ON FINITE GK-DIMENSIONAL NICHOLS ALGEBRAS OVER ABELIAN GROUPS

NICOLÁS ANDRUSKIEWITSCH, IVÁN ANGIONO, ISTVÁN HECKENBERGER

Abstract. We describe

Contents

1. Yetter-Drinfeld modules of dimension 3

1.1. **The setting.** Let Γ be an abelian group. In this Section we consider $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, $\dim V = 3$, such that the corresponding braided vector space is not of diagonal type. So, V is not semisimple and we have two possibilities that we discuss in §1.1.1 and **??**.

1.1.1. *A block and a point.* $V = \mathcal{V}_{g_1}(\chi_1, \eta) \oplus \Bbbk^{\chi_2}_{g_2}$, where $g_1, g_2 \in \Gamma$, $\chi_1, \chi_2 \in \widehat{\Gamma}$ and $\eta : \Gamma \to \Bbbk$ is a (χ_1, χ_1)-derivation. Here $\mathcal{V}_{g_1}(\chi_1, \eta) \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ is indecomposable with basis $(x_i)_{i\in\mathbb{I}_2}$ and action given by (**??**); while $\Bbbk^{\chi_2}_{g_2} \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ is irreducible with base (x_3). Also $\eta(g_1) \neq 0$, otherwise V would be of diagonal type, and then we may suppose that $\eta(g_1) = 1$ by normalizing x_1. Let

$$q_{ij} = \chi_j(g_i), \qquad i, j \in \mathbb{I}_2; \qquad \epsilon = q_{11}; \qquad a = q_{21}^{-1}\eta(g_2).$$

Then the braiding is given in the basis $(x_i)_{i\in\mathbb{I}_3}$ by

$$(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_3} = \begin{pmatrix} \epsilon x_1 \otimes x_1 & (\epsilon x_2 + x_1) \otimes x_1 & q_{12} x_3 \otimes x_1 \\ \epsilon x_1 \otimes x_2 & (\epsilon x_2 + x_1) \otimes x_2 & q_{12} x_3 \otimes x_2 \\ q_{21} x_1 \otimes x_3 & q_{21}(x_2 + a x_1) \otimes x_3 & q_{22} x_3 \otimes x_3 \end{pmatrix}. \tag{1.1}$$

Let $V_1 = \mathcal{V}_{g_1}(\chi_1, \eta)$, $V_2 = \Bbbk^{\chi_2}_{g_2}$. If $\epsilon^2 = 1$, then

$$c^2_{|V_1\otimes V_2} = \mathrm{id} \iff q_{12}q_{21} = 1 \text{ and } a = 0. \tag{1.2}$$

2000 *Mathematics Subject Classification.* 16W30.

The work was partially supported by CONICET, FONCyT-ANPCyT, Secyt (UNC).

The scalar $q_{12}q_{21}$ will be called the *interaction* between the block and the point. The interaction is

weak if $q_{12}q_{21} = 1$, mild if $q_{12}q_{21} = -1$, strong if $q_{12}q_{21} \notin \{\pm 1\}$.

So $c^2_{|V_1 \otimes V_2}$ is determined by the interaction and the (somewhat hidden) parameter a. We introduce a normalized version of a, called the *ghost*:

$$\mathscr{G} = \begin{cases} -2a, & \epsilon = 1, \\ a, & \epsilon = -1. \end{cases} \tag{1.3}$$

If $\mathscr{G} \in \mathbb{N}$, then we say that the ghost is *discrete*.

Theorem 1.1. *Let V be a braided vector space with braiding* (1.1). *Assume that* $\operatorname{GKdim} \mathcal{B}(V) < \infty$. *Then V is as in Table 1.*

Table 1. Nichols algebras of a block and a point with finite GKdim

interaction	ϵ	q_{22}	$\mathscr{G}$	$\mathcal{B}(V)$, §	GKdim
weak	± 1	1 or $\notin \mathbb{G}_\infty$	0	$\mathcal{B}(\mathcal{V}(\epsilon, 1)) \underline{\otimes} \mathcal{B}(\Bbbk x_3)$	3
		$\in \mathbb{G}_\infty - \{1\}$			2
	1	1	discrete	$\mathcal{B}(\mathfrak{L}(1, \mathscr{G}))$, 1.2.1	$\mathscr{G} + 3$
		-1	discrete	$\mathcal{B}(\mathfrak{L}(-1, \mathscr{G}))$, 1.2.2	2
		$\in \mathbb{G}'_3$	1	$\mathcal{B}(\mathfrak{L}(\omega, \mathscr{G}))$, 1.2.5	2
	-1	1	discrete	$\mathcal{B}(\mathfrak{L}_-(1, \mathscr{G}))$, 1.2.3	$\mathscr{G} + 3$
		-1	discrete	$\mathcal{B}(\mathfrak{L}_-(-1, \mathscr{G}))$, 1.2.4	$\mathscr{G} + 2$
mild	-1	-1	1	$\mathcal{B}(\mathfrak{C}_1)$, 1.2.6	2

1.2. **The Nichols algebras with finite** GKdim**.** Here we describe a presentation by generators and relations and exhibit an explicit PBW basis of the Nichols algebras in Theorem 1.1. We denote the braided vector space with braiding (1.1) by

$\mathfrak{L}(q_{22}, \mathscr{G})$, if the interaction is weak, $\epsilon = 1$;
$\mathfrak{L}_-(q_{22}, \mathscr{G})$, if the interaction is weak, $\epsilon = -1$;
$\mathfrak{C}_1$, if the interaction is mild, $\epsilon = q_{22} = -1$, $\mathscr{G} = 1$.

Recall the relations of the Jordan and super Jordan planes:

$$x_2 x_1 - x_1 x_2 + \frac{1}{2} x_1^2, \tag{??}$$

$$x_1^2, \tag{??}$$

$$x_2 x_{12} - x_{12} x_2 - x_1 x_{12}. \tag{??}$$

Lemma 1.2. *Assume that* $\epsilon^2 = q_{22}^2 = 1$. *In* $\mathcal{B}(\mathfrak{L}(q_{22}, \mathscr{G}))$, *or correspondingly* $\mathcal{B}_-(\mathfrak{L}(q_{22}, \mathscr{G}))$

$$z_{|2a|+1} = 0, \tag{1.4}$$

$$z_t z_{t+1} = q_{21} q_{22} z_{t+1} z_t \qquad t \in \mathbb{N}_0, t < |2a|, \tag{1.5}$$

$$z_t^2 = 0 \qquad t \in \mathbb{N}_0,\ \epsilon^t q_{22} = -1. \tag{1.6}$$

$$\partial_3(z_t^{n+1}) = \mu_t q_{21}^{nt} q_{22}^n n\, y_t z_t^n, \qquad n, t \in \mathbb{N}_0, \epsilon^t q_{22} = 1. \tag{1.7}$$

Lemma 1.3. *Let* $\mathcal{B}$ *be a quotient algebra of* $T(V)$. *Assume that* $x_1 x_3 = q_{12} x_3 x_1$, *and either*

(a) (**??**), *or else*
(b) (**??**), $x_{12} x_3 = q_{12}^2 x_3 x_{12}$

hold in $\mathcal{B}$. *Then for all* $n \in \mathbb{N}_0$, $x_1 z_n = \epsilon^n q_{12} z_n x_1$ *and* $x_{12} z_n = q_{12}^2 z_n x_{12}$.

Lemma 1.4. *Let* $\mathcal{B}$ *be a quotient algebra of* $T(V)$.

(i) *Assume that* (1.5) *holds in* $\mathcal{B}$. *Then for* $0 \le t < k \le 2|a|$,

$$z_t z_k - \epsilon^{tk} q_{21}^{k-t} q_{22} z_k z_t = \sum_{j=0}^{\frac{t+k}{2}} \nu_{tk}(j)\, z_{t+k-j} z_j, \quad \text{for some } \nu_{tk}(j) \in \Bbbk. \tag{1.8}$$

(ii) *Assume that* $z_t^2 = 0$ *in* $\mathcal{B}$ *for* $t \in \mathbb{N}_0$ *such that* $\epsilon^t q_{22} = -1$. *Then* $z_t z_{t+1} = q_{21} q_{22} z_{t+1} z_t$ *in* $\mathcal{B}$.

1.2.1. *The Nichols algebra* $\mathcal{B}(\mathfrak{L}(1, \mathscr{G}))$. Recall that $z_n = (ad_c x_2)^n x_3$.

Proposition 1.5. *Let* $\mathscr{G} \in \mathbb{N}$. *The algebra* $\mathcal{B}(\mathfrak{L}(1, \mathscr{G}))$ *is presented by generators* x_1, x_2, x_3 *and relations* (**??**),

$$x_1 x_3 = q_{12}\, x_3 x_1, \tag{1.9}$$

$$z_{1+\mathscr{G}} = 0, \tag{1.10}$$

$$z_t z_{t+1} = q_{21} q_{22}\, z_{t+1} z_t, \qquad 0 \le t < \mathscr{G}. \tag{1.11}$$

The set

$$B = \{x_1^{m_1} x_2^{m_2} z_{\mathscr{G}}^{n_{\mathscr{G}}} \dots z_1^{n_1} z_0^{n_0} : m_i, n_j \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}(\mathfrak{L}(1, \mathscr{G}))$ *and* $\operatorname{GKdim} \mathcal{B}(\mathfrak{L}(1, \mathscr{G})) = 3 + \mathscr{G}$.

Proof. Relations (1.9), (1.10) are 0 in $\mathcal{B}(\mathfrak{L}(1, \mathscr{G}))$ being annihilated by ∂_i, $i = 1, 2, 3$, and (1.11) holds by Lemma 1.2. Hence the quotient $\widetilde{\mathcal{B}}$ of $T(V)$ by (**??**), (1.9), (1.10) and (1.11) projects onto $\mathcal{B}(\mathfrak{L}(1, \mathscr{G}))$. Then (1.8) holds in $\widetilde{\mathcal{B}}$.

We claim that the subspace I spanned by B is a right ideal of $\widetilde{\mathcal{B}}$. Indeed,

- $I x_1 \subseteq I$ follows by Lemma 1.3,
- $I x_2 \subseteq I$ since $z_t x_2 = \epsilon^t q_{21}(x_2 z_t - z_{t+1})$, so we use (1.10), (1.8),

and $Ix_3 \subseteq I$ by definition. Since $1 \in I$, $\widetilde{\mathcal{B}}$ is spanned by B.

To prove that $\widetilde{\mathcal{B}} \simeq \mathcal{B}(\mathfrak{L}(1,\mathscr{G}))$, it remains to show that B is linearly independent in $\mathcal{B}(\mathfrak{L}(1,\mathscr{G}))$. For, suppose that there is a non-trivial linear combination $\mathtt{S}$ of elements of B in $\mathcal{B}(\mathfrak{L}(1,\mathscr{G}))$, say of minimal degree. Now

$$\partial_1(x_1^{m_1}x_2^{m_2}z_{\mathscr{G}}^{n_{\mathscr{G}}}\dots z_1^{n_1}z_0^{n_0}) = m_1\, q_{12}^{\sum n_i}\, x_1^{m_1-1}x_2^{m_2}z_{\mathscr{G}}^{n_{\mathscr{G}}}\dots z_1^{n_1}z_0^{n_0},$$
$$\partial_2(x_1^{m_1}x_2^{m_2}z_{\mathscr{G}}^{n_{\mathscr{G}}}\dots z_1^{n_1}z_0^{n_0}) = m_2\, q_{12}^{\sum n_i}\, x_1^{m_1}x_2^{m_2-1}z_{\mathscr{G}}^{n_{\mathscr{G}}}\dots z_1^{n_1}z_0^{n_0},$$

since ∂_1, ∂_2 are skew derivations, so we apply Lemma **??** and $\partial_2(z_t) = 0$. Then such linear combination does not have terms with m_1 or m_2 greater than 0. Let k be maximal such that $z_k^{n_k}\dots z_1^{n_1}z_0^{n_0}$ has non-zero coefficient in $\mathtt{S}$ for some $k \geq 1$, and for such k fix the maximal n_k. By (1.7), $y_k z_k^{n_k-1}\dots z_1^{n_1}z_0^{n_0}$ has non-zero coefficient in $\partial_3(\mathtt{S})$, and $\partial_3(\mathtt{S})$ is also a non-trivial linear combination of elements of B, a contradiction. Then B is a basis of $\mathcal{B}(\mathfrak{L}(1,\mathscr{G}))$ and $\widetilde{\mathcal{B}} = \mathcal{B}(\mathfrak{L}(1,\mathscr{G}))$. The computation of GKdim follows from the Hilbert series at once. □

1.2.2. *The Nichols algebra* $\mathcal{B}(\mathfrak{L}(-1,\mathscr{G}))$.

Proposition 1.6. *Let* $\mathscr{G} \in \mathbb{N}$. *The algebra* $\mathcal{B}(\mathfrak{L}(-1,\mathscr{G}))$ *is presented by generators* x_1, x_2, x_3 *and relations* (**??**), (1.9), (1.10) *and*

$$z_t^2 = 0, \qquad 0 \leq t \leq \mathscr{G}. \tag{1.12}$$

The set

$$B = \{x_1^{m_1}x_2^{m_2}z_{\mathscr{G}}^{n_{\mathscr{G}}}\dots z_1^{n_1}z_0^{n_0} : n_i \in \{0,1\}, m_j \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}(\mathfrak{L}(-1,\mathscr{G}))$ *and* $\operatorname{GKdim}\mathcal{B}(\mathfrak{L}(-1,\mathscr{G})) = 2$.

1.2.3. *The Nichols algebra* $\mathcal{B}(\mathfrak{L}_-(1,\mathscr{G}))$.

Proposition 1.7. *Let* $\mathscr{G} \in \mathbb{N}$. *The algebra* $\mathcal{B}(\mathfrak{L}_-(1,\mathscr{G}))$ *is presented by generators* x_1, x_2, x_3 *and relations* (**??**), (**??**), (1.9) *and*

$$z_{1+2\mathscr{G}} = 0, \tag{1.13}$$
$$x_{12}z_0 = q_{12}^2\, z_0x_{12}, \tag{1.14}$$
$$z_{2k+1}^2 = 0, \qquad 0 \leq k < \mathscr{G}, \tag{1.15}$$
$$z_{2k}z_{2k+1} = q_{21}q_{22}\, z_{2k+1}z_{2k}, \qquad 0 \leq k < \mathscr{G}. \tag{1.16}$$

The set

$$B = \{x_1^{m_1}x_{12}^{m_2}x_2^{m_3}z_{2\mathscr{G}}^{n_{2\mathscr{G}}}\dots z_1^{n_1}z_0^{n_0} : m_1, n_{2k+1} \in \{0,1\}, m_2, m_3, n_{2k} \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}(\mathfrak{L}_-(1,\mathscr{G}))$ *and* $\operatorname{GKdim}\mathcal{B}(\mathfrak{L}_-(1,\mathscr{G})) = \mathscr{G} + 3$.

1.2.4. *The Nichols algebra $\mathcal{B}(\mathfrak{L}_{-}(-1,\mathscr{G}))$.*

Proposition 1.8. *Let $\mathscr{G} \in \mathbb{N}$. The algebra $\mathcal{B}(\mathfrak{L}_{-}(-1,\mathscr{G}))$ is presented by generators x_1, x_2, x_3 and relations* (**??**), (**??**), (1.9), (1.13), (1.14) *and*

$$z_{2k}^2 = 0, \qquad 0 \leq k \leq \mathscr{G}, \tag{1.17}$$

$$z_{2k-1}z_{2k} = q_{21}q_{22}z_{2k}z_{2k-1}, \qquad 0 < k \leq \mathscr{G}. \tag{1.18}$$

The set

$$B = \{x_1^{m_1}x_{12}^{m_2}x_2^{m_3}z_{2\mathscr{G}}^{n_{2\mathscr{G}}} \dots z_1^{n_1}z_0^{n_0} : m_1, n_{2k} \in \{0,1\}, m_2, m_3, n_{2k-1} \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}(\mathfrak{L}_{-}(-1,\mathscr{G}))$ and $\operatorname{GKdim} \mathcal{B}(\mathfrak{L}_{-}(-1,\mathscr{G})) = \mathscr{G} + 2$.

1.2.5. *The Nichols algebra $\mathcal{B}(\mathfrak{L}(\omega,1))$.*

Remark 1.9. As in the previous cases, (1.9) and

$$z_2 = 0 \tag{1.19}$$

hold in $\mathcal{B}(\mathfrak{L}(\omega,1))$. As $q_{22} = \omega \in \mathbb{G}_3'$ we also have

$$z_0^3 = 0. \tag{1.20}$$

Let $z_{1,0} := z_1z_0 - q_{12}q_{22}z_0z_1$.

Remark 1.10. The following equations hold in $\mathcal{B}(\mathfrak{L}(\omega,1))$ by Lemma **??**

$$g_1 \cdot z_{1,0} = q_{12}^2z_{1,0}, \qquad g_2 \cdot z_{1,0} = q_{21}q_{22}^2z_{1,0}, \tag{1.21}$$

$$\partial_1(z_{1,0}) = \partial_2(z_{1,0}) = 0, \qquad \partial_3(z_{1,0}) = (1 - q_{22}^2)z_{1,0}. \tag{1.22}$$

Lemma 1.11. *Let $\mathcal{B}$ be a quotient algebra of $T(V)$. Assume that* (1.9), (1.19) *and* (1.20) *hold in $\mathcal{B}$. Then the following relations also hold:*

$$z_1z_{1,0} = q_{12}\omega^2z_{1,0}z_1, \qquad z_{1,0}z_0 = q_{12}\omega^2z_0z_{1,0}, \tag{1.23}$$

$$x_2z_{1,0} = q_{12}^2z_{1,0}x_2 + q_{12}(1-\omega)z_1^2, \qquad x_1z_{1,0} = q_{12}^2z_{1,0}x_1. \tag{1.24}$$

Lemma 1.12. *In $\mathcal{B}(\mathfrak{L}(\omega,1))$,*

$$z_1^3 = z_{1,0}^3 = 0. \tag{1.25}$$

Proposition 1.13. *Let $\omega \in \mathbb{G}_3'$. The algebra $\mathcal{B}(\mathfrak{L}(\omega,1))$ is presented by generators x_1, x_2, x_3 and relations* (**??**), (1.9), (1.19), (1.20) *and* (1.25). *The set*

$$B = \{x_1^{m_1}x_2^{m_2}z_1^{n_1}z^{n_2}z_0^{n_3} : m_i \in \mathbb{N}_0, 0 \leq n_j \leq 2\}$$

is a basis of $\mathcal{B}(\mathfrak{L}(\omega,1))$ and $\operatorname{GKdim} \mathcal{B}(\mathfrak{L}(\omega,1)) = 2$.

1.2.6. *The Nichols algebra* $\mathcal{B}(\mathfrak{C}_1)$. Recall that $f_i = (\operatorname{ad}_c x_1) z_i$.

Remark 1.14. The following relations hold in $\mathcal{B}(\mathfrak{C}_1)$:

$$x_{12} z_0 = q_{12}^2 z_0 x_{12}, \tag{1.26}$$

$$x_2 z_1 + q_{12} z_1 x_2 = q_{12} f_0 x_2 + \frac{1}{2} f_1, \tag{1.27}$$

$$z_i^2 = 0, \qquad i = 0, 1, \tag{1.28}$$

$$f_1^2 = 0. \tag{1.29}$$

Indeed (1.26) follows from the proof of Lemma **??**, while (1.27), (1.28) and (1.29) follows from the proof of Lemma **??**.

Lemma 1.15. *Let $\mathcal{B}$ be a quotient algebra of $T(V)$. Assume that* (**??**)*,* (**??**)*,* (1.26)*,* (1.27)*,* (1.28) *and* (1.29) *hold in $\mathcal{B}$. Then the following relations also hold: $f_0^2 = 0$,*

$$x_1 f_0 = -q_{12} f_0 x_1, \qquad x_1 f_1 = q_{12} f_1 x_1, \tag{1.30}$$

$$x_2 f_0 + q_{12} f_0 x_2 = -f_1, \qquad x_2 f_1 = -q_{12} f_1 x_2, \tag{1.31}$$

$$z_1 z_0 = -q_{12} z_1 z_0, \qquad f_1 f_0 = q_{12} f_0 f_1, \tag{1.32}$$

$$f_1 z_0 + q_{12}^2 z_0 f_1 = -2 q_{12} f_0 z_1, \qquad f_0 z_0 = -q_{12} z_0 f_0, \tag{1.33}$$

$$f_1 z_1 - q_{12}^2 z_1 f_1 = -2 q_{12} f_0 z_1, \qquad f_0 z_1 = -z_1 f_0, \tag{1.34}$$

$$x_{12} f_0 = q_{12}^2 f_0 x_{12}, \qquad x_{12} f_1 = q_{12}^2 f_1 x_{12}, \tag{1.35}$$

$$x_{12} z_1 - q_{12}^2 z_1 x_{12} = 2 x_2 f_1 - x_1 f_1 - 2 x_{12} f_0. \tag{1.36}$$

Proposition 1.16. *The algebra $\mathcal{B}(\mathfrak{C}_1))$ is presented by generators x_1, x_2, x_3 and relations* (**??**)*,* (**??**)*,* (1.26)*,* (1.27)*,* (1.28) *and* (1.29)*. The set*

$$B = \{x_1^{m_1} x_{12}^{m_2} x_2^{m_3} f_1^{n_1} f_0^{n_2} z_1^{n_3} z_0^{n_4} : m_1, n_i \in \{0,1\}, m_2, m_3 \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}(\mathfrak{C}_1)$ and $\operatorname{GKdim} \mathcal{B}(\mathfrak{C}_1) = 2$.

2. One block and several points

2.1. **The setting.** Let Γ be an abelian group. In this Section and the next we consider $V \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$, $\dim V > 3$, such that the corresponding braided vector space is a direct sum of blocks and points, but we assume that the underlying braided vector space is not of diagonal type. We seek to determine when $\operatorname{GKdim} \mathcal{B}(V) < \infty$. By Theorem **??**, we may assume that the blocks are of the form $\mathcal{V}(\epsilon, 2)$, with $\epsilon^2 = 1$. So, V is not semisimple and we need to consider various possibilities:

- The direct sum of one block and several points.
- The direct sum of two blocks and possibly several points.

In this Section we deal with one block and several points. For a more suggestive presentation, we introduce the notation

$$\mathbb{I}_{2,\theta} = \mathbb{I}_\theta - \{1\}, \qquad \mathbb{I}_\theta^\dagger = \mathbb{I}_\theta \cup \{\tfrac{3}{2}\}, \qquad \theta \in \mathbb{N}.$$

Let $g_1, \dots, g_\theta \in \Gamma$, $\chi_1, \dots, \chi_\theta \in \widehat{\Gamma}$ and $\eta : \Gamma \to \Bbbk$ a (χ_1, χ_1)-derivation. Let $\mathcal{V}_{g_1}(\chi_1, \eta) \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ be the indecomposable with basis $(x_i)_{i \in \mathbb{I}_1^\dagger}$ and action given by (**??**)– but with $\frac{3}{2}$ instead of 2; while $\Bbbk_{g_j}^{\chi_j} \in {}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ is irreducible with basis (x_j), $j \in \mathbb{I}_{2,\theta}$. Let

$$V = \mathcal{V}_{g_1}(\chi_1, \eta) \oplus \Bbbk_{g_2}^{\chi_2} \oplus \dots \oplus \Bbbk_{g_\theta}^{\chi_\theta}.$$

Thus $(x_i)_{i \in \mathbb{I}_\theta^\dagger}$ is a basis of V. We suppose that V is not of diagonal type, hence $\eta(g_1) \neq 0$; we may assume that $\eta(g_1) = 1$ by normalizing x_1. Let

$$q_{ij} = \chi_j(g_i), \qquad i, j \in \mathbb{I}_\theta; \qquad a_j = q_{j1}^{-1}\eta(g_j), \qquad j \in \mathbb{I}_\theta.$$

Let $\lfloor i \rfloor$ be the largest integer $\leq i$. Then the braiding in the basis $(x_i)_{i \in \mathbb{I}_\theta^\dagger}$ is

$$c(x_i \otimes x_j) = \begin{cases} q_{\lfloor i \rfloor j} x_j \otimes x_i, & i \in \mathbb{I}_\theta^\dagger,\ j \in \mathbb{I}_\theta; \\ q_{\lfloor i \rfloor 1}(x_{\frac{3}{2}} + a_{\lfloor i \rfloor} x_1) \otimes x_i, & i \in \mathbb{I}_\theta^\dagger,\ j = \frac{3}{2}. \end{cases} \tag{2.1}$$

Let $\epsilon := q_{11}$. Notice that $\mathcal{B}(\mathcal{V}_{g_1}(\chi_1, \eta) \oplus \Bbbk_{g_j}^{\chi_j}) \hookrightarrow \mathcal{B}(V)$ for all $\in \mathbb{I}_{2,\theta}$, thus we may apply the results from §5, 6. By Theorem **??**, we may assume that $\epsilon^2 = 1$, thus $a_1 = \epsilon$.

The *interaction* and the *ghost* between the block and the points are the vectors

$$(q_{1h}q_{h1})_{h \in \mathbb{I}_{2,\theta}}, \qquad \mathscr{G} = (\mathscr{G}_j)_{j \in \mathbb{I}_{2,\theta}} = \begin{cases} -2(a_j)_{j \in \mathbb{I}_{2,\theta}}, & \epsilon = 1, \\ (a_j)_{j \in \mathbb{I}_{2,\theta}}, & \epsilon = -1. \end{cases} \tag{2.2}$$

The interaction is strong if there exists $h \in \mathbb{I}_{2,\theta}$ such that $q_{1h}q_{h1} \notin \{\pm 1\}$; when it is not strong, it is

$$\text{weak if } q_{1h}q_{h1} = 1, \qquad \forall h \in \mathbb{I}_{2,\theta}; \qquad \text{mild, otherwise.}$$

We say that the ghost is *discrete* if $\mathscr{G} \in \mathbb{N}_0^{\mathbb{I}_{2,\theta}} - \{0\}$.

We can present our main object of interest in the language of braided vector spaces. Given $(q_{ij})_{i,j \in \mathbb{I}_\theta}$, with $q_{11}^2 = 1$, and $\mathscr{G} \in \Bbbk^{\mathbb{I}_{2,\theta}}$, we set $a_1 = \epsilon = q_{11}$ and consider the braided vector space (V, c) of dimension $\theta + 1$, with a basis $(x_i)_{i \in \mathbb{I}_\theta^\dagger}$ and braiding given by (2.1). This braided vector space (V, c) can be realized as a Yetter-Drinfeld module $\mathcal{V}_{g_1}(\chi_1, \eta) \oplus \bigoplus_{j \in \mathbb{I}_\theta} \Bbbk_{g_j}^{\chi_j}$ over some abelian group Γ as described above; for instance $\Gamma = \mathbb{Z}^\theta$ would do. Such a realization will be called *principal*.

The braided subspace V_1 spanned by $x_1, x_{\frac{3}{2}}$ is $\simeq \mathcal{V}(\epsilon, 2)$, while V_2 spanned by $(x_i)_{i \in \mathbb{I}_{2,\theta}}$ is of diagonal type. Obviously,

$$V = V_1 \oplus V_2. \tag{2.3}$$

Let $\mathcal{X}$ be the set of connected components of the generalized Dynkin diagram of the matrix $\mathbf{q} = (q_{ij})_{i,j \in \mathbb{I}_{2,\theta}}$. If $J \in \mathcal{X}$, then we set $J' = \mathbb{I}_{2,\theta} - J$,

$$V_J = \sum_{j \in J} \Bbbk_{g_j}^{\chi_j}, \qquad \mathscr{G}_J = (\mathscr{G}_j)_{j \in J}, \qquad \text{interaction of } J = (q_{1h}q_{h1})_{h \in J}.$$

As before, J could have weak, mild or strong interaction.

TABLE 2. A block and several points, finite GKdim, weak interaction, $\epsilon = 1$; here $\omega \in \mathbb{G}_3'$ and $\mathfrak{d}_J = \operatorname{GKdim} \mathcal{B}(K_J)$

V_J	type	$\mathscr{G}_J$	K_J	$\mathfrak{d}_J$
$\overset{1}{\circ}$	A_1	discrete	$(A_1)^{\mathscr{G}_J+1}$	$\mathscr{G}_J + 1$
$\overset{-1}{\circ}$	A_1	discrete	$(A_1)^{\mathscr{G}_J+1}$	0
$\overset{\omega}{\circ}$	A_1	1	A_2	0
$\overset{-1}{\circ}\overset{-1}{\text{------}}\overset{-1}{\circ}\ \dots\ \overset{-1}{\circ}\overset{-1}{\text{------}}\overset{-1}{\circ}$	$A_{\theta-1}$	$(1,0,\dots,0)$	A_3, $\theta = 3$ / D_θ, $\theta > 3$	0
$\overset{-1}{\circ}\overset{-1}{\text{------}}\overset{-1}{\circ}$	A_2	$(2,0)$	D_4	0
$\overset{-1}{\circ}\overset{\omega}{\text{------}}\overset{-1}{\circ}$	super A	$(1,0)$	$\mathfrak{g}(2,3)$	0
$\overset{-1}{\circ}\overset{\omega^2}{\text{------}}\overset{\omega}{\circ}$	super A	$(1,0)$	super A	0
		$(0,1)$	$\mathfrak{g}(2,3)$	0
$\overset{-1}{\circ}\overset{\omega}{\text{------}}\overset{\omega^2}{\circ}\overset{\omega}{\text{------}}\overset{\omega^2}{\circ}$	super A	$(1,0,0)$	$\mathfrak{g}(3,3)$	0
$\overset{-1}{\circ}\overset{\omega}{\text{------}}\overset{\omega^2}{\circ}\overset{\omega^2}{\text{------}}\overset{\omega}{\circ}$	super $\mathfrak{osp}$	$(1,0,0)$	$\mathfrak{g}(3,3)$	0
$\overset{-1}{\circ}\overset{r^{-1}}{\text{------}}\overset{r}{\circ}$, $r \notin \mathbb{G}_\infty$	super A	$(1,0)$	$D(2,1;\alpha)$	2
$\overset{-1}{\circ}\overset{r^{-1}}{\text{------}}\overset{r}{\circ}$, $r \in \mathbb{G}_N'$, $N > 3$	super A	$(1,0)$	$D(2,1;\alpha)$	0

Theorem 2.1. *Let V be a braided vector space with braiding* (2.1). *Assume that $\epsilon = 1$; then the interaction is weak. Then the following are equivalent:*

(i) $\operatorname{GKdim} \mathcal{B}(V) < \infty$.

(ii) *For $J \in \mathcal{X}$, either $\mathscr{G}_J = 0$, or else V_J is as in Table 4.*

Furthermore, if (ii) *holds, then*

$$\operatorname{GKdim} \mathcal{B}(V) = 2 + \sum_{J \in \mathcal{X}} \operatorname{GKdim} \mathcal{B}(K_J). \tag{2.4}$$

Theorem 2.2. *Let V be a braided vector space with braiding* (2.1). *Assume that $\epsilon = -1$. Then the following are equivalent:*

(i) $\operatorname{GKdim} \mathcal{B}(V) < \infty$.

(ii) *For $J \in \mathcal{X}$, either of the following holds:*

(a) *The interaction of J is weak and $\mathscr{G}_J = 0$.*

(b) *The interaction of J is weak, $J = \{i\}$, $\mathscr{G}_i$ discrete and $q_{ii} = \pm 1$.*

(c) *The interaction of J is $(-1, 1, \dots, 1)$, $\mathscr{G} = (1, 0, \dots, 0)$ and the Dynkin diagram of V_J is* $\overset{-1}{\circ}\overset{-1}{\text{------}}\overset{-1}{\circ}\ \dots\ \overset{-1}{\circ}\overset{-1}{\text{------}}\overset{-1}{\circ}$.

Furthermore, if (ii) *holds, then*

$$\operatorname{GKdim}\mathcal{B}(V) = 2 + \sum_{J\in\mathcal{X}} \operatorname{GKdim}\mathcal{B}(K_J). \tag{2.5}$$

The meaning of K_J in Table 4 is explained in §**??** below.

2.2. **The Nichols algebras with finite** GKdim**.** Let $V = V_1 \oplus V_2$ as in (2.3) and assume that the Dynkin diagram of V_2 is connected, i.e. $\mathcal{X} = \{J\}$, where $J = \mathbb{I}_{2,\theta}$. Assume that $|J| > 1$. We provide a presentation by generators and relations and exhibit an explicit PBW basis of $\mathcal{B}(V)$, cf. Theorem **??**.

The subspace $V_1 \oplus \Bbbk x_2$ is a braided vector subspace of type either of type $\mathcal{B}(\mathfrak{L}(-1,2))$ when V is of type $\mathfrak{L}(A_2,2)$, or else $\mathcal{B}(\mathfrak{L}(\omega,1))$ when V is of type $\mathfrak{L}(A(1|0)_3;\omega)$, or $\mathcal{B}(\mathfrak{L}(-1,1))$ for all the other cases. Thus the subalgebra generated by $V_1 \oplus \Bbbk x_2$ is a Nichols algebra of the corresponding type.

Recall the defining relations of $\mathcal{B}(\mathfrak{L}(-1,1))$:

$$x_{\frac{3}{2}}x_1 - x_1x_{\frac{3}{2}} + \frac{1}{2}x_1^2, \tag{??}$$

$$x_1x_2 - q_{12}\,x_2x_1, \tag{1.9}$$

$$(\operatorname{ad}_c x_{\frac{3}{2}})^2 x_2, \tag{1.10}$$

$$x_2^2,\ x^2_{\frac{3}{2}2}. \tag{1.12}$$

Remark 2.3. Let $j \in \mathbb{I}_{3,\theta}$. As $q_{1j}q_{j1} = 1$ and $\mathscr{G}_j = 0$,

$$x_1x_j = q_{1j}x_jx_1, \qquad\qquad x_{\frac{3}{2}}x_j = q_{1j}x_jx_{\frac{3}{2}}. \tag{2.6}$$

TABLE 3. $V = V(1,2) \oplus V_2$, weak int., $\mathscr{G} = (1,0,\dots,0)$

V_2	V
$\overset{-1}{\circ}\overset{-1}{\rule[3pt]{30pt}{0.4pt}}\overset{-1}{\circ}\ \dots\ \overset{-1}{\circ}\overset{-1}{\rule[3pt]{30pt}{0.4pt}}\overset{-1}{\circ}$	$\mathfrak{L}(A_{\theta-1})$, $\theta > 2$
$\overset{-1}{\circ}\overset{-1}{\rule[3pt]{30pt}{0.4pt}}\overset{-1}{\circ}$, $\mathscr{G} = (2,0)$	$\mathfrak{L}(A_2,2)$
$\overset{-1}{\circ}\overset{r^{-1}}{\rule[3pt]{30pt}{0.4pt}}\overset{r}{\circ}$; $r \in \Bbbk^\times$	$\mathfrak{L}(A(1\|0)_1;r)$
$\overset{-1}{\circ}\overset{\omega}{\rule[3pt]{30pt}{0.4pt}}\overset{-1}{\circ}$; $\omega \in \mathbb{G}_3'$	$\mathfrak{L}(A(1\|0)_2;\omega)$
$\overset{\omega}{\circ}\overset{\omega^2}{\rule[3pt]{30pt}{0.4pt}}\overset{-1}{\circ}$; $\omega \in \mathbb{G}_3'$	$\mathfrak{L}(A(1\|0)_3;\omega)$
$\overset{-1}{\circ}\overset{\omega}{\rule[3pt]{30pt}{0.4pt}}\overset{\omega^2}{\circ}\overset{\omega}{\rule[3pt]{30pt}{0.4pt}}\overset{\omega^2}{\circ}$; $\omega \in \mathbb{G}_3'$	$\mathfrak{L}(A(2\|0)_1;\omega)$
$\overset{-1}{\circ}\overset{\omega}{\rule[3pt]{30pt}{0.4pt}}\overset{\omega^2}{\circ}\overset{\omega^2}{\rule[3pt]{30pt}{0.4pt}}\overset{\omega}{\circ}$; $\omega \in \mathbb{G}_3'$	$\mathfrak{L}(D(2\|1);\omega)$

2.2.1. *The Nichols algebra* $\mathcal{B}(\mathfrak{L}(A_{\theta-1}))$.

Proposition 2.4. *The algebra* $\mathcal{B}(\mathfrak{L}(A_{\theta-1}))$ *is presented by generators* $x_1, \dots$ *and relations* (**??**),

$$x_1x_3 = q_{12}\, x_3x_1, \tag{2.7}$$

$$z_{1+\mathscr{G}} = 0, \tag{2.8}$$

$$z_tz_{t+1} = q_{21}q_{22}\, z_{t+1}z_t, \qquad 0 \leq t < \mathscr{G}. \tag{2.9}$$

The set

$$B = \{x_1^{m_1}x_2^{m_2}z_{\mathscr{G}}^{n_{\mathscr{G}}}\dots z_1^{n_1}z_0^{n_0} : m_i, n_j \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}(\mathfrak{L}(A_{\theta-1}))$ *and* $\operatorname{GKdim}\mathcal{B}(\mathfrak{L}(A_{\theta-1})) = 2$.

2.2.2. *The Nichols algebra* $\mathcal{B}(\mathfrak{L}(A_2, 2))$.

Proposition 2.5. *The algebra* $\mathcal{B}(\mathfrak{L}(A_2, 2))$ *is presented by generators* $x_1, \dots$ *and relations* (**??**),

$$x_1x_3 = q_{12}\, x_3x_1, \tag{2.10}$$

$$z_{1+\mathscr{G}} = 0, \tag{2.11}$$

$$z_tz_{t+1} = q_{21}q_{22}\, z_{t+1}z_t, \qquad 0 \leq t < \mathscr{G}. \tag{2.12}$$

The set

$$B = \{x_1^{m_1}x_2^{m_2}z_{\mathscr{G}}^{n_{\mathscr{G}}}\dots z_1^{n_1}z_0^{n_0} : m_i, n_j \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}(\mathfrak{L}(A_2, 2))$ *and* $\operatorname{GKdim}\mathcal{B}(\mathfrak{L}(A_2, 2)) = 2$.

2.2.3. *The Nichols algebra* $\mathcal{B}(\mathfrak{L}(A(1|0)_1; r))$. Let r be a root of unity of order $N \geq 3$. The subalgebra generated by x_2, x_3 is a Nichols algebra of type $A(1|0)_1$. Thus,

$$(\operatorname{ad}_c x_3)^2 x_2 = 0, \qquad x_3^N = 0. \tag{2.13}$$

Let W be the braided vector space with basis y_1, y_2, y_3 and Dynkin diagram $\overset{-1}{\circ}\overset{r^{-1}}{\text{------}}\overset{r}{\circ}\overset{r^{-1}}{\text{------}}\overset{-1}{\circ}$. By [An3], $\mathcal{B}(W)$ is presented by generators y_1, y_2, y_3 and relations

$$(\operatorname{ad}_c y_2)^2 y_1, \quad (\operatorname{ad}_c y_2)^2 y_3, \quad (\operatorname{ad}_c y_1) y_3, \quad y_1^2, \quad y_3^2, \quad y_2^N, \quad y_{123}^N. \tag{2.14}$$

The set

$$B_W = \{y_1^{n_1}y_{12}^{n_{12}}y_{123}^{n_{123}}y_2^{n_2}y_{23}^{n_{23}}y_3^{n_3} : n_1, n_{12}, n_{23}, n_3 \in \{0,1\}, 0 \leq n_2, n_{123} < N\}$$

is a basis of $\mathcal{B}(W)$.

Remark 2.6. By Lemma **??** K^1 is isomorphic to W as braided vector spaces. Moreover there exists an algebra isomorphism $\psi : \mathcal{B}(W) \to K$ such that $\psi(y_1) = x_{3/2,2}$, $\psi(y_2) = x_3$, $\psi(y_3) = x_2$. Let

$$\mathtt{z}_1 = [x_{\frac{3}{2}2}, x_3]_c = \psi(y_{12}), \qquad \mathtt{z}_2 = [x_{\frac{3}{2}2}, x_{23}]_c = \psi(y_{123}).$$

Thus, in $\mathcal{B}(\mathfrak{L}(q_{22},\mathscr{G}))$,

$$\mathsf{z}_2^N = 0, \tag{2.15}$$

and the set

$$B_K = \{x_{\frac{3}{2}2}^{n_1}\mathsf{z}_1^{n_2}\mathsf{z}_2^{n_3}x_3^{n_4}x_{23}^{n_5}x_2^{n_6} : n_1,n_2,n_5,n_6 \in \{0,1\}, 0 \le n_3,n_4 < N\}$$

is a basis of K.

Lemma 2.7. *Let $\mathcal{B}$ be a quotient algebra of $T(V)$. Assume that* (1.10), (1.12), (2.6), (2.13), (2.15) *hold in $\mathcal{B}$. Then there exists an algebra map $\phi : \mathcal{B}(W) \to \mathcal{B}$ such that $\phi(y_1) = x_{\frac{3}{2},2}$, $\phi(y_2) = x_3$, $\phi(y_3) = x_2$.*

Proof. Let $\overline{\phi} : T(W) \to \mathcal{B}$ be the algebra map defined as ϕ on the y_i's. We claim that $\overline{\phi}$ annihilates all the relations in (2.14), and the Lemma follows. The second and the sixth relations are annihilated by (2.13) while the last is (2.15). The fourth and the fifth relations are annihilated because of (1.12), and for the third relation we apply Lemma 1.4 (ii). Finally,

$$\overline{\phi}\left((\operatorname{ad}_c y_2)^2 y_1\right) = (\operatorname{ad}_c x_3)^2 x_{\frac{3}{2},2} = q_{31}^2(\operatorname{ad}_c x_{\frac{3}{2}})(\operatorname{ad}_c x_3)^2 x_2 = 0,$$

where we use (2.6) and (2.13). □

Proposition 2.8. *The algebra $\mathcal{B}(\mathfrak{L}(A(1|0)_1;r))$ is presented by generators x_i, $i \in \mathbb{I}_\theta^\dagger$, and relations* (**??**), (1.9), (1.10), (1.12), (2.6), (2.13), (2.15).

The set

$$B = \{x_1^{m_1}x_{\frac{3}{2}}^{m_2}x_{\frac{3}{2}2}^{n_1}\mathsf{z}_1^{n_2}\mathsf{z}_2^{n_3}x_3^{n_4}x_{23}^{n_5}x_2^{n_6} : n_1,n_2,n_5,n_6 \in \{0,1\},$$
$$0 \le n_3,n_4 < N, m_i \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}(\mathfrak{L}(A(1|0)_1;r))$ and $\operatorname{GKdim}\mathcal{B}(\mathfrak{L}(A(1|0)_1;r)) = 2$.

Proof. The set B is a basis because of the isomorphism $\mathcal{B}(\mathfrak{L}(A(1|0)_1;r)) \simeq K\#\mathcal{B}(V_1)$ as in §**??**, and Remark 2.6. The computation of GKdim follows from the Hilbert series at once.

Relations (**??**), (1.9), (1.10), (1.12), (2.6), (2.13), (2.15) hold as we have discussed at the beginning of the subsection. Hence the quotient $\widetilde{\mathcal{B}}$ of $T(V)$ by these relations projects onto $\mathcal{B}(\mathfrak{L}(A(1|0)_1;r))$.

We claim that the subspace I spanned by B is a left ideal of $\widetilde{\mathcal{B}}$. Indeed, $x_1 I \subseteq I$ by definition, and $x_{\frac{3}{2}} I \subseteq I$ by (**??**). By Lemma 2.7,

$$x_3\phi(B_W) = \phi(y_2 B_W) \subset \phi(B_W), \qquad x_{\frac{3}{2}2}\phi(B_W) = \phi(y_1 B_W) \subset \phi(B_W)$$
$$x_2\phi(B_W) = \phi(y_3 B_W) \subset \phi(B_W).$$

As $I = \sum_{m_1,m_2} \Bbbk\, x_1^{m_1} x_{\frac{3}{2}}^{m_2} \phi(B_W)$, we have that

$$\begin{aligned}
x_3 I &= \sum_{m_1,m_2} \Bbbk\, x_3 x_1^{m_1} x_{\frac{3}{2}}^{m_2} \phi(B_W) = \sum_{m_1,m_2} \Bbbk\, x_1^{m_1} x_{\frac{3}{2}}^{m_2} x_3 \phi(B_W) \subset I,\\
x_2 I &= \sum_{m_1,m_2} \Bbbk\, x_2 x_1^{m_1} x_{\frac{3}{2}}^{m_2} \phi(B_W)\\
&= \sum_{m_1,m_2} \Bbbk\, x_1^{m_1} x_{\frac{3}{2}}^{m_2} x_2 \phi(B_W) + \Bbbk\, x_1^{m_1} x_{\frac{3}{2}}^{m_2-1} x_{\frac{3}{2}2} \phi(B_W) \subset I,
\end{aligned}$$

by (2.6), (1.12). Since $1 \in I$, $\widetilde{\mathcal{B}}$ is spanned by B. Thus $\widetilde{\mathcal{B}} = \mathcal{B}(\mathfrak{L}(1,\mathscr{G}))$ since B is a basis of $\mathcal{B}(\mathfrak{L}(1,\mathscr{G}))$. □

Dudas: donde fijar la notacion $y_{i_1\dots y_k} = (\operatorname{ad}_c y_{i_1}) y_{i_2\dots y_k}$? como trabajar con otros sistemas de raices mas complejos?

2.2.4. *The Nichols algebra $\mathcal{B}(\mathfrak{L}(A(1|0)_2;\omega))$.*

Proposition 2.9. *The algebra $\mathcal{B}(\mathfrak{L}(A(1|0)_2;\omega))$ is presented by generators $x_1, \dots$ and relations (**??**),*

$$x_1 x_3 = q_{12}\, x_3 x_1, \tag{2.16}$$
$$z_{1+\mathscr{G}} = 0, \tag{2.17}$$
$$z_t z_{t+1} = q_{21} q_{22}\, z_{t+1} z_t, \qquad 0 \le t < \mathscr{G}. \tag{2.18}$$

The set

$$B = \{x_1^{m_1} x_2^{m_2} z_{\mathscr{G}}^{n_{\mathscr{G}}} \dots z_1^{n_1} z_0^{n_0} : m_i, n_j \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}(\mathfrak{L}(A(1|0)_2;\omega))$ and $\operatorname{GKdim} \mathcal{B}(\mathfrak{L}(A(1|0)_2;\omega)) = 2$.

2.2.5. *The Nichols algebra $\mathcal{B}(\mathfrak{L}(A(1|0)_2;\omega))$.*

Proposition 2.10. *The algebra $\mathcal{B}(\mathfrak{L}(A(1|0)_2;\omega))$ is presented by generators $x_1, \dots$ and relations (**??**),*

$$x_1 x_3 = q_{12}\, x_3 x_1, \tag{2.19}$$
$$z_{1+\mathscr{G}} = 0, \tag{2.20}$$
$$z_t z_{t+1} = q_{21} q_{22}\, z_{t+1} z_t, \qquad 0 \le t < \mathscr{G}. \tag{2.21}$$

The set

$$B = \{x_1^{m_1} x_3^{m_3} z_{\mathscr{G}}^{n_{\mathscr{G}}} \dots z_1^{n_1} z_0^{n_0} : m_i, n_j \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}(\mathfrak{L}(A(1|0)_3;\omega))$ and $\operatorname{GKdim} \mathcal{B}(\mathfrak{L}(A(1|0)_3;\omega)) = 2$.

2.2.6. *The Nichols algebra* $\mathcal{B}(\mathfrak{L}(A(2|0)_1;\omega))$.

Proposition 2.11. *The algebra* $\mathcal{B}(\mathfrak{L}(A(2|0)_1;\omega))$ *is presented by generators* $x_1,\dots$ *and relations* (**??**),

$$x_1x_3 = q_{12}\,x_3x_1, \tag{2.22}$$
$$z_{1+\mathscr{G}} = 0, \tag{2.23}$$
$$z_tz_{t+1} = q_{21}q_{22}\,z_{t+1}z_t, \qquad 0 \le t < \mathscr{G}. \tag{2.24}$$

The set

$$B = \{x_1^{m_1}x_2^{m_2}z_{\mathscr{G}}^{n_{\mathscr{G}}}\dots z_1^{n_1}z_0^{n_0} : m_i, n_j \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}(\mathfrak{L}(A(2|0)_1;\omega))$ *and* $\operatorname{GKdim}\mathcal{B}(\mathfrak{L}(A(2|0)_1;\omega)) = 2$.

2.2.7. *The Nichols algebra* $\mathcal{B}(\mathfrak{L}(D(2|1);\omega))$.

Proposition 2.12. *The algebra* $\mathcal{B}(\mathfrak{L}(D(2|1);\omega))$ *is presented by generators* $x_1,\dots$ *and relations* (**??**),

$$x_1x_3 = q_{12}\,x_3x_1, \tag{2.25}$$
$$z_{1+\mathscr{G}} = 0, \tag{2.26}$$
$$z_tz_{t+1} = q_{21}q_{22}\,z_{t+1}z_t, \qquad 0 \le t < \mathscr{G}. \tag{2.27}$$

The set

$$B = \{x_1^{m_1}x_2^{m_2}z_{\mathscr{G}}^{n_{\mathscr{G}}}\dots z_1^{n_1}z_0^{n_0} : m_i, n_j \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}(\mathfrak{L}(D(2|1);\omega))$ *and* $\operatorname{GKdim}\mathcal{B}(\mathfrak{L}(D(2|1);\omega)) = 2$.

2.2.8. *The Nichols algebra* $\mathcal{B}(\mathfrak{C}_n)$.

Lemma 2.13. *Let* $\mathscr{I}_J = (-1, 1, \dots, 1)$, $\mathscr{G}_J = (1, 0, \dots, 0)$, $n \ge 3$, *with Dynkin diagram* $\overset{-1}{\circ}\overset{-1}{\text{———}}\overset{-1}{\circ}\ \dots\ \overset{-1}{\circ}\overset{-1}{\text{———}}\overset{-1}{\circ}$. *Then* $\operatorname{GKdim}\mathcal{B}(V) = \infty$.

Proof. We may assume that $n = 2$. Fix $J = \{2,3\}$, $\mathcal{B} = \mathcal{B}(V_1 \oplus V_J)$, and set $u = [x_{\frac{3}{2}23}, x_2]_c$, $v = [x_1, u]_c \in \mathcal{B}$. As $\partial_2(x_{\frac{3}{2}23}) = 0$, we have

$$\begin{aligned}\partial_2(u) &= \partial_2(x_{\frac{3}{2}23}x_2 + q_{12}q_{32}x_2x_{\frac{3}{2}23}) = x_{\frac{3}{2}23} + q_{12}q_{32}g_2\cdot x_{\frac{3}{2}23}\\ &= x_{\frac{3}{2}23} + q_{12}q_{32}q_{21}q_{22}q_{23}(x_{\frac{3}{2}23} + x_{123}) = -x_{123}.\end{aligned}$$

As x_1, x_2, x_3 span a braided vector space of Cartan type A_3, we have that $[x_{123}, x_2]_c = x_1^2 = 0$. From the first relation,

$$g_1\cdot u = q_{12}^2q_{13}(-u + [x_{123}, x_2]_c) = -q_{12}^2q_{13}\,u,$$

and from the second, $x_1x_{123} = -q_{12}q_{13}\,x_{123}x_1$, so

$$\begin{aligned}\partial_2(v) &= \partial_2(x_1u + q_{12}^2q_{13}ux_1) = x_1\partial_2(u) + q_{12}^2q_{13}\partial_2(u)g_2\cdot x_1\\ &= -x_1x_{123} + q_{12}q_{13}\,x_{123}x_1 = 2q_{12}q_{13}\,x_{123}x_1.\end{aligned}$$

Thus $v \neq 0$. As $x_2^2 = (\mathrm{ad}_c\, x_{\frac{3}{2}})x_3 = 0$,

$$\begin{aligned}\Delta(x_{\frac{3}{2}23}) &= x_{\frac{3}{2}23} \otimes 1 + 2x_{\frac{3}{2}2} \otimes x_3 + (2x_{\frac{3}{2}} + x_1) \otimes x_{23} + 1 \otimes x_{\frac{3}{2}23},\\ \Delta(u) &= u \otimes 1 - 2x_{12} \otimes x_3x_2 + 1 \otimes u,\\ \Delta(v) &= v \otimes 1 + 4q_{12}x_{12}x_1 \otimes x_3x_2 + 2q_{12}x_{12} \otimes x_3x_{12} + 1 \otimes v,\end{aligned}$$

so v is a primitive element in $\mathcal{B}^{\mathrm{diag}}$. Let $\widetilde{\mathcal{B}}_1$ be the subalgebra of $\mathcal{B}^{\mathrm{diag}}$ generated by v and the x_i's. and $\widetilde{\mathcal{B}}_2$ the Nichols algebra whose degree one part is isomorphic (as a braided vector space) to $\Bbbk u \oplus V$. Arguing as in Theorem **??**, let $\tilde{\mathcal{B}}_2$ be the graded braided Hopf algebra associated to the natural Hopf algebra filtration of $\tilde{\mathcal{B}}_1$, where the generators v and x_i have degree one. Let $\tilde{\mathcal{B}}_3$ be the Nichols algebra quotient of $\tilde{\mathcal{B}}_2$. Then

$$\mathrm{GKdim}\, \tilde{\mathcal{B}}_3 \leq \mathrm{GKdim}\, \tilde{\mathcal{B}}_2 \leq \mathrm{GKdim}\, \tilde{\mathcal{B}}_1 \leq \mathrm{GKdim}\, \mathcal{B}^{\mathrm{diag}} = \mathrm{GKdim}\, \mathcal{B}(V).$$

The Dynkin diagram of the degree one part of $\tilde{\mathcal{B}}_3$ is of Cartan type $D_4^{(1)}$ (with vertices labeled by -1), so $\mathrm{GKdim}\, \tilde{\mathcal{B}}_3 = \infty$ by Theorem **??**, and then $\mathrm{GKdim}\, \mathcal{B}(V) = \infty$. □

Lemma 2.14. *Let* $\mathscr{I}_J = (-1, 1, \dots, 1)$, $\mathscr{G}_J = (1, 0, \dots, 0)$, $n \geq 3$, *with Dynkin diagram* $\overset{-1}{\circ} \overset{-1}{\text{——}} \overset{-1}{\circ} \dots \overset{-1}{\circ} \overset{-1}{\text{——}} \overset{-1}{\circ}$. *Then* $\mathrm{GKdim}\, \mathcal{B}(V) = \infty$.

Proof. We may assume that $n = 3$. Fix $J = \{2, 3, 4\}$, $\mathcal{B} = \mathcal{B}(V_1 \oplus V_J)$, and set $u = [x_{\frac{3}{2}23}, x_2]_c \in \mathcal{B}$. As $\partial_2(x_{\frac{3}{2}23}) = 0$, we have

$$\begin{aligned}\partial_2(u) &= \partial_2(x_{\frac{3}{2}23}x_2 + q_{12}q_{32}x_2x_{\frac{3}{2}23}) = x_{\frac{3}{2}23} + q_{12}q_{32}g_2 \cdot x_{\frac{3}{2}23}\\ &= x_{\frac{3}{2}23} + q_{12}q_{32}q_{21}q_{22}q_{23}(x_{\frac{3}{2}23} + x_{123}) = -x_{123}.\end{aligned}$$

Thus $u \neq 0$. As $x_2^2 = (\mathrm{ad}_c\, x_{\frac{3}{2}})x_3 = 0$,

$$\begin{aligned}\Delta(x_{\frac{3}{2}23}) &= x_{\frac{3}{2}23} \otimes 1 + 2x_{\frac{3}{2}2} \otimes x_3 + (2x_{\frac{3}{2}} + x_1) \otimes x_{23} + 1 \otimes x_{\frac{3}{2}23},\\ \Delta(u) &= u \otimes 1 - 2x_{12} \otimes x_3x_2 + 1 \otimes u,\end{aligned}$$

so u is a primitive element in $\mathcal{B}^{\mathrm{diag}}$. Let $\widetilde{\mathcal{B}}_1$ be the subalgebra of $\mathcal{B}^{\mathrm{diag}}$ generated by u, x_4, and $\widetilde{\mathcal{B}}_2$ the Nichols algebra whose degree one part is isomorphic (as a braided vector space) to $\Bbbk u \oplus \Bbbk x_4$. The Dynkin diagram of $\widetilde{\mathcal{B}}_2$ is $\overset{1}{\circ} \overset{-1}{\text{——}} \overset{-1}{\circ}$, thus $\mathrm{GKdim}\, \widetilde{\mathcal{B}}_2 = \infty$. Arguing as in Theorem **??**, $\mathrm{GKdim}\, \mathcal{B} = \infty$. □

As x_1, $x_{\frac{3}{2}}$, x_2 span a braided vector space of type $\mathfrak{C}_1$, the following relations hold in $\mathcal{B}(\mathfrak{C}_n)$:

$$x_{1\frac{3}{2}}x_2 - q_{12}^2 x_2 x_{1\frac{3}{2}} = 0, \tag{1.26}$$

$$x_{\frac{3}{2}}x_{\frac{3}{2}2} + q_{12}x_{\frac{3}{2}2}x_{\frac{3}{2}} = q_{12}x_{12}x_{\frac{3}{2}} + \frac{1}{2}x_{1\frac{3}{2}2}, \tag{1.27}$$

$$x_{\frac{3}{2}}x_{1\frac{3}{2}} - x_{1\frac{3}{2}}x_{\frac{3}{2}} = x_1 x_{1\frac{3}{2}}. \tag{??}$$

Lemma 2.15. *The following relations hold in* $\mathcal{B}(\mathfrak{C}_n))$*:*

$$(\operatorname{ad}_c x_i)x_j, \qquad i < j-1, \tag{2.28}$$

$$[x_{\alpha_{i-1\,i+1}}, x_i]_c, \qquad 2 \le i \le n-1, \tag{2.29}$$

$$[x_{\beta_3}, x_2]_c, \tag{2.30}$$

$$[x_{12\frac{3}{2}}, x_2]_c. \tag{2.31}$$

Proof. As x_i, $i \in \mathbb{I}_n$, span a braided vector space of A_n type, (2.28) for $i \neq \frac{3}{2}$ and (2.29) hold in $\mathcal{B}(\mathfrak{C}_n))$; (2.28), $i = \frac{3}{2}$, follows since $q_{1j}q_{j1} = 1$ $\mathscr{G}_j = 0$.

For (2.30), we claim that $\partial_j([x_{\beta_3}, x_2]_c) = 0$ for all $j \in \mathbb{I}_n^{\dagger}$. Indeed, it holds for $j \neq 2, 3$ since $\partial_j(x_{\beta_3}) = \partial_j(x_2) = 0$. As $\partial_3(x_{\beta_3}) = 2x_{\beta_2}$, $\partial_2(x_{\beta_3}) = 0$,

$$\partial_3([x_{\beta_3}, x_2]_c) = \partial_3(x_{\beta_3}x_2 + q_{12}q_{32}x_2x_{\beta_3}) = 2q_{32}(x_{\beta_2}x_2 + q_{12}x_2x_{\beta_2}) = 0,$$

$$\partial_2([x_{\beta_3}, x_2]_c) = \partial_2(x_{\beta_3}x_2 + q_{12}q_{32}x_2x_{\beta_3}) = (1 - q_{12}q_{32}q_{21}q_{23})x_{\beta_3} = 0.$$

(2.31) □

Lemma 2.16. *For each* $\alpha \in \Delta_+^{\mathbf{q}}$*, set* x_α *with the same recursive definition as* y_α*. Then*

$$y_\alpha^2 = 0, \qquad \alpha \in \Delta_+^{\mathbf{q}} - \{\alpha_{\frac{3}{2}}\}. \tag{2.32}$$

Proof.

$$x_2^2 = x_{\frac{3}{2}2}^2 = 0, \qquad i = 0, 1, \tag{1.28}$$

$$x_{12}^2 = x_{1\frac{3}{2}3}^2 = 0, \qquad i = 0, 1, \tag{1.29}$$

$$x_1^2 = 0. \tag{??}$$

□

Proposition 2.17. *The algebra* $\mathcal{B}(\mathfrak{C}_2))$ *is presented by generators* x_1, $x_{\frac{3}{2}}$, x_2, x_3 *and relations* (**??**), (**??**), (1.26), (1.27), (1.28) *and* (1.29)*. The set*

$$B = \{x_1^{m_1}x_{12}^{m_2}x_2^{m_3}f_1^{n_1}f_0^{n_2}z_1^{n_3}z_0^{n_4} : m_1, n_i \in \{0,1\}, m_2, m_3 \in \mathbb{N}_0\}$$

is a basis of $\mathcal{B}(\mathfrak{C}_2)$ *and* $\operatorname{GKdim}\mathcal{B}(\mathfrak{C}_2) = 2$.

Proof. □

TABLE 4. PBW generators of $\mathcal{B}(\mathfrak{C}_2)$

	x_1	$x_{1\frac{3}{2}}$	$x_{\frac{3}{2}}$	$x_{1\frac{3}{2}2}$	$x_{\frac{3}{2}2}$	x_{12}	x_2	$x_{1\frac{3}{2}232}$	$x_{1\frac{3}{2}23}$	x_{123}	$x_{\frac{3}{2}23}$	$x_{\frac{3}{2}232}$	x_{23}	x_3
x_1	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$	$\blacksquare$
$x_{1\frac{3}{2}}$	$\checkmark$	$\blacksquare$												
$x_{\frac{3}{2}}$	$\checkmark$	$\checkmark$												
$x_{1\frac{3}{2}2}$	$\checkmark$	$\checkmark$												
$x_{\frac{3}{2}2}$	$\checkmark$	$\checkmark$												
x_{12}	$\checkmark$	$\checkmark$												
x_2	$\checkmark$	$\checkmark$												
$x_{1\frac{3}{2}232}$	$\checkmark$													
$x_{1\frac{3}{2}23}$	$\checkmark$													
x_{123}	$\checkmark$													
$x_{\frac{3}{2}23}$	$\checkmark$													
$x_{\frac{3}{2}232}$	$\checkmark$													
x_{23}	$\checkmark$	$\checkmark$												
x_3	$\checkmark$	$\checkmark$												
height	2	∞	∞	2	2	2	2	???	???	2	???	∞	2	2

REFERENCES

[AA] N. Andruskiewitsch, I. Angiono. *On Nichols algebras with generic braiding*, in *Modules and Comodules*, Trends in Mathematics. Brzezinski, T.; Gómez Pardo, J.L.; Shestakov, I.; Smith, P.F. (Eds.), pp. 47–64 (2008).

[AAG] Andruskiewitsch, N., Angiono, I., García Iglesias, A. *Liftings of Nichols algebras of diagonal type I. Cartan type A*, in preparation.

[A+] Andruskiewitsch, N., Angiono, I., García Iglesias, A., Masuoka, A., Vay, C., *Lifting via cocycle deformation*, *J. Pure Appl. Alg.* **218** (2014), 684–703.

[AAH1] N. Andruskiewitsch, I. Angiono and I. Heckenberger. *On finite GK-dimensional Nichols algebras of diagonal type.* In preparation.

[AAR] Andruskiewitsch, N., Angiono, I., Rossi Bertone, F. *The divided powers algebra of a finite-dimensional Nichols algebra of diagonal type*, `arXiv:1501.04518`.

[ACE] N. Andruskiewitsch, J. Cuadra and P. Etingof, *On two finiteness conditions for Hopf algebras with nonzero integral.* Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5), to appear.

[AD] N. Andruskiewitsch and S. Dăscălescu, *Co-Frobenius Hopf algebras and the coradical filtration.* Math. Z. **243** (2003), 145-154.

[AHS] Andruskiewitsch, N., Heckenberger, I., Schneider, H.-J., The Nichols algebra of a semisimple Yetter-Drinfeld module, *Amer. J. Math.* **132** (2010), 1493–1547.

[AS1] N. Andruskiewitsch, H.-J. Schneider, *Pointed Hopf algebras*, New directions in Hopf algebras, MSRI series Cambridge Univ. Press; 1–68 (2002).

[AS2] ———. *A characterization of quantum groups.* J. Reine Angew. Math. **577** (2004) 81–104.

[AS3] ———, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. Math. **171**, 375–417, (2010).

[ASc] N. Andruskiewitsch, C. Schweigert, *The unrolled quantum double of a finite-dimensional Nichols algebra of diagonal type*, in preparation.

[An1] I. Angiono. *Basic quasi-Hopf algebras over cyclic groups*. Adv. Math. 225 (2010), 3545-3575.

[An2] ———. *A presentation by generators and relations of Nichols algebras of diagonal type and convex orders on root systems*, J. Europ. Math. Soc. 17 (2015), 2643–2671.

[An3] ———. *On Nichols algebras of diagonal type*, J. Reine Angew. Math. 683 (2013), 189–251.

[An4] ———. *Distinguished pre-Nichols algebras*. `arXiv:1405.6681`.

[C+] L. Carbone, S. Chung, C. Cobbs, R. McRae, D. Nandi, Y. Naqvi, D. Penta, *Classification of hyperbolic Dynkin diagrams, root lengths and Weyl group orbits*. J. Phys. A: Math. Theor. 43 (2010) (15): 155209.

[D+] E. E. Demidov, Yu. I. Manin, E. E. Mukhin, D. V. Zhdanovich, *Nonstandard quantum deformations of GL(n) and constant solutions of the Yang-Baxter equation*. Common trends in mathematics and quantum field theories (Kyoto, 1990). Progr. Theoret. Phys. Suppl. No. 102 (1990), 203–218 (1991).

[FG] D. Flores de Chela, J. A. Green, *Quantum Symmetric Algebras*. Algebras and Representation Theory 4 (2001), 55–76.

[G] M. Graña, *A freeness theorem for Nichols algebras*. J. Algebra **231** (2000), 235–257.

[Gr] M. Gromov, *Groups of polynomial growth and expanding maps*. Inst. Hautes Études Sci. Publ. Math. **1981** (2000), 53–73.

[GH] M. Graña, I. Heckenberger, *On a factorization of graded Hopf algebras using Lyndon words*. J. Algebra **314** (2007), 324–343.

[Gu] R. Günther, *Crossed products for pointed Hopf algebras*. Comm. Algebra, **27** (1999), 4389–4410.

[G] D. I. Gurevich, *The Yang-Baxter equation and the generalization of formal Lie theory*. Dokl. Akad. Nauk SSSR 288 (1986), no. 4, 797–801.

[H1] I. Heckenberger. *The Weyl groupoid of a Nichols algebra of diagonal type*, Inventiones Math. **164**, 175–188 (2006).

[H2] ———. *Classification of arithmetic root systems*, *Adv. Math.* **220** (2009), 59–124.

[Dr] Drinfeld, V. G. *Quantum groups*, Proc. Int. Congr. Math., Berkeley 1986, Vol. 1, 798–820 (1987).

[EG] Etingof, Pavel; Gelaki, Shlomo Quasisymmetric and unipotent tensor categories. Math. Res. Lett. 15, No. 5-6, 857-866 (2008).

[K] V. Kac, *Infinite-dimensional Lie algebras*. Third edition. Cambridge University Press, Cambridge, 1990. xxii+400 pp.

[KL] G. Krause, T. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*. Revised edition. Graduate Studies in Mathematics, **22**. American Mathematical Society, Providence, RI, 2000. x+212 pp

[LM] A. Yu. Lazarev, M. V. Movshev, *Quantization of some Lie groups and algebras*. Russian Math. Surveys 46 (1991), no. 6, 225–226.

[Mo] S. Montgomery, *Hopf algebras and their action on rings*, CBMS Regional Conference Series **82** (1993).

[O] C. Ohn, A $*$-product on $SL(2)$ and the corresponding nonstandard quantum $U(sl(2))$, Lett. Math. Phys. 25 (1992), 85–88.

[R] M. Rosso, *Quantum groups and quantum shuffles*. Invent. Math. **133** (1998), 399–416.

[S] H.J. Schneider, *Normal basis and transitivity of crossed products for Hopf algebras*, J. Algebra **152** (1992) 289–312.

[U] S. Ufer, *PBW bases for a class of braided Hopf algebras*. J. Algebra **280** (2004), 84–119.

[Y] H. Yamane, *epresentations of a Z/3Z-quantum group*. Publ. Res. Inst. Math. Sci. **43** (2007), 75–93.

[WZZ] W. Wu, S. Zhang, Y.-Z. Zhang, *Finite dimensional Nichols algebras over finite cyclic groups*, Journal of Lie Theory 24 (2014) 351-372.

[Z] S. Zakrzewski, A Hopf star-algebra of polynomials on the quantum SL(2, R) for a unitary R-matrix, Lett. Math. Phys. 22, 287–289 (1991).

FaMAF-CIEM (CONICET), Universidad Nacional de Córdoba, Medina Allende s/n, Ciudad Universitaria, 5000 Córdoba, República Argentina.

E-mail address: (andrus|angiono)@mate.uncor.edu