On Neighborhood-Helly Graphs

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Abstract

A family $\mathcal{F}$ of subsets of some set is *intersecting* when sets of $\mathcal{F}$ pairwise intersect. The family $\mathcal{F}$ is *Helly* when every intersecting subfamily of it contains a common element. In this paper we examine the families of vertex neighborhoods of a graph, with the aim of determining whether or not they are Helly, and also whether or nor they are *hereditary Helly*, that is, each of the induced subgraphs of the graph is Helly. We examine the cases where the neighborhoods are all open, or all closed, or mixed, that is, some open and some closed. For mixed neighborhoods there are two different kinds of choice of the neighborhood of each vertex to be considered: fixed or arbitrary choice. By fixed mixed neighborhood, we mean that the choice, open or closed, for the neighborhood of a vertex is known in advance, that is part of the input. On the other hand, an arbitrary choice implies that the choice can be made along the process. For the cases of open, closed and fixed mixed neighborhoods, we describe characterizations, both for the neighborhoods to be Helly and hereditary Helly. The characterizations are of two types: based on the concept of extensions, or, for the hereditary cases, by forbidden induced subgraphs. Polynomial time recognition algorithms follow directly from the characterizations. In contrast, for arbitrary mixed neighborhoods, we prove that it is NP-complete to decide whether the family of neighborhoods is Helly or hereditary Helly.

Keywords: Complexity, Extensions, Helly property, NP-hardness

1. Introduction

Denote by G a finite simple graph, with vertex set $V(G)$ and edge set $E(G)$. We use n and m to denote $|V(G)|$ and $|E(G)|$. A *complete set* is a subset $V' \subseteq V(G)$ formed by pairwise adjacent vertices. A *triangle* is a complete set of size 3 and a subset of vertices is a *co-triangle* when it is a triangle in $\overline{G}$, the complement of G. Denote by $N(v_i) = \{v_j \in V(G) | (v_i, v_j) \in E(G)\}$, and $N[v_i] = N(v_i) \cup \{v_i\}$, the *open* and *closed neighborhoods* of G, respectively. The degree of a vertex v_i, $d(v_i)$, is $|N(v_i)|$ and the maximum degree of G is denoted by Δ. For $V' \subseteq V(G)$, $G[V']$ is the subgraph of G induced by V'. Let $\mathcal{F}$ be a family of subsets

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of some set. Say that $\mathcal{F}$ is *intersecting* when the subsets of $\mathcal{F}$ pairwise intersect. On the other hand, when every intersecting subfamily of $\mathcal{F}$ has a common element then $\mathcal{F}$ is a *Helly* family.

The Helly property in the context of graphs and hypergraphs has been considered in many papers. Among them, we can mention [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35, 36].

A graph G is *open neighborhood-Helly (closed neighborhood-Helly)* when its family of open neighborhoods (closed neighborhoods) is Helly. Finally, G is *hereditary open neighborhood-Helly (hereditary closed neighborhood-Helly)* when every of its induced subgraphs is open neighborhood-Helly (closed neighborhood-Helly).

Different characterizations were given for these graph classes and most of them lead to polynomial-time recognition algorithms: closed neighborhood-Helly graphs [15, 25]; open neighborhood-Helly graphs [25]; hereditary closed neighborhood-Helly graphs [26]; hereditary open neighborhood-Helly graphs [26].

A natural extension of these classes is the case where the neighborhoods are not necessarily all open or all closed, that is they are of some mixed type. In this situation, there is a partition of the vertices, according how each neighborhood is to be considered, open or closed. Such a partition can be fixed or variable. Say that G is *fixed mixed neighborhood-Helly* when for a given partition of its vertices, into open and closed, the corresponding neighborhoods satisfy the Helly property. On the other hand, G is *arbitrary mixed neighborhood-Helly* when there exists some partition of the vertices which turns the neighborhoods to be Helly. Accordingly, define the concepts of *fixed hereditary mixed neighborhood-Helly* and *arbitrary hereditary mixed neighborhood-Helly*.

In this work, we describe characterizations, based on the concept of extensions, for the classes of neighborhood-Helly graphs: open, closed and mixed, both fixed and arbitrary. In addition, we also describe characterizations for their corresponding hereditary classes. As a consequence, we can obtain naturally polynomial time recognition algorithms for the classes into consideration, except for two cases. The exceptions are those of the arbitrary mixed classes. We prove that it is NP-complete to recognize arbitrary mixed neighborhood-Helly graphs and arbitrary hereditary mixed neighborhood-Helly graphs. The NP-completeness remains even for graphs of maximum degree 4.

We describe two different types of characterizations, those based on the concept of *extensions*, and, for hereditary classes, forbidden subgraph characterizations.

In Section 2, we present some terminology relevant to this work, open neighborhoods and closed neighborhoods are respectively considered in Sections 3 and 4, whereas fixed and arbitrary mixed neighborhoods are the subjects of Sections 5 and 6, respectively.

2. Preliminaries

First, we describe additional notation and definitions.

In the case of mixed neighborhoods, we may employ the notation $N\{v\}$ to mean that the neighborhood to be considered for vertex v, open or closed, is not determined.

Denote by $N[v,w]$ the intersection of $N[v]$ and $N[w]$, i.e. $N[v,w] = N[v] \cap N[w]$. On the other hand, we define the universal set of v, w as: $U[v,w]= \{u \in V(G)/N[v,w] \subseteq N[u]\}$. The universal set of u, v, w is defined as: $U[u,v,w]= U[u,v] \cap U[v,w] \cap U[u,w]$.

Define the *extension* of vertices u, v, w, as $E(u,v,w)= N[u,v] \cup N[v,w] \cup N[u,w]$, whenever $N[u,v]$, $N[v,w], N[u,w] \neq \emptyset$, and $E(u,v,w) = \emptyset$ otherwise.

A *chord* of cycle C is an edge of the graph between two vertices v_i, v_j not consecutive in C. If v_i, v_j are at distance two in C then the edge v_iv_j is called a *short chord*.

There is a polynomial-time general algorithm to check the Helly property for a family of subsets of polynomial size, based on the following characterization.

Theorem 2.1. *[5] Let $\mathcal{F}$ be a family of subsets of some set U. Given three different elements $u, v, w \in U$, let $\mathcal{F}_{\{u,v,w\}}$ be the subfamily of $\mathcal{F}$ formed by the sets containing at least two of these three elements. $\mathcal{F}$ is Helly if and only if for every triple $\{u, v, w\} \subseteq U$, $\bigcap S \neq \emptyset$, where $S \in \mathcal{F}_{\{u,v,w\}}$.*

3. Open Neighborhood

Open neighborhood-Helly graphs have been characterized in terms of extensions.

Theorem 3.1. *[25] A graph G is open neighborhood-Helly if and only if it does not contain triangles, and for every 3-independent set {u,v,w} there exists a vertex $z \in V(G)$ such that the extension $E(u,v,w) \subseteq N[z]$, i.e, $U(u,v,w) \neq \emptyset$ or $E(u,v,w) = \emptyset$.*

For hereditary open neighborhood-Helly graphs, we describe two characterizations. The first is by forbidden induced subgraphs.

Theorem 3.2. *[26] Let G be a graph. Then G is hereditary open neighborhood-Helly if and only if G does not contain C_6 nor triangles as induced subgraphs.*

Next, we formulate a characterization of hereditary open neighborhood-Helly graphs in terms of extensions.

Theorem 3.3. *A graph G is hereditary open neighborhood-Helly if and only if it does not contain triangles and for every triple of pairwise non-adjacent vertices u, v, w such that $N[u,v], N[v,w], N[u,w] \neq \emptyset$ then either $E(u,v,w) \subseteq N[u]$, $E(u,v,w) \subseteq N[v]$ or $E(u,v,w) \subseteq N[w]$, i.e $U(u,v,w) \cap \{u,v,w\} \neq \emptyset$ or $E(u,v,w) = \emptyset$.*

Proof Let G be a hereditary open neighborhood-Helly graph. By Theorem 3.2, G has no triangles. Then, we analyze the case u, v, w are non adjacent vertices. Suppose that the Theorem is not true for the triple of non adjacent vertices u, v, w, where $N[u,v], N[v,w], N[u,w] \neq \emptyset$. Then, there exist vertices x, y, z such that: $x \in N[u,v]$, $x \notin N[w]$, $y \in N[u,w]$, $y \notin N[v]$ and $z \in N[v,w]$, $z \notin N[u]$.

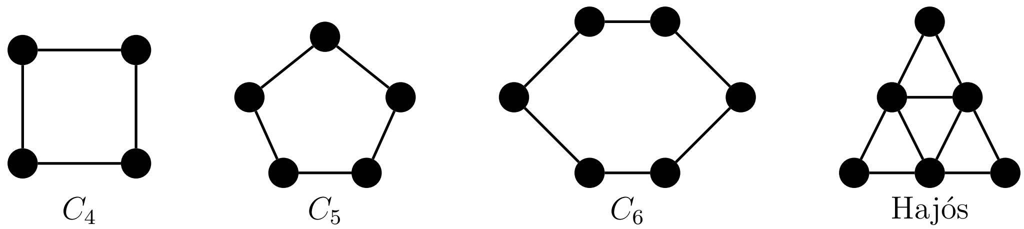

Figure 1: Forbidden induced subgraphs for hereditary closed neighborhood-Helly graphs

If it is this situation, any of xy, xz and yz is an edge of G, then the graph contains a triangle. Otherwise, u, v, w, x, y, z induce the graph C_6. Either cases contradict Theorem 3.2.

Conversely, we prove that a graph that satisfies the conditions of the Theorem, does not contain a C_6 as an induced subgraph. By contrary, suppose that G contains vertices u, x, v, y, z, w inducing a cycle. Consider the triple u, v, w of non-adjacent vertices. It follows that $N[u, w]$ is not included in $N[v]$, $N[v, w]$ is not included in $N[u]$, and $N[u, v]$ is not included in $N[w]$. That is, $E(u, v, w)$ is contained in neither $N[u]$, $N[v]$ nor $N[w]$, contradicting the hypothesis □

4. Closed Neighborhood

In this section, we describe characterizations of closed neighborhood-Helly graphs and hereditary closed neighborhood-Helly graphs.

The theorem below characterizes closed neighborhood-Helly graphs, in terms of extensions.

Theorem 4.1. *A graph G is closed neighborhood-Helly if and only if for every u, v, w there exists a vertex $z \in V(G)$ such that the extension $E(u, v, w) \subseteq N[z]$, i.e, $U(u, v, w) \neq \emptyset$ or $E(u, v, w) = \emptyset$.*

Proof It follows directly from Theorem 2.1.□

This theorem is a special case of Theorem 5.1.

In what follows, we describe characterizations of hereditary closed neighborhood-Helly graphs. The first one is by forbidden induced subgraphs.

Theorem 4.2. *[26] A graph G is hereditary closed neighborhood-Helly if and only if it does not contain C_4, C_5, C_6 nor the Hajós graph as induced subgraphs.*

Next, we present a characterization in terms of extensions.

Theorem 4.3. *A graph is hereditary closed neighborhood-Helly graph if and only if for every triple* u, v, w *such that* $N[u, v], N[v, w], N[u, w] \neq \emptyset$*, then either* $E(u, v, w) \subseteq N[u]$*,* $E(u, v, w) \subseteq N[w]$ *or* $E(u, v, w) \subseteq N[v]$*, i.e.,* $U(u, v, w) \neq \emptyset$ *or* $E(u, v, w) = \emptyset$.

Proof Let G be a hereditary closed neighborhood-Helly graph. Consider u, v, w such that $N[u, v]$, $N[v, w]$, $N[u, w] \neq \emptyset$. Examine the neighborhoods of u, v, w.

Case 1: $\{u, v, w\}$ induces a triangle:

Suppose that the Theorem is not true for u, v, w. Then, there exist vertices x, y, z such that: $x \in N[u, v]$, $x \notin N[w]$, $y \in N[u, w]$, $y \notin N[v]$ and $z \in N[v, w]$, $z \notin N[u]$. First, consider the situation where there is no edge between x, y , z. Then $\{u, v, w, x, y, z\}$ induces an Hajós graph which contradicts Theorem 4.2. Hence, at least one of xy,xz,yz is edge of G, this implies the existence of the subgraph C_4. Again, this contradicts Theorem 4.2.

Case 2: $\{u, v, w\}$ induces a P_3:

Consider u, w as the non-adjacent vertices. Suppose that $N[u, w]$ is not included in $N[v]$. Then, there is a vertex $x \in N[u, w]$ such that u, x, w, v induces the graph C_4, what is a contradiction, according to Theorem 4.3. We conclude that $N[u, w] \subseteq N[v]$. Hence $E(u, v, w) \subseteq N[v]$.

Case 3: $\{u, v, w\}$ induces a $\overline{P_3}$:

Let u be the isolated vertex of the $\overline{P_3}$. Suppose that $N[u, w]$ is not included in $N[v]$ and let $x \in N[u, w]$, $x \notin N[v]$. Analogously, if $N[u, v]$ is not included in $N[w]$, consider $y \in N[u, v]$, $y \notin N[w]$. Then, depending on whether x, y are adjacent or not, vertices v, w, x, y or v, w, x, u, y form the graph C_4 or C_5 respectively, a contradiction, by Theorem 4.3. It follows that either $N[u, w] \subseteq N[v]$ or $N[u, v] \subseteq N[w]$. Hence $E(u, v, w) \subseteq N[v]$ or $E(u, v, w) \subseteq N[w]$.

Finally, consider the case where u, v, w form an independent set. Suppose that the Theorem is not true for u, v, w. Then, there exist vertices x, y, z such that: $x \in N[u, v]$, $x \notin N[w]$, $y \in N[u, w]$, $y \notin N[v]$ and $z \in N[v, w]$, $z \notin N[u]$. First, examine the situation where xy, xz, yz are edges of G. Then $u, v, w \in E(x, y, z)$. We can then apply Case 1 and conclude that $N[x]$, $N[y]$ or $N[z]$ contains $E(x, y, z)$, contradicting $x \notin N[w]$, $y \notin N[v]$ or $z \notin N[u]$, respectively. The next alternatives, where one or two of the pairs xy, xz,yz are edges of G, imply the existence of the subgraphs C_5 or C_4, respectively. In the remaining case, $\{x, y, z\}$ induces an independent set, meaning that $\{u, x, v, z, w, y\}$ induces the graph C_6. Either cases contradicts Theorem 4.2.

Conversely, we prove that a graph that satisfies the conditions of the Theorem, can not contain any of the graphs C_4, C_5, C_6 nor the Hajós graph as induced subgraphs. We choose a convenient triple of vertices u, v, w, as follows. Suppose G contains an induced C_4. Let u, v, w be any consecutive vertices of this cycle. Suppose C_5 is an induced subgraph of G. Choose u, v, w to be vertices of C_5 that induce the complement of P_3. Finally, for both C_6 and the Hajós, consider the triple u, v, w of non adjacent vertices. In every case, it follows that $N[u, w]$ is not included in $N[v]$, $N[v, w]$ is not included in $N[u]$, and $N[u, v]$ is not included in $N[w]$. Consequently, it follows from the hypothesis,that G does not contain any of the graphs C_4, C_5, C_6 nor the Hajós graph. by Theorem 4.2, G is hereditary closed neighborhood-Helly.□

5. Fixed Mixed Neighborhood

In this section, we consider fixed neighborhood-Helly graphs. For a graph G, there is a fixed partition $O \cup C = V(G)$ of its vertex set, where the neighborhoods of O and C are to be considered as open and closed, respectively. The aim is to determine whether such neighborhoods satisfy the Helly Property. Usually in figures of this section and the next section, the vertices of O are colored with white color and the vertices of C with black color. Sometimes, there are vertices which can belong to any partition (O or C). In this case, they are colored with gray color.

The following theorem characterizes fixed mixed neighborhood-Helly graphs, based on extensions.

Theorem 5.1. *A graph G is mixed neighborhood-Helly for a bipartition $V(G) = O \cup C$ if and only if for every extension $E(u, v, w)$, there exists a vertex $z \in V(G)$ such that $E(u, v, w) \subseteq N\{z\}$, i.e. $U[u, v, w] \neq \emptyset$ or $E(u, v, w) = \emptyset$.*

Proof Let $S = \{v_1, v_2, v_3\} \subseteq V(G)$ and $E(v_1, v_2, v_3)$ its extension. If $E(v_1, v_2, v_3) = \emptyset$, there is nothing to do. Otherwise, since each vertex of $E(v_1, v_2, v_3)$ is adjacent to a pair of vertices of S, it is clear that the family of the neighborhoods of vertices of $E(v_1, v_2, v_3)$ intersect. By hypothesis, since G is mixed neighborhood-Helly, there is a vertex z which belongs to every neighborhood of the family which implies $E(u, v, w) \subseteq N\{z\}$. Note that $x \in N\{y\}$ is equivalent to $y \in N\{x\}$ because either $x = y$ or $xy \in E$.

Conversely, let G be a graph satisfying the hypothesis. By way of contradiction, assume that G is not mixed neighborhood-Helly. For $v_i \in V(G)$, denote by $N_i = N\{v_i\}$ the neighborhood of v_i.

Let $\mathcal{N} = \{N_1,\ N_2, ..., N_l\}$, $l \geq 3$, be a minimal subfamily of neighborhoods of G which is not Helly. Then $l \geq 3$. We know that, for each $i = 1, ..., l$, $\mathcal{N} \setminus \{N_i\}$ is a Helly family, and therefore has a non empty intersection.

Consider $w_i \in \cap_{j \neq i} N_j$, and examine $E(w_1, w_2, w_3)$. Since $w_i \in N\{v_j\}$ for $i \neq j$, $v_j \in N[w_i]$. Then $N[w_1, w_2] \neq \emptyset$, $N[w_2, w_3] \neq \emptyset$ and $N[w_3, w_1] \neq \emptyset$. It follows $E(w_1, w_2, w_3) \neq \emptyset$. By hypothesis, there is a vertex z such that $E(w_1, w_2, w_3) \subseteq N\{z\}$.

Observe that for every $k \in \{1, \ldots, l\}$, $v_k \in N[w_1, w_2], v_k \in N[w_2, w_3]$ or $v_k \in N[w_3, w_1]$. Consequently, $v_k \in E(w_1, w_2, w_3)$. Since $E(w_1, w_2, w_3) \subseteq N\{z\}$, z belongs to N_k for $1 \leq k \leq l$, which is a contradiction. Consequently, G is indeed mixed neighborhood-Helly. □

Next, we consider hereditary fixed mixed neighborhood-Helly graphs. First, we describe some necessary conditions for minimal non hereditary fixed mixed neighborhood-Helly graphs. A graph G is minimal non hereditary fixed mixed neighborhood-Helly graph if all its induced subgraphs are mixed neighborhood-Helly except itself.

Lemma 5.2. *If a graph G with a fixed bipartition $O \cup C = V(G)$ is minimal non hereditary mixed neighborhood-Helly graph then any universal vertex must belong to O.*

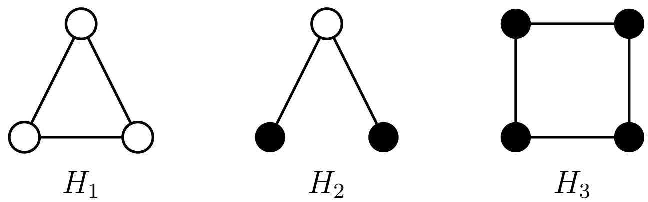

Figure 2: Minimal non hereditary fixed mixed neighborhood-Helly graphs with at most 4 vertices

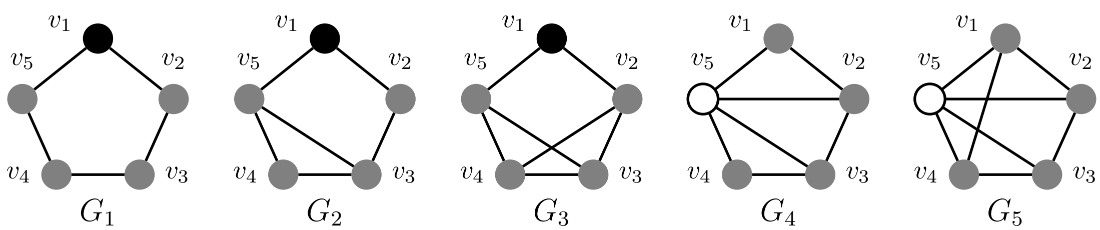

Figure 3: Some non fixed mixed neighborhood-Helly graphs with 5 vertices

Proof If a universal vertex v belongs to C, then v is a common vertex for any family of neighborhoods. Hence, G is mixed neighborhood-Helly which is a contradiction. □

The following lemma describes some non hereditary fixed mixed neighborhood-Helly graphs with at most 4 vertices.

Lemma 5.3. *H_1,H_2 or H_3 (see Figure 2) are not mixed neighborhood-Helly graphs.*

Proof For each graph, consider $\mathcal{N}$ as the family of all neighborhoods. Clearly, $\mathcal{N}$ is an intersecting family having no common vertices. □

Lemma 5.4. *G_1, G_2, G_3, G_4 and G_5 (see Figure 3) are not mixed neighborhood-Helly graphs.*

Proof There is an intersecting family of neighborhoods $\mathcal{N}$ having no common vertices for each graph of Figure 3.

For G_1 and G_2: $\mathcal{N} = \{N[v_1], N\{v_2\}, N\{v_4\}\}$.

For G_3, G_4 and G_5 : $\mathcal{N} = \{N\{v_1\}, N\{v_2\}, N\{v_3\}, N\{v_4\}, N\{v_5\}\}$.

□

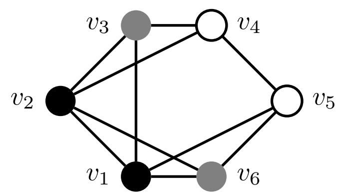

Figure 4: Graph F

The following lemma is useful to prove Theorem 5.7.

Lemma 5.5. *Let G be a cycle $v_1, v_2, \ldots, v_6$, such that v_i is not adjacent to $v_{i+3 \ mod\ 6}$ which means this cycle can have only short chords.*

(a) *If G is mixed-neighborhood-Helly graph for a bipartition $V(G) = O \cup C$, then G is isomorphic to F (see Figure 4).*

(b) *If G is hereditary mixed-neighborhood-Helly graph for a bipartition $V(G) = O \cup C$, then G is isomorphic to F^* (see Figure 5).*

Proof Suppose G is a mixed neighborhood-Helly graph for some bipartition of vertices $O \cup C = V(G)$. Consider $\{N\{v_1\}, N\{v_3\}, N\{v_5\}\}$. It is an intersecting family of neighborhoods, independently of the partition $O \cup C$. Then, it must have a common vertex v. Because, $v_1v_4, v_2v_5, v_3v_6 \notin E(G)$, it follows that $v \in \{v_1, v_3, v_5\}$. The latter implies that v must be a closed vertex. Without loss of generality, let $v = v_1$. The latter implies that v_1 is a closed vertex and $v_1v_3, v_1v_5 \in E(G)$

Similarly, we known that $\{N\{v_2\}, N\{v_4\}, N\{v_6\}\}$ is an intersecting family. Since $v_1v_3 \in E(G)$, we conclude that the enlarged family $\{N[v_1], N\{v_2\}, N\{v_4\}, N\{v_6\}\}$ is also intersecting. Therefore, the corresponding neighborhoods have a common vertex. The only possible alternatives for a common vertex are v_2 or v_6. Wihout loss of generality, suppose $v_2 \in N\{v_i\}$ for $i = 1, 2, 4, 6$. Then v_2 is a closed vertex and $v_2v_4, v_2v_6 \in E(G)$.

On the other hand, if v_4v_6 or v_3v_5 are edges of G, then the intersecting family $\{N[v_1], N[v_2], N\{v_3\}, N\{v_4\}, N\{v_5\}, N\{v_6\}\}$ does not have a common vertex which is a contradiction. Consequently, $v_4v_6, v_3v_5 \notin E(G)$.

Next, suppose v_4 is a closed vertex. With this assumption, $\{N[1], N[v2], N\{v_3\}, N[v_4], N\{v_5\}\}$ form an intersecting family with no common vertex, a contradiction. Therefore, v_4 must be open. By symmetry, v_5 also must be open.

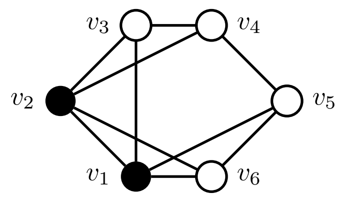

Figure 5: Graph F^*

In this situation, $\{N[v_1], N[v_2], N\{v_3\}, N(v_4), N\{v_6\}\}$ and $\{N[v_1], N[v_2], N\{v_3\}, N(v_5), N\{v_6\}\}$ are two maximal intersecting families, each one containing a common vertex. Therefore, G is isomorphic to F.

If G is also hereditary mixed neighborhood-Helly graph using the same bipartition $V(G) = O \cup C$ then v_3 and v_6 must be open vertices. Suppose v_3 is a closed vertex. Then $\{v_3, v_4, v_5, v_6, v_1\}$ induces a subgraph that matches to G_2 which is not a mixed neighborhood-Helly graph, a contradiction. Hence, v_3 must be an open vertex. By symmetry, v_6 also must be a open vertex. Therefore, G is isomorphic to F^*.

Finally, we prove that F^* is hereditary mixed neighborhood-Helly graph. Suppose that F^* is non hereditary mixed neighborhood-Helly graph, then F^* contains some minimal non hereditary mixed neighborhood-Helly graph as induced subgraphs. F^* is mixed neighborhood-Helly graph because F is so, and F^* does not contain H_1, H_2 nor H_3. In consequence, the minimal non hereditary mixed neighborhood-Helly induced subgraphs of F^* must have exactly 5 vertices. There are 3 non isomorphic induced subgraphs of 5 vertices.

$F^* \setminus \{v_2\}$: In this case, consider an intersecting family of neighborhoods $\mathcal{N}$ having no common vertices. If $N(v_4) \notin \mathcal{N}$, then v_1 is a common vertex for $\mathcal{N}$ which is a contradiction. Hence, $N(v_4) \in \mathcal{N}$. In this situation, $N(v_3), N(v_5) \notin \mathcal{N}$ meaning $\mathcal{N} = \{N[v_1], N(v_4), N(v_6)\}$ and v_5 is a common vertex for $\mathcal{N}$. Again, this is a contradiction. In consequence, $F^* \setminus \{v_2\}$ is a mixed neighborhood-Helly graph.

$F^* \setminus \{v_3\}$: Again, consider an intersecting family of neighborhoods $\mathcal{N}$ having no common vertices. If $N(v_4) \notin \mathcal{N}$, then v_1 is a common vertex for $\mathcal{N}$ which is a contradiction. Hence, $N(v_4) \in \mathcal{N}$. In this situation, $N(v_5) \notin \mathcal{N}$ and v_2 is a common vertex for $\mathcal{N}$, a contradiction. Consequently, $F^* \setminus \{v_3\}$ is a mixed neighborhood-Helly graph.

$F^* \setminus \{v_4\}$: In this case, v_1 is a closed universal vertex. By Lemma 5.2, $F^* \setminus \{v_4\}$ is not a minimal non hereditary mixed neighborhood-Helly graph.

We conclude that F^* does not contain any minimal non hereditary mixed neighborhood-Helly induced subgraph which means F^* is a hereditary mixed neighborhood-Helly graph. □

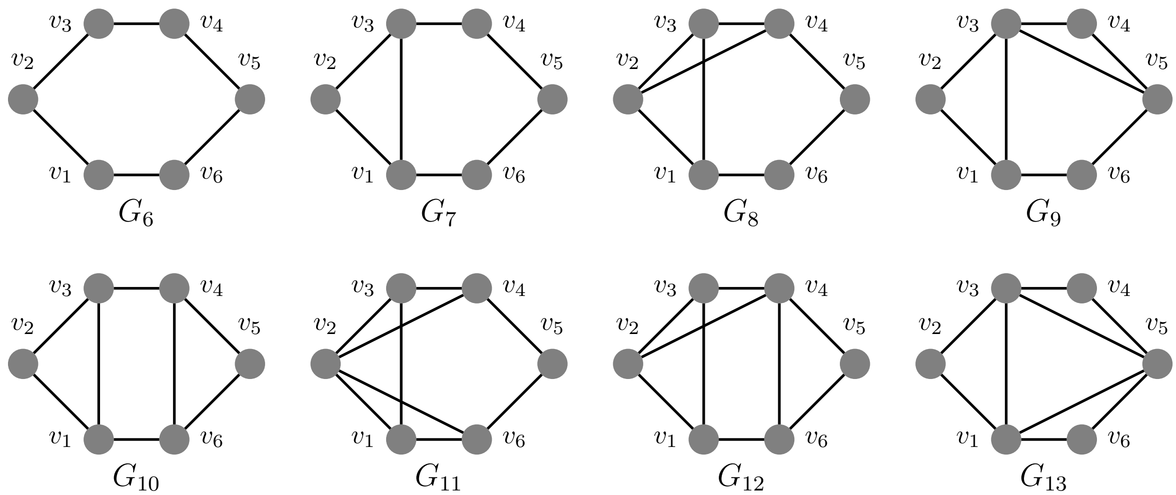

Figure 6: Graphs of Corollary 5.6 Case (1)

Corollary 5.6. *Let G be a cycle $v_1, v_2, \ldots, v_6$, such that v_i is not adjacent to $v_{i+3 \bmod 6}$. If G is not hereditary mixed-neighborhood-Helly graph for a bipartition $V(G) = O \cup C$, then G is isomorphic to some of the following graphs.*

1. *G has at most three short chords for any arbitrary bipartition $O \cup C$ (see Figure 6).*
2. *G has at least five short chords for any arbitrary bipartition $O \cup C$ (G_{14} and G_{15} of Figure 7).*
3. *G has exactly four short chords, we consider two cases:*
   (a) *G has not exactly two adjacent degree 4 vertices for any arbitrary bipartition $O \cup C$ (G_{16} and G_{17} of Figure 7).*
   (b) *G has exactly two adjacent degree 4 vertices and it is not isomorphic to F^* (G_{18}, G_{19} and G_{20} of Figure 8).*

The following is a characterization of fixed hereditary mixed neighborhood-Helly graphs, by forbidden induced subgraphs.

Theorem 5.7. *G is hereditary mixed neighborhood-Helly for a bipartiton $V(G) = O \cup C$ if and only if G does not contain an induced subgraph isomorphic to $H_1, H_2, H_3, G_1, G_2, \ldots, G_{20}$.*

Proof By Lemmas 5.3 and 5.4 and Corollary 5.6, $H_1, H_2, H_3, G_1, G_2, \ldots, G_{20}$ are not hereditary mixed neighborhood-Helly graphs and if G contains some of them as induced subgraphs then G is not hereditary mixed neighborhood-Helly graph.

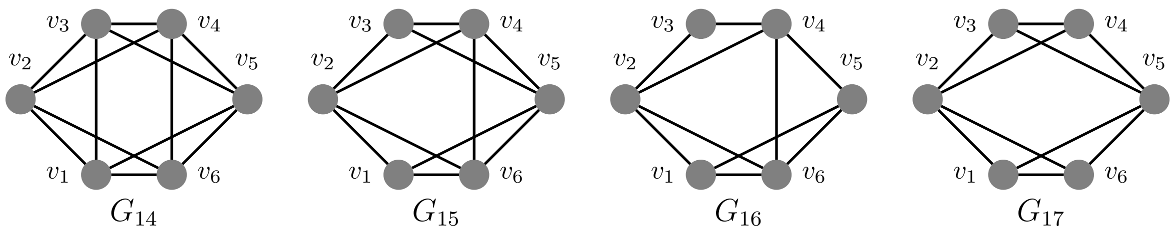

Figure 7: Graphs of Corollary 5.6 Cases (2) and (3.a)

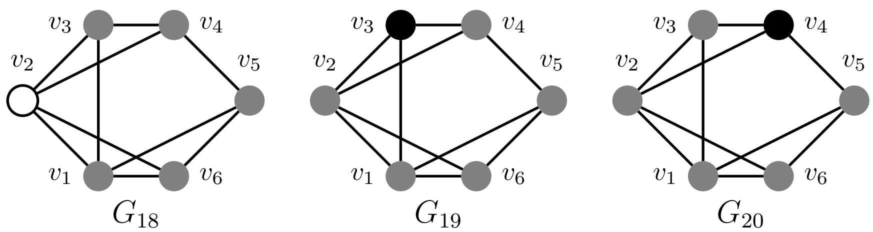

Figure 8: Graphs of Corollary 5.6 Case (3.b)

Conversely, G does not contain an induced subgraph isomorphic to $H_1, H_2, H_3, G_1, G_2, \ldots, G_{20}$ and suppose that G is not hereditary mixed neighborhood-Helly for a bipartiton $V(G) = O \cup C$. Without loss of generality, we can assume that G is a minimal non hereditary mixed neighborhood-Helly graph. Then, there is a minimal intersecting family of neighborhoods of G, $\mathcal{N} = \{N\{v_1\}, N\{v_2\}, \ldots, N\{v_m\}\}$, which is not Helly. Clearly, $m \geq 3$. Consider the subfamily $\mathcal{N} \setminus \{N\{v_i\}\}, 1 \leq i \leq m$ and let w_i be the common vertex of this subfamily. Clearly, $w_i \notin N\{v_i\}$ $(1 \leq i \leq m)$ and w_i, w_j $(1 \leq i < j \leq m)$ are different vertices since there is no common vertex for the family $\mathcal{N}$. Hence, if $v_i = w_j$ $(1 \leq i, j \leq m)$ then $v_i \in O$ iff $i = j$ ($v_i \in C$ iff $i \neq j$). We consider the size of $\{v_1, v_2, v_3, w_1, w_2, w_3\}$.

1. $|\{v_1, v_2, v_3, w_1, w_2, w_3\}| = 3$. In this case, $v_1 = w_j$ for some $j \in \{1, 2, 3\}$.
   (a) $j = 1$, then $v_1 \in O$. There are only two alternatives.
      i. $v_2 = w_2$ and $v_3 = w_3$. Clearly, $v_2, v_3 \in O$ and $\{v_1, v_2, v_3\}$ induces an H_1, a contradiction.
      ii. $v_2 = w_3$ and $v_3 = w_2$. Clearly, $v_2, v_3 \in C$ and $\{v_1, v_2, v_3\}$ induces an H_2, a contradiction.
   (b) $j \neq 1$, then $v_1 \in C$. Without loss of generality, $j = 2$, which means v_1 and v_2 are not adjacent vertices. If $v_2 = w_3$ then v_2 must be a neighbor of v_1, a contradiction. Hence, $v_2 = w_1$ and $v_3 = w_3$. Therefore, $v_2 \in C$, $v_3 \in O$ and $\{v_1, v_2, v_3\}$ induces an H_2, a contradiction.
2. $|\{v_1, v_2, v_3, w_1, w_2, w_3\}| = 4$. Without loss of generality, $v_1 = w_j$ for some $j \in \{1, 2, 3\}$.

(a) $j = 1$, then $v_1 \in O$. There is some vertex $v_i = w_j$, $i, j \in \{2, 3\}$, w.l.o.g., we can assume $v_2 = w_j$.

i. $v_2 = w_2$ and $v_3 \neq w_3$. Clearly, $v_2 \in O$ and $\{v_1, v_2, v_3\}$ induces a K_3. Hence, $v_3 \in C$, otherwise $\{v_1, v_2, v_3\}$ induces an H_1 which is a contradiction. As $N(w_3) \cap \{v_1, v_2, v_3\} = \{v_1, v_2\}$, $\{v_1, v_2, w_3\}$ induces an H_1 if $w_3 \in O$ or $\{v_1, v_3, w_3\}$ induces an H_2 if $w_3 \in C$. In any case, there is a contradiction.

ii. $v_2 = w_3$ and $v_3 \neq w_2$. Clearly, $v_2 \in C$, $N(v_2) \cap \{v_1, w_2, v_3\} = \{v_1\}$ and $\{v_1, w_2, v_3\}$ induces a K_3. If $w_2, v_3 \in O$ then $\{v_1, w_2, v_3\}$ induces an H_1 which is a contradiction. Hence, $w_2 \in C$ or $v_3 \in C$. Therefore, $\{v_1, v_2, w_3\}$ or $\{v_1, v_2, v_3\}$ induces an H_2. Again, we have a contradiction.

(b) $j \neq 1$, then $v_1 \in C$. Without loss of generality, $j = 2$, which means $v_3, w_3 \in N(v_1)$ and $v_2, w_1 \notin N(v_1)$. There are two alternatives.

i. $v_3 = w_3$. This is a symmetric case of (2.a.ii).

ii. $v_2 = w_1$. Clearly, $v_2 \in C$ and $\{v_1, v_3, v_2, w_3\}$ induces a C_4. If $v_3, w_3 \in C$ then $\{v_1, v_3, v_2, w_3\}$ induces an H_3 which is a contradiction. Hence, $v_3 \in O$ or $w_3 \in O$ which implies $\{v_1, v_3, v_2\}$ or $\{v_1, w_3, v_2\}$ induces a H_2. Again, this is a contradiction.

3. $|\{v_1, v_2, v_3, w_1, w_2, w_3\}| = 5$. Without loss of generality, $v_1 = w_j$ for some $j \in \{1, 2, 3\}$.

(a) $j = 1$, then $v_1 \in O$. Clearly, $\{v_2, v_3, w_2, w_3\} \subseteq N(v_1)$ and $\{v_1, v_2, w_3\}$ ($\{v_1, v_3, w_2\}$) induces a K_3. Hence, $v_2 \in C$ or $w_3 \in C$ ($v_3 \in C$ or $w_2 \in C$). Otherwise, there is an induced H_1. On the other hand, v_2w_2 and v_3w_3 are not edges of G. If $v_2, w_2 \in C$ ($v_3, w_3 \in C$) then $\{v_1, v_2, w_2\}$ ($\{v_1, v_3, w_3\}$) induces an H_2. Hence, the only valid alternatives are:

i. $v_2, v_3 \in C$. In this case, v_2v_3 must be an edge of G, otherwise, $\{v_1, v_2, v_3\}$ induces an H_2, a contradiction. Then, $(v_1, w_2, v_3, v_2, w_3, v_1)$ is a 5-cycle and $\{v_1, w_2, v_3, v_2, w_3\}$ induces a subgraph which is isomorphic to G_4 (w_2w_3 is not an edge of G) or G_5 (w_2w_3 is an edge of G) taking v_1 as as the distinguished open vertex in G_4 or G_5. This is a contradiction.

ii. $w_2, w_3 \in C$. It is symmetric to above case.

(b) $j \neq 1$, then $v_1 \in C$. Without loss of generality, $j = 2$, which means $v_3, w_3 \in N(v_1)$ and $v_2, w_1 \notin N(v_1)$. Clearly, $(v_1, w_3, v_2, w_1, v_3, v_1)$ is a 5-cycle and $\{v_1, w_3, v_2, w_1, v_3\}$ induces a subgraph which is isomorphic to G_1 (w_3w_1 and v_2v_3 are not edges of G), G_2 (exactly one of w_3w_1 and v_2v_3 is an edge of G) or G_3 (w_3w_1 and v_2v_3 are edges of G) taking v_1 as the distinguished closed vertex in G_1, G_2 or G_3. This is a contradiction.

4. $|\{v_1, v_2, v_3, w_1, w_2, w_3\}| = 6$. Clearly, $(v_1, w_3, v_2, w_1, v_3, w_2, v_1)$ is a 6-cycle which can have only short chords. If G has only 6 vertices, since G is not hereditary mixed neighborhood-Helly graph then by Corollary 5.6, G must be isomorphic to $G_6, G_7, \ldots, G_{19}$ or G_{20} which is a contradiction. Therefore, G has more than 6 vertices and $\{v_1, v_2, v_3, w_1, w_2, w_3\}$ induces a proper induced subgraph G' of G. Since G

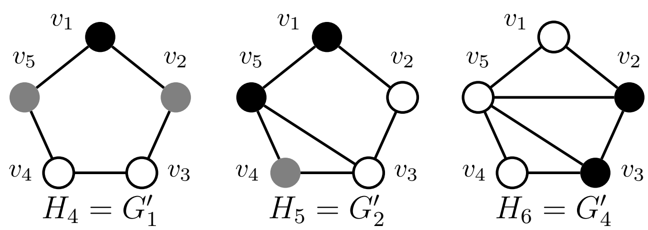

Figure 9: Minimal non fixed mixed neighborhood-Helly graphs with 5 vertices

is a minimal non hereditary mixed neighborhood-Helly graph, G' is a hereditary mixed neighborhood-Helly graph and it must be isomorphic to F^* by Lemma 5.5. It is clear that exactly one of v_1, v_2 and v_3 is a closed vertex. Without loss of generality, we assume $v_1 \in C$. As $v_1 \in N[v_1] \cap N(v_2) \cap N(v_3)$, there is some $4 \leq i \leq m$ such that $v_1 \notin N\{v_i\}$ because $\mathcal{N}$ has not a common vertex. It is clear that $w_1, w_2, w_3 \in N(v_i)$, $(v_i, w_2, v_1, v_2, w_1, v_i)$ is a 5-cycle and $\{v_i, w_2, v_1, v_2, w_1\}$ induces G_1 (v_iv_2 and w_1w_2 are not edges of G), G_2 (exactly one of v_iv_2 and w_1w_2 is an edge of G) or G_3 (v_iv_2 and w_1w_2 are edges of G) taking v_1 as the distinguished closed vertex in G_1, G_2 or G_3. This is a contradiction.

Consequently, G must be hereditary mixed neighborhood-Helly graph. $\square$

The family of forbidden induced subgraphs of Theorem 5.7 is not minimal, which implies there is another characterization forbidding a proper minimal subset $\mathcal{H} \subset \mathcal{F}$. Clearly, $\mathcal{H}$ is formed by all minimal non hereditary fixed mixed neighborhood-Helly graphs. In this sense, H_1, H_2 and H_3 are members of this subset because H_1 and H_2 are smallest members of $\mathcal{F}$ and H_3 does not have H_1 nor H_2 as induced subgraphs. Next lemmas describe the other members of $\mathcal{H}$.

Lemma 5.8. *Minimal non hereditary fixed mixed neighborhood-Helly graphs with 5 vertices are exactly* $H_4 = G'_1$, $H_5 = G'_2$ *and* $H_6 = G'_4$ *where* G'_i *meaning that it is derived from* G_i *(see Figure 9).*

Proof It is easy to see that H_4, H_5 and H_6 are all graphs from $\{G_1, G_2, G_3, G_4, G_5\}$ which do not contain any H_1, H_2 nor H_3 as induced subgraphs. $\square$

Lemma 5.9. *The minimal non hereditary fixed mixed neighborhood-Helly graphs with 6 vertices are exactly* $H_7 = G^1_6$, $H_8 = G^2_6$, $H_9 = G^1_7$, $H_{10} = G^1_8$, $H_{11} = G^1_9$, $H_{12} = G^1_{10}$, $H_{13} = G^1_{11}$, $H_{14} = G^1_{13}$ *and* $H_{15} = G^1_{16}$ *where* G^j_i *meaning that it is derived from* G_i *(see Figure 10).*

Proof It is easy to see that $H_7, H_8, \ldots, H_{15}$ are all graphs from $\{G_6, G_7, \ldots, G_{20}\}$ which do not contain any $H_1, H_2, H_3, G_1, G_2, G_3, G_4$ nor G_5 as induced subgraphs. $\square$

As a consequence, we have the following minimal characterization by forbidden induced subgraphs.

Corollary 5.10. *G is hereditary mixed neighborhood-Helly for a bipartiton $V(G) = O \cup C$ if and only if G does not contain an induced subgraph isomorphic to $H_1, H_2, \ldots, H_{15}$.*

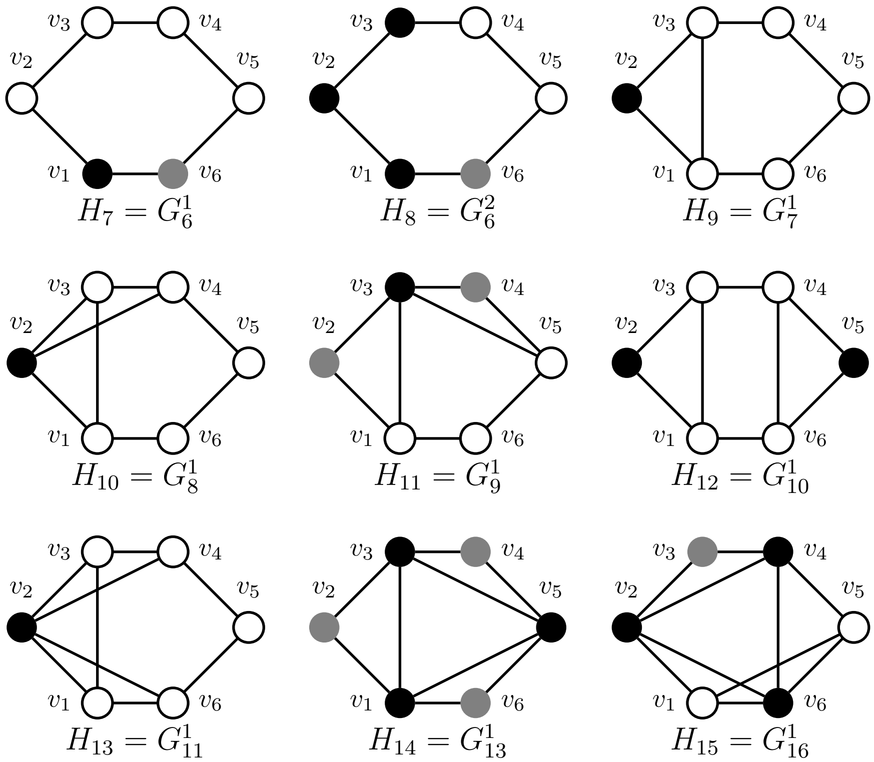

Figure 10: Minimal non fixed mixed neighborhood-Helly graphs with 6 vertices

The following theorem gives another characterization for hereditary mixed neighborhood-Helly graphs based on extensions.

First, we call a 3-set $S = \{u, v, w\}$ as *valid* if S does not induce a P_3 where u, w are non-adjacent open vertices and v is closed.

Theorem 5.11. *A graph G is a fixed hereditary mixed neighborhood-Helly for a bipartition $V(G) = O \cup C$ if and only if for every extension $E(u, v, w)$ where $\{u, v, w\}$ is a valid 3-set, there exists a vertex $z \in \{u, v, w\}$ satisfying $E(u, v, w) \subseteq N\{z\}$, i.e. $U[u, v, w] \cap \{u, v, w\} \neq \emptyset$ or $E(u, v, w) = \emptyset$.*

The proof is ommited, as it follows from similar arguments as presented troughout the paper.

6. Arbitrary Mixed Neighborhood-Helly Graphs

In this section we study the arbitrary mixed neighborhood-Helly (ARBITRARY MNH) problem. It consists of deciding whether a graph is mixed neighborhood-Helly (MNH) for some partition of its vertices, into open and closed neighborhoods. In addition, we also study the hereditary arbitrary mixed neighborhood-Helly (ARBITRARY HMNH) problem, which consists of deciding if there exists a bipartition of the vertices of a given graph G, into open and closed neighborhoods, such that G is mixed neighborhood-Helly for any induced subgraph of it. We employ a similar notation as in the last section. That is, when applying the

Helly property, assume a bi-coloring of the vertices of G using colors *black, white*, such that black vertices correspond to those whose neighborhoods are to be considered as closed, while the white ones are those with open neighborhoods.

We prove that both the above problems are NP-complete using a reduction from a special version of 3-SAT, called $3 - SAT_{2+1}$. The $3 - SAT_{2+1}$ problem consists of determining if the variables of a given boolean formula written in a conjunctive normal form, having 2 or 3 literals per clause, where each literal is an occurrence of some variable x_i (positive literal) or its negation $\neg x_i$ (negative literal), can be assigned values *true* or *false*, in such a way that the formula is *true.* In this restricted instance, each variable x_i occurs at most 3 times, twice positive and once negative. Without loss of generality, we assume that each variable x_i appears at most once in each clause. NP-completeness of such problem follows from [32].

Lemma 6.1. *If a graph G is mixed neighborhood-Helly then the bi-coloring of its vertices must satisfy the following conditions.*

(a) *$G[\{v_1, v_2, v_3, v_4, v_5\}]$ is an induced C_5 of G and at least 4 of these vertices have degree exactly 2. Then the color of all of them is white.*

(b) *$G[\{v_1, v_2, v_3\}]$ is an induced P_3 of G and it is not part of an induced diamond. If the color of v_2 (middle vertex of P_3) is white, then v_1 or v_3 must have color white.*

(c) *$G[\{v_1, v_2, v_3\}]$ is an induced triangle of G and it is not part of an induced K_4. Then at least one of these 3 vertices must have color black.*

Proof (a) Suppose that v_i is a black vertex, let v_j a neighbor of v_i in the induced C_5 and v_k the unique vertex in the induced C_5 such that $N(v_k) \cap \{v_i, v_j\} = \emptyset$. Clearly, the neighborhoods of these three vertices do not verify the Helly property which is a contradiction.

(b) Suppose that v_1 and v_3 are black vertices. Clearly, the neighborhoods of v_1, v_2 and v_3 do not verify the Helly property which is a contradiction.

(c) Suppose that v_1, v_2 and v_3 are white vertices. Clearly, the neighborhoods of v_1, v_2 and v_3 do not verify the Helly property which is a contradiction.

□

Let B be a given boolean formula, input to the $3 - SAT_{2+1}$ problem. Let $x_1, ..., x_n$ be the variables of B and $C_1, ..., C_k$ its clauses. We construct a graph $G(B)$ as follows.

1. for every variable x_i, construct a subgraph G_i, called *variable gadget*, consisting of 7 vertices, three of them, a_i, b_i, c_i form a triangle (a_i is the positive pole of G_i and b_i the negative pole of G_i), c_i and the other 4 vertices form a C_5 (see Figure 11).

2. for each clause C_j, add a triangle T_j

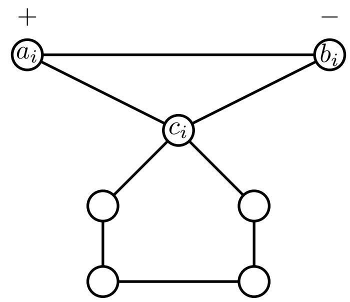

Figure 11: gadget G_i for variable x_i

- for each literal x_i or $\neg x_i$ of C_j, choose a different vertex v in T_j, create a new C_5 with a distinguished vertex $w_{i,j}$ and add edges
  - $vw_{i,j}$ and $a_i w_{i,j}$ if the literal is positive (x_i)
  - $vw_{i,j}$ and $b_i w_{i,j}$ if the literal is negative ($\neg x_i$)

3. for each clause C_j of size two, there is a unique vertex v of T_j that still has degree 2. In this case, add 4 new vertices to form a C_5, together with v.

See Figure 12, for a complete example. It is easy to see that:

- $G(B)$ has no induced K_4 nor diamond.
- Every induced cycle in $G(B)$ has length 3, 5 or at least 10.
- In consequence, every 5-cycle in $G(B)$ is an induced C_5.
- There are no 6-cycles in $G(B)$.
- In every C_5, at least 4 of its vertices have degree exactly 2 in $G(B)$.
- The maximum degree among the vertices in $G(B)$ is 4.

Lemma 6.2. *If $G(B)$ is ARBITRARY MNH then B is satisfiable.*

Proof Since $G(B)$ is ARBITRARY MNH, there is a bi-coloring for $G(B)$ that makes it mixed neighborhood-Helly. We assign convenient values to the variables $x_1, \ldots, x_n$ as follow: x_i is true if and only if a_i is white vertex. Let us to prove that this assignment turns B to be true. That is, we have to prove that each clause C_j has the value true. Some vertex v of the triangle T_j corresponding to C_j must be black, by Lemma 6.1.(c). Clearly, v is not part of any induced C_5 because in that case, v would be white, by Lemma 6.1.(a) since it is part of some induced C_5, a contradiction. Consequently, v must be a neighbor of some vertex $w_{i,j}$ by construction of $G(B)$ and $w_{i,j}$ is a white vertex because it is part of some induced C_5. As $w_{i,j}$ is the

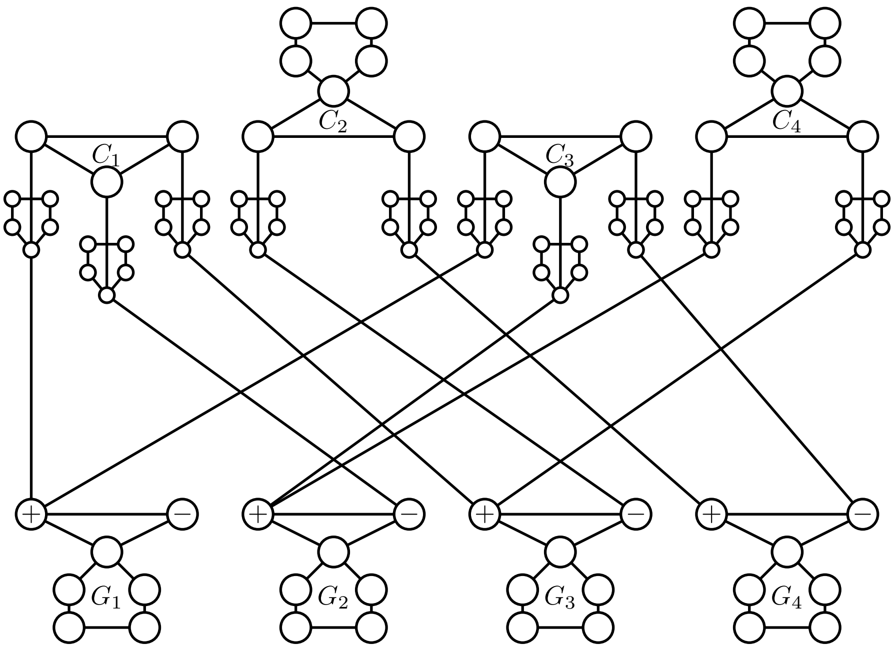

Figure 12: Transformed graph for $B = C_1 \wedge C_2 \wedge C_3 \wedge C_4$, $C_1 = x_1 \vee \neg x_2 \vee x_3$, $C_2 = \neg x_3 \vee x_4$, $C_3 = x_1 \vee x_2 \vee \neg x_4$ and $C_4 = x_2 \vee x_3$

middle vertex of some P_3 and one extreme vertex of this P_3 is v, by Lemma 6.1.(b), the other extreme vertex must be white. There are two possibilities for this extreme vertex:

(i) to be a_i when x_i is a literal of C_j. The value of x_i is true because a_i is a white vertex, implying that C_j is true.

(ii) to be b_i when $\neg x_i$ is a literal of C_j. Clearly, c_i is a white vertex, since it is part of an induced C_5. By Lemma 6.1.(c), a_i must be black. So, the value of x_i is false and $\neg x_i$ is true implying C_j is true.

In any of these cases, C_j always is true. In consequence, B is true using this assignment and B is satisfiable. □

Corollary 6.3. *If $G(B)$ is ARBITRARY HMNH then B is satisfiable.*

Lemma 6.4. *B is satisfiable then $G(B)$ is ARBITRARY HMNH.*

Proof Consider an assignment *true* or *false* for the variables of B, such that the formula is satisfiable and call this assignment as V. We consider the following coloring for the vertices. If variable x_i is *true* then vertex b_i is black, and all the remaining vertices of G_i are white. If x_i is *false*, then a_i is black and the remaining are white. Further, for each clause C_j, a vertex v of its corresponding triangle T_j is black if and only if v is a neighbor of some vertex $w_{i,j}$ and $w_{i,j}$ has a white neighbor in G_i. The remaining vertices of $G(B)$ are white.

We prove that the above colors makes $G(B)$ hereditary MNH. With this purpose, apply Theorem 5.7 and examine its forbidden induced subgraphs.

1. First, examine the triangles of G. There are two types of triangles:

   (i) those formed by a_i, b_i and c_i in G_i. In this case, b_i or a_i is a black vertex because x_i is true or false in V.

   (ii) a triangle T_j whose corresponding clause is C_j. As C_j is true in V, there is some true literal x_i ($\neg x_i$) of C_j. In this case, a_i (b_i) is a white neighbor of $w_{i,j}$ in G_i and there is a black neighbor v of $w_{i,j}$ in T_j.

   Therefore, no triangle of $G(B)$ is formed by white vertices which implies that G does not have H_1 as induced subgraphs.

2. $G(B)$ does not contain 6-cycles, hence G does not have $G_6, \ldots, G_{20}$ as induced subgraphs.

3. Every 5-cycle in $G(B)$ is an induced C_5 and all its vertices are white vertices. Therefore, none of $G_1, \ldots, G_5$ is an induced subgraph of $G(B)$.

4. $G(B)$ does not contain induced C_4s, hence H_3 is not an induced subgraph of $G(B)$.

5. Two black vertices of $G(B)$ are either adjacent or at distance at least three, implying that no P_3 of G has black extremes and a white middle vertex and H_2 is not an induced subgraph of $G(B)$.

The above conditions imply that $G(B)$ does not contain any forbidden induced subgraph of Theorem 5.7. That is, $G(B)$ is hereditary MNH $\square$

Corollary 6.5. *If B is satisfiable then $G(B)$ is ARBITRARY MNH.*

Theorem 6.6. *ARBITRARY MNH and ARBITRARY HMNH are NP-complete.*

Proof For establishing that both problems belong to NP, let G be a graph and consider a coloring $\mathcal{C}$ of the vertices of G, using colors *black* or *white*. By employing Theorem 5.7 or Corollary 5.10 and observing that the described forbidden subgraphs are of fixed size, we can check in polynomial time, whether $\mathcal{C}$ makes the collection of the neighborhoods to be Helly or not. Consequently, the problem ARBITRARY HMNH is in NP. On the other hand, by employing Berge's algorithm [5], we can check in polynomial time, if the collection of neighborhoods, open or closed according to the colors, of the vertices of G satisfy the Helly property. Therefore ARBITRARY MNH also belongs to NP.

The reduction proof is from the $3 - SAT_{2+1}$ problem. Let B be a given boolean formula, input to the $3 - SAT_{2+1}$ problem and $G(B)$ its transformed graph.

As a consequence of Lemma 6.2 and Corollary 6.5, B is satisfiable if and only if $G(B)$ is ARBITRARY MNH and as a consequence of Lemma 6.4 and Corollary 6.3, B is satisfiable if and only if $G(B)$ is ARBITRARY HMNH which completes the proof $\square$

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