Journal of Algebra and Its Applications
Vol. 15, No. 10 (2016) 1650191 (19 pages)

DOI: 10.1142/S0219498816501917

Minimal representations for 6-dimensional nilpotent Lie algebra

Nadina Rojas
CIEM-CONICET and FCEFyN
Universidad Nacional de Córdoba, Argentina
nrojas@efn.unc.edu.ar

Received 19 November 2013
Accepted 24 November 2015
Published 26 January 2016

Communicated by Efim Zelmanov

Given a Lie algebra $\mathfrak{g}$, let $\mu(\mathfrak{g})$ and $\mu_{\rm nil}(\mathfrak{g})$ be the minimal dimension of a faithful representation and nilrepresentation of $\mathfrak{g}$, respectively. In this paper, we give $\mu(\mathfrak{g})$ and $\mu_{\rm nil}(\mathfrak{g})$ for each nilpotent Lie algebra $\mathfrak{g}$ of dimension 6 over a field $\mathbb{K}$ of characteristic zero. We also give a minimal faithful representation and nilrepresentation in each case.

Keywords: Nilpotent Lie algebras; Ado's theorem; minimal faithful representation; nilrepresentation.

Mathematics Subject Classification 2010: 17B10, 17B30, 17B45

1. Introduction

In this paper, Lie algebras and representations are finite-dimensional over a field $\mathbb{K}$ of characteristic zero. Ado's theorem states that every Lie algebra has a finite-dimensional faithful representation [14, p. 202]. So, given a Lie algebra $\mathfrak{g}$, let

$$\mu(\mathfrak{g}) = \min\{\dim V : (\pi, V) \text{ is a faithful representation of } \mathfrak{g}\},$$

and additionally, if $\mathfrak{g}$ is nilpotent let

$$\mu_{\rm nil}(\mathfrak{g}) = \min\{\dim V : (\pi_{\rm nil}, V) \text{ is a faithful nilrepresentation of } \mathfrak{g}\}.$$

In general, it is very difficult to determine $\mu(\mathfrak{g})$ and it is also hard to obtain suitable estimations for $\mu(\mathfrak{g})$. More generally, to give faithful representations of nilpotent Lie algebras is a challenging problem in the theory of finite-dimensional Lie algebras ([4, 8]) which is related to another well-known problems, such as Milnor's Conjecture (see [2, 16]). Some results on μ can be found in [2, 5, 7, 13, 18, 19]. An upper bound for $\mu(\mathfrak{g})$ is given in [6] for any finite-dimensional complex Lie

Table 1. μ and $\mu_{\rm nil}$ for all nilpotent Lie algebras of dim = 6.

De Graaf's classification	μ	$\mu_{\rm nil}$
$L_{6,3}$, $L_{6,4}$, $L_{6,5}$, $L_{6,8}$	4	5
$L_{6,1}$, $L_{6,2}$, $L_{6,6}$, $L_{6,7}$, $L_{6,10}$, $L_{6,11}$,$L_{6,12}$, $L_{6,13}$ $L_{6,20}$, $L_{6,21}(\epsilon)$, $L_{6,22}(\epsilon)$, $L_{6,23}$, $L_{6,25}$, $L_{6,26}$	5	5
$L_{6,9}$	5	6
$L_{6,14}$, $L_{6,15}$, $L_{6,16}$, $L_{6,17}$, $L_{6,18}$	6	6
$L_{6,19}(\epsilon)$ if $\sqrt{-\epsilon} \in \mathbb{K}^*$	4	4
$L_{6,19}(\epsilon)$ if $\sqrt{-\epsilon} \notin \mathbb{K}^*$	5	5
$L_{6,24}(\epsilon)$ if $\sqrt{\epsilon} \in \mathbb{K}^*$	5	5
$L_{6,24}(\epsilon)$ if $\sqrt{\epsilon} \notin \mathbb{K}^*$	6	6

algebras. In particular, it is known that $\mu(\mathfrak{g}) < \frac{3}{\sqrt{\dim \mathfrak{g}}} 2^{\dim \mathfrak{g}}$ for any nilpotent Lie algebra $\mathfrak{g}$ (see for instance [3]) and if $\dim \mathfrak{g} \leq 7$ then $\mu(\mathfrak{g}) \leq \dim \mathfrak{g} + 1$ [2, Lemma 6].

Many authors have tried to determine minimal faithful representations for low-dimensional nilpotent Lie algebras starting from a known classification of nilpotent Lie algebras. For instance in [15], μ is computed by the Lie algebras of dimension ≤ 4. Faithful representations of nilpotent Lie algebras of dimension 5 are given in [10] but the minimality problem was studied independently in [1]. On the other hand, a faithful representation was given for each indecomposable nilpotent Lie algebras of dimension 6 in [11] but the minimality problem is not addressed in this.

Our purpose is to give μ and $\mu_{\rm nil}$ for each nilpotent Lie algebra of dimension ≤ 6 over any field $\mathbb{K}$ of characteristic zero. In fact, we give a minimal faithful representation and minimal faithful nilrepresentation for this family of Lie algebras. Throughout the paper, we use the classification of nilpotent Lie algebras up to dimension 6 by De Graaf [9]. According to the notation used in [9], the values of μ and $\mu_{\rm nil}$ of all Lie algebras of dimension 6 are given in Table 1.

The problems that occur in the 7-dimensional nilpotent Lie algebra are similar to the 6-dimensional case, except for the number of Lie algebras. For instance, the set of the 7-dimensional complex nilpotent Lie algebras can be seen as six curves of pairwise nonisomorphic nilpotent Lie algebras and 148 complex nilpotent Lie algebras (see [12]).

2. Preliminaries

We recall some results that will be needed throughout the paper.

If $\mathfrak{g}$ is a nilpotent Lie algebra and $(\pi_{\rm nil}, V)$ is a nilrepresentation of $\mathfrak{g}$, let $\pi : \mathfrak{g} \oplus \mathbb{K} \to \mathfrak{gl}(V)$ be the linear mapping given by $\pi(X, a) = \pi_{\rm nil}(X) + aI$ where I is the identity map on V. It is easy to see that if $(\pi_{\rm nil}, V)$ is a faithful nilrepresentation then (π, V) is a faithful representation of $\mathfrak{g}$ so

$$\mu(\mathfrak{g} \oplus \mathbb{K}) \leq \mu_{\rm nil}(\mathfrak{g}). \tag{2.1}$$

On the other hand, it follows from [7, Theorems 2.1 and 2.2] that if $\mathfrak{g}$ is nilpotent and the center $\mathfrak{z}(\mathfrak{g})$ such that $\mathfrak{z}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$ then

$$\mu(\mathfrak{g}) = \mu_{\text{nil}}(\mathfrak{g}). \tag{2.2}$$

Let us now summarize well-known results on μ.

Proposition 2.1. *Let $\mathfrak{g}$ be a Lie algebra of finite dimension over a field $\mathbb{K}$ of characteristic zero.*

(1) *If $\mathfrak{g}$ is an abelian Lie algebra then $\mu(\mathfrak{g}) = \lceil 2\sqrt{\dim \mathfrak{g} - 1}\rceil$ and $\mu_{\text{nil}}(\mathfrak{g}) = \lceil 2\sqrt{\dim \mathfrak{g}}\rceil$ (see [3, 13, 17, 19]).*
(2) *If $\mathfrak{g}$ is a filiform Lie algebra then $\mu(\mathfrak{g}) \geq \dim \mathfrak{g}$ and if $\dim \mathfrak{g} < 10$ then $\mu(\mathfrak{g}) = \dim \mathfrak{g}$ (see [3]). We also have $\mu_{\text{nil}}(\mathfrak{g}) = \mu(\mathfrak{g})$ (see (2.2)).*
(3) *If $\mathfrak{g}$ is a nilpotent Lie algebra then $\mu(\mathfrak{g} \oplus \mathbb{K}) \leq \mu_{\text{nil}}(\mathfrak{g})$ and if $\mathfrak{z}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$ then $\mu(\mathfrak{g} \oplus \mathbb{K}) = \mu(\mathfrak{g})$ (see (2.1) and (2.2)).*

In the remainder of this section, we summarize the results obtained in [1, 18] for the Lie algebras of dimension 5.

The following list contains the classification of all nilpotent Lie algebras of dimension 5.

(1) $L_{5,1}$ is the abelian Lie algebra of dimension 5.
(2) $L_{5,2} : [X_1, X_2] = Z_1$.
(3) $L_{5,3} : [X_1, X_2] = X_3, [X_1, X_3] = Z_1$.
(4) $L_{5,4} : [X_1, X_2] = Z_1, [X_3, X_4] = Z_1$.
(5) $L_{5,5} : [X_1, X_2] = X_3, [X_1, X_3] = Z_1, [X_2, X_4] = Z_1$.
(6) $L_{5,6} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = Z_1, [X_2, X_3] = Z_1$.
(7) $L_{5,7} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = Z_1$.
(8) $L_{5,8} : [X_1, X_2] = Z_1, [X_1, X_3] = Z_1$.
(9) $L_{5,9} : [X_1, X_2] = X_3, [X_1, X_3] = Z_1, [X_2, X_3] = Z_2$.

Tables 2 and 3 contain a minimal faithful representation and a minimal faithful nilrepresentation for each nilpotent Lie algebra of dimension 5 and in the last column contains the necessary reference for the lower bound of μ and μ_{nil}. For example, where it says "by Engel's Theorem" means that if (π, V) is a faithful nilrepresentation of $L_{5,j}$ (or $L_{6,j}$) there exists a basis B of V such that $[\pi(X)]_B$ is a strictly upper triangular matrix for all $X \in L_{5,j}$. Therefore, since $\dim L_{5,j} = 5$ (or $\dim L_{6,j} = 6$), we obtain $\mu_{\text{nil}}(L_{5,j}) \geq 4$.

In Table 3 all the Lie algebras, except $L_{5,3}$, verify that $\mathfrak{z}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$. From (2.2) we have $\mu_{\text{nil}}(\mathfrak{g}) = \mu(\mathfrak{g})$.

The matrices appearing in the tables are given by

$$\pi\left(\sum x_i X_i + \sum z_i Z_i + \sum a_i A_i\right),$$

Table 2. Lie algebras such that $\mu < \mu_{\text{nil}}$.

$\mathfrak{g}$	μ_{nil}	μ	Faith. nilrep.	Faith. rep.	Ref.
$L_{5,1}$	5	4	$\begin{bmatrix} 0 & 0 & a_1 & a_2 & a_3 \\ 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} a_1 & 0 & a_2 & a_3 \\ 0 & a_1 & a_4 & a_5 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_1 \end{bmatrix}$	Proposition 2.1(1)
$L_{5,2}$	5	4	$\begin{bmatrix} 0 & x_1 & z_1 & a_1 & a_2 \\ 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} a_1 & x_1 & z_1 & 0 \\ 0 & a_1 & x_2 & 0 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 \end{bmatrix}$	[18], since $L_{5,2}$ is $L_{3,2} \oplus L_{2,1}$

Table 3. Lie algebras such that $\mu = \mu_{\text{nil}}$.

$\mathfrak{g}$	μ	Faithful nilrepresentation	Ref.
$L_{5,3}$	4	$\begin{bmatrix} 0 & x_1 & x_3 + a_1 & -2z_1 \\ 0 & 0 & x_2 & -x_3 + a_1 \\ 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	Proposition 2.1(3)
$L_{5,4}$	4	$\begin{bmatrix} 0 & x_1 & x_3 & z_1 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & x_4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	By Engel's theorem, we have $\mu_{\text{nil}}(L_{5,4}) \geq 4$
$L_{5,5}$	4	$\begin{bmatrix} 0 & x_1 & -x_4 & z_1 \\ 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	By Engel's theorem, we have $\mu_{\text{nil}}(L_{5,4}) \geq 4$
$L_{5,6}$	5	$\begin{bmatrix} 0 & x_1 & \frac{1}{2}x_2 & -\frac{1}{2}x_3 & z_1 \\ 0 & 0 & x_1 & 0 & x_4 \\ 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Proposition 2.1(2)
$L_{5,7}$	5	$\begin{bmatrix} 0 & x_1 & 0 & 0 & z_1 \\ 0 & 0 & x_1 & 0 & x_4 \\ 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Proposition 2.1(2)
$L_{5,8}$	4	$\begin{bmatrix} 0 & x_1 & z_1 & z_2 \\ 0 & 0 & x_2 & x_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	By Engel's Theorem, we have $\mu_{\text{nil}}(L_{5,4}) \geq 4$
$L_{5,9}$	5	$\begin{bmatrix} 0 & 0 & \frac{1}{2}x_2 & -\frac{1}{2}x_3 & z_2 \\ 0 & 0 & x_1 & 0 & z_1 \\ 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Remark 2.1

such that $\{Z_1, \ldots, Z_q, A_1, \ldots, A_r\}$ is a basis of $\mathfrak{z}(\mathfrak{g})$, $\{Z_1, \ldots, Z_q\}$ is a basis of $\mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ and $\{A_1, \ldots, A_r\}$ is a basis of a linear complement of $\mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{z}(\mathfrak{g})$.

Remark 2.1. If $\mu_{\text{nil}}(L_{5,9}) = 4$ we obtain $\dim[L_{5,9}, [L_{5,9}, L_{5,9}]] \leq 1$. But $\dim[L_{5,9}, [L_{5,9}, L_{5,9}]] = 2$, therefore $\mu_{\text{nil}}(L_{5,9}) \geq 5$.

3. Minimal Faithful Representations for the Nilpotent Lie Algebras of Dimension 6

Let us now recall De Graaf's classification of the nilpotent Lie algebras of dimension 6 (see [9]).

(1) $L_{6,j} = L_{5,j} \oplus \mathbb{K}$ for all $j = 1, \ldots, 9$.
(2) $L_{6,10} : [X_1, X_2] = X_3, [X_1, X_3] = Z_1, [X_4, X_5] = Z_1$.
(3) $L_{6,11} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = Z_1, [X_2, X_3] = Z_1, [X_2, X_5] = Z_1$.
(4) $L_{6,12} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = Z_1, [X_2, X_5] = Z_1$.
(5) $L_{6,13} : [X_1, X_2] = X_3, [X_1, X_3] = X_5, [X_2, X_4] = X_5, [X_1, X_5] = Z_1, [X_3, X_4] = Z_1$.
(6) $L_{6,14} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_2, X_3] = X_5, [X_2, X_5] = Z_1, [X_3, X_4] = -Z_1$.
(7) $L_{6,15} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_2, X_3] = X_5, [X_1, X_5] = Z_1, [X_2, X_4] = Z_1$.
(8) $L_{6,16} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_2, X_5] = Z_1, [X_3, X_4] = -Z_1$.
(9) $L_{6,17} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_1, X_5] = Z_1, [X_2, X_3] = Z_1$.
(10) $L_{6,18} : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_1, X_4] = X_5, [X_1, X_5] = Z_1$.
(11) $L_{6,19}(\epsilon) : [X_1, X_2] = X_4, [X_1, X_3] = X_5, [X_2, X_4] = Z_1, [X_3, X_5] = \epsilon Z_1$. Isomorphism: $L_{6,19}(\epsilon) \cong L_{6,19}(\delta)$ if and only if there is $\alpha \in \mathbb{K}^*$ such that $\epsilon = \alpha^2\delta$.
(12) $L_{6,20} : [X_1, X_2] = X_4, [X_1, X_3] = X_5, [X_1, X_5] = Z_1, [X_2, X_4] = Z_1$.
(13) $L_{6,21}(\epsilon) : [X_1, X_2] = X_3, [X_1, X_3] = X_4, [X_2, X_3] = X_5, [X_1, X_4] = Z_1, [X_2, X_5] = \epsilon Z_1$. Isomorphism: $L_{6,21}(\epsilon) \cong L_{6,21}(\delta)$ if and only if there is $\alpha \in \mathbb{K}^*$ such that $\epsilon = \alpha^2\delta$.
(14) $L_{6,22}(\epsilon) : [X_1, X_2] = Z_1, [X_1, X_3] = Z_2, [X_2, X_4] = \epsilon Z_2, [X_3, X_4] = Z_1$. Isomorphism: $L_{6,22}(\epsilon) \cong L_{6,22}(\delta)$ if and only if there is $\alpha \in \mathbb{K}^*$ such that $\epsilon = \alpha^2\delta$.
(15) $L_{6,23} : [X_1, X_2] = X_3, [X_1, X_3] = Z_1, [X_1, X_4] = Z_2, [X_2, X_4] = Z_1$.
(16) $L_{6,24}(\epsilon) : [X_1, X_2] = X_3, [X_1, X_3] = Z_1, [X_1, X_4] = \epsilon Z_2, [X_2, X_3] = Z_2, [X_2, X_4] = Z_1$. Isomorphism: $L_{6,24}(\epsilon) \cong L_{6,24}(\delta)$ if and only if there is $\alpha \in \mathbb{K}^*$ such that $\epsilon = \alpha^2\delta$.
(17) $L_{6,25} : [X_1, X_2] = X_3, [X_1, X_3] = Z_1, [X_1, X_4] = Z_2$.
(18) $L_{6,26} : [X_1, X_2] = Z_1, [X_1, X_3] = Z_2, [X_2, X_3] = Z_3$.

Since $\dim L_{6,j} = 6$ and by Engel's Theorem, we obtain $\mu_{\text{nil}}(L_{6,j}) \geq 4$ for $j = 1, \ldots, 26$.

It is easy to see that $\mathfrak{z}(L_{6,j}) \subseteq [L_{6,j}, L_{6,j}]$ for $j = 10, \ldots, 26$. From (2.2) we have $4 \leq \mu(L_{6,j}) = \mu_{\text{nil}}(L_{6,j})$ for $j = 10, \ldots, 26$. By other hand, $L_{6,j} = L_{5,j} \oplus \mathbb{K}$ for $j = 1, \ldots, 9$. From Tables 2 and 3, we obtain $4 \leq \mu(L_{5,j}) \leq \mu(L_{6,j})$ for $j = 1, \ldots, 9$.

It follows that

$$4 \leq \mu(L_{6,j}) \quad \text{and} \quad 4 \leq \mu_{\text{nil}}(L_{6,j}) \tag{3.1}$$

for all $j = 1, \ldots, 26$.

3.1. *Minimal faithful representation and minimal faithful nilrepresentation*

Tables 4 and 5 contain a minimal faithful representation and a minimal faithful nilrepresentation for each Lie algebra $L_{6,j}$, $j = 1, \ldots, 26$. The last column contains the necessary reference for the lower bound of μ and μ_{nil}. The matrices in the tables are given by

$$\pi\left(\sum x_i X_i + \sum z_i Z_i + \sum a_i A_i\right),$$

where $\{Z_1, \ldots, Z_q, A_1, \ldots, A_r\}$ is a basis of $\mathfrak{z}(\mathfrak{g})$, $\{Z_1, \ldots, Z_q\}$ is a basis of $\mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ and $\{A_1, \ldots, A_r\}$ is a basis of a linear complement of $\mathfrak{z}(\mathfrak{g}) \cap [\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{z}(\mathfrak{g})$.

In Table 5, all the Lie algebras, except the Lie algebras $L_{6,1}, L_{6,2}, L_{6,6}$ and $L_{6,7}$, verify that $\mathfrak{z}(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$ and so, from (2.2), we have $\mu_{\text{nil}}(\mathfrak{g}) = \mu(\mathfrak{g})$.

Table 4. Lie algebras such that $\mu < \mu_{\text{nil}}$.

$\mathfrak{g}$	μ_n	μ	Faith. nilreps.	Faith. reps.	Ref.
$L_{6,3}$	5	4	$\begin{bmatrix} 0 & x_1 & x_3 + a_1 & -2z_1 & a_2 \\ 0 & 0 & x_2 & -x_3 + a_1 & 0 \\ 0 & 0 & 0 & x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} a_2 & x_1 & x_3 + a_1 & -2z_1 \\ 0 & a_2 & x_2 & -x_3 + a_1 \\ 0 & 0 & a_2 & x_1 \\ 0 & 0 & 0 & a_2 \end{bmatrix}$	Eq. (3.4) Eq. (3.1)
$L_{6,4}$	5	4	$\begin{bmatrix} 0 & x_1 & x_2 & z_1 & a_1 \\ 0 & 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} a_1 & x_1 & x_2 & z_1 \\ 0 & a_1 & 0 & x_3 \\ 0 & 0 & a_1 & x_4 \\ 0 & 0 & 0 & a_1 \end{bmatrix}$	Eq. (3.4) Eq. (3.1)
$L_{6,5}$	5	4	$\begin{bmatrix} 0 & x_1 & -x_4 & z_1 & a_1 \\ 0 & 0 & x_1 & x_3 & 0 \\ 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} a_1 & x_1 & -x_4 & z_1 \\ 0 & a_1 & x_1 & x_3 \\ 0 & 0 & a_1 & x_2 \\ 0 & 0 & 0 & a_1 \end{bmatrix}$	Eq. (3.4) Eq. (3.1)
$L_{6,8}$	5	4	$\begin{bmatrix} 0 & x_1 & z_1 & z_2 & a_1 \\ 0 & 0 & x_2 & x_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} a_1 & x_1 & z_1 & z_2 \\ 0 & a_1 & x_2 & x_3 \\ 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_1 \end{bmatrix}$	Eq. (3.4) Eq. (3.1)
$L_{6,9}$	6	5	$\begin{bmatrix} 0 & 0 & \frac{1}{2}x_2 & -\frac{1}{2}x_3 & z_2 & a_1 \\ 0 & 0 & x_1 & 0 & z_1 & 0 \\ 0 & 0 & 0 & x_1 & x_3 & 0 \\ 0 & 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} a_1 & 0 & \frac{1}{2}x_2 & -\frac{1}{2}x_3 & z_2 \\ 0 & a_1 & x_1 & 0 & z_1 \\ 0 & 0 & a_1 & x_1 & x_3 \\ 0 & 0 & 0 & a_1 & x_2 \\ 0 & 0 & 0 & 0 & a_1 \end{bmatrix}$	Corollary 3.1 Proposition 2.1(3)

Table 5. Lie algebras such that $\mu = \mu_{\mathrm{nil}}$.

$\mathfrak{g}$	μ	Faith. nilreps.	Ref.
$L_{6,1}$	5	$\begin{bmatrix} 0 & 0 & a_1 & a_2 & a_3 \\ 0 & 0 & a_4 & a_5 & a_6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Proposition 2.1(1)
$L_{6,2}$	5	$\begin{bmatrix} 0 & 0 & x_1 & z_1 & a_1 \\ 0 & 0 & 0 & a_2 & a_3 \\ 0 & 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	[18], since $L_{6,2}$ is $L_{3,2} \oplus L_{3,1}$
$L_{6,6}$	5	$\begin{bmatrix} 0 & x_1 & 3x_2 & x_4 + a_1 & -3z_1 \\ 0 & 0 & x_1 & x_3 & -2x_4 + a_1 \\ 0 & 0 & 0 & x_2 & -x_3 \\ 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Proposition 2.1(3)
$L_{6,7}$	5	$\begin{bmatrix} 0 & x_1 & 0 & x_4 + a_1 & -3z_1 \\ 0 & 0 & x_1 & x_3 & -2x_4 + a_1 \\ 0 & 0 & 0 & x_2 & -x_3 \\ 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Proposition 2.1(3)
$L_{6,10}$	5	$\begin{bmatrix} 0 & x_1 & 0 & x_4 & z_1 \\ 0 & 0 & x_1 & 0 & x_3 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & x_5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Eq. (3.4)
$L_{6,11}$	5	$\begin{bmatrix} 0 & x_1 & x_2 & -x_5 & z_1 \\ 0 & 0 & x_1 & x_2 & x_4 \\ 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Eq. (3.4)
$L_{6,12}$	5	$\begin{bmatrix} 0 & x_1 & 0 & -x_5 & z_1 \\ 0 & 0 & x_1 & 0 & x_4 \\ 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Eq. (3.4)
$L_{6,13}$	5	$\begin{bmatrix} 0 & x_1 & -x_4 & 0 & z_1 \\ 0 & 0 & x_1 & -x_4 & x_5 \\ 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Eq. (3.4)
$L_{6,14}$	6	$\begin{bmatrix} 0 & x_2 & -x_3 & 0 & 0 & z_1 \\ 0 & 0 & x_1 & \frac{1}{2}x_2 & -\frac{1}{2}x_3 & x_5 \\ 0 & 0 & 0 & x_1 & 0 & x_4 \\ 0 & 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Proposition 2.1(2)
$L_{6,15}$	6	$\begin{bmatrix} 0 & x_1 & \frac{1}{2}x_2 & 0 & -\frac{1}{2}x_4 & z_1 \\ 0 & 0 & x_1 & \frac{1}{2}x_2 & -\frac{1}{2}x_3 & x_5 \\ 0 & 0 & 0 & x_1 & 0 & x_4 \\ 0 & 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Proposition 2.1(2)

(*Continued*)

Table 5. (*Continued*)

$\mathfrak{g}$	μ	Faith. nilreps.	Ref.
$L_{6,16}$	6	$\begin{bmatrix} 0 & \frac{1}{2}x_2 & -\frac{1}{2}x_3 & \frac{1}{2}x_4 & -\frac{1}{2}x_5 & z_1 \\ 0 & 0 & x_1 & 0 & 0 & x_5 \\ 0 & 0 & 0 & x_1 & 0 & x_4 \\ 0 & 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Proposition 2.1(2)
$L_{6,17}$	6	$\begin{bmatrix} 0 & x_1 & 0 & \frac{1}{2}x_2 & -\frac{1}{2}x_3 & z_1 \\ 0 & 0 & x_1 & 0 & 0 & x_5 \\ 0 & 0 & 0 & x_1 & 0 & x_4 \\ 0 & 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Proposition 2.1(2)
$L_{6,18}$	6	$\begin{bmatrix} 0 & x_1 & 0 & 0 & 0 & z_1 \\ 0 & 0 & x_1 & 0 & 0 & x_5 \\ 0 & 0 & 0 & x_1 & 0 & x_4 \\ 0 & 0 & 0 & 0 & x_1 & x_3 \\ 0 & 0 & 0 & 0 & 0 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Proposition 2.1(2)
$L_{6,19}(\epsilon)$	4	$\begin{bmatrix} 0 & -\frac{1}{\sqrt{-\epsilon}}x_2 + x_3 & \frac{1}{\sqrt{-\epsilon}}x_4 - x_5 & -\frac{2}{\sqrt{-\epsilon}}z_1 \\ 0 & 0 & x_1 & x_4 + \sqrt{-\epsilon}x_5 \\ 0 & 0 & 0 & x_2 + \sqrt{-\epsilon}x_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$	Proposition 3.1; if there exists $\alpha \in \mathbb{K}^*$ such that $\epsilon = -\alpha^2$
	5	$\begin{bmatrix} 0 & x_1 & x_4 & x_5 & z_1 \\ 0 & 0 & x_2 & x_3 & 0 \\ 0 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 & -\epsilon x_3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Proposition 3.1; if $\epsilon \neq -\alpha^2$ for all $\alpha \in \mathbb{K}^*$
$L_{6,20}$	5	$\begin{bmatrix} 0 & x_1 & 0 & x_4 & z_1 \\ 0 & 0 & x_1 & x_2 & x_5 \\ 0 & 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Eq. (3.4)
$L_{6,21}(\epsilon)$	5	$\begin{bmatrix} 0 & -x_1 + x_2 & (\epsilon+1)x_3 & -x_4 - \epsilon x_5 & 3z_1 \\ 0 & 0 & x_1 - (\epsilon+2)x_2 & x_3 & -2x_4 + \epsilon x_5 \\ 0 & 0 & 0 & x_2 & -x_3 \\ 0 & 0 & 0 & 0 & 2x_2 + x_1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Eq. (3.4); if $\epsilon \neq 0$
	5	$\begin{bmatrix} 0 & -x_1 + x_2 + x_3 & -x_3 - 2x_4 - x_5 & x_4 - x_5 & 3z_1 \\ 0 & 0 & x_2 & -x_3 & x_4 - x_5 \\ 0 & 0 & 0 & x_1 + x_2 & -x_3 \\ 0 & 0 & 0 & 0 & x_1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Eq. (3.4); if $\epsilon = 0$.
$L_{6,22}(\epsilon)$	5	$\begin{bmatrix} 0 & x_1 & x_4 & z_1 & z_2 \\ 0 & 0 & 0 & x_2 & x_3 \\ 0 & 0 & 0 & x_3 & \epsilon x_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Eq. (3.4)
$L_{6,23}$	5	$\begin{bmatrix} 0 & x_1 & -x_4 & z_2 & z_1 \\ 0 & 0 & x_1 & 0 & x_3 \\ 0 & 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Eq. (3.4)
$L_{6,24}(\epsilon)$	5	Remark 3.1	Corollary 3.2; if there exists $\alpha \in \mathbb{K}$ such that $\epsilon = \alpha^2$

Table 5. (*Continued*)

$\mathfrak{g}$	μ	Faith. nilreps.	Ref.
	6	$\begin{bmatrix} 0 & x_2 & x_1 & x_3 & -2z_1 & -z_2 \\ 0 & 0 & 0 & 0 & -2x_4 & 0 \\ 0 & 0 & 0 & x_2 & -x_3 & -\epsilon x_4 \\ 0 & 0 & 0 & 0 & x_1 & x_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Corollary 3.2; if $\epsilon \neq \alpha^2$ for all $\alpha \in \mathbb{K}$
$L_{6,25}$	5	$\begin{bmatrix} 0 & x_1 & x_3 & 2z_1 & z_2 \\ 0 & 0 & x_2 & x_3 & x_4 \\ 0 & 0 & 0 & -x_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Eq. (3.4)
$L_{6,26}$	5	$\begin{bmatrix} 0 & 0 & x_1 & z_1 & z_2 \\ 0 & 0 & x_2 & 0 & z_3 \\ 0 & 0 & 0 & x_2 & x_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	Eq. (3.4)

Remark 3.1. For space reasons, we give a minimal faithful representation of the Lie algebra $L_{6,24}(\epsilon)$ (if there exists $a \in \mathbb{K}$ such that $\epsilon = \alpha^2$) in this remark:

$$\begin{bmatrix} 0 & \sqrt{\epsilon}x_1 + x_2 & (-\epsilon + \sqrt{\epsilon})x_4 + (3\epsilon - 1)x_3 & (3\sqrt{\epsilon} - 1)z_1 + (3 - \sqrt{\epsilon})z_2 & (-\sqrt{\epsilon} + 1)z_1 + (-\sqrt{\epsilon} + 1)z_2 \\ 0 & 0 & x_1 + x_2 & (-\epsilon + 4\sqrt{\epsilon} - 1)x_4 + 2x_3 & (-\epsilon + 1)x_4 \\ 0 & 0 & 0 & -x_1 + x_2 & x_1 + x_2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The faithful representation of $L_{6,19}(\epsilon), L_{6,21}(\epsilon)$ and $L_{6,24}(\epsilon)$ was obtained using Maple.

3.2. *Computing μ and μ_{nil} of $L_{6,19}(\epsilon)$ for $\epsilon \in \mathbb{K}$*

A basis of the Lie algebra $L_{6,19}(\epsilon)$ is $B = \{X_1, X_2, X_3, X_4, X_5, Z_1\}$ which is such that the brackets are given by

$$[X_1, X_2] = X_4, \quad [X_1, X_3] = X_5, \quad [X_2, X_4] = Z_1, \quad [X_3, X_5] = \epsilon Z_1. \tag{3.2}$$

The Lie algebras $L_{6,19}(\epsilon)$ and $L_{6,19}(\delta)$ are isomorphic if and only if there exists $\alpha \in \mathbb{K}^*$ such that

$$\epsilon = \alpha^2 \delta. \tag{3.3}$$

Now we prove the following result.

Proposition 3.1. *Let $\mathbb{K}$ be a field of characteristic $\neq 2$ then*

$$\mu_{\mathrm{nil}}(L_{6,19}(\epsilon)) = \begin{cases} 4 & \text{if } \exists\, \alpha \in \mathbb{K}^* : \epsilon = -\alpha^2; \\ 5 & \text{if } \forall\, \alpha \in \mathbb{K}^* : \epsilon \neq -\alpha^2. \end{cases}$$

Proof. Let $\pi_1 : L_{6,19}(\epsilon) \to \mathfrak{gl}(5)$ be a linear map defined by

$$\pi_1\left(\sum_{i=1}^{5} x_i X_i + z_1 Z_1\right) = \begin{bmatrix} 0 & x_1 & x_4 & x_5 & z_1 \\ 0 & 0 & x_2 & x_3 & 0 \\ 0 & 0 & 0 & 0 & -x_2 \\ 0 & 0 & 0 & 0 & -\epsilon x_3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to check that $(\pi_1, \mathbb{K}^5)$ is a faithful representation for all $\epsilon \in \mathbb{K}$. Then $\mu_{\rm nil}(L_{6,19}(\epsilon)) \leq 5$ for all $\epsilon \in \mathbb{K}$.

Let $\pi_2 : L_{6,19}(-1) \to \mathfrak{gl}(4)$ be a linear map given by

$$\pi_2\left(\sum_{i=1}^{5} x_i X_i + z_1 Z_1\right) = \begin{bmatrix} 0 & x_2 + x_3 & -x_4 - x_5 & 2z_1 \\ 0 & 0 & x_1 & x_4 - x_5 \\ 0 & 0 & 0 & x_2 - x_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By a straightforward calculation we have $(\pi_2, \mathbb{K}^4)$ a faithful representation. Hence $\mu_{\rm nil}(L_{6,19}(-1)) \leq 4$ and from (3.1), we obtain

$$\mu_{\rm nil}(L_{6,19}(-1)) = 4.$$

The Lie algebra $L_{6,19}(-1)$ is isomorphic to $L_{6,19}(\epsilon)$ if and only if there is $\alpha \in \mathbb{K}^*$ such that $\epsilon = -\alpha^2$. Therefore, if $\epsilon \neq -\alpha^2$ for all $\alpha \in \mathbb{K}^*$, we obtain $5 \leq \mu_{\rm nil}(L_{6,19}(\epsilon))$. □

Suppose that $\mu_{\rm nil}(L_{6,j}) = 4$. By Engel's Theorem, we have $L_{6,j} \cong \mathfrak{n}_4(\mathbb{K})$ and by Proposition 3.1 we obtain $L_{6,j} \cong L_{6,19}(\epsilon)$ for some $\epsilon \in \mathbb{K}$. From [9], with $j = 19$, it follows that,

$$5 \leq \mu_{\rm nil}(L_{6,j}). \tag{3.4}$$

3.3. *The lower bound of $\mu_{\rm nil}$ for $L_{6,9}$ and $L_{6,24}(\epsilon)$ for all $\epsilon \in \mathbb{K}$*

The aim of this subsection is to prove the following results.

Theorem 3.1. *Let $n \in \mathbb{N}$ and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{n}_n(\mathbb{K})$ isomorphic to $L_{6,9}$ then $n \geq 6$.*

Theorem 3.2. *Let $\epsilon \in \mathbb{K}, n \in \mathbb{N}$ and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{n}_n(\mathbb{K})$ isomorphic to $L_{6,24}(\epsilon)$. Then*

(1) *$n = 5$, if there exists $\alpha \in \mathbb{K}$ such that $\epsilon = \alpha^2$ and*
(2) *$n \geq 6$, if $\epsilon = \alpha^2$ for all $\alpha \in \mathbb{K}$.*

Let $B = \{X_1, X_2, X_3, X_4, Z_1, Z_2\}$ be a basis of $L_{6,24}(\epsilon)$ such that the only nonzero brackets are

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = Z_1, \quad [X_2, X_3] = Z_2,$$
$$[X_1, X_4] = \epsilon Z_2, \quad [X_2, X_4] = Z_1.$$

The Lie algebra $\mathfrak{h}$ generated by the set $\{X_1, X_2, X_3, Z_1, Z_2\}$ is a Lie subalgebra of $L_{6,24}(\epsilon)$ isomorphic to $L_{5,9}$. On the other hand, $L_{6,9} = L_{5,9} \oplus L_{1,1}$ then $L_{5,9}$ is an ideal of $L_{6,9}$ of codimension 1.

Let V be a vector space of dimension 3 and let W be a subspace of V of dimension 2. If $B = \{v_1, v_2, v_3\}$ is a basis of V then, by a straightforward calculation using linear algebra, it follows that W has a basis given by one of the following ways:

(1) $\{v_1 + av_3, v_2 + bv_3\}$ with $a, b \in \mathbb{K}$;
(2) $\{v_1 + cv_2, v_3\}$ with $c \in \mathbb{K}$;
(3) $\{v_2, v_3\}$.

In order to prove Theorems 3.1 and 3.2 we need the following results. By E_{ij} we denote the $n \times n$ matrix with a 1 on position (i, j) and zeros elsewhere.

Lemma 3.1. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{n}_5(\mathbb{K})$ isomorphic to $L_{5,9}$. Then the center $\mathfrak{z}(\mathfrak{h})$ is given by one of the following ways*

(1) $\mathfrak{z}(\mathfrak{h}) = \mathrm{span}_{\mathbb{K}}\{E_{14} + cE_{25}, E_{15}\}$ *with* $c \in \mathbb{K}$;
(2) $\mathfrak{z}(\mathfrak{h}) = \mathrm{span}_{\mathbb{K}}\{E_{25}, E_{15}\}$.

Proof. Since $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{n}_5(\mathbb{K})$ isomorphic to $L_{5,9}$, we have $\mathfrak{z}(\mathfrak{h}) = [\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]]$ and $\dim \mathfrak{z}(\mathfrak{h}) = 2$. Hence $\mathfrak{z}(\mathfrak{h}) \subseteq \mathrm{span}_{\mathbb{K}}\{E_{14}, E_{25}, E_{15}\}$ then a basis of $\mathfrak{z}(\mathfrak{h})$ is given by one of the following ways

(1) $\{E_{14} + aE_{15}, E_{25} + bE_{15}\}$ with $a, b \in \mathbb{K}$;
(2) $\{E_{14} + cE_{25}, E_{15}\}$ with $c \in \mathbb{K}$;
(3) $\{E_{25}, E_{15}\}$.

If the center is $\mathfrak{z}(\mathfrak{h}) = \mathrm{span}_{\mathbb{K}}\{E_{14} + aE_{15}, E_{25} + bE_{15}\}$, it follows that

$$\mathfrak{h} \subseteq \left\{ \begin{bmatrix} 0 & 0 & x_{13} & x_{14} & x_{15} \\ 0 & 0 & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 0 & x_{34} & x_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} : x_{ij} \in \mathbb{K} \right\}.$$

Therefore $\mathfrak{h}$ is k-step nilpotent Lie algebra with $k \leq 2$, which contradicts that $\mathfrak{h}$ is isomorphic to $L_{5,9}$. □

Lemma 3.2. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{n}_5(\mathbb{K})$ isomorphic to $L_{5,9}$. Let $X_1 = [x_{ij}]_{1 \leq i < j \leq 5}, X_2 = [y_{ij}]_{1 \leq i < j \leq 5} \in \mathfrak{n}_5(\mathbb{K})$ such that the set*

$$\{X_1, X_2, X_3, Z_1, Z_2\} \subseteq \mathfrak{n}_5(\mathbb{K})$$

is a basis of $\mathfrak{h}$ that verified $[X_1, X_2] = X_3, [X_1, X_3] = Z_1$ and $[X_2, X_3] = Z_2$.

(1) *If $\mathfrak{z}(\mathfrak{h}) = \mathrm{span}_{\mathbb{K}}\{E_{14} + cE_{25}, E_{15}\}$ with $c \neq 0$ then $y_{34} \neq 0, x_{23} = \frac{y_{23}}{y_{34}} x_{34}$ and $y_{23}(x_{12}y_{34} - x_{34}y_{12}) \neq 0$.*

(2) *If* $\mathfrak{z}(\mathfrak{h}) = \text{span}_{\mathbb{K}}\{E_{14}, E_{15}\}$ *then* $(y_{12}x_{23} - x_{12}y_{23})(x_{34}y_{35} - x_{35}y_{34}) \neq 0$.
(3) *If* $\mathfrak{z}(\mathfrak{h}) = \text{span}_{\mathbb{K}}\{E_{25}, E_{15}\}$ *then* $(x_{23}y_{12} - x_{12}y_{23})(x_{34}y_{35} - x_{35}y_{34}) \neq 0$.

Proof. Since $X_3 = [X_1, X_2], Z_1 = [X_1, X_3]$ and $Z_2 = [X_2, X_3]$, we get

$$Z_1 = \begin{bmatrix} 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & a_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Z_2 = \begin{bmatrix} 0 & 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 & b_3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

with

$$\begin{cases} a_1 = x_{12}(x_{23}y_{34} - y_{23}x_{34}) - (x_{12}y_{23} - y_{12}x_{23})x_{34}, \\ a_2 = x_{12}(x_{23}y_{35} + x_{24}y_{45} - y_{23}x_{35} - y_{24}x_{45}) + x_{13}(x_{34}y_{45} - y_{34}x_{45}) \\ \qquad - x_{35}(x_{12}y_{23} - y_{12}x_{23}) - x_{45}(x_{12}y_{24} + x_{13}y_{34} - y_{12}x_{24} - y_{13}x_{34}), \\ a_3 = x_{23}(x_{34}y_{45} - y_{34}x_{45}) - x_{45}(x_{23}y_{34} - y_{23}x_{34}), \\ b_1 = y_{12}(x_{23}y_{34} - y_{23}x_{34}) - y_{34}(x_{12}y_{23} - y_{12}x_{23}), \\ b_2 = y_{12}(x_{23}y_{35} + x_{24}y_{45} - y_{23}x_{35} - y_{24}x_{45}) + y_{13}(x_{34}y_{45} - y_{34}x_{45}) \\ \qquad - y_{35}(x_{12}y_{23} - y_{12}x_{23}) - y_{45}(x_{12}y_{24} + x_{13}y_{34} - y_{12}x_{24} - y_{13}x_{34}), \\ b_3 = y_{23}(x_{34}y_{45} - y_{34}x_{45}) - y_{45}(x_{23}y_{34} - y_{23}x_{34}). \end{cases}$$

(1) Let $c \neq 0$, if $\mathfrak{z}(\mathfrak{h}) = \text{span}_{\mathbb{K}}\{E_{14} + cE_{25}, E_{15}\}$, we obtain

$$\mathfrak{h} \subseteq \left\{ \begin{bmatrix} 0 & m_{12} & m_{13} & m_{14} & m_{15} \\ 0 & 0 & m_{23} & m_{24} & m_{25} \\ 0 & 0 & 0 & m_{34} & m_{35} \\ 0 & 0 & 0 & 0 & c\, m_{12} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} : m_{ij} \in \mathbb{K} \right\}.$$

Therefore

(1) $x_{45} = cx_{12}$, $y_{45} = cy_{12}$ and
(2) $Z_1, Z_2 \in \text{span}_{\mathbb{K}}\{E_{14} + cE_{25}, E_{15}\}$.

By item (a), we obtain

$$\begin{cases} a_1 = x_{12}(x_{23}y_{34} - y_{23}x_{34}) - x_{34}(x_{12}y_{23} - y_{12}x_{23}), \\ a_3 = c(x_{23}(x_{34}y_{12} - y_{34}x_{12}) - x_{12}(x_{23}y_{34} - y_{23}x_{34})), \\ b_1 = y_{12}(x_{23}y_{34} - y_{23}x_{34}) - y_{34}(x_{12}y_{23} - y_{12}x_{23}), \\ b_3 = c(y_{23}(x_{34}y_{12} - y_{34}x_{12}) - y_{12}(x_{23}y_{34} - y_{23}x_{34})). \end{cases}$$

Since $c \neq 0$ and by item (b), we have $a_3 = ca_1$ and $b_3 = ca_1$. It follows that

$$x_{12}(x_{23}y_{34} - y_{23}x_{34}) = 0,$$

$$y_{12}(x_{23}y_{34} - y_{23}x_{34}) = 0.$$

By a straightforward calculation, we have the only solution that it makes the set $\{Z_1, Z_2\}$ a linearly independent set i.e. $y_{34} \neq 0$ and $x_{23} = \frac{y_{23}}{y_{34}} x_{34}$. Since

$$Z_1 = \begin{bmatrix} 0 & 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & 0 & ca_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad Z_2 = \begin{bmatrix} 0 & 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & 0 & cb_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we have $\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \neq 0$ with

$$\begin{cases} a_2 = x_{12}(x_{23}y_{35} - y_{23}x_{35} + cx_{24}y_{12} - cy_{24}x_{12}) + cx_{13}(x_{34}y_{12} - y_{34}x_{12}) \\ \qquad - x_{35}(x_{12}y_{23} - y_{12}x_{23}) - cx_{12}(x_{12}y_{24} + x_{13}y_{34} - y_{12}x_{24} - y_{13}x_{34}), \\ b_2 = y_{12}(x_{23}y_{35} - y_{23}x_{35} + cx_{24}y_{12} - cy_{24}x_{12}) + cy_{13}(x_{34}y_{12} - y_{34}x_{12}) \\ \qquad - y_{35}(x_{12}y_{23} - y_{12}x_{23}) - cy_{12}(x_{12}y_{24} + x_{13}y_{34} - y_{12}x_{24} - y_{13}x_{34}). \end{cases}$$

Therefore

$$\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \frac{-2y_{23}(y_{34}x_{12} - x_{34}y_{12})^2 a}{{y_{34}}^2} \neq 0$$

with $a = cy_{34}(y_{24}x_{12} + x_{13}y_{34} - x_{34}y_{13} - x_{24}y_{12}) + y_{23}(x_{35}y_{34} - x_{34}y_{35})$.

(2) If $c = 0$ and $\mathfrak{z}(\mathfrak{h}) = \operatorname{span}_{\mathbb{K}}\{E_{14}, E_{15}\}$, we get

$$\mathfrak{h} \subseteq \left\{ \begin{bmatrix} 0 & m_{12} & m_{13} & m_{14} & m_{15} \\ 0 & 0 & m_{23} & m_{24} & m_{25} \\ 0 & 0 & 0 & m_{34} & m_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} : m_{ij} \in \mathbb{K} \right\}.$$

Hence $x_{45} = y_{45} = 0$, thus $a_3 = b_3 = 0$. Since the set $\{Z_1, Z_2\}$ is linearly independent, we obtain $\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \neq 0$ with

$$\begin{cases} a_2 = x_{12}(x_{23}y_{35} - y_{23}x_{35}) - x_{35}(x_{12}y_{23} - y_{12}x_{23}), \\ b_2 = y_{12}(x_{23}y_{35} - y_{23}x_{35}) - y_{35}(x_{12}y_{23} - y_{12}x_{23}). \end{cases}$$

It follows that $\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = 2(y_{12}x_{23} - x_{12}y_{23})^2(x_{34}y_{35} - x_{35}y_{34}) \neq 0$.

(3) Analysis similar to that in the proof of (2) shows that

$$(y_{12}x_{23} - x_{12}y_{23})^2(x_{34}y_{35} - x_{35}y_{34}) \neq 0.$$

The proof is now completed. □

Lemma 3.3. *Let* $\mathfrak{h}$ *be a Lie subalgebra of* $\mathfrak{n}_5(\mathbb{K})$ *isomorphic to* $L_{5,9}$. *Let* $X_1 = [x_{ij}]_{1 \le i < j \le 5}, X_2 = [y_{ij}]_{1 \le i < j \le 5} \in \mathfrak{n}_5(\mathbb{K})$ *such that the set*

$$\{X_1, X_2, X_3, Z_1, Z_2\} \subseteq \mathfrak{n}_5(\mathbb{K})$$

is a basis of $\mathfrak{h}$ *that verified* $[X_1, X_2] = X_3, [X_1, X_3] = Z_1, [X_2, X_3] = Z_2$ *and let* $X_4 = [a_{ij}]_{1\le i<j\le 5} \in \mathfrak{n}_5(\mathbb{K})$ *be nonzero matrix such that* $[X_3, X_4] = 0$.

(1) *If* $Z_1, Z_2, [X_1, X_4], [X_2, X_4] \in \mathrm{span}_{\mathbb{K}}\{E_{14} + cE_{25}, E_{15}\}$ *and* $a_{45} = ca_{12}$ *then* $a_{12} = a_{23} = a_{34} = 0$ *and* $a_{35} = -ca_{13}\frac{y_{34}}{y_{23}}$;
(2) *if* $Z_1, Z_2, [X_1, X_4], [X_2, X_4] \in \mathrm{span}_{\mathbb{K}}\{E_{25}, E_{15}\}$ *and* $a_{12} = 0$ *then* $a_{23} = a_{34} = a_{45} = 0$.

Proof. (1) If $Z_1, Z_2 \in \mathbb{K}\{E_{14} + cE_{25}, E_{15}\}$, we get

$$\mathfrak{h} \subseteq \left\{ \begin{bmatrix} 0 & m_{12} & m_{13} & m_{14} & m_{15} \\ 0 & 0 & m_{23} & m_{24} & m_{25} \\ 0 & 0 & 0 & m_{34} & m_{35} \\ 0 & 0 & 0 & 0 & cm_{12} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} : m_{ij} \in \mathbb{K} \right\}$$

for every $c \in \mathbb{K}$. We first assume that $c \neq 0$. By Lemma 3.2(1), we get

$$X_1 = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 0 & \frac{y_{23}}{y_{34}}x_{34} & x_{24} & x_{25} \\ 0 & 0 & 0 & x_{34} & x_{35} \\ 0 & 0 & 0 & 0 & cx_{12} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & y_{12} & y_{13} & y_{14} & y_{15} \\ 0 & 0 & y_{23} & y_{24} & y_{25} \\ 0 & 0 & 0 & y_{34} & y_{35} \\ 0 & 0 & 0 & 0 & cy_{12} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$y_{23}(x_{12}y_{34} - x_{34}y_{12}) \neq 0. \tag{3.5}$$

Since $0 = [X_3, X_4] = [[X_1, X_2], X_4]$, we obtain

(1) $a_{34}\frac{y_{23}}{y_{34}}(y_{34}x_{12} - x_{34}y_{12}) = 0$,
(2) $a_{23}c(y_{34}x_{12} - x_{34}y_{12}) = 0$,
(3) $a_{35}\frac{y_{23}}{y_{34}}(y_{34}x_{12} - x_{34}y_{12}) + a_{13}c(y_{34}x_{12} - x_{34}y_{12}) - a_{12}m = 0$ with $m = 2c(x_{12}y_{24} - y_{12}x_{24}) + c(x_{13}y_{34} - y_{13}x_{34}) + \frac{y_{23}}{y_{34}}(x_{35}y_{34} - x_{34}y_{35})$.

Since $[X_2, X_4] \in \mathrm{span}_{\mathbb{K}}\{E_{14} + cE_{25}, E_{15}\}$, we have

(1) $y_{12}a_{23} - a_{12}y_{23} = 0$.

From (a), (b), (d) and (3.5), we obtain $a_{34} = a_{23} = a_{12} = 0$. Finally, by (c) and Eq. (3.5), it follows that $a_{35}\frac{y_{23}}{y_{34}} + ca_{13} = 0$.

Now assume $c = 0$. From Lemma 3.2(2), we obtain

$$(y_{12}x_{23} - x_{12}y_{23})^2(x_{34}y_{35} - x_{35}y_{34}) \neq 0. \tag{3.6}$$

Since $[X_1, X_4], [X_2, X_4] \in \mathrm{span}_{\mathbb{K}}\{E_{14}, E_{15}\}$, we have the following equations

$$\begin{cases} x_{12}a_{23} - a_{12}x_{23} = 0, & y_{12}a_{23} - a_{12}y_{23} = 0, \\ x_{23}a_{34} - a_{23}x_{34} = 0, & y_{23}a_{34} - a_{23}y_{34} = 0, \\ x_{23}a_{35} - a_{23}x_{35} = 0, & y_{23}a_{35} - a_{23}y_{35} = 0. \end{cases}$$

From Eq. (3.6), we get $a_{12} = a_{23} = a_{34} = a_{35} = 0$.

(2) Analysis similar to that in (1) with $c = 0$ shows that $a_{13} = a_{23} = a_{34} = 0$ and the proof is complete. $\square$

Proposition 3.2. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{n}_5(\mathbb{K})$ isomorphic to $L_{5,9}$. Let $X_1 = [x_{ij}]_{1\le i<j\le 5}, X_2 = [y_{ij}]_{1\le i<j\le 5} \in \mathfrak{n}_5(\mathbb{K})$ such that the set*

$$\{X_1, X_2, X_3, Z_1, Z_2\} \subseteq \mathfrak{n}_5(\mathbb{K})$$

is a basis of $\mathfrak{h}$ that verified $[X_1, X_2] = X_3, [X_1, X_3] = Z_1, [X_2, X_3] = Z_2$ and let $X_4 \in \mathfrak{n}_5(\mathbb{K})$ such that $[X_i, X_4] = 0$ for $i = 1, 2, 3$. Then $X_4 \in \mathrm{span}_{\mathbb{K}}\{X_1, X_2, X_3, Z_1, Z_2\}$.

Proof. From Lemma 3.1, we have the center $\mathfrak{z}(\mathfrak{h})$ given by any of the following ways:

(1) $\mathfrak{z}(\mathfrak{h}) = \mathrm{span}_{\mathbb{K}}\{E_{14} + cE_{25}, E_{15}\}$ with $c \in \mathbb{K}$;
(2) $\mathfrak{z}(\mathfrak{h}) = \mathrm{span}_{\mathbb{K}}\{E_{25}, E_{15}\}$.

We first assume (1). If $c \neq 0$, by Lemmas 3.2(1) and 3.3(1), we get

$$X_1 = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 0 & \dfrac{y_{23}}{y_{34}}x_{34} & x_{24} & x_{25} \\ 0 & 0 & 0 & x_{34} & x_{35} \\ 0 & 0 & 0 & 0 & cx_{12} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & y_{12} & y_{13} & y_{14} & y_{15} \\ 0 & 0 & y_{23} & y_{24} & y_{25} \\ 0 & 0 & 0 & y_{34} & y_{35} \\ 0 & 0 & 0 & 0 & cy_{12} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$y_{23}(y_{34}x_{12} - x_{34}y_{12}) \neq 0 \tag{3.7}$$

and $X_4 = \begin{bmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & 0 & 0 & a_{24} & a_{25} \\ 0 & 0 & 0 & 0 & -ca_{13}\frac{y_{34}}{y_{23}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. From $[X_1, X_4] = [X_2, X_4] = 0$, we obtain the following equations:

(1) $a_{24}x_{12} - a_{13}x_{34} = 0$;
(2) $a_{24}y_{12} - a_{13}y_{34} = 0$;
(3) $(a_{25} - ca_{14})x_{12} - a_{13}(\frac{cx_{13}y_{34}}{y_{23}} - x_{35}) = 0$;
(4) $(a_{25} - ca_{14})y_{12} - a_{13}(\frac{cy_{13}y_{34}}{y_{23}} - y_{35}) = 0$.

From items (a), (b) and from Eq. (3.7), we obtain $a_{13} = a_{24} = 0$. Hence, from items (c), (d) and from Eq. (3.7) we have

$$X_4 = \begin{bmatrix} 0 & 0 & 0 & a_{14} & a_{15} \\ 0 & 0 & 0 & 0 & ca_{14} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $c = 0$, by Lemmas 3.2(2) and 3.3(1), we get

$$X_1 = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 0 & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 0 & x_{34} & x_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & y_{12} & y_{13} & y_{14} & y_{15} \\ 0 & 0 & y_{23} & y_{24} & y_{25} \\ 0 & 0 & 0 & y_{34} & y_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.8}$$

$$(y_{12}x_{23} - x_{12}y_{23})(x_{34}y_{35} - x_{35}y_{34}) \neq 0$$

and $X_4 = \begin{bmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & 0 & 0 & a_{24} & a_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. From $[X_1, X_4] = [X_2, X_4] = 0$, we obtain the following equations

$$\begin{cases} x_{12}a_{24} - a_{13}x_{34} = 0, & x_{12}a_{25} - a_{13}x_{35} = 0, \\ y_{12}a_{24} - a_{13}y_{34} = 0, & y_{12}a_{25} - a_{13}y_{35} = 0. \end{cases}$$

By Eq. (3.8), it is not difficult to see that

$$X_4 = \begin{bmatrix} 0 & 0 & 0 & a_{14} & a_{15} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

(2) Analysis similar to that in (1) with $c = 0$ shows that

$$X_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & a_{15} \\ 0 & 0 & 0 & 0 & a_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and the proposition is proved. □

Proposition 3.3. *Let $\epsilon \in \mathbb{K}$ and let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{n}_5(\mathbb{K})$ isomorphic to $L_{5,9}$. Let $X_1 = [x_{ij}]_{1\leq i<j\leq 5}, X_2 = [y_{ij}]_{1\leq i<j\leq 5} \in \mathfrak{n}_5(\mathbb{K})$ such that the set*

$$\{X_1, X_2, X_3, Z_1, Z_2\} \subseteq \mathfrak{n}_5(\mathbb{K})$$

is a basis of $\mathfrak{h}$ that verified $[X_1, X_2] = X_3$, $[X_1, X_3] = Z_1$, $[X_2, X_3] = Z_2$. Let $X_4 \in \mathfrak{n}_5(\mathbb{K})$ such that

(1) *$\{X_1, X_2, X_3, X_4, Z_1, Z_2\}$ is linearly independent and*
(2) *$[X_1, X_4] = \epsilon Z_2, [X_2, X_4] = Z_1, [X_3, X_4] = [Z_j, X_4] = 0$ for $j = 1, 2$.*

Then there exists $a \in \mathbb{K}$ such that $a^2 - \epsilon = 0$.

Proof. From Lemma 3.1, we have the center $\mathfrak{z}(\mathfrak{h})$ given by one of the following ways:

(1) $\mathfrak{z}(\mathfrak{h}) = \mathrm{span}_{\mathbb{K}}\{E_{14} + cE_{25}, E_{15}\}$ with $c \in \mathbb{K}$;
(2) $\mathfrak{z}(\mathfrak{h}) = \mathrm{span}_{\mathbb{K}}\{E_{25}, E_{15}\}$.

We first assume (1). If $c \neq 0$, by Lemmas 3.2(1) and 3.3(1), we get

$$X_1 = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 0 & \frac{y_{23}}{y_{34}}x_{34} & x_{24} & x_{25} \\ 0 & 0 & 0 & x_{34} & x_{35} \\ 0 & 0 & 0 & 0 & cx_{12} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & y_{12} & y_{13} & y_{14} & y_{15} \\ 0 & 0 & y_{23} & y_{24} & y_{25} \\ 0 & 0 & 0 & y_{34} & y_{35} \\ 0 & 0 & 0 & 0 & cy_{12} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.9}$$

$$y_{23}(y_{34}x_{12} - x_{34}y_{12}) \neq 0$$

and $X_4 = \begin{bmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & 0 & 0 & a_{24} & a_{25} \\ 0 & 0 & 0 & 0 & -ca_{13}\frac{y_{34}}{y_{23}} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. From $[X_2, X_4] = Z_1$, $[X_1, X_4] = \epsilon Z_2$, we obtain the following equations

(1) $a_{24}x_{12} - a_{13}x_{34} = -a_{13}x_{34} - a_{24}x_{12}$,
(2) $a_{24}y_{12} - a_{13}y_{34} = -a_{13}y_{34} - a_{24}y_{12}$,
(3) $a_{13}y_{34} = (x_{12}y_{23} - \frac{y_{12}y_{23}x_{34}}{y_{34}})x_{34}$,
(4) $a_{13}x_{34} = \epsilon(x_{12}y_{23} - \frac{y_{12}y_{23}x_{34}}{y_{34}})y_{34}$.

By (a), (b) and Eq. (3.9), we have $a_{24} = 0$. By (c), (d) and Eq. (3.9), we obtain $\epsilon = \frac{x_{34}^2}{y_{34}^2}$.

If $c = 0$, by Lemmas 3.2(2) and 3.3(1), we have

$$X_1 = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ 0 & 0 & x_{23} & x_{24} & x_{25} \\ 0 & 0 & 0 & x_{34} & x_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & y_{12} & y_{13} & y_{14} & y_{15} \\ 0 & 0 & y_{23} & y_{24} & y_{25} \\ 0 & 0 & 0 & y_{34} & y_{35} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{3.10}$$

$$(y_{12}x_{23} - x_{12}y_{23})(x_{34}y_{35} - x_{35}y_{34}) \neq 0$$

and $X_4 = \begin{bmatrix} 0 & 0 & a_{13} & a_{14} & a_{15} \\ 0 & 0 & 0 & a_{24} & a_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. From $[X_1, X_4] = \epsilon Z_2$, $[X_2, X_4] = Z_1$, we obtain the following equations

$$\begin{cases} x_{12}a_{24} - a_{13}x_{34} = \epsilon(y_{12}(x_{23}y_{34} - y_{23}x_{34}) - (x_{12}y_{23} - y_{12}x_{23})y_{34}), \\ x_{12}a_{25} - a_{13}x_{35} = \epsilon(y_{12}(x_{23}y_{35} - y_{23}x_{35}) - (x_{12}y_{23} - y_{12}x_{23})y_{35}), \\ y_{12}a_{24} - a_{13}y_{34} = x_{12}(x_{23}y_{34} - y_{23}x_{34}) - (x_{12}y_{23} - y_{12}x_{23})x_{34}, \\ y_{12}a_{25} - a_{13}y_{35} = x_{12}(x_{23}y_{35} - y_{23}x_{35}) - (x_{12}y_{23} - y_{12}x_{23})x_{35}. \end{cases}$$

By Eq. (3.10), we verify computationally that $\{X_1, X_2, X_3, X_4, Z_1, Z_2\} \subseteq \mathfrak{n}_5(\mathbb{K})$ is a linearly independent set if there exists $\alpha \in \mathbb{K}$ such that $\alpha^2 - \epsilon = 0$.

(2) Analysis similar to that in (1) with $c = 0$ shows that $\{X_1, X_2, X_3, X_4, Z_1, Z_2\} \subseteq \mathfrak{n}_5(\mathbb{K})$ is linearly independent if there exists $a \in \mathbb{K}$ such that $a^2 - \epsilon = 0$. It completes the proof. □

Proof of Theorem 3.1. Since $\mathfrak{g}$ is a Lie algebra isomorphic to $L_{6,9}$, we have

(1) $n \geq 5$, by Eq. (3.4) and
(2) there exists a basis $\{X_1, X_2, X_3, X_4, Z_1, Z_2\}$ of $\mathfrak{g}$ such that $[X_1, X_2] = X_3, [X_1, X_3] = Z_1, [X_2, X_3] = Z_2$ and $[X_i, X_4] = [Z_j, X_4] = 0$ for $i = 1, 2, 3$ and $j = 1, 2$.

By Proposition 3.2, we obtain $n \geq 6$. □

Proof of Theorem 3.2. Since $\mathfrak{g}$ is a Lie algebra isomorphic to $L_{6,24}(\epsilon)$, we have

(1) $n \geq 5$, by Eq. (3.4) and
(1) there exists a basis $\{X_1, X_2, X_3, X_4, Z_1, Z_2\}$ of $\mathfrak{g}$ such that $[X_1, X_2] = X_3, [X_1, X_3] = Z_1, [X_2, X_3] = Z_2$ and $[X_1, X_4] = \epsilon Z_2$, $[X_2, X_4] = Z_1$ and $[X_4, Z_j] = 0$ for $j = 1, 2$.

If $n = 5$, by Proposition 3.3, there exists $a \in \mathbb{K}$ such that $a^2 - \epsilon = 0$. It completes the proof. □

By Theorems 3.1 and 3.2, we easily obtain the following corollaries.

Corollary 3.1. $\mu_{\mathrm{nil}}(L_{6,9}) \geq 6$.

Corollary 3.2. *Let* $\epsilon \in \mathbb{K}$ *then*

$$\mu_{\mathrm{nil}}(L_{6,24}(\epsilon)) \geq \begin{cases} 5 & \textit{if } \exists\, \alpha \in \mathbb{K} : \epsilon = \alpha^2, \\ 6 & \textit{if } \epsilon \neq \alpha^2 \textit{ for all } \alpha \in \mathbb{K}. \end{cases}$$

Acknowledgments

This is part of the author's Ph.D. thesis, which was written under supervision of Professor Leandro Cagliero at the University of Córdoba, Argentina.

References

[1] J. C. Benjumea-Acevedo, J. Nuñez-Valdes and A. F. Tenorio-Villalón, Minimal linear representations of the low-dimensional nilpotent Lie algebras, *Math. Scand.* **102**(1) (2008) 17–26.
[2] Y. Benoist, Une nilvariete non affine, *J. Diff. Geom.* **41** (1995) 21–52.
[3] D. Burde, A refinement of Ado's theorem, *Arch. Math.* **70** (1998) 118–127.
[4] D. Burde, B. Eick and W. de Graaf, Computing faithful representations for nilpotent Lie algebras, *J. Algebra* **322**(3) (2009) 602–612.

[5] D. Burde and W. Moens, Minimal faithful representations of reductive Lie algebras, *Arch. Math.* **89**(6) (2007) 513–523.

[6] D. Burde and W. Moens, Faithful Lie algebra modules and quotients of the universal enveloping algebra, *J. Alg.* **325**(1) (2011) 440–460.

[7] L. Cagliero and N. Rojas, Faithful representation of minimal dimension of current Heisenberg Lie algebras, *Internat. J. Math.* **20**(11) (2009) 1347–1362.

[8] W. A. De Graaf, Constructing faithful matrix representations of Lie algebras, in *Proc. 1997 Int. Symp. Symbolic and Algebraic Computation*, ISSAC97 (ACM Press, New York, 1997), pp. 54–59.

[9] W. A. De Graaf, Classification of 6-dimensional nilpotent Lie algebras over fields of characteristic not 2, *J. Algebra* **309** (2007) 640–653.

[10] R. Ghanam, I. Strugar and G. Thompson, Matrix representations for low dimensional Lie algebras, *Ext. Math.* **20**(2) (2005) 151–184.

[11] R. Ghanam, G. Thompson and S. Tonon, Representations for six-dimensional nilpotent Lie algebras, *Hadronic J.* **29**(3) (2006) 299–317.

[12] M.-P. Gong, Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed fields and $\mathbb{R}$), Ph.D. thesis, University of Waterloo, Waterloo, Canada (1998).

[13] N. Jacobson, Schur's theorem on commutative matrices, *Bull. Amer. Math. Soc.* **50** (1944) 431–436.

[14] N. Jacobson, *Lie Algebras* (Interscience Publishers, New York, 1962).

[15] Y. Kang and C. Bai, Refinement of Ado's theorem in low dimensions and application in affine geometry, *Comm. Algebra* **36**(1) (2008) 82–93.

[16] J. Milnor, On fundamental groups of complete affinely flat manifolds, *Adv. Math.* **25** (1977) 178–187.

[17] M. Mirzakhani, A simple proof of a theorem of Schur, *Amer. Math. Monthly* **105**(3) (1992) 260–262.

[18] N. Rojas, Minimal faithful representation of the Heisenberg lie algebra with abelian factor, *J. Lie Theory* **23**(4) (2013) 1105–1114.

[19] I. Schur, Zur theorie vertauschbarer matrizen, *J. Reine Angew. Math.* **130** (1905) 66–76.