Let $\Delta^{\mathfrak{q}} = \Delta_+^{\mathfrak{q}} \cup -\Delta_+^{\mathfrak{q}}$. Let

$$\mathcal{X} := \{\rho_{j_1} \dots \rho_{j_N}(\mathfrak{q}) : j_1, \dots, j_N \in \mathbb{I}, N \in \mathbb{N}\}.$$

Then the generalized root system of $\mathfrak{q}$ is the fibration $\Delta \to \mathcal{X}$, where the fiber of $\rho_{j_1} \dots \rho_{j_N}(\mathfrak{q})$ is $\Delta^{\rho_{j_1}\cdots\rho_{j_N}(\mathfrak{q})}$. The Weyl groupoid of $\mathcal{B}_{\mathfrak{q}}$ is a groupoid, denoted $\mathcal{W}_{\mathfrak{q}}$, that acts on this fibration, generalizing the classical Weyl group, see [H1]. We know from *loc. cit.* that $\mathcal{W}_{\mathfrak{q}}$ is finite (and this characterizes finite-dimensional Nichols algebras of diagonal type).

Here is a useful description of $\Delta_+^{\mathfrak{q}}$. Let $w \in \mathcal{W}_{\mathfrak{q}}$ be an element of maximal length. We fix a reduced expression $w = \sigma_{i_1}^{\mathfrak{q}} \sigma_{i_2} \cdots \sigma_{i_M}$. For $1 \leq k \leq M$ set

$$\beta_k = s_{i_1}^{\mathfrak{q}} \cdots s_{i_{k-1}}(\alpha_{i_k}), \tag{4}$$

Then $\Delta_+^{\mathfrak{q}} = \{\beta_k | 1 \leq k \leq M\}$ [CH, Prop. 2.12]; in particular $|\Delta_+^{\mathfrak{q}}| = M$.

The notion of Cartan root is instrumental for the definitions of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ and $\mathcal{L}_{\mathfrak{q}}$. First, following [A5] we say that $i \in \mathbb{I}$ is a *Cartan vertex* of $\mathfrak{q}$ if

$$q_{ij} q_{ji} = q_{ii}^{c_{ij}^{\mathfrak{q}}}, \qquad \text{for all } j \neq i, \tag{5}$$

Then the set of *Cartan roots* of $\mathfrak{q}$ is

$$\mathfrak{O}_{\mathfrak{q}} = \{s_{i_1}^{\mathfrak{q}} s_{i_2} \dots s_{i_k}(\alpha_i) \in \Delta_+^{\mathfrak{q}} : i \in \mathbb{I} \text{ is a Cartan vertex of } \rho_{i_k} \dots \rho_{i_2}\rho_{i_1}(\mathfrak{q})\}.$$

Given a positive root $\beta \in \Delta_+^{\mathfrak{q}}$, there is an associated root vector $x_\beta \in \mathcal{B}_{\mathfrak{q}}$ defined via the so-called Lusztig isomorphisms [H3]. Set $N_\beta = \operatorname{ord} q_{\beta\beta} \in \mathbb{N}$, $\beta \in \Delta_+^{\mathfrak{q}}$. Also, for $\mathbf{h} = (h_1, \dots, h_M) \in \mathbb{N}_0^M$ we write

$$x^{\mathbf{h}} = x_{\beta_M}^{h_M} x_{\beta_{M-1}}^{h_{M-1}} \cdots x_{\beta_1}^{h_1}.$$

Let $\widetilde{N}_k = \begin{cases} N_{\beta_k} & \text{if } \beta_k \notin \mathcal{O}_{\mathfrak{q}}, \\ \infty & \text{if } \beta_k \in \mathcal{O}_{\mathfrak{q}}. \end{cases}$ For simplicity, we introduce

$$\mathtt{H} = \{\mathbf{h} \in \mathbb{N}_0^M : 0 \le h_k < \widetilde{N}_k, \text{ for all } k \in \mathbb{I}_M\}. \tag{6}$$

By [A5, Theorem 3.6] the set $\{x^{\mathbf{h}} \,|\, \mathbf{h} \in \mathtt{H}\}$ is a basis of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$.

As said in the Introduction, the Lusztig algebra associated to $\mathcal{B}_{\mathfrak{q}}$ is the braided Hopf algebra $\mathcal{L}_{\mathfrak{q}}$ which is the graded dual of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$. Thus, it comes equipped with a bilinear form $\langle\,,\,\rangle : \widetilde{\mathcal{B}}_{\mathfrak{q}} \times \mathcal{L}_{\mathfrak{q}} \to \mathbf{k}$, which satisfies for all $x, x' \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$, $y, y' \in \mathcal{L}_{\mathfrak{q}}$

$$\langle y, xx'\rangle = \langle y^{(2)}, x\rangle\langle y^{(1)}, x'\rangle \qquad \text{and} \qquad \langle yy', x\rangle = \langle y, x^{(2)}\rangle\langle y', x^{(1)}\rangle.$$

If $\mathbf{h} \in \mathtt{H}$, then define $\mathbf{y_h} \in \mathcal{L}_{\mathfrak{q}}$ by $\langle \mathbf{y_h}, x^{\mathbf{j}}\rangle = \delta_{\mathbf{h},\mathbf{j}}$, $\mathbf{j} \in \mathtt{H}$. Let $(\mathbf{h}_k)_{k\in\mathbb{I}_M}$ denote the canonical basis of $\mathbb{Z}^M$. If $k \in \mathbb{I}_M$ and $\beta = \beta_k \in \Delta_+^{\mathfrak{q}}$, then we denote the element $\mathbf{y}_{n\mathbf{h}_k}$ by $y_\beta^{(n)}$. Then the algebra $\mathcal{L}_{\mathfrak{q}}$ is generated by

$$\{y_\alpha : \alpha \in \Pi_{\mathfrak{q}}\} \cup \{y_\alpha^{(N_\alpha)} : \alpha \in \mathfrak{O}_{\mathfrak{q}},\ x_\alpha^{N_\alpha} \in \mathcal{P}(\widetilde{\mathcal{B}}_{\mathfrak{q}})\},$$

by [AAR]. Moreover, by [AAR, 4.6], the following set is a basis of $\mathcal{L}_{\mathfrak{q}}$:

$$\{y_{\beta_1}^{(h_1)} \cdots y_{\beta_M}^{(h_M)} \,|\, (h_1, \dots, h_M) \in \mathtt{H}\}.$$

2.2. **Lyndon words, convex order and PBW-basis.** For the computations in Section 4 we need some preliminaries on Kharchenko's PBW-basis. Let $(V, \mathfrak{q})$ be as above and let $\mathbb{X}$ be the set of words with letters in $X = \{x_1, \dots, x_\theta\}$ (our fixed basis of V); the empty word is 1 and for $u \in \mathbb{X}$ we write $\ell(u)$ the length of u. We can identify $\mathbf{k}\mathbb{X}$ with $T(V)$.

Definition 2.1. Consider the lexicographic order in $\mathbb{X}$. We say that $u \in \mathbb{X} - \{1\}$ is a *Lyndon word* if for every decomposition $u = vw$, $v, w \in \mathbb{X} - \{1\}$, then $u < w$. We denote by L the set of all Lyndon words.

A well-known theorem, due to Lyndon, established that any word $u \in \mathbb{X}$ admits a unique decomposition, named *Lyndon decomposition*, as a non-increasing product of Lyndon words:

$$u = l_1 l_2 \dots l_r, \qquad l_i \in L, l_r \le \dots \le l_1. \tag{7}$$

Also, each $l_i \in L$ in (7) is called a *Lyndon letter* of u.

Now each $u \in L - X$ admits at least one decomposition $u = v_1 v_2$ with $v_1, v_2 \in L$. Then the *Shirshov decomposition* of u is the decomposition $u = u_1 u_2$, $u_1, u_2 \in L$, such that u_2 is the smallest end of u between all possible decompositions of this form.

For any braided vector space V, the *braided bracket* of $x, y \in T(V)$ is

$$[x, y]_c := \text{multiplication } \circ (\mathrm{id} - c)(x \otimes y). \tag{8}$$

Using the identification $T(V) = \mathbf{k}\mathbb{X}$ and the decompositions described above, we can define a $\mathbf{k}$-linear endomorphism $[-]_c$ of $T(V)$ as follows:

$$[u]_c := \begin{cases} u, & \text{if } u = 1 \text{ or } u \in X; \\ [[v]_c, [w]_c]_c, & \text{if } u \in L - X,\ u = vw \text{ its Shirshov decomposition;} \\ [u_1]_c \dots [u_t]_c, & \text{if } u \in \mathbb{X} - L, u = u_1 \dots u_t \text{ its Lyndon decomposition.} \end{cases}$$

We will describe PBW-bases using this endomorphism.

Definition 2.2. For $l \in L$, the element $[l]_c$ is the corresponding *hyperletter*. A word written in hyperletters is an *hyperword*; a *monotone hyperword* is an hyperword $W = [u_1]_c^{k_1} \dots [u_m]_c^{k_m}$ such that $u_1 > \cdots > u_m$.

Consider now a different order on $\mathbb{X}$, called *deg-lex order* [K]: For each pair $u, v \in \mathbb{X}$, we have that $u \succ v$ if $\ell(u) < \ell(v)$, or $\ell(u) = \ell(v)$ and $u > v$ for the lexicographical order. This order is total, the empty word 1 is the maximal element and it is invariant by left and right multiplication.

Let I be a Hopf ideal of $T(V)$ and $R = T(V)/I$. Let $\pi : T(V) \to R$ be the canonical projection. We set:

$$G_I := \{u \in \mathbb{X} : u \notin \mathbf{k}\mathbb{X}_{\succ u} + I\}.$$

Thus, if $u \in G_I$ and $u = vw$, then $v, w \in G_I$. So, each $u \in G_I$ is a non-increasing product of Lyndon words of G_I.

Let $S_I := G_I \cap L$ and let $h_I : S_I \to \{2, 3, \dots\} \cup \{\infty\}$ be defined by:

$$h_I(u) := \min\left\{t \in \mathbb{N} : u^t \in \mathbf{k}\mathbb{X}_{\succ u^t} + I\right\}. \tag{9}$$

Theorem 2.3. [K] *The following set is a PBW-basis of $R = T(V)/I$:*

$$\{[u_1]_c^{k_1} \dots [u_m]_c^{k_m} : m \in \mathbb{N}_0, u_1 > \dots > u_m, u_i \in S_I, 0 < k_i < h_I(u_i)\}. \qquad \square$$

We refer to this base as *Kharchenko's PBW-basis* of $T(V)/I$ (it depends on the order of X).

Definition 2.4. [A2, 2.6] Let $\Delta_{\mathfrak{q}}^+$ be as above and let $<$ be a total order on $\Delta_{\mathfrak{q}}^+$. We say that the order is *convex* if for each $\alpha, \beta \in \Delta_{\mathfrak{q}}^+$ such that $\alpha < \beta$ and $\alpha + \beta \in \Delta_{\mathfrak{q}}^+$, then $\alpha < \alpha + \beta < \beta$. The order is called *strongly convex* if for each ordered subset $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_k$ of elements of $\Delta_{\mathfrak{q}}^+$ such that $\alpha = \sum_i \alpha_i \in \Delta_{\mathfrak{q}}^+$, then $\alpha_1 < \alpha < \alpha_k$.

Theorem 2.5. [A2, 2.11] *The following statements are equivalent:*

- *The order is convex.*
- *The order is strongly convex.*
- *The order arises from a reduced expression of a longest element $w \in \mathcal{W}_{\mathfrak{q}}$, cf.* (4). $\square$

Now, we have two PBW-basis of $\mathcal{B}_q$ (and correspondingly of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$), namely Kharchenko's PBW-basis and the PBW-basis defined from a reduced expression of a longest element of the Weyl groupoid. But both basis are reconciled by [AY, Theorem 4.12], thanks to [A2, 2.14]. Indeed, each generator of Kharchenko's PBW-basis is a multiple scalar of a generator of the secondly mentioned PBW-basis. So, for ease of calculations, in the rest of this work we shall use the Kharchenko generators.

The following proposition is used to compute the hyperword $[l_\beta]_c$ associated to a root $\beta \in \Delta_{\mathfrak{q}}^+$:

Proposition 2.6. [A2, 2.17] *For* $\beta \in \Delta_{\mathfrak{q}}^+$,

$$l_\beta = \begin{cases} x_{\alpha_i}, & \text{if } \beta = \alpha_i,\ i \in \mathbb{I}; \\ \max\{l_{\delta_1} l_{\delta_2} : \delta_1, \delta_2 \in \Delta_{\mathfrak{q}}^+, \delta_1 + \delta_2 = \beta, l_{\delta_1} < l_{\delta_2}\}, & \text{if } \beta \neq \alpha_i,\ i \in \mathbb{I}. \end{cases} \qquad \square$$

We give a list of the hyperwords appearing in the next section:

Root	Hyperword	Notation
α_i	x_i	x_i
$n\alpha_1 + \alpha_2$	$(\mathrm{ad}_c\, x_1)^n x_2$	$x_{1\dots12}$
$\alpha_1 + 2\alpha_2$	$[x_{\alpha_1+\alpha_2}, x_2]_c$	$[x_{12}, x_2]_c$
$3\alpha_1 + 2\alpha_2$	$[x_{2\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2}]_c$	$[x_{112}, x_{12}]_c$
$4\alpha_1 + 3\alpha_2$	$[x_{3\alpha_1+2\alpha_2}, x_{\alpha_1+\alpha_2}]_c$	$[[x_{112}, x_{12}]_c, x_{12}]_c$
$5\alpha_1 + 3\alpha_2$	$[x_{2\alpha_1+\alpha_2}, x_{3\alpha_1+2\alpha_2}]_c$	$[x_{112}, [x_{112}, x_{12}]_c]_c$

We use an analogous notation for the elements of $\mathcal{L}_{\mathfrak{q}}$: for example we write $y_{112,12}$ when we refer to the element of $\mathcal{L}_{\mathfrak{q}}$ which corresponds to $[x_{112}, x_{12}]_c$.

3. Extensions of braided Hopf algebras

We recall the definition of braided Hopf algebra extensions given in [AN]; we refer to [BD, GG] for more general definitions. Below we denote by $\underline{\Delta}$ the coproduct of a braided Hopf algebra A and by A^+ the kernel of the counit.

First, if $\pi : C \to B$ is a morphism of Hopf algebras in ${}^H_H\mathcal{YD}$, then we set

$$C^{\operatorname{co}\pi} = \{c \in C \mid (\mathrm{id} \otimes \pi)\underline{\Delta}(c) = c \otimes 1\},$$
$${}^{\operatorname{co}\pi}C = \{c \in C \mid (\pi \otimes \mathrm{id})\underline{\Delta}(c) = 1 \otimes c\}.$$

Definition 3.1. [AN, §2.5] Let H be a Hopf algebra. A sequence of morphisms of Hopf algebras in ${}^H_H\mathcal{YD}$

$$\mathbf{k} \to A \xrightarrow{\iota} C \xrightarrow{\pi} B \to \mathbf{k} \tag{10}$$

is an *extension of braided Hopf algebras* if

(i) ι is injective,
(ii) π is surjective,
(iii) $\ker \pi = C\iota(A^+)$ and
(iv) $A = C^{\operatorname{co}\pi}$, or equivalently $A = {}^{\operatorname{co}\pi}C$.

For simplicity, we shall write $A \overset{\iota}{\hookrightarrow} C \overset{\pi}{\twoheadrightarrow} B$ instead of (10).

This Definition applies in our context: recall that $\mathcal{B}_{\mathfrak{q}} \simeq \widetilde{\mathcal{B}}_{\mathfrak{q}}/\langle x_\beta^{N_\beta},\ \beta \in \mathfrak{O}_{\mathfrak{q}}\rangle$. Let $Z_{\mathfrak{q}}$ be the subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ generated by $x_\beta^{N_\beta}$, $\beta \in \mathfrak{O}_{\mathfrak{q}}$. Then

- The inclusion $\iota : Z_{\mathfrak{q}} \to \widetilde{\mathcal{B}}_{\mathfrak{q}}$ is injective and the projection $\pi : \widetilde{\mathcal{B}}_{\mathfrak{q}} \to \mathcal{B}_{\mathfrak{q}}$ is surjective.
- [A5, Theorem 4.10] $Z_{\mathfrak{q}}$ is a *normal* Hopf subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$; since $\ker \pi$ is the two-sided ideal generated by $\iota(Z_{\mathfrak{q}}^+)$, $\ker \pi = \widetilde{\mathcal{B}}_{\mathfrak{q}} \iota(Z_{\mathfrak{q}}^+)$.
- [A5, Theorem 4.13] $Z_{\mathfrak{q}} = {}^{\operatorname{co} \pi}\widetilde{\mathcal{B}}_{\mathfrak{q}}$.

Hence we have an extension of braided Hopf algebras

$$Z_{\mathfrak{q}} \overset{\iota}{\hookrightarrow} \widetilde{\mathcal{B}}_{\mathfrak{q}} \overset{\pi}{\twoheadrightarrow} \mathcal{B}_{\mathfrak{q}}. \tag{11}$$

The morphisms ι and π are graded. Thus, taking graded duals, we obtain a new sequence of morphisms of braided Hopf algebras

$$\mathcal{B}_{\mathfrak{q}} \overset{\pi^*}{\hookrightarrow} \mathcal{L}_{\mathfrak{q}} \overset{\iota^*}{\twoheadrightarrow} \mathfrak{Z}_{\mathfrak{q}}. \tag{2}$$

Proposition 3.2. *The sequence* (2) *is an extension of braided Hopf algebras.*

Proof. The argument of [A, 3.3.1] can be adapted to the present situation, or more generally to extensions of braided Hopf algebras that are graded with finite-dimensional homogeneous components. The map $\pi^* : \mathcal{B}_{\mathfrak{q}} \to \mathcal{L}_{\mathfrak{q}}$ is injective because $\mathcal{B}_{\mathfrak{q}} \simeq \mathcal{B}_{\mathfrak{q}}^*$; $\iota^* : \mathcal{L}_{\mathfrak{q}} \overset{\iota^*}{\to} \mathfrak{Z}_{\mathfrak{q}}$ is surjective being the transpose of a graded monomorphism between two locally finite graded vector spaces. Now, since $Z_{\mathfrak{q}} = {}^{\operatorname{co} \pi}\widetilde{\mathcal{B}}_{\mathfrak{q}} = \widetilde{\mathcal{B}}_{\mathfrak{q}}^{\operatorname{co} \pi}$, we have

$$\ker \iota^* = \mathcal{L}_{\mathfrak{q}} \mathcal{B}_{\mathfrak{q}}^+ = \mathcal{B}_{\mathfrak{q}}^+ \mathcal{L}_{\mathfrak{q}}. \tag{12}$$

Similarly $\mathcal{L}_{\mathfrak{q}}^{\operatorname{co} \iota^*} = \mathcal{B}_{\mathfrak{q}}^*$ because $\ker \pi^\perp = \mathcal{B}_{\mathfrak{q}}$. □

From now on, we assume the condition (1) on the matrix $\mathfrak{q}$ mentioned in the Introduction, that is

$$q_{\alpha\beta}^{N_\beta} = 1, \qquad \forall \alpha, \beta \in \mathfrak{O}_{\mathfrak{q}}.$$

The following result is our basic tool to compute the Lie algebra $\mathfrak{n}_{\mathfrak{q}}$.

Theorem 3.3. *The braided Hopf algebra $\mathfrak{Z}_{\mathfrak{q}}$ is an usual Hopf algebra, isomorphic to the universal enveloping algebra of the Lie algebra $\mathfrak{n}_{\mathfrak{q}} = \mathcal{P}(\mathfrak{Z}_{\mathfrak{q}})$. The elements $\xi_\beta := \iota^*(y_\beta^{(N_\beta)})$, $\beta \in \mathfrak{O}_{\mathfrak{q}}$, form a basis of $\mathfrak{n}_{\mathfrak{q}}$.*

Proof. Let $A_{\mathfrak{q}}$ be the subspace of $\mathcal{L}_{\mathfrak{q}}$ generated by the ordered monomials $y_{\beta_{i_1}}^{(r_1 N_{\beta_{i_1}})} \dots y_{\beta_{i_k}}^{(r_k N_{\beta_{i_k}})}$ where $\beta_{i_1} < \dots < \beta_{i_k}$ are all the Cartan roots of $\mathcal{B}_{\mathfrak{q}}$ and $r_1, \dots, r_k \in \mathbb{N}_0$. We claim that the restriction of the multiplication $\mu : \mathcal{B}_{\mathfrak{q}} \otimes A_{\mathfrak{q}} \to \mathcal{L}_{\mathfrak{q}}$ is an isomorphism of vector spaces. Indeed, μ is surjective by the commuting relations in $\mathcal{L}_{\mathfrak{q}}$. Also, the Hilbert series of $\mathcal{L}_{\mathfrak{q}}$, $\mathcal{B}_{\mathfrak{q}}$ and $A_{\mathfrak{q}}$ are respectively:

$$\mathcal{H}_{\mathcal{L}_{\mathfrak{q}}} = \prod_{\beta_k \in \mathfrak{O}_{\mathfrak{q}}} \frac{1}{1-T^{\deg\beta}} \cdot \prod_{\beta_k \notin \mathfrak{O}_{\mathfrak{q}}} \frac{1-T^{N_\beta \deg\beta}}{1-T^{\deg\beta}};$$

$$\mathcal{H}_{\mathcal{B}_{\mathfrak{q}}} = \prod_{\beta_k \in \Delta_{\mathfrak{q}}^+} \frac{1-T^{N_\beta \deg\beta}}{1-T^{\deg\beta}};$$

$$\mathcal{H}_{A_{\mathfrak{q}}} = \prod_{\beta_k \in \mathfrak{O}_{\mathfrak{q}}} \frac{1}{1-T^{N_\beta \deg\beta}}.$$

Since the multiplication is graded and $\mathcal{H}_{\mathcal{L}_{\mathfrak{q}}} = \mathcal{H}_{\mathcal{B}_{\mathfrak{q}}}\mathcal{H}_{A_{\mathfrak{q}}}$, μ is injective. The claim follows and we have

$$\mathcal{L}_{\mathfrak{q}} = A_{\mathfrak{q}} \oplus \mathcal{B}_{\mathfrak{q}}^+ A_{\mathfrak{q}}. \tag{13}$$

We next claim that $\iota^* : A_{\mathfrak{q}} \to \mathfrak{Z}_{\mathfrak{q}}$ is an isomorphism of vector spaces. Indeed, by (12), $\ker\iota^* = \mathcal{B}_{\mathfrak{q}}^+\mathcal{L}_{\mathfrak{q}} = \mathcal{B}_{\mathfrak{q}}^+(\mathcal{B}_{\mathfrak{q}}A_{\mathfrak{q}}) = \mathcal{B}_{\mathfrak{q}}^+A_{\mathfrak{q}}$. By (13), the claim follows.

By (1), $Z_{\mathfrak{q}}$ is a commutative Hopf algebra, see [A5]; hence $\mathfrak{Z}_{\mathfrak{q}}$ is a cocommutative Hopf algebra. Now the elements $\xi_\beta := \iota^*(y_\beta^{(N_\beta)})$, , $\beta \in \mathfrak{O}_{\mathfrak{q}}$, are primitive, i.e. belong to $\mathfrak{n}_{\mathfrak{q}} = \mathcal{P}(\mathfrak{Z}_{\mathfrak{q}})$. The monomials $\xi_{\beta_{i_1}}^{r_1} \dots \xi_{\beta_{i_k}}^{r_k}$, $\beta_{i_1} < \dots < \beta_{i_k} \in \mathfrak{O}_{\mathfrak{q}}$, $r_1, \dots, r_k \in \mathbb{N}_0$ form a basis of $\mathfrak{Z}_{\mathfrak{q}}$, hence

$$\mathfrak{Z}_{\mathfrak{q}} = \mathbf{k}\langle \xi_\beta : \beta \in \mathfrak{O}_{\mathfrak{q}}\rangle \subseteq \mathcal{U}(\mathfrak{n}_{\mathfrak{q}}) \subseteq \mathfrak{Z}_{\mathfrak{q}}.$$

We conclude that $(\xi_\beta)_{\beta\in\mathfrak{O}_{\mathfrak{q}}}$ is a basis of $\mathfrak{n}_{\mathfrak{q}}$ and that $\mathfrak{Z}_{\mathfrak{q}} = \mathcal{U}(\mathfrak{n}_{\mathfrak{q}})$. □

4. Proof of Theorem 1.1

In this section we consider all indecomposable matrices $\mathfrak{q}$ of rank 2 whose associated Nichols algebra $\mathcal{B}_{\mathfrak{q}}$ is finite-dimensional; these are classified in [H2] and we recall their diagrams in Table 1. For each $\mathfrak{q}$ we obtain an isomorphism between $\mathfrak{Z}_{\mathfrak{q}}$ and $\mathcal{U}(\mathfrak{g}^+)$, the universal enveloping algebra of the positive part of $\mathfrak{g}$. Here $\mathfrak{g}$ is the semisimple Lie algebra of the last column of Table 1, with Cartan matrix $A = (a_{ij})_{1\le i,j\le 2}$. By simplicity we denote $\mathfrak{g}$ by its type, e.g. $\mathfrak{g} = A_2$.

We recall that we assume (1) and that $\xi_\beta = \iota^*(y_\beta^{(N_\beta)}) \in \mathfrak{Z}_{\mathfrak{q}}$. Thus,

$$[\xi_\alpha, \xi_\beta]_c = \xi_\alpha\xi_\beta - \xi_\beta\xi_\alpha = [\xi_\alpha, \xi_\beta], \qquad \text{for all } \alpha, \beta \in \mathfrak{O}_{\mathfrak{q}}.$$

The strategy to prove the isomorphism $\mathfrak{F} : \mathcal{U}(\mathfrak{g}^+) \to \mathfrak{Z}_{\mathfrak{q}}$ is the following:

(1) If $\mathfrak{O}_{\mathfrak{q}} = \emptyset$, then $\mathfrak{g}^+ = 0$. If $|\mathfrak{O}_{\mathfrak{q}}| = 1$, then $\mathfrak{g} = \mathfrak{sl}_2$, i.e. of type A_1.
(2) If $|\mathfrak{O}_{\mathfrak{q}}| = 2$, then $\mathfrak{g}$ is of type $A_1 \oplus A_1$. Indeed, let $\mathfrak{O}_{\mathfrak{q}} = \{\alpha, \beta\}$. As $\mathfrak{Z}_{\mathfrak{q}}$ is $\mathbb{N}_0^\theta$-graded, $[\xi_\alpha, \xi_\beta] \in \mathfrak{n}_{\mathfrak{q}}$ has degree $N_\alpha\alpha + N_\beta\beta$. Thus $[\xi_\alpha, \xi_\beta] = 0$.
(3) Now assume that $|\mathfrak{O}_{\mathfrak{q}}| > 2$. We recall that $\mathfrak{Z}_{\mathfrak{q}}$ is generated by

$$\{\xi_\beta | x_\beta^{N_\beta} \text{ is a primitive element of } \widetilde{\mathcal{B}}_{\mathfrak{q}}\}.$$

We compute the coproduct of all $x_\beta^{N_\beta}$ in $\widetilde{\mathcal{B}}_{\mathfrak{q}}$, $\beta \in \mathfrak{O}_{\mathfrak{q}}$, using that $\underline{\Delta}$ is a graded map and $Z_{\mathfrak{q}}$ is a Hopf subalgebra of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$. In all cases we get two primitive elements $x_{\beta_1}^{N_{\beta_1}}$ and $x_{\beta_2}^{N_{\beta_2}}$, thus $\mathfrak{Z}_{\mathfrak{q}}$ is generated by ξ_{β_1} and ξ_{β_2}.

(4) Using the coproduct again, we check that

$$(\operatorname{ad} \xi_{\beta_i})^{1-a_{ij}} \xi_{\beta_j} = 0, \qquad 1 \le i \ne j \le 2. \tag{14}$$

To prove (14), it is enough to observe that $\mathfrak{n}_{\mathfrak{q}}$ has a trivial component of degree $N_{\beta_i}(1-a_{ij})\beta_i + N_{\beta_j}\beta_j$. Now (14) implies that there exists a surjective map of Hopf algebras $\mathfrak{F} : \mathcal{U}(\mathfrak{g}^+) \twoheadrightarrow \mathfrak{Z}_{\mathfrak{q}}$ such that $e_i \mapsto \xi_{\beta_i}$.

(5) To prove that $\mathfrak{F}$ is an isomorphism, it suffices to see that the restriction $\mathfrak{g}^+ \xrightarrow{*} \mathfrak{n}_{\mathfrak{q}}$ is an isomorphism; but in each case we see that $*$ is surjective, and $\dim \mathfrak{g}^+ = \dim \mathfrak{n}_{\mathfrak{q}} = |\mathfrak{O}_{\mathfrak{q}}|$.

We refer to [A1, AAY, A4] for the presentation, root system and Cartan roots of braidings of standard, super and unidentified type respectively.

Row 1. Let $q \in \mathbb{G}'_N$, $N \ge 2$. The diagram $\overset{q}{\circ}\!\!\overset{q^{-1}}{\text{------}}\!\!\overset{q}{\circ}$ corresponds to a braiding of Cartan type A_2 whose set of positive roots is $\Delta_{\mathfrak{q}}^+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$. In this case $\mathfrak{O}_{\mathfrak{q}} = \Delta_{\mathfrak{q}}^+$ and $N_\beta = N$ for all $\beta \in \mathfrak{O}_{\mathfrak{q}}$. By hypothesis, $q_{12}^N = q_{21}^N = 1$. The elements $x_1, x_2 \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$ are primitive and

$$\underline{\Delta}(x_{12}) = x_{12} \otimes 1 + 1 \otimes x_{12} + (1-q^{-1}) x_1 \otimes x_2.$$

Then the coproducts of the elements $x_1^N, x_{12}^N, x_2^N \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$ are:

$$\underline{\Delta}(x_1^N) = x_1^N \otimes 1 + 1 \otimes x_1^N; \qquad \underline{\Delta}(x_2^N) = x_2^N \otimes 1 + 1 \otimes x_2^N;$$
$$\underline{\Delta}(x_{12}^N) = x_{12}^N \otimes 1 + 1 \otimes x_{12}^N + (1-q^{-1})^N q_{21}^{\frac{N(N-1)}{2}} x_1^N \otimes x_2^N.$$

As $[\xi_2, \xi_{12}]$, $[\xi_1, \xi_{12}] \in \mathfrak{n}_{\mathfrak{q}}$ have degree $N\alpha_1 + 2N\alpha_2$, respectively $2N\alpha_1 + N\alpha_2$, and the components of these degrees of $\mathfrak{n}_{\mathfrak{q}}$ are trivial, we have

$$[\xi_2, \xi_{12}] = [\xi_1, \xi_{12}] = 0.$$

Again by degree considerations, there exists $c \in \mathbf{k}$ such that $[\xi_2, \xi_1] = c\xi_{12}$. By the duality between $\mathfrak{Z}_{\mathfrak{q}}$ and $Z_{\mathfrak{q}}$ we have that

$$[\xi_2, \xi_1] = (1-q^{-1})^N q_{21}^{\frac{N(N-1)}{2}} \xi_{12}.$$

Then there exists a morphism of algebras $\mathfrak{F} : \mathcal{U}(A_2^+) \to \mathfrak{Z}_{\mathfrak{q}}$ given by

$$e_1 \mapsto \xi_1, \qquad e_2 \mapsto \xi_2.$$

This morphism takes a basis of A_2^+ to a basis of $\mathfrak{n}_{\mathfrak{q}}$, so $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}(A_2^+)$.

Row 2. Let $q \in \mathbb{G}'_N$, $N \ge 3$. These diagrams correspond to braidings of super type A with positive roots $\Delta_{\mathfrak{q}}^+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_2\}$.

The first diagram is $\overset{q}{\circ}\overset{q^{-1}}{\text{———}}\overset{-1}{\circ}$. In this case the unique Cartan root is α_1 with $N_{\alpha_1} = N$. The element $x_1^N \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$ is primitive and $\mathfrak{Z}_{\mathfrak{q}}$ is generated by ξ_1. Hence $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}(A_1^+)$.

The second diagram gives a similar situation, since $\mathfrak{O}_{\mathfrak{q}} = \{\alpha_1 + \alpha_2\}$.

Row 3. Let $q \in \mathbb{G}'_N$, $N \geq 3$. The diagram $\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{q^2}{\circ}$ corresponds to a braiding of Cartan type B_2 with $\Delta_{\mathfrak{q}}^+ = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}$. In this case $\mathfrak{O}_{\mathfrak{q}} = \Delta_{\mathfrak{q}}^+$. The coproducts of the generators of $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ are:

$$\begin{aligned}
\underline{\Delta}(x_1) =& x_1 \otimes 1 + 1 \otimes x_1; \qquad \underline{\Delta}(x_2) = x_2 \otimes 1 + 1 \otimes x_2;\\
\underline{\Delta}(x_{12}) =& x_{12} \otimes 1 + 1 \otimes x_{12} + (1 - q^{-2})\, x_1 \otimes x_2;\\
\underline{\Delta}(x_{112}) =& x_{112} \otimes 1 + 1 \otimes x_{112} + (1 - q^{-1})(1 - q^{-2})\, x_1^2 \otimes x_2\\
& + q(1 - q^{-2})\, x_1 \otimes x_{12}.
\end{aligned}$$

We have two different cases depending on the parity of N.

(1) If N is odd, then $N_\beta = N$ for all $\beta \in \Delta_{\mathfrak{q}}^+$. In this case,

$$\begin{aligned}
\underline{\Delta}(x_1^N) =& x_1^N \otimes 1 + 1 \otimes x_1^N; \qquad \underline{\Delta}(x_2^N) = x_2^N \otimes 1 + 1 \otimes x_2^N;\\
\underline{\Delta}(x_{12}^N) =& x_{12}^N \otimes 1 + 1 \otimes x_{12}^N + (1 - q^{-2})^N x_1^N \otimes x_2^N;\\
\underline{\Delta}(x_{112}^N) =& x_{112}^N \otimes 1 + 1 \otimes x_{112}^N + (1 - q^{-1})^N (1 - q^{-2})^N x_1^{2N} \otimes x_2^N\\
& + C\, x_1^N \otimes x_{12}^N,
\end{aligned}$$

for some $C \in \mathbf{k}$. Hence, in $\mathfrak{Z}_{\mathfrak{q}}$ we have the relations

$$\begin{aligned}
&[\xi_1, \xi_2] = (1 - q^{-2})^N \xi_{12};\\
&[\xi_{12}, \xi_1] = C\, \xi_{112};\\
&[\xi_1, \xi_2]_c = (1 - q^{-1})^N (1 - q^{-2})^N \xi_{112} + (1 - q^{-2})^N \xi_1 \xi_{12};\\
&[\xi_1, \xi_{112}] = [\xi_2, \xi_{12}] = 0.
\end{aligned}$$

Thus there exists an algebra map $\mathfrak{F} : \mathcal{U}(B_2^+) \to \mathfrak{Z}_{\mathfrak{q}}$ given by $e_1 \mapsto \xi_1$, $e_2 \mapsto \xi_2$. Moreover, $\mathfrak{F}$ is an isomorphism, and so $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}(B_2^+)$. Using the relations of $\mathcal{U}(B_2^+)$ we check that $C = 2(1 - q^{-1})^N (1 - q^{-2})^N$.

(2) If $N = 2M > 2$, then $N_{\alpha_1} = N_{\alpha_1 + \alpha_2} = N$ and $N_{2\alpha_1 + \alpha_2} = N_{\alpha_2} = M$. In this case we have

$$\begin{aligned}
\underline{\Delta}(x_1^N) =& x_1^N \otimes 1 + 1 \otimes x_1^N; \qquad \underline{\Delta}(x_2^M) = x_2^M \otimes 1 + 1 \otimes x_2^M;\\
\underline{\Delta}(x_{12}^N) =& x_{12}^N \otimes 1 + 1 \otimes x_{12}^N + (1 - q^{-2})^N q_{21}^{M(N-1)}\, x_1^N \otimes x_2^{2M}\\
& + (1 - q^{-2})^M q_{21}^{M^2} x_{112}^M \otimes x_2^M;\\
\underline{\Delta}(x_{112}^M) =& x_{112}^M \otimes 1 + 1 \otimes x_{112}^M + (1 - q^{-1})^M (1 - q^{-2})^M q_{21}^{M(M-1)} x_1^N \otimes x_2^M.
\end{aligned}$$

Hence, the following relations hold in $\mathfrak{Z}_{\mathfrak{q}}$:

$$[\xi_2, \xi_1] = (1 - q^{-1})^M (1 - q^{-2})^M q_{21}^{M(M-1)} \xi_{112};$$
$$[\xi_{112}, \xi_2] = (1 - q^{-2})^M q_{21}^{M^2} \xi_{12};$$
$$[\xi_1, \xi_{112}] = [\xi_2, \xi_{12}] = 0.$$

Thus $\mathfrak{F} : \mathcal{U}(C_2^+) \to \mathfrak{Z}_{\mathfrak{q}}$, $e_1 \mapsto \xi_1$, $e_2 \mapsto \xi_2$, is an isomorphism of algebras. (Of course $C_2 \simeq B_2$ but in higher rank we will get different root systems depending on the parity of N).

Row 4. Let $q \in \mathbb{G}'_N$, $N \neq 2, 4$. These diagrams correspond to braidings of super type B with $\Delta_{\mathfrak{q}}^+ = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}$.

If the diagram is $\overset{q}{\circ}\overset{q^{-2}}{\text{———}}\overset{-1}{\circ}$, then the Cartan roots are α_1 and $\alpha_1 + \alpha_2$, with $N_{\alpha_1} = N$, $N_{\alpha_1+\alpha_2} = M$; here, $M = N$ if N is odd and $M = \frac{N}{2}$ if N is even. The elements $x_1^N, x_{12}^M \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$ are primitive in $\widetilde{\mathcal{B}}_{\mathfrak{q}}$. Thus, in $\mathfrak{Z}_{\mathfrak{q}}$, $[\xi_{12}, \xi_1] = 0$ and $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}((A_1 \oplus A_1)^+)$.

If we consider the diagram $\overset{-q^{-1}}{\circ}\overset{q^{2}}{\text{———}}\overset{-1}{\circ}$, then $\mathfrak{O}_{\mathfrak{q}} = \{\alpha_1, \alpha_1+\alpha_2\}$, $N_{\alpha_1} = M$ and $N_{\alpha_1+\alpha_2} = N$. The elements $x_1^M, x_{12}^N \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$ are primitive, so $[\xi_{12}, \xi_1] = 0$ and $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}((A_1 \oplus A_1)^+)$.

Row 5. Let $q \in \mathbb{G}'_N$, $N \neq 3$, $\zeta \in \mathbb{G}'_3$. The diagram $\overset{\zeta}{\circ}\overset{q^{-1}}{\text{———}}\overset{q}{\circ}$ corresponds to a braiding of standard type B_2, so $\Delta_{\mathfrak{q}}^+ = \{\alpha_1, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2, \alpha_2\}$. The other diagram $\overset{\zeta}{\circ}\overset{q\zeta^{-1}}{\text{———}}\overset{\zeta q^{-1}}{\circ}$ is obtained by changing the parameter $q \leftrightarrow \zeta q^{-1}$.

The Cartan roots are $2\alpha_1 + \alpha_2$ and α_2, with $N_{2\alpha_1+\alpha_2} = M := \operatorname{ord}(\zeta q^{-1})$ and $N_{\alpha_2} = N$. The elements x_{112}^M, $x_2^N \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$ are primitive. Thus, in $\mathfrak{Z}_{\mathfrak{q}}$, we have $[\xi_{112}, \xi_2] = 0$. Hence, $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}((A_1 \oplus A_1)^+)$.

Row 6. Let $\zeta \in \mathbb{G}'_3$. The diagrams $\overset{\zeta}{\circ}\overset{-\zeta}{\text{———}}\overset{-1}{\circ}$ and $\overset{\zeta^{-1}}{\circ}\overset{-\zeta^{-1}}{\text{———}}\overset{-1}{\circ}$ correspond to braidings of standard type B, thus $\Delta_{\mathfrak{q}}^+ = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\}$. In both cases $\mathfrak{O}_{\mathfrak{q}}$ is empty so the corresponding Lie algebras are trivial.

Row 7. Let $\zeta \in \mathbb{G}'_{12}$. The diagrams of this row correspond to braidings of type $\mathfrak{ufo}(7)$. In all cases $\mathfrak{O}_{\mathfrak{q}} = \emptyset$ and the associated Lie algebras are trivial.

Row 8. Let $\zeta \in \mathbb{G}'_{12}$. The diagrams of this row correspond to braidings of type $\mathfrak{ufo}(8)$. For $\overset{-\zeta^{2}}{\circ}\overset{\zeta}{\text{———}}\overset{-\zeta^{2}}{\circ}$, $\Delta_{\mathfrak{q}}^+ = \{\alpha_1, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}$. In this case $\mathfrak{O}_{\mathfrak{q}} = \{\alpha_1 + \alpha_2\}$, $N_{\alpha_1+\alpha_2} = 12$. Hence $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}(A_1^+)$. The same result holds for the other braidings in this row.

Row 9. Let $\zeta \in \mathbb{G}'_9$. The diagrams of this row correspond to braidings of type $\mathfrak{brj}(2; 3)$. If $\mathfrak{q}$ has diagram $\overset{-\zeta}{\circ}\overset{\zeta^{7}}{\text{———}}\overset{\zeta^{3}}{\circ}$, then

$$\Delta_{\mathfrak{q}}^+ = \{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}.$$

In this case $\mathfrak{O}_{\mathfrak{q}} = \{\alpha_1, \alpha_1 + \alpha_2\}$ and $N_{\alpha_1} = N_{\alpha_1+\alpha_2} = 18$. Thus $[\xi_{12}, \xi_1] = 0$, so $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}((A_1 \oplus A_1)^+)$.

If $\mathfrak{q}$ has diagram $\overset{\zeta^3}{\circ}\overset{\zeta^8}{\text{——}}\overset{-1}{\circ}$, $\overset{-\zeta^2}{\circ}\overset{\zeta}{\text{——}}\overset{-1}{\circ}$ the set of positive roots are, respectively,

$$\{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_2\},$$
$$\{\alpha_1, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2, \alpha_2\};$$

the Cartan roots are, respectively, $\alpha_1+\alpha_2, 2\alpha_1+\alpha_2$ and $\alpha_1, 2\alpha_1+\alpha_2$. Hence, in both cases, $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}((A_1 \oplus A_1)^+)$.

Row 10. Let $q \in \mathbb{G}'_N$, $N \geq 4$. The diagram $\overset{q}{\circ}\overset{q^{-3}}{\text{——}}\overset{q^3}{\circ}$ corresponds to a braiding of Cartan type G_2, so $\mathfrak{O}_{\mathfrak{q}} = \Delta_{\mathfrak{q}}^+ = \{\alpha_1, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_2\}$. The coproducts of the PBW-generators are:

$$\begin{aligned}
&\underline{\Delta}(x_1) = x_1 \otimes 1 + 1 \otimes x_1; \qquad \underline{\Delta}(x_2) = x_2 \otimes 1 + 1 \otimes x_2;\\
&\underline{\Delta}(x_{12}) = x_{12} \otimes 1 + 1 \otimes x_{12} + (1 - q^{-3})\, x_1 \otimes x_2;\\
&\underline{\Delta}(x_{112}) = x_{112} \otimes 1 + 1 \otimes x_{112} + (1 + q)(1 - q^{-2})\, x_1 \otimes x_{12}\\
&\quad + (1 - q^{-2})(1 - q^{-3})\, x_1^2 \otimes x_2;\\
&\underline{\Delta}(x_{1112}) = x_{1112} \otimes 1 + 1 \otimes x_{1112} + q^2(1 - q^{-3})\, x_1 \otimes x_{112}\\
&\quad + (q^2 - 1)(1 - q^{-3})\, x_1^2 \otimes x_{12} + (1 - q^{-3})(1 - q^{-2})(1 - q^{-1})\, x_1^3 \otimes x_2;\\
&\underline{\Delta}([x_{112}, x_{12}]_c) = [x_{112}, x_{12}]_c \otimes 1 + 1 \otimes [x_{112}, x_{12}]_c + (q - q^{-1})\, x_{112} \otimes x_{12}\\
&\quad + (1 - q^{-3})(1 + q)(1 - q^{-1} + q)\, x_{112}x_1 \otimes x_2\\
&\quad - q q_{21}(1 - q^{-3})(1 + q - q^2)\, x_{1112} \otimes x_2 + q^2 q_{21}(1 - q^{-3})\, x_1 \otimes [x_{112}, x_2]_c\\
&\quad + (1 - q^{-3})^2(q^2 - 1)\, x_1^2 \otimes x_2 x_{12}\\
&\quad + q_{21}(1 - q^{-3})^2(1 - q^{-2})(1 - q^{-1})\, x_1^3 \otimes x_2^2.
\end{aligned}$$

We have two cases.

(1) If 3 does not divide N, then $N_\beta = N$ for all $\beta \in \Delta_{\mathfrak{q}}^+$. Thus, in $\widetilde{\mathcal{B}}_{\mathfrak{q}}$,

$$\begin{aligned}
&\underline{\Delta}(x_1^N) = x_1^N \otimes 1 + 1 \otimes x_1^N; \qquad \underline{\Delta}(x_2^N) = x_2^N \otimes 1 + 1 \otimes x_2^N;\\
&\underline{\Delta}(x_{12}^N) = x_{12}^N \otimes 1 + 1 \otimes x_{12}^N + a_1\, x_1^N \otimes x_2^N;\\
&\underline{\Delta}(x_{112}^N) = x_{112}^N \otimes 1 + 1 \otimes x_{112}^N + a_2\, x_1^N \otimes x_{12}^N + a_3\, x_1^{2N} \otimes x_2^N;\\
&\underline{\Delta}(x_{1112}^N) = x_{1112}^N \otimes 1 + 1 \otimes x_{1112}^N + a_4\, x_1^N \otimes x_{112}^N + a_5\, x_1^{2N} \otimes x_{12}^N\\
&\quad + a_6\, x_1^{3N} \otimes x_2^N;\\
&\underline{\Delta}([x_{112}, x_{12}]_c^N) = [x_{112}, x_{12}]_c^N \otimes 1 + 1 \otimes [x_{112}, x_{12}]_c^N + a_7\, x_{112}^N \otimes x_{12}^N\\
&\quad + a_8\, x_{1112}^N \otimes x_2^N + a_9\, x_1^N \otimes x_{12}^{2N} + a_{10}\, x_1^{2N} \otimes x_2^N x_{12}^N\\
&\quad + a_{11}\, x_{112}^N x_1^N \otimes x_2^N + a_{12}\, x_1^{3N} \otimes x_2^{2N};
\end{aligned}$$

for some $a_i \in \mathbf{k}$. Since

$$\begin{aligned}
a_1 &= (1-q^{-3})^N q_{21}^{\frac{N(N-1)}{2}} \neq 0,\\
a_3 &= (1-q^{-2})^N (1-q^{-3})^N \neq 0,\\
a_6 &= (1-q^{-1})^N (1-q^{-2})^N (1-q^{-3})^N q_{21}^{\frac{3N(N-1)}{2}} \neq 0,\\
a_{12} &= (1-q^{-1})^N (1-q^{-2})^N (1-q^{-3})^{2N} \neq 0,
\end{aligned}$$

the elements x_{12}^N, x_{112}^N, x_{1112}^N and $[x_{112}, x_{12}]_c^N$ are not primitive. Hence $\mathfrak{Z}_{\mathfrak{q}}$ is generated by ξ_1 and ξ_2; also

$$\begin{aligned}
[\xi_2, \xi_1] &= a_1\, \xi_{12}; & [\xi_{12}, \xi_1] &= a_2\, \xi_{112};\\
[\xi_{112}, \xi_1] &= a_4\, \xi_{1112}; & [\xi_1, \xi_{1112}] &= [\xi_2, \xi_{12}] = 0.
\end{aligned}$$

Thus, we have $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}(G_2^+)$.

(2) If $N = 3M$, then $N_{\alpha_1} = N_{\alpha_1+\alpha_2} = N_{2\alpha_1+\alpha_2} = N$ and $N_{3\alpha_1+\alpha_2} = N_{3\alpha_1+2\alpha_2} = N_{\alpha_2} = M$. In this case we have

$$\begin{aligned}
&\underline{\Delta}(x_1^N) = x_1^N \otimes 1 + 1 \otimes x_1^N; \qquad \underline{\Delta}(x_2^M) = x_2^M \otimes 1 + 1 \otimes x_2^M;\\
&\underline{\Delta}(x_{12}^N) = x_{12}^N \otimes 1 + 1 \otimes x_{12}^N + (1-q^{-3})^M q_{21}^{\frac{N(M-1)}{2}} [x_{112}, x_{12}]_c^M \otimes x_2^M\\
&\quad + (1-q^{-3})^{2M} x_{1112}^M \otimes x_2^{2M} + (1-q^{-3})^N q_{21}^{\frac{N(N-1)}{2}} x_1^N \otimes x_2^{3M};\\
&\underline{\Delta}(x_{112}^N) = x_{112}^M \otimes 1 + 1 \otimes x_{112}^M + b_1\, x_1^N \otimes x_{12}^N + b_2\, x_{1112}^M \otimes [x_{112}, x_{12}]_c^M\\
&\qquad + b_3\, x_1^{2N} \otimes x_2^{3M} + b_4\, x_{1112}^{2M} \otimes x_2^M\\
&\qquad + b_5\, x_{1112}^M x_1^N \otimes x_2^{2M} + b_6\, x_1^N \otimes x_2^M;\\
&\underline{\Delta}(x_{1112}^M) = x_{112}^M \otimes 1 + 1 \otimes x_{112}^M + b_7\, x_1^N \otimes x_2^M [x_{112}, x_{12}]_c^M;\\
&\underline{\Delta}([x_{112}, x_{12}]_c^M) = x_{112}^M \otimes 1 + 1 \otimes x_{112}^M + b_8\, x_1^N \otimes x_2^{2M} + b_9\, x_{1112}^M \otimes x_2^M;
\end{aligned}$$

for some $b_i \in \mathbf{k}$. We compute some of them explicitly:

$$\begin{aligned}
b_2 &= (1+q)^M (1-q^{-2})^M q^{2M} q_{21}^{\frac{N(M-1)}{2}},\\
b_7 &= (1-q^{-3})^M (1-q^{-2})^M (1-q^{-1})^M q_{21}^{\frac{N(M-1)}{2}},\\
b_8 &= (1-q^{-3})^{2M} (1-q^{-2})^M (1-q^{-1})^M q_{21}^M.
\end{aligned}$$

As these scalars are not zero, the elements x_{12}^N, x_{112}^N, x_{1112}^M and $[x_{112}, x_{12}]_c^M$ are not primitive. Thus $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}(G_2^+)$.

Row 11. Let $\zeta \in \mathbb{G}'_8$. The diagrams of this row correspond to braidings of standard type G_2, so $\Delta_{\mathfrak{q}}^+ = \{\alpha_1, 3\alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, \alpha_1+\alpha_2, \alpha_2\}$.

If $\mathfrak{q}$ has diagram $\overset{\zeta^2}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^{-1}}{\circ}$, then the Cartan roots are $2\alpha_1+\alpha_2$ and α_2 with $N_{2\alpha_1+\alpha_2} = N_{\alpha_2} = 8$. The elements $x_{112}^8, x_2^8 \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$ are primitive and $[\xi_2, \xi_{112}] = 0$ in $\mathfrak{Z}_{\mathfrak{q}}$. Hence $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}((A_1 \oplus A_1)^+)$. An analogous result holds for the other diagrams of the row.

Row 12. Let $\zeta \in \mathbb{G}'_{24}$. This row corresponds to type $\mathfrak{ufo}(9)$. If $\mathfrak{q}$ has diagram $\overset{\zeta^6}{\circ}\overset{\zeta^{11}}{\text{———}}\overset{\zeta^8}{\circ}$, then

$$\Delta_{\mathfrak{q}}^{+} = \{\alpha_1, 3\alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, 4\alpha_1+3\alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_2\}$$

and $\mathfrak{O}_{\mathfrak{q}} = \{\alpha_1+\alpha_2, 3\alpha_1+\alpha_2\}$. Here, $N_{\alpha_1+\alpha_2} = N_{3\alpha_1+\alpha_2} = 24$, and $x_{12}^{24}, x_{1112}^{24} \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$ are primitive. In $\mathfrak{Z}_{\mathfrak{q}}$ we have the relation $[\xi_{12}, \xi_{1112}] = 0$; thus $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}((A_1 \oplus A_1)^+)$.

For the other diagrams, $\overset{\zeta^6}{\circ}\overset{\zeta}{\text{———}}\overset{\zeta^{-1}}{\circ}$, $\overset{\zeta^8}{\circ}\overset{\zeta^5}{\text{———}}\overset{-1}{\circ}$ and $\overset{\zeta}{\circ}\overset{\zeta^{19}}{\text{———}}\overset{-1}{\circ}$, the sets of positive roots are, respectively,

$$\{\alpha_1, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, 5\alpha_1+2\alpha_2, 5\alpha_1+3\alpha_2, \alpha_2\},$$
$$\{\alpha_1, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, 4\alpha_1+3\alpha_2, 5\alpha_1+3\alpha_2, 5\alpha_1+4\alpha_2, \alpha_2\},$$
$$\{\alpha_1, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 4\alpha_1+\alpha_2, 5\alpha_1+\alpha_2, 5\alpha_1+2\alpha_2, \alpha_2\}.$$

The Cartan roots are, respectively, $2\alpha_1+\alpha_2, \alpha_2$; $\alpha_1+\alpha_2, 5\alpha_1+3\alpha_2$; $\alpha_1, 5\alpha_1+2\alpha_2$. Hence, in all cases, $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}((A_1 \oplus A_1)^+)$.

Row 13. Let $\zeta \in \mathbb{G}'_5$. The braidings in this row are associated to the Lie superalgebra $\mathfrak{brj}(2;5)$ [A5, §5.2]. If $\mathfrak{q}$ has diagram $\overset{\zeta}{\circ}\overset{\zeta^2}{\text{———}}\overset{-1}{\circ}$, then $\Delta_{\mathfrak{q}}^{+} = \{\alpha_1, 3\alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 5\alpha_1+3\alpha_2, 3\alpha_1+2\alpha_2, 4\alpha_1+3\alpha_2, \alpha_1+\alpha_2, \alpha_2\}$. In this case the Cartan roots are α_1, $\alpha_1+\alpha_2$, $2\alpha_1+\alpha_2$ and $3\alpha_1+\alpha_2$, with $N_{\alpha_1} = N_{3\alpha_1+2\alpha_2} = 5$ and $N_{\alpha_1+\alpha_2} = N_{2\alpha_1+\alpha_2} = 10$. In $\widetilde{\mathcal{B}}_{\mathfrak{q}}$,

$$\begin{aligned}
&\underline{\Delta}(x_1) = x_1 \otimes 1 + 1 \otimes x_1;\\
&\underline{\Delta}(x_{12}) = x_{12} \otimes 1 + 1 \otimes x_{12} + (1-\zeta^2)\, x_1 \otimes x_2;\\
&\underline{\Delta}(x_{112}) = x_{112} \otimes 1 + 1 \otimes x_{112} + (1+\zeta)(1-\zeta^3)\, x_1 \otimes x_{12}\\
&\quad + (1-\zeta^2)(1-\zeta^3)\, x_1^2 \otimes x_2;\\
&\underline{\Delta}([x_{112}, x_{12}]_c) = [x_{112}, x_{12}]_c \otimes 1 + 1 \otimes [x_{112}, x_{12}]_c\\
&\quad - \zeta^3(1-\zeta^3)(1+\zeta)^2\, x_1 \otimes x_{12}^2 - \zeta q_{21}\, x_1 x_{112} \otimes x_2\\
&\quad + (1 + q_{21} + \zeta^3 q_{21})\, x_{112} x_1 \otimes x_2 + \zeta(1-\zeta^2)\, x_1 x_{12} x_1 \otimes x_2\\
&\quad + (1-\zeta^2)(1-\zeta^3)^2\, x_1^2 \otimes x_2 x_{12}.
\end{aligned}$$

Hence the coproducts of $x_1^5, x_{12}^{10}, x_{112}^{10}, [x_{112}, x_{12}]_c^5, \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$ are:

$$\begin{aligned}
&\underline{\Delta}(x_1^5) = x_1^5 \otimes 1 + 1 \otimes x_1^5; \qquad \underline{\Delta}(x_{12}^{10}) = x_{12}^{10} \otimes 1 + 1 \otimes x_{12}^{10};\\
&\underline{\Delta}(x_{112}^{10}) = x_{112}^{10} \otimes 1 + 1 \otimes x_{112}^{10} + a_1\, x_1^{10} \otimes x_{12}^{10} + a_2\, x_1^5 \otimes [x_{112}, x_{12}]_c^5;\\
&\underline{\Delta}([x_{112}, x_{12}]_c^5) = [x_{112}, x_{12}]_c^5 \otimes 1 + 1 \otimes [x_{112}, x_{12}]_c^5 + a_3\, x_1^5 \otimes x_{12}^{10}.
\end{aligned}$$

for some $a_i \in \mathbf{k}$. Thus, the following relations hold in $\mathfrak{Z}_{\mathfrak{q}}$

$$[\xi_{12}, \xi_1] = a_3\, \xi_{112,12}; \quad [\xi_{112,12}, \xi_1] = a_2\, \xi_{112}; \quad [\xi_1, \xi_{112,12}] = [\xi_{12}, \xi_{112}] = 0.$$

Since

$$\begin{aligned} a_1 &= -(1-\zeta^3)^5(1+\zeta)^5(1+62\zeta-15\zeta^2-87\zeta^3+70\zeta^4) \neq 0;\\ a_3 &= -(1-\zeta^3)^5(1+\zeta)^8(4-8\zeta-19\zeta^2-3\zeta^3-50\zeta^4) \neq 0, \end{aligned}$$

the elements $x_{112}^{10}, [x_{112}, x_{12}]_c^5$ are not primitive, so ξ_1, ξ_{12} generate $\mathfrak{Z}_{\mathfrak{q}}$. Hence, $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}(B_2^+)$.

If $\mathfrak{q}$ has diagram $\overset{-\zeta^3}{\circ}\overset{\zeta^3}{\text{———}}\overset{-1}{\circ}$, then

$\Delta_{\mathfrak{q}}^+ = \{\alpha_1, 4\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 5\alpha_1+2\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, \alpha_1+\alpha_2, \alpha_2\}$,
$\mathfrak{O}_{\mathfrak{q}} = \{\alpha_1, 3\alpha_1+\alpha_2, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2\}$,

with $N_{\alpha_1} = N_{\alpha_1+\alpha_2} = 10$, $N_{3\alpha_1+\alpha_2} = N_{\alpha_1+\alpha_2} = 5$. The generators of $\mathfrak{Z}_{\mathfrak{q}}$ are ξ_1 and ξ_{12} and they satisfy the following relations

$$[\xi_{12}, \xi_1] = b_1\, \xi_{1112}, \quad [\xi_{1112}, \xi_{12}] = b_2\, \xi_{112}, \quad [\xi_1, \xi_{1112}] = [\xi_{12}, \xi_{112}] = 0,$$

for some $b_1, b_2 \in \mathbf{k}^\times$. Hence $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}(C_2^+)$.

Row 14. Let $\zeta \in \mathbb{G}'_{20}$. This row corresponds to type $\mathfrak{ufo}(10)$. If $\mathfrak{q}$ has diagram $\overset{\zeta}{\circ}\overset{\zeta^{17}}{\text{———}}\overset{-1}{\circ}$, then $\Delta_{\mathfrak{q}}^+ = \{\alpha_1, 3\alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 5\alpha_1+3\alpha_2, 3\alpha_1+2\alpha_2, 4\alpha_1+3\alpha_2, \alpha_1+\alpha_2, \alpha_2\}$. The Cartan roots are α_1 and $3\alpha_1+2\alpha_2$ with $N_{\alpha_1} = N_{3\alpha_1+2\alpha_2} = 20$. The elements $x_1^{20}, [x_{112}, x_{12}]_c^{20} \in \widetilde{\mathcal{B}}_{\mathfrak{q}}$ are primitive; thus $[\xi_{12}, \xi_{112,12}] = 0$ in $\mathfrak{Z}_{\mathfrak{q}}$ and $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}((A_1 \oplus A_1)^+)$. The same holds when the diagram of $\mathfrak{q}$ is another one in this row: $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}((A_1 \oplus A_1)^+)$.

Row 15. Let $\zeta \in \mathbb{G}'_{15}$. This row corresponds to type $\mathfrak{ufo}(11)$. If $\mathfrak{q}$ has diagram $\overset{-\zeta}{\circ}\overset{-\zeta^{12}}{\text{———}}\overset{\zeta^5}{\circ}$, then $\Delta_{\mathfrak{q}}^+ = \{\alpha_1, 3\alpha_1+\alpha_2, 5\alpha_1+2\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, \alpha_1+\alpha_2, \alpha_1+2\alpha_2, \alpha_2\}$. The Cartan roots are α_1 and $3\alpha_1+2\alpha_2$ with $N_{\alpha_1} = N_{3\alpha_1+2\alpha_2} = 30$. In $\mathfrak{Z}_{\mathfrak{q}}$ we have $[\xi_{12}, \xi_{112,12}] = 0$, thus $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}((A_1 \oplus A_1)^+)$. The same result holds if we consider the other diagrams of this row.

Row 16. Let $\zeta \in \mathbb{G}'_7$. This row corresponds to type $\mathfrak{ufo}(12)$. If $\mathfrak{q}$ has diagram $\overset{-\zeta^5}{\circ}\overset{-\zeta^3}{\text{———}}\overset{-1}{\circ}$, then

$$\begin{aligned} \Delta_{\mathfrak{q}}^+ = \{&\alpha_1, 5\alpha_1+\alpha_2, 4\alpha_1+\alpha_2, 7\alpha_1+2\alpha_2, 3\alpha_1+\alpha_2, 8\alpha_1+3\alpha_2,\\ &5\alpha_1+2\alpha_2, 7\alpha_1+3\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2, \alpha_1+\alpha_2, \alpha_2\}. \end{aligned}$$

Also, $\mathfrak{O}_{\mathfrak{q}} = \{\alpha_1, 4\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 5\alpha_1+2\alpha_2, 2\alpha_1+\alpha_2, \alpha_1+\alpha_2\}$ with $N_\beta = 14$ for all $\beta \in \mathfrak{O}_{\mathfrak{q}}$. In $\widetilde{\mathcal{B}}_{\mathfrak{q}}$ we have

$$\begin{aligned} &\underline{\Delta}(x_1) = x_1 \otimes 1 + 1 \otimes x_1;\\ &\underline{\Delta}(x_{12}) = x_{12} \otimes 1 + 1 \otimes x_{12} + (1+\zeta^3)\, x_1 \otimes x_2;\\ &\underline{\Delta}(x_{112}) = x_{112} \otimes 1 + 1 \otimes x_{112} + (1-\zeta)(1-\zeta^5)\, x_1 \otimes x_{12}\\ &\qquad + (1-\zeta)(1+\zeta^3)\, x_1^2 \otimes x_2; \end{aligned}$$

$$\begin{aligned}
&\underline{\Delta}(x_{1112}) = x_{1112}\otimes 1 + 1\otimes x_{1112} + (1+\zeta^3-\zeta^5)(1+\zeta^6)\, x_1\otimes x_{112}\\
&\quad + \zeta(\zeta^3-1)\, x_1^2\otimes x_{12} + \zeta^6(1-\zeta^2)(1+\zeta^3)\, x_1^3\otimes x_2;\\
&\underline{\Delta}(x_{11112}) = x_{11112}\otimes 1 + 1\otimes x_{11112} - \zeta(1-\zeta)(1-\zeta^2)\, x_1\otimes x_{1112}\\
&\quad + (1-\zeta^4)\, x_1^2\otimes x_{112} - (1-\zeta)(1-\zeta^2)^2\, x_1^3\otimes x_{12}\\
&\quad + \zeta^2(1-\zeta)(1-\zeta^2)\, x_1^4\otimes x_2;\\
&\underline{\Delta}([x_{1112},x_{112}]_c) = [x_{1112},x_{112}]_c\otimes 1 + 1\otimes [x_{1112},x_{112}]_c\\
&\quad - \frac{(1-\zeta^5)}{(1+\zeta)}(1-\zeta^3+2\zeta^4)\, x_1\otimes x_{112}^2\\
&\quad - q_{21}(1-\zeta)(1-\zeta^3)\, x_1^2\otimes [x_{112},x_{12}]_c\\
&\quad - (1-\zeta)^2(4+4\zeta+\zeta^2-2\zeta^3-3\zeta^4)\, x_1^2\otimes x_{12}x_{112}\\
&\quad + q_{21}(1-\zeta^2)^2\zeta^4(1-2\zeta-3\zeta^4-2\zeta^5+\zeta^6)\, x_1^3\otimes x_{12}^2\\
&\quad + (1-\zeta)^2(1+\zeta^3)^2(1+\zeta^6)\, x_1^3\otimes x_2x_{112} - \zeta(1-\zeta)(1-\zeta^2)\, x_{1112}\otimes x_{112}\\
&\quad - q_{21}\zeta^6(1-\zeta)^2(1-\zeta^2)(1+2\zeta)\, x_1^4\otimes x_2x_{12}\\
&\quad + q_{21}^2\zeta^2(1-\zeta)^2(1-\zeta^2)(1+\zeta^3)\, x_1^5\otimes x_2^2\\
&\quad - q_{12}^2(1+\zeta^3)(1-\zeta)(1-\zeta^4+\zeta^6)\, x_{111112}\otimes x_2\\
&\quad + \zeta q_{21}(1+\zeta^3)(1-\zeta)(1-\zeta^2)(1+\zeta-\zeta^2)\, x_{11112}x_1\otimes x_2\\
&\quad - \zeta(1-\zeta)^2(1+\zeta^3)(1-\zeta-2\zeta^2-\zeta^3)\, x_{1112}x_1^2\otimes x_2\\
&\quad + (1-\zeta)(1+\zeta^2+\zeta^3-\zeta^4-\zeta^5)\, x_{1112}x_1\otimes x_{12}\\
&\quad + \zeta q_{21}(1-\zeta)^2(2+\zeta-\zeta^3)\, x_{11112}\otimes x_{12}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\underline{\Delta}(x_1^{14}) = x_1^{14}\otimes 1 + 1\otimes x_1^{14}; \qquad \underline{\Delta}(x_{12}^{14}) = x_{12}^{14}\otimes 1 + 1\otimes x_{12}^{14};\\
&\underline{\Delta}(x_{112}^{14}) = x_{112}^{14}\otimes 1 + 1\otimes x_{112}^{14} + a_1\, x_1^{14}\otimes x_{12}^{14};\\
&\underline{\Delta}(x_{1112}^{14}) = x_{1112}^{14}\otimes 1 + 1\otimes x_{1112}^{14} + a_2\, x_1^{14}\otimes x_{112}^{14} + a_3\, x_1^{28}\otimes x_{12}^{14};\\
&\underline{\Delta}(x_{11112}^{14}) = x_{11112}^{14}\otimes 1 + 1\otimes x_{11112}^{14} + a_4\, x_1^{14}\otimes x_{1112}^{14}\\
&\qquad + a_5\, x_1^{28}\otimes x_{112}^{14} + a_6\, x_1^{42}\otimes x_{12}^{14};\\
&\underline{\Delta}([x_{1112},x_{112}]_c^{14}) = [x_{1112},x_{112}]_c^{14}\otimes 1 + 1\otimes [x_{1112},x_{112}]_c^{14} + a_7\, x_{1112}^{14}\otimes x_{12}^{14}\\
&\qquad + a_8\, x_{11112}^{14}\otimes x_{12}^{14} + a_9\, x_1^{42}\otimes x_{12}^{28} + a_{10}\, x_1^{14}\otimes x_{112}^{28}\\
&\qquad + a_{11}\, x_1^{28}\otimes x_{12}^{14}x_{112}^{14} + a_{12}\, x_{1112}^{14}x_1^{14}\otimes x_{12}^{14};
\end{aligned}$$

with $a_i \in \mathbf{k}$. For instance,

$$\begin{aligned}
a_1 = q_{21}^7(1-\zeta)^7(1-\zeta^5)^7\big(&4059 - 7124\zeta + 35105\zeta^2 + 31472\zeta^3 - 17431\zeta^4\\
&+19299\zeta^5 + 40124\zeta^6\big) \neq 0,
\end{aligned}$$

because $\zeta \in \mathbb{G}_7'$. Also,

$$a_3 = 26686268 + 39070423\zeta - 42643895\zeta^2 - 19103336\zeta^3 + 52678504\zeta^4$$
$$-4378676\zeta^5 - 51111858\zeta^6 \neq 0.$$

Since $a_1, a_3, a_6, a_{12} \neq 0$ then x_{112}^{14}, x_{1112}^{14}, x_{11112}^{14} and $[x_{1112}, x_{112}]_c^{14}$ are not primitive elements in $\widetilde{\mathcal{B}}_{\mathfrak{q}}$. Thus, ξ_1 and ξ_{12} generates $\mathfrak{Z}_{\mathfrak{q}}$.

Also, in $\mathfrak{Z}_{\mathfrak{q}}$ we have

$$[\xi_{12}, \xi_1] = a_1\, \xi_{112}; \qquad [\xi_1, \xi_{112}] = a_2\, \xi_{1112};$$
$$[\xi_1, \xi_{1112}] = a_4\, \xi_{11112}; \qquad [\xi_1, \xi_{11112}] = [\xi_{12}, \xi_{112}] = 0.$$

So, $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}(G_2^+)$.

In the case of the diagram $\overset{-\zeta}{\circ}\!\overset{-\zeta^4}{\text{———}}\!\overset{-1}{\circ}$ $\mathfrak{Z}_{\mathfrak{q}}$ is generated by ξ_1, ξ_{12} and

$$[\xi_{12}, \xi_1] = b_1\, \xi_{112}; \qquad [\xi_{12}, \xi_{112}] = b_2\, \xi_{112,12};$$
$$[\xi_{12}, \xi_{112,12}] = b_3\, \xi_{(112,12),12}; \qquad [\xi_1, \xi_{112}] = [\xi_{12}, \xi_{(112,12),12}] = 0,$$

where $b_1, b_2, b_3 \in \mathbf{k}^\times$. Hence, we also have $\mathfrak{Z}_{\mathfrak{q}} \simeq \mathcal{U}(G_2^+)$.

Remark 4.1. The results of this paper are part of the thesis of one of the authors [RB], where missing details of the computations can be found.

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