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Influence of the saturation–suction relationship in the formulation of non-saturated soil consolidation models

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ABSTRACT

The main scope of this paper is to present a fully coupled numerical model for isothermal soil consolidation analysis based on a combination of different stress states. Being originally a non-symmetric problem, it may be straightforward reduced to a symmetric one, and general guidelines for the conditions in which this reduction may be carried out, are addressed. Non-linear saturation–suction and permeability-suction functions were incorporated into a Galerkin approach of the non-saturated soil consolidation problem, which was solved using the finite element method.

In order to validate the model, various examples, for which previous solutions are known, were solved. The use of either a strongly non-linear and non-symmetric formulation or a simple symmetric formulation with accurate prediction in deformation and pore-pressures is extremely dependent on the soil characteristic curves and their derivatives and this aspect is taken into account in the present mathematical approach. The emergent coupling effects may be easily uncoupled in the computer model by merely recasting some coefficients of the discrete equation system.

1. Introduction

Soil consolidation research is commonly carried out by the application of the finite element method to specific mathematical models. These models have undergone a continuous evolution during the last years, so that important achievements towards the prediction of porous media behaviour were attained, based on a robust mathematical framework [1,2].

From Biot [3] pioneer work, many approaches for soil consolidation analysis have been made. For the non-saturated case, Ghaboussi et al. [4] presented a two-phase model; Lewis et al. [5] and Pietraszezak et al. [6] developed some of the earlier three phase models, and, Ng et al. [7] tackled ordinary problems using similar methods. The non-isothermal analysis in saturated models is included in Masters et al. [8], while in Yang et al. [9] the non-isothermal case is extended to the unsaturated situation.

More recently and based on Hassanizadeh et al. [10–12] environmental geomechanics topics were faced by Schrefler [13], among some others. Klubertantz et al. [14] worked on models with miscible and immiscible pore fluids, addressing domains of applicability for each case and Ehlers and Blome [15] developed a triphasic model based on porous media theory.

With respect to constitutive models, one of the most referenced for the solid phase was developed by Alonso et al. [16]. Khalili et al. [17] presented a non-associative plasticity model and Sun et al. [18] developed a model based on suction

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controlled triaxial tests, whereas Graziano & Lancellotta [19] dealt with the derivation of an evolution constitutive equation for deformable porous media.

Regarding with the mathematical framework of the aforementioned models, one controversial topic is the degree of saturation as the main coupling element between water–air fields [17] and the induced matric suction variation. From the present review, it comes up that the suction change gives to the governing equations a highly non-linear [17] characteristic and leads to the loss of symmetry in the isothermal case, which is the main scope of the present paper. However, it was not clearly stated throughout the reviewed papers, when it is crucial to take on the additional costs in terms of processing time and computational memory due to the appearance of non-symmetric matrices and how reliable the approach is without the inclusion of the suction rate.

One straightforward way to deal with this problem may be to select among any of the aforementioned three phase models and neglect all those terms that involve any suction rate and solve the remainder system, if possible. Nevertheless, this procedure may be tedious and cumbersome with plenty of hesitance and uncertainties. Moreover, the remainder equations may not be totally consistent.

Then a different line of attack was taken on. Khalili et al. [20] approach for non-saturated consolidation analysis was followed. It relies on the combination of different components (stress decomposition) and lead to a simple symmetric equation system. In Ref. [21], the mathematical framework was debugged and also implemented using the finite element method with good results in certain conditions. Throughout the present article, the original model [21] was extended by means of the addition of new components and the inclusion of suction changes with the saturation levels (widely used nowadays). Thereby, a new model is attained and with further finite element implementation, the results agree with those given in Refs. [5,22,23] in isothermal conditions. Besides, one noteworthy feature of the model is that it may be used to establish whether or not the additional costs in terms of processing time and computational memory due to the loss of symmetry are justified, based on a consistent mathematical background.

Related to the mechanical behaviour model for the soil skeleton, it must be pointed out that it may be elastic or alternatively elastoplastic [24,25]. For the elastoplastic and isothermal case, in order to preserve the symmetry, it should be noted that – consistent with the critical state theory [26] – the volumetric plastic strains must vanish at the critical state line, independently of the suction value; therefore, it is not convenient to take the plastic potential as a function of the suction [17]. This situation leads to a non-associative flow rule and the elimination of the symmetry; hence some considerations on the plastic potential and stress function will be addressed in future publications.

One last issue was considered. The non-saturated soil consolidation analysis without thermal effects is of great interest for civil constructions like buildings and earth dams, especially when the location area is placed in the north east region of Argentine or south of Paraguay and Brazil. In these locations, many important cities are situated in ancient river valleys where clay, lime-clay or even sandy soil type with a degree of saturation over 70% (generally due to the groundwater table position) are commonly found.

This last fact and the possibility of counting with a simple, open and reliable code were the principal motivations for the present work. Additionally, a different perspective for a well-known problem was addressed and consequently some hidden aspects were appropriately discussed.

2. The governing equations

2.1. Introduction

In this section, a review of Ref. [20] as well as some fundamental modifications to what was proposed there will be included. The present approach may be depicted as a three element set, namely, a deformation model, a water flux model and an air flux model.

The constitutive equations are written in terms of effective stress rather than conventional stress variables used in constitutive models for unsaturated soils, such as net stress [16]; this allows handling a single variable in the elastic deformation model [17] besides the fact that it furnishes a direct bridge between the two-phase and three-phase system.

The flow model is based on two interacting continua: the liquid flow and the air flow. Two pressures are introduced: the average water pressure and the average air pressure.

The coupling between the flow and the deformation fields is established by means of the introduction of parameters that connect the water and air phase pressures to the change in the deformation matrix.

2.2. Mechanical equilibrium

For the solid phase description in an unsaturated medium, it can be considered a differential volume of the solid medium subjected to external total stress components σ_{ij}. The linear momentum balance, disregarding inertial effects, is given by:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + F_i = 0 \quad \text{with } i, j = 1, 2, 3. \tag{1}$$

As it was previously pointed out in paragraph Section 2.1, it is convenient for this work general purpose to extend the Terzaghi's effective stress concept [20,22,27] to unsaturated soils. Considering the effects of pore air and pore water pressures, the total stress components and the effective stress components, may be expressed as (using infinitesimal notation):

$$\mathrm{d}\sigma'_{ij} = \mathrm{d}\sigma_{ij} - a_1 \mathrm{d}p^w \delta_{ij} - a_2 \mathrm{d}p^g \delta_{ij} \tag{2}$$

being $\mathrm{d}p^w$ and $\mathrm{d}p^g$ the differential pore water and pore air pressures respectively, a_1 and a_2 are the effective stress parameters and δ_{ij} is the Kronecker delta. This equation may be also expressed in terms of time derivatives, as:

$$\dot{\sigma}'_{ij} = \dot{\sigma}_{ij} - a_1 \dot{p}^w \delta_{ij} - a_2 \dot{p}^g \delta_{ij} \tag{3}$$

with:

$$a_1 = \frac{c_m}{c} - \frac{c_s}{c} \tag{4}$$

$$a_2 = 1 - \frac{c_m}{c} \tag{5}$$

where

$c_m = \frac{1}{K_m}$ is the compressibility of soil structure with respect to a change in matric suction p^c, ($p^c = p^g - p^w$).

$c = \frac{1}{K_T}$ is the drained compressibility of the soil structure.

$c_s = \frac{1}{K_s}$ is the compressibility of the soil grains.

$K_T = (1 - \alpha) K_s$ is the bulk modulus of the overall skeleton[5] with "α" being the Biot constant[3].

The following expression is suggested for the soil skeleton compressibility with respect to matric suction, based on the compressibility of each single component, on some laboratory tests [21] and on the water saturation–suction relationship (see Appendix B):

$$K_m = \frac{K_T K_s}{S_w K_s + S_g K_T} \tag{6}$$

being S_w and S_g the water and gas phase saturation, respectively. Consider that $S_w = V_w/V_v$ and $S_g = V_g/V_v$ where V_w, is the pore-water volume,V_g, is the pore-air volume and V_v, is the void volume.

Expression (6) has the following restriction: when $S_w \to 1$; $K_m \to K_T$, (fully saturated case).

2.3. Flux model: Water phase

The water flow in saturated, as well as in unsaturated soils, can be described by the combination of Darcy's law and the conservation of fluid mass.

According to Darcy's law [28]:

$$v_{wi} = -\frac{k_{wi}}{\gamma^w} \frac{\partial \rho^w}{\partial x_i} \tag{7}$$

and as the fluid mass continuity is given by:

$$-\frac{\partial}{\partial x_i} \left(\rho^w n_w v_{fi} \right) = \frac{\partial}{\partial t} \left(n_w \rho^w \right) \tag{8}$$

the following expression is obtained [20]:

$$-\frac{1}{\rho^w} \frac{\partial}{\partial x_i} \left(\rho^w \frac{k_{wi}}{\gamma_w} \frac{\partial p^w}{\partial x_i} \right) = -n_w c_f \dot{p}^w + \frac{1}{V} \dot{V}_w \tag{9}$$

where k_{wi} is the coefficient of permeability, v_{wi} is the relative discharge velocity of the water flux, γ_w is the water specific weight, ρ^w is the water density, v_{fi} is the absolute fluid velocity, $n_w = V_w/V$ stands for the relative water porosity, V is the total volume and c_f is the fluid compressibility.

2.4. Flux model: Air phase

The Fick's law is commonly used to describe the flow of air through unsaturated soils. According to this law, the rate of mass transfer for a diffusing substance across a unit area (J_{gi}) is proportional to the concentration of the diffusing substance (C)[29]. This definition renders the following expression:

$$J_{gi} = -D_i \frac{\partial C}{\partial x_i} \tag{10}$$

where D_i, is the diffusion coefficient.

For isothermal conditions and satisfying the conservation of air mass, using a similar procedure than the one employed for the water phase, the following expression is obtained [20]:

$$-\frac{1}{\rho^g}\frac{\partial}{\partial x_i}\left(D_i^*\frac{\partial p^g}{\partial x_i}\right)=-\frac{n_g}{P}\dot{p}^g+\frac{1}{V}\dot{V}_g \quad (11)$$

in which D_i^* is the transmission coefficient for the air phase, ρ^g is the air density, P is the absolute pressure, $n_g = V_g/V$ stands for the relative air porosity and V is the total volume.

Eqs. (9) and (11) are the governing differential equations describing flow of water and air through unsaturated porous media, respectively. Note that we have two equations for four unknowns (p^w, p^g, V_w and V_g), therefore two additional equations are required. To tackle the problem, a relationship between the rate of water and air volume and the primary variables water pore pressure, air pore pressure and displacement components (p^w, p^g, u_i), is addressed in the upcoming paragraph. Once this relationship was attained, the Eq. (3) should be formulated in terms of the displacement field through the addition of an adequate constitutive equation (for the elastic case, see Ref. [20]). Hence (3), (9) and (11) could be solved simultaneously.

2.5. The originally proposed soil states

To set up the required variables link, consider a representative volume of unsaturated porous media subjected to the following stress conditions [20] (see Fig. 1):

The state (1) corresponds to an external isotropic pressure, $d\overline{\sigma}$, an internal pore-water pressure dp^w and an internal pore-air pressure dp^g.

The state number (2) shows a soil portion with identical internal and external pressure dp^w and the state number (3) matches an identical external and internal pore-air dp^w plus a pore-water pressure equal to zero.

According to the above situation and after applying Betty's reciprocal law to the states [(1)–(2)], [(1)–(3)], and [(2)–(3)], Khalili & Khabbaz[20] have obtained the following expression:

$$\frac{dV_w}{V}=\left(\frac{1}{K_m}-\frac{1}{K_s}\right)d\overline{\sigma}-\left[\left(1-n_g\right)\frac{1}{K_m}-\left(1+n_w-n_g\right)\frac{1}{K_s}\right]dp^w-n_g\left(\frac{1}{K_m}-\frac{1}{K_s}\right)dp^g \quad (12)$$

or, in a more compact form, expression (12) may be written as follows,

$$\frac{dV_w}{V}=(c_m-c_s)\,d\overline{\sigma}-\left[\left(1-n_g\right)c_m-\left(1+n_w-n_g\right)c_s\right]dp^w-n_g\,(c_m-c_s)\,dp^g. \quad (13)$$

Regarding the following equation [20]

$$d\overline{\sigma}=\frac{d\epsilon_{ii}}{c}+a_1 dp^w+a_{12} dp^g \quad (14)$$

and, substituting (6) in (4) and (5), the subsequent relationships are yielded:

$$a_1=\alpha S_w \quad (15)$$

$$a_2=\alpha S_g. \quad (16)$$

Considering (14)–(16); and after some algebraic manipulation, Eq. (12) becomes:

$$\frac{dV_w}{V}=\alpha S_w\epsilon_{ii}+\left(\frac{1}{K_m}-\frac{1}{K_s}\right)(\alpha-n)\,S_g dp^g-\left[\frac{S_w}{K_s}(\alpha-n)+S_g(\alpha-n)\left(\frac{1}{K_m}-\frac{1}{K_s}\right)\right]dp^w \quad (17)$$

or

$$\frac{dV_w}{V}=\alpha S_w\epsilon_{ii}-\frac{(\alpha-n)}{K_s}\left(S_wS_w dp^w+S_gS_w dp^g\right)+\frac{(\alpha-n)}{K_T}S_gS_w\left(dp^g-dp^w\right). \quad (18)$$

Substituting (18) in (9) and taking the time derivative, it is obtained:

$$\frac{1}{\rho^w}\frac{\partial}{\partial x_i}\left(\rho^w\frac{k_{wi}}{\gamma_w}\frac{\partial p^w}{\partial x_i}\right)-\frac{nS_w}{K_w}\dot{p}^w-\frac{(\alpha-n)}{K_s}S_w^2\dot{p}^w+\alpha S_w\dot{\epsilon}_{ii}-\frac{(\alpha-n)}{K_s}S_gS_w\dot{p}^g$$
$$-\left(\frac{(\alpha-n)}{K_s}S_w dp^w-\frac{(\alpha-n)}{K_s}S_w dp^g\right)\dot{S}_w+S_w\frac{(\alpha-n)}{K_T}\left(S_g\dot{p}^c+dp^c\dot{S}_g\right)=0 \quad (19)$$

where $\dot{p}^c=\dot{p}^g-\dot{p}^w$ and n is the porosity ($n=n_g/S_g=n_w/S_w$). An important issue which must be outlined here is that in Ref. [20] the rate of water saturation with time was not considered whereas in this work, this aspect was taken into account trough the derivative mentioned above.

Moreover, to attain the goals of the present article a different proposal with respect to states 1 to 3 shown in Fig. 1 will be addressed in the next paragraph. Through this rephrasing, a convenient relationship between V_g and the primary variables may be achieved.

2.6. Additional soils states

Accordingly to what was previously mentioned, two new states are introduced and they are shown in Fig. 2.

The state (4) shows a soil portion with identical internal and external pressure equal to dp^g. The state (5) corresponds to a soil portion with external and pore-water pressure equal to dp^g, plus a null pore-air pressure.

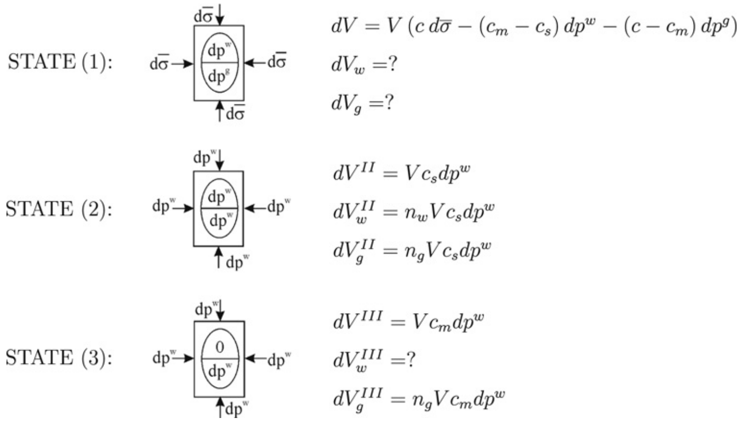

Fig. 1. Set of soil states proposed in Ref. [20].

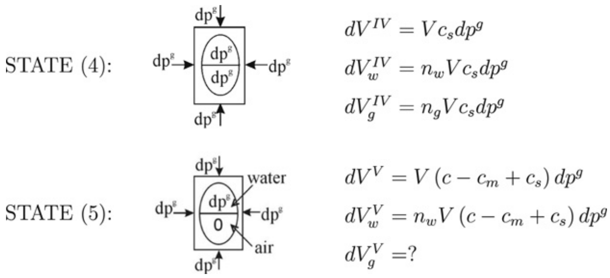

Fig. 2. Additional soil states.

From this new situation, applying Betty's reciprocal law to the states [(1)–(5)], [(4)–(5)] and [(1)–(4)], the subsequent relationships are addressed:

$$(\mathrm{d}V - \mathrm{d}V_w)\,\mathrm{d}p^g = -\mathrm{d}V_g^V\,\mathrm{d}p^g + V\left[(c - c_m + c_s)\,\mathrm{d}\overline{\sigma} - n_w\,(c - c_m + c_s)\,\mathrm{d}p^w\right]\mathrm{d}p^g \tag{20}$$

$$V\,(c - c_m + c_s)\left(\mathrm{d}p^g - n_w\mathrm{d}p^g\right)\mathrm{d}p^g - \mathrm{d}V_g^V\,\mathrm{d}p^g = V\left(c_s\mathrm{d}p^g - n_w c_s\mathrm{d}p^g\right)\mathrm{d}p^g. \tag{21}$$

Recalling the expression

$$\left(\mathrm{d}V - \mathrm{d}V_w - \mathrm{d}V_g\right) = Vc_s\left(\mathrm{d}\overline{\sigma} - n_w\mathrm{d}p^w - n_g\mathrm{d}p^g\right) \tag{22}$$

obtained in Ref. [20] from state 1 and state 2 (Fig. 1), and solving Eqs. (20) and (21) and (22) for $\mathrm{d}V_g$, it leads to:

$$\frac{\mathrm{d}V_g}{V} = \left[(c_m - c_s)\,(n_w - 1) + c_s n_g\right]\mathrm{d}p^g - \left[(c - c_m)\,n_w\right]\mathrm{d}p^w + \left[c - c_m\right]\mathrm{d}\overline{\sigma}. \tag{23}$$

The above equation may be easily compared with (13) using the following relationship:

$$\hat{c}_m = c - c_m + c_s. \tag{24}$$

Including (24) in (23), it leads to

$$\frac{\mathrm{d}V_g}{V} = \left(\hat{c}_m - c_s\right)\mathrm{d}\overline{\sigma} - \left[(1 - n_w)\,\hat{c}_m - \left(1 - n_w + n_g\right)c_s\right]\mathrm{d}p^g - n_w\left(\hat{c}_m - c_s\right)\mathrm{d}p^w. \tag{25}$$

Both Eqs. (13) and (25), show the consistency of the new added states, for they denote relationships between the fluid phases and the principal variables in a tantamount manner since they have the same form, apart from merely a parameter (i.e. c_m and $\hat{c}_m$) and the order of some factors.

The expression (23), will be used in place of the Eq. (33b) of Ref. [20] since it will indirectly show to be a more general description, provided that the system of equations furnished in Khalili & Kabbaz[20] is a particular case of the system yielded in the next section.

Once more, considering (14)–(16), and after some algebraic manipulation, Eq. (23) becomes:

$$\frac{\mathrm{d}V_g}{V} = \alpha S_g \epsilon_{ii} + \left[(c - c_m)\,\alpha S_w - nS_w\,(c - c_m)\right]\mathrm{d}p^w + \left[(c - c_m)\,\alpha S_g + c_s n S_g - c\,(1 - nS_w) + c_m\,(1 - nS_w)\right]\mathrm{d}p^g \tag{26}$$

or

$$\frac{\mathrm{d}V_g}{V} = S_g\left[\alpha\epsilon_{ii} + \frac{(\alpha-n)}{K_s}\left(S_w\mathrm{d}p^w + S_g\mathrm{d}p^g\right) - \frac{(\alpha-n)}{K_T}S_w\mathrm{d}p^c\right]. \tag{27}$$

Likewise Eq. (19), substituting (27) in (11) and taking the time derivative, it is obtained:

$$-\frac{1}{\rho^g}\frac{\partial}{\partial x_i}\left(D_i^*\frac{\partial p^g}{\partial x_i}\right) = -\frac{nS_g}{P}\dot{p}^g + \alpha S_g\dot{\epsilon}_{ii} - \frac{(\alpha-n)}{K_s}S_wS_g\dot{p}^w - \frac{(\alpha-n)}{K_s}S_g^2\dot{p}^g$$
$$-\frac{(\alpha-n)}{K_s}S_g\mathrm{d}p^w\dot{S}_w - \frac{(\alpha-n)}{K_s}S_g\mathrm{d}p^g\dot{S}_g - S_g\frac{(\alpha-n)}{K_T}\left(S_w\dot{p}^c + \mathrm{d}p^c\dot{S}_g\right). \tag{28}$$

3. Full system of equation for unsaturated consolidation analysis

Eqs. (6), (19) and (28) constitute a set of three independent equations. In order to implement the finite element method, some transformations in these equations are necessary:

3.1. Mechanical equilibrium

Considering small strains and small displacement (the geometrically non-linear analysis will be left out of the scope of this article), the stress-strain relationship may be expressed as follows:

$$\dot{\sigma}'_{ij} = C_{ijkl}\dot{\epsilon}_{kl} \quad \text{with } \dot{\epsilon}_{kl} = \frac{1}{2}\left(\frac{\partial\dot{u}_k}{\partial x_l} + \frac{\partial\dot{u}_l}{\partial x_k}\right). \tag{29}$$

Initially, no restrictions will be imposed to C_{ijkl}, i.e., it should be elastic or alternatively elastoplastic with the limitations pointed out in paragraph Section 1. The equilibrium equations in rate form are given by:

$$\nabla\dot{\sigma}_{ij} + \dot{F}_i = 0. \tag{30}$$

Combining Eqs. (3), (29) and (30); the following equation governing the soil deformation is obtained:

$$\nabla\left(C_{ijkl}\dot{\epsilon}_{kl}\right) + a_1\nabla\dot{p}^w + a_2\nabla\dot{p}^g + \dot{F}_i = 0. \tag{31}$$

3.2. Water phase

After some manipulation and considering the relationships $\dot{S}_w = \frac{\mathrm{d}S_w}{\mathrm{d}p^c}\dot{p}^c$ and $\dot{p}^c = \dot{p}^g - \dot{p}^w$ the Eq. (19) may be written as follows:

$$\frac{1}{\rho^w}\frac{\partial}{\partial x_i}\left(\rho^w\frac{k_{wi}}{\gamma_w}\frac{\partial p^w}{\partial x_i}\right) + \alpha S_w\dot{\epsilon}_{ii} - \left\{n\frac{S_w}{K_w} - \left[\frac{(\alpha-n)}{K_s}S_w\left(-S_w - \frac{\mathrm{d}S_w}{\mathrm{d}p^c}\left(\mathrm{d}p^g - \mathrm{d}p^w\right)\right)\right.\right.$$
$$\left.\left. - \frac{(\alpha-n)}{K_T}S_w\left(S_g - \mathrm{d}p^c\frac{\mathrm{d}S_w}{\mathrm{d}p^c}\right)\right]\right\}\dot{p}^w - \left\{\frac{(\alpha-n)}{K_s}S_gS_w\right.$$
$$\left. - \left(\frac{(\alpha-n)}{K_s}S_w\left(\mathrm{d}p^g - \mathrm{d}p^w\right)\right)\frac{\mathrm{d}S_w}{\mathrm{d}p^c} - \frac{(\alpha-n)}{K_T}S_w\left(S_g - \mathrm{d}p^c\frac{\mathrm{d}S_w}{\mathrm{d}p^c}\right)\right\}\dot{p}^g = 0. \tag{32}$$

3.3. Air phase

Carrying out similar modifications than those performed in the water phase and adding the relationship $\dot{S}_g = \frac{\mathrm{d}S_g}{\mathrm{d}p^c}\dot{p}^c = -\frac{\mathrm{d}S_w}{\mathrm{d}p^c}\dot{p}^c$, the Eq. (28) may be written as follows:

$$\frac{1}{\rho^g}\frac{\partial}{\partial x_i}\left(D_i^*\frac{\partial p^g}{\partial x_i}\right) + \alpha S_g\dot{\epsilon}_{ii} - \left\{\frac{(\alpha-n)}{K_s}S_wS_g + S_g\left[\frac{(\alpha-n)}{K_s}\left(\mathrm{d}p^g - \mathrm{d}p^w\right)\frac{\mathrm{d}S_w}{\mathrm{d}p^c} - \frac{(\alpha-n)}{K_T}\left(S_w - \mathrm{d}p^c\frac{\mathrm{d}S_w}{\mathrm{d}p^c}\right)\right]\right\}\dot{p}^w$$
$$-\left\{\frac{nS_g}{P} + \frac{(\alpha-n)}{K_s}S_g^2 - S_g\left[\frac{(\alpha-n)}{K_s}\left(\mathrm{d}p^g - \mathrm{d}p^w\right)\frac{\mathrm{d}S_w}{\mathrm{d}p^c} - \frac{(\alpha-n)}{K_T}\left(S_w - \mathrm{d}p^c\frac{\mathrm{d}S_w}{\mathrm{d}p^c}\right)\right]\right\}\dot{p}^g = 0. \tag{33}$$

3.4. Coupling of the mechanical equilibrium with the fluid phases

Gathering Eqs. (31)–(33), it is obtained:

$$\begin{aligned}
&\nabla\left(C_{ijkl}\dot{\epsilon}_{kl}\right)+a_1\nabla\dot{p}^w+a_2\nabla\dot{p}^g+\dot{F}_i=0\\
&a_1\dot{\epsilon}_{ii}-a_{11}\dot{p}^w-a_{12}\dot{p}^g+\frac{1}{\rho^w}\frac{\partial}{\partial x_i}\left(\rho^w\frac{k_{wi}}{\gamma_w}\frac{\partial p^w}{\partial x_i}\right)=0\\
&a_2\dot{\epsilon}_{ii}-a_{21}\dot{p}^w-a_{22}\dot{p}^g+\frac{1}{\rho^g}\frac{\partial}{\partial x_i}\left(D_i^*\frac{\partial p^g}{\partial x_i}\right)=0
\end{aligned}\tag{34}$$

where strain displacement components are given by Eq. (29).

The parameters in the above equation are:

$$\begin{aligned}
&a_1=\alpha S_w,\qquad a_2=\alpha S_g\\
&a_{11}=\left\{\frac{nS_w}{K_w}+\frac{(\alpha-n)}{K_s}S_w\left(S_w+\left(\mathrm{d}p^g-\mathrm{d}p^w\right)\frac{C_s}{n}\right)-C_s^w\right\}\\
&a_{12}=\left\{\frac{(\alpha-n)}{K_s}S_w\left(S_g-\left(\mathrm{d}p^g-\mathrm{d}p^w\right)\frac{C_s}{n}\right)+C_s^w\right\}\\
&a_{21}=\left\{\frac{(\alpha-n)}{K_s}S_g\left(S_w+\left(\mathrm{d}p^g-\mathrm{d}p^w\right)\frac{C_s}{n}\right)+C_s^g\right\}\\
&a_{22}=\left\{\frac{nS_g}{P}+\frac{(\alpha-n)}{K_s}S_g\left(S_g-\left(\mathrm{d}p^g-\mathrm{d}p^w\right)\frac{C_s}{n}\right)-C_s^g\right\}
\end{aligned}\tag{35}$$

with

$$\begin{aligned}
&C_s=n\frac{\mathrm{d}S_w}{\mathrm{d}p^c}\\
&C_s^w=-S_w\frac{(\alpha-n)}{K_T}\left(S_g-\mathrm{d}p^c\frac{\mathrm{d}S_w}{\mathrm{d}p^c}\right)\\
&C_s^g=-S_g\frac{(\alpha-n)}{K_T}\left(S_w-\mathrm{d}p^c\frac{\mathrm{d}S_w}{\mathrm{d}p^c}\right).
\end{aligned}\tag{36}$$

Eq. (34) stands for a system of partial differential equations for the solution of isothermal soil consolidation problems, obtained by a combination of several stress situations applied on a soil system. This formulation leads to non-symmetrical matrices when the finite elements method is applied and may be straightforward reduced to a formulation with symmetric matrices adjusting only the saturation indicator. The simplicity of this fact eliminates any possibility of conceptual drawbacks and it will be remarked in the next paragraph.

4. General reliability

To analyze the reliability of this approach, it is important to be careful with the limit conditions. Firstly, consider the fully saturated case. This means a unit value in water saturation and a null value in air saturation, leading to the following values of the different coefficients:

$$S_w=1\Longrightarrow a_1=\alpha\wedge a_2=a_{12}=a_{21}=a_{22}=0\wedge a_{11}=\frac{nS_w}{K_w}-\frac{(\alpha-n)}{K_s}.\tag{37}$$

The substitution of these values in Eq. (34), leads to a set of equation equivalent to those corresponding to the saturated case [27,30].

Another interesting possibility is a non-saturated soil, in which the saturation change with suction is disregarded. Then, the coefficients turn into:

$$\begin{aligned}
&a_1=\alpha S_w,\quad a_2=\alpha S_g\\
&a_{11}=\left\{\frac{nS_w}{K_w}+(\alpha-n)\,S_w\left[\frac{S_w}{K_s}+\frac{S_g}{K_T}\right]\right\}\\
&a_{22}=\left\{\frac{nS_g}{P}+(\alpha-n)\,S_g\left[\frac{S_g}{K_s}+\frac{S_w}{K_T}\right]\right\}\\
&a_{12}=a_{21}=\left\{(\alpha-n)\,S_wS_g\left[\frac{1}{K_s}-\frac{1}{K_T}\right]\right\}.
\end{aligned}\tag{38}$$

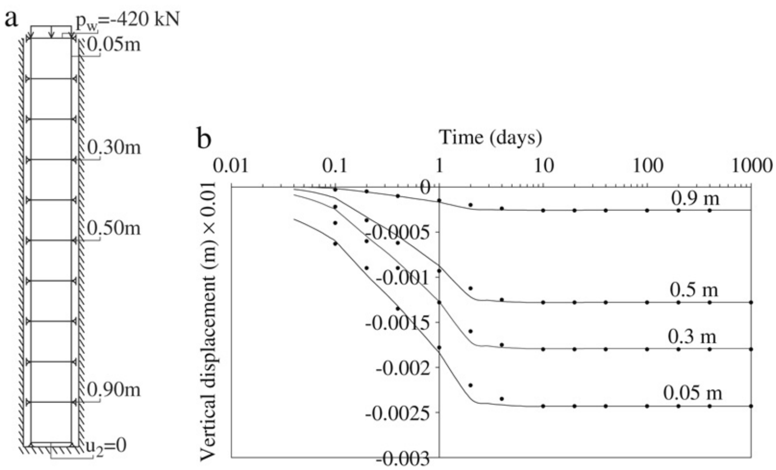

Fig. 3. (a) The one dimensional consolidation problem. Geometry, finite element mesh, load, boundary conditions and selected points to study displacements and water saturation behaviour. (b) Curve of Vertical Displacement vs. Time. (Present work = solid line, Lewis et al. [31] = dotted line).

All values exactly match those obtained in the symmetric case [20,21]. This situation may be regarded as a reduction of the non-symmetric system of Eq. (34) to a symmetric system. Further insights about aspects in which this condition may be exploitable will be given in Section 6. Furthermore, this is a feature of utmost importance because it reveals the crucial role that the saturation variation with suction renders to the whole model. In the isothermal case, the loss of symmetry is exclusively due to this fact.

5. Finite element implementation

Applying the Galerkin method to the system of Eq. (34) and using the finite element technique, it is obtained the following system of ordinary differential equations:

$$\begin{aligned} &\mathbf{K}\dot{\mathbf{u}} + \mathbf{C}_{sw}\dot{\mathbf{p}}^{w} + \mathbf{C}_{sg}\dot{\mathbf{p}}^{g} = \dot{\mathbf{F}}_{s} \\ &\mathbf{C}_{ws}\dot{\mathbf{u}} + \mathbf{P}_{ww}\dot{\mathbf{p}}^{w} + \mathbf{Q}_{wg}\dot{\mathbf{p}}^{g} + \mathbf{H}_{ww}\mathbf{p}^{w} = \dot{\mathbf{F}}_{w} \\ &\mathbf{C}_{gs}\dot{\mathbf{u}} + \mathbf{Q}_{wg}\dot{\mathbf{p}}^{w} + \mathbf{P}_{gg}\dot{\mathbf{p}}^{g} + \mathbf{H}_{gg}\mathbf{p}^{g} = \dot{\mathbf{F}}_{g} \end{aligned} \tag{39}$$

or

$$\begin{bmatrix} \mathbf{K} & \mathbf{C}_{sw} & \mathbf{C}_{sg} \\ \mathbf{C}_{ws} & \mathbf{P}_{ww} & \mathbf{Q}_{wg} \\ \mathbf{C}_{gs} & \mathbf{Q}_{gw} & \mathbf{P}_{gg} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{p}}^{w} \\ \dot{\mathbf{p}}^{g} \end{Bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{H}_{ww} & 0 \\ 0 & 0 & \mathbf{H}_{gg} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{p}^{w} \\ \mathbf{p}^{g} \end{Bmatrix} = \begin{Bmatrix} \dot{\mathbf{F}}_{s} \\ \dot{\mathbf{F}}_{w} \\ \dot{\mathbf{F}}_{g} \end{Bmatrix}. \tag{40}$$

All the matrices involved in Eqs. (39) and (40) are summarized in Appendix A.

6. Model validation

6.1. Example 1: A one dimensional consolidation problem

In order to validate the numerical model obtained in the previous paragraph, a one dimensional consolidation problem was solved and the outcomes were checked against those obtained in Lewis et al. [31]. The soil column was assumed to be unsaturated with initial water saturation S_w equal to 0.52. The initial pore water pressure p_w was taken equal to −280 kPa and the boundary pore water pressure was instantaneously changed to −420 kPa at the surface. The column height was taken equal to 1 m.

The material parameters are: Young's modulus $E = 173\,000$ kPa, Poisson's ratio $\nu = 0.4$, Permeability $k = 0.11456$ m/day, Void ratio $e = 0.4$, and the parameters for the Eq. (B.1) are: $a = 427.0$, $n = 0.794$, $m = 0.613$, $p_r^c = 3000.0$ Kpa and $S_{w0} = 1.0$. In Fig. 3(a) the geometry, the finite element mesh, the load and the boundary conditions are shown. Points where displacements and water saturation evolution will be analyzed, are indicated. The boundary conditions are: (1) Lateral surface: $u_1 = 0.0$. (2) Top surface: $p_w = -420$ kPa, p_g= atmospheric pressure (3) Bottom surface: $u_2 = 0.0$. Fig. 3(b) shows the vertical displacement vs. time for some selected points and Fig. 4 presents the distribution of water saturation vs. time at different points within the soil column.

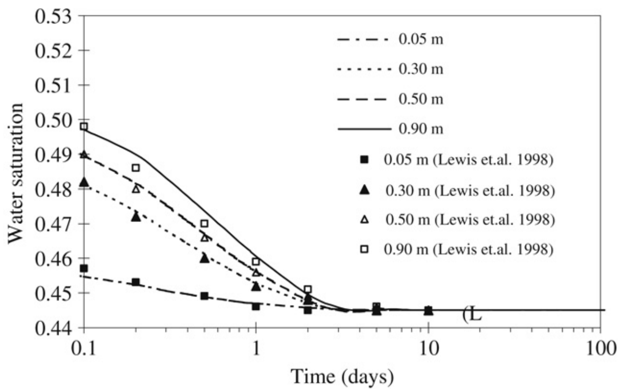

Fig. 4. Curve of Water Saturation vs. Time.

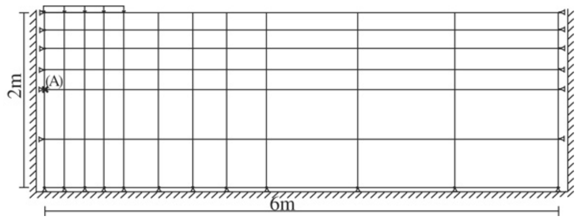

Fig. 5. Strip footing: mesh and dimensions.

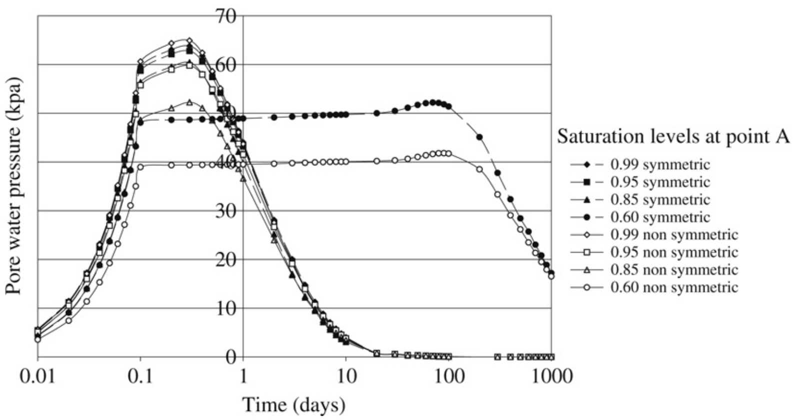

Fig. 6. Pore water pressure vs. time for different levels of saturation at point A.

6.2. Example 2: A strip footing

Another useful example is a strip footing under uniform load solved in Ref. [5]. The data are (see Fig. 5): Width: 6 m, Depth: 2 m,Young's modulus: $E = 13\,000$ kPa, Poisson's ratio: $\nu = 0.4$, Permeability: $k = 3.4 * 10^{-4}$ m/day, Void ratio: 0.9 and Load: $Q = 1$ kN/m.

In first place, in Fig. 6, the pore water pressure for different saturation levels is presented for the symmetric and for the non-symmetric schemes.

Fig. 6 allows confirming, for this case, the strong influence that the saturation level and indirectly the suction exerts over the values of pore pressure at a point located 1m below the centre of the strip footing. For high values of saturation, the outcomes are almost not affected by the suction changes.

Another important issue that must be emphasized is the dependence of the permeability coefficient on saturation [29]. At the outset, as the soil becomes unsaturated, the air replaces the water in the large pores and this causes the water to flow through the smaller pores with an increase of the tortuosity in the flow path. A further increase in the matric suction of the soil, which may be understood as a consequence of the increment in the curvature of the air-water interface, leads to a further decrease in the pore volume occupied by the water feeding back the process. Thus, as long as the degree of saturation

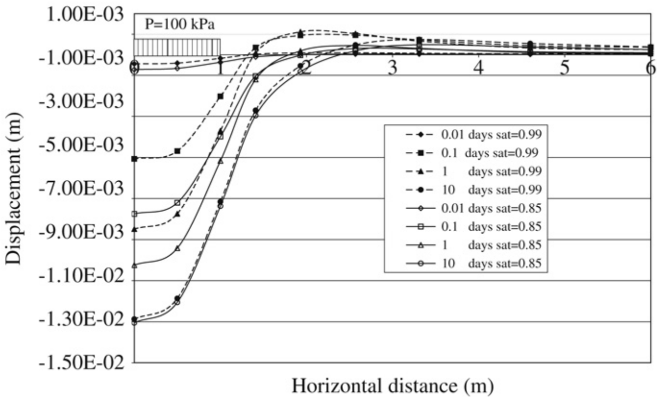

Fig. 7. Surface settlement for different time values.

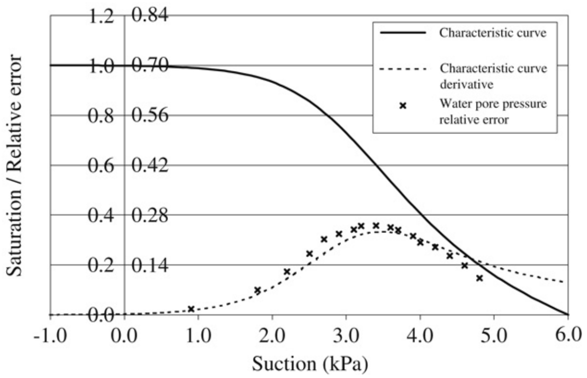

Fig. 8. Characteristic curve derivative and the relative error of pore water pressure vs. Suction.

decreases the coefficient of permeability declines as well, and the dissipation of pore pressures occurs taking a large amount of extra time. Finally, in Fig. 7, the surface settlements versus horizontal distances at different time values, are presented.

7. Field where the symmetric approach is valid

Eqs. (35) and (36) show the strong dependence of system (34) with respect to saturation values through the matrices coefficients, something stated hereinafter. A straightforward way to relate the water saturation to the matric suction is through the saturation–suction characteristic curve. The complete mathematical model for these curves derived from experimental results may be found in Ref. [32], however, a summary of this article was carried out in Appendix B.

It must be underscored the importance of having a smooth expression for the saturation–suction relationship, i.e. a mathematical model, since the slope of this curve plays a fundamental roll in this article, as it will be shortly seen. The expression of the characteristic curve derivative was furnished here from the function proposed in Ref. [32] (this function was also repeated in Appendix B), and it is given by the following equation:

$$\frac{\partial S_w}{\partial p^c} = \left\{1 - \frac{\ln\left(1+\frac{p^c}{p_r^c}\right)}{\ln\left(1+\frac{10^6}{p_r^c}\right)}\right\}\left\{Sw_0(-m)\left[\ln\left(e+\left(\frac{p^c}{a}\right)^n\right)\right]^{(-m-1)}\left[\frac{\left(n\left(\frac{p^c}{a}\right)^{(n-1)}\left(\frac{1}{a}\right)\right)}{\left(e+\left(\frac{p^c}{a}\right)^n\right)}\right]\right\}$$
$$-\left\{\frac{\left(\frac{1}{p_r^c}\right)}{\left(\ln\left(1+\frac{10^6}{p_r^c}\right)\right)\left(1+\frac{p^c}{p_r^c}\right)}\right\}\left\{\frac{S_{w_0}}{\left(\ln\left(e+\left(\frac{p^c}{a}\right)^n\right)\right)^m}\right\}. \quad (41)$$

All the parameters involved in the above equation may be found in the aforementioned appendix and for the sake of brevity, they will not be repeated here.

For the one dimensional consolidation problem considered formerly, the pore pressure vs. suction, taken at the point with depth equal to 0.30 m, was calculated using the symmetric approach presented in Refs. [20,21] and using Eq. (34), for a soil which characteristic curve matches those of the sort used in Fig. 8. Afterwards, the relative error between both approaches with respect to the suction was calculated. Comparing the derivative of the characteristic curve and the relative error, it can be easily noticed that both lines grow and decrease in a similar form, which allows us to bring the relative error into correspondence with saturation derivative. Hence, the higher the values of the slope, the higher the relative error and this fact gives a criterion for deciding whether to assume the computational cost of the approach using Eq. (34) or to assume a rational error and use the same line of attack presented in Refs. [20,21].

Specifically, for the example of Fig. 8, as long as an error of about 10% was tolerated, the entire soil portion with saturation above 90 % or, i.e. a derivative value above 0.14 may be treated disregarding the suction influence. On the other hand, the soil portion with lower water content must be solved with the general case addressed in (34). However, what it seems to be a complication, with the system (34) can be straightforward done by merely setting the matrix coefficients according to Eq. (37) and then some domain decomposition strategies [33] may be implemented later on for the different soil portions.

Finally, we must underscored one fact about the relative error curve. Provided that the tendency of this curve was already settled down with the plotted points and low saturation values are not a priority for this article, the relative error curve was no longer assessed for suction values over 5 kPa.

8. Final remarks

A general formulation and the numerical solution for non-saturated soils consolidation were presented. A non-linear saturation–suction and permeability-suction functions were incorporated into a Galerkin finite element model. The governing equation, in terms of displacement and fluid pressures, result in a coupled non-linear partial differential equation system.

A one dimensional numerical simulation was presented to validate the model and an alternative to avoid non-symmetrical systems was addressed. Furthermore, a study of the settlement and pore pressures evolution for a strip footing was carried out.

When the rate of matric suction is brought in the mathematical framework, a strong non-linearity is induced to the equation system as well as an important coupling effect involving the degree of saturation and the principal variables. Within these considerations, the relevance of the coupling effect may be evaluated through the soil characteristic curve, for the decision of using either complex and non-symmetric system or simple symmetric formulations with accurate prediction in deformation and pore-pressures may be aided by the analysis of the soil characteristic curves and their derivatives.

Furthermore, using the one dimensional example, some limit values for the degree of saturation of a selected soil-type were provided and, using the aforementioned curve, it allows foreseeing whether the simple symmetric formulation will be accurate enough to be selected for the solution, provided that this selection may be straightforward undertaken. Through the computer modelling of the mathematical approach presented here, it is possible to easily uncouple the suction effects, i.e. to switch between both kinds of formulations, by merely recasting some coefficients of the discrete equation system.

It would be interesting to introduce in future works, a subdomain decomposition technique in order to handle with the symmetric and the non-symmetric formulations simultaneously.

Appendix A. Matrices and vectors given in Eq. (34)

$$\dot{u}_i = \mathbf{N}^u\dot{\mathbf{u}}, \qquad \dot{p}^g = \mathbf{N}^p\dot{\mathbf{p}}^g, \qquad \dot{p}^w = \mathbf{N}^p\dot{\mathbf{p}}^w, \qquad \mathbf{m} = \{1, 1, 1, 0, 0, 0\}$$

$$\mathbf{K} = \int_\Omega \mathbf{B}^{u^T}\mathbf{D}\mathbf{B}^u \mathrm{d}\Omega \qquad \dot{\mathbf{F}}_s = \int_\Omega \mathbf{N}^{u^T}\dot{b}\mathrm{d}\Omega + \int_{\Gamma_g} \mathbf{N}^{u^T}\dot{t}\mathrm{d}\Omega$$

$$\mathbf{C}_{sg} = \int_\Omega \mathbf{B}^{u^T}a_2 m\mathbf{N}^p \mathrm{d}\Omega \qquad \mathbf{C}_{gs} = \int_\Omega \mathbf{N}^{p^T} m a_2\mathbf{B}^u \mathrm{d}\Omega$$

$$\mathbf{C}_{ws} = \int_\Omega \mathbf{N}^{p^T} m a_1\mathbf{B}^u \mathrm{d}\Omega \qquad \mathbf{C}_{sw} = \int_\Omega \mathbf{B}^{u^T}a_1 m\mathbf{N}^p \mathrm{d}\Omega$$

$$\mathbf{Q}_{wg} = -\int_\Omega \mathbf{N}^{p^T}a_{12}\mathbf{N}^p \mathrm{d}\Omega \qquad \mathbf{Q}_{gw} = -\int_\Omega \mathbf{N}^{p^T}a_{21}\mathbf{N}^p \mathrm{d}\Omega$$

$$\mathbf{H}_{ww} = \int_\Omega \nabla\mathbf{N}^{p^T}\frac{\mathbf{k}_{wi}}{\gamma_w}\nabla\mathbf{N}^p \mathrm{d}\Omega \qquad \mathbf{H}_{gg} = \int_\Omega \nabla\mathbf{N}^{p^T}\frac{D_i}{P}(1-S_w)\, n\nabla\mathbf{N}^p \mathrm{d}\Omega$$

$$\mathbf{P}_{ww} = -\int_\Omega \mathbf{N}^{p^T}a_{11}\mathbf{N}^p \mathrm{d}\Omega \qquad \mathbf{P}_{gg} = -\int_\Omega \mathbf{N}^{p^T}a_{22}\mathbf{N}^p \mathrm{d}\Omega$$

$$\dot{\mathbf{F}}_g = -\int_{\Gamma_g} \mathbf{N}^{p^T}\dot{\mathbf{q}}_g \mathrm{d}\Gamma \qquad \dot{\mathbf{F}}_w = -\int_{\Gamma_w} \mathbf{N}^{p^T}\dot{\mathbf{q}}_w \mathrm{d}\Gamma$$

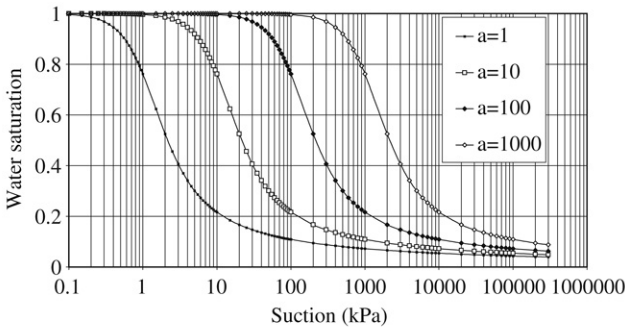

Fig. B.1. Modelled clay type characteristic curves.

where:

- $\mathbf{N}^u$: Vector containing the interpolation functions of the displacements components.
- $\mathbf{N}^p$: Vector containing the interpolation functions of the water and gas pressures.
- $\mathbf{B}^u$: Matrix relating strain and displacement components.
- $\mathbf{D}$: Constitutive matrix.
- $\mathbf{b}$: Body force vector acting on the element domain.
- $\mathbf{t}$: Surface force vector acting on the element boundary.
- $\nabla\mathbf{N}^u$: Gradient of the interpolation functions for water and gas pore pressures.
- $\mathbf{q}_w$: Water flux through the boundary element.
- $\mathbf{q}_g$: Air flux through the boundary element.

Appendix B. Characteristic Curve: Saturation-Suction relationship

In the present work, to relate the saturation values to suction, the soil characteristic curve is used. This curve may adopt different shapes depending on soils attributes and several authors refer to its different aspects and usefulness [32,34,35]. Moreover, in Ref. [32] an approach to obtain a mathematical model from experimental curves is addressed, and, since it is specifically important here, it will be briefly described.

Characteristic curves may be obtained from the following formulae:

$$S_w = \left[1 - \left\{\frac{\ln\left(1+\frac{p^c}{p_r^c}\right)}{\ln\left(1+\frac{10^6}{p_r^c}\right)}\right\}\right]\left[\frac{S_{w_0}}{\left(\ln\left(e+(p^c/a)^n\right)\right)^m}\right]. \tag{B.1}$$

The soil parameters used above are:

p^c	Suction values (kPa)
p_r^c	Residual water content suction (kPa)
S_w	Saturation Values
S_{w_0}	Saturation initial value (Here this value was taken equal to 1)
$a = p_i^c$	Suction value at the inflection point.

$m = 3.67 \ln\left(\frac{S_{w_0}}{S_{w_i}}\right) p_r^c$ With S_{w_i}, being the saturation value at the inflection point.

$n = \frac{1.31^{m+1}}{mS_{w_0}} 3.72 s\, p_i^c$

$s = \frac{S_{w_0}}{p_p^c - p_i^c}$ Slope of the tangent line.

Different characteristic curves are presented in Fig. B.1 standing for the evolution of the saturation of water towards matric suction for clay-type soils.

As may be noticed, the Eq. (B.1) relies on some experimental values. These experimental quantities, i.e. p_r^c, p_p^c, p_i^c, S_{w_i}; are indicated in Fig. B.2:

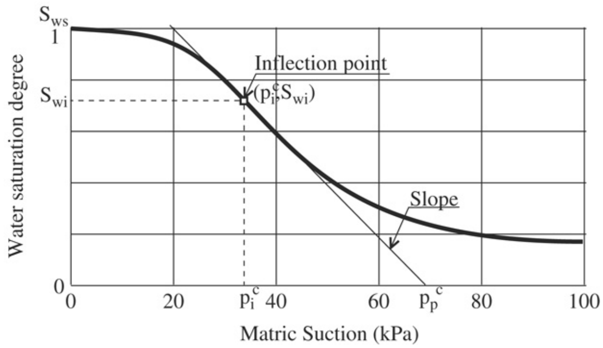

Fig. B.2. Sample plot showing how to obtain values of p_i^c, p_p^c, S_{w_i} and S_{w_0} in order to compute the three parameters a, n and m, used in expressions (41) and (B.1) [32].

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