Early Child Development and Care

ISSN: 0300-4430 (Print) 1476-8275 (Online) Journal homepage: http://www.tandfonline.com/loi/gecd20

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**To cite this article:** Nora Scheuer, Flavia Irene Santamaria & Mónica Haydée Echenique (2016): Exploring children's thinking with and about numbers from a resources-based approach, Early Child Development and Care, DOI: 10.1080/03004430.2016.1157789

**To link to this article:** http://dx.doi.org/10.1080/03004430.2016.1157789

EARLY CHILD DEVELOPMENT AND CARE, 2016
http://dx.doi.org/10.1080/03004430.2016.1157789

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# Exploring children's thinking with and about numbers from a resources-based approach

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**ABSTRACT**
The aim of this study is to achieve a deeper understanding of the repertoire of cognitive resources children can display in the process of learning numbers. Forty-two children attending Age 4 Kindergarten Class or Year-One in Argentina were individually interviewed, based on a semi-structured script requiring them to represent definite and indefinite quantities orally and notationally. Children's responses throughout the interview were categorized according to Scope of conventional knowledge, Non-conventional responses, Playfulness and Progress. Results of a series of Classification Analyses showed three main classes, which capture the tension between using what is known and exploring novelties in order to represent numerical meanings.

**ARTICLE HISTORY**
Received 15 January 2016
Accepted 21 February 2016

**KEYWORDS**
Number; children; cognitive resources; zone of proximal development; playfulness



## Introduction

The purpose of this paper is to inform educational researchers and practitioners about the rich repertoire of cognitive resources children can display in the process of learning numbers. We propose that one of the keys to improve the quality of childhood education in the field of early mathematics is to consider children as active thinkers whose numerical learning is enhanced by providing them meaningful and playful opportunities to use, communicate and reflect with and about numbers.

Children's numerical knowledge develops by adapting cultural numerical forms (e.g. oral and notational number series, counting and addition procedures) and functions (e.g. enumeration, quantification, numerical comparison and addition) in order to accomplish personal goals in contexts of activity, interaction and reflection (Nunes & Bryant, 1996; Saxe, Guberman, & Gearhart, 1987). As children gradually appropriate established cultural knowledge, they generate plenty of ideas and procedures. Even when such ideas and procedures do not match conventions strictly, they reveal a search for meaning and transient organizations of emerging knowledge. The tensions between what is established and what is possible (Bruner, 2010), or between conventions and inventions (Brizuela, 2004), may turn into sources of cognitive conflict and transformation. Hammer, Elby, Scherr, and Redish (2005) have proposed a 'resources-based framework of the cognitive structure', according to which the ideas and procedures students display in problem spaces are dynamical and context-embedded cognitive resources (from now on, resources). This framework motivates a shift in the kind of aims for research of learning and cognitive development; namely, from verifying the onset of a given piece or structure of knowledge to capturing when and how learners bring those resources into play in problem spaces and how such resources unfold. Social interaction and semiotic

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Supplemental data for this article can be accessed here http://dx.doi.org/10.1080/03004430.2016.1157789.

mediation are two inter-related factors that contribute to trajectories of cognitive progress (Vygotsky, 1978, 1986). Vygotsky's formulation of the zone of proximal or potential development allows to understand how subjective cognitive novelties emerge within semiotically mediated spaces of social interaction. In such spaces, intersubjective functioning operates as a precursor of future independent functioning (Corral Ruso, 2001). In order for educators or researchers to build this kind of socio-cognitive space when interacting with a child, they should be capable of proposing contents the child can load with sense on the basis of his/her current standpoint, as well as of being sensitive to the child's perspective and of detecting the logic of his/her attempts and solutions.

Achieving a deeper knowledge of the ways in which children deal with numerical forms and functions and use them in situations is essential to design and implement educational practices that improve learning for all students from the very beginnings of formal education. Based on these concerns, in this paper we present a qualitative, in-depth study of how children in early years of compulsory education think with and about numbers. Before turning to the study, we briefly sketch our general psycho-educational conceptual frame and review previous research of children's numerical development and learning.



## *Adequacy to standards or enhancing learners' resources*

Since the early 1980s, the notion of quality of products and processes has landed in the terrain of education and became mainstream in the following decades up to date. The notion of quality comes from the field of management, based on models of economic efficiency in terms of final products. In the educational field, the idea of efficiency has usually been understood as a matter of measuring students' 'school performance' (Aguerrondo, n.d.), according to an assembly-line model of instruction (Rogoff, 2012).

The implantation of this view of quality of education has succeeded in making citizens and policymakers aware that access to school is not enough for learning to occur (Marchesi & Martín, 2014; Rivas, 2015). However, borrowing the concept of quality from theories of management acritically has led to apply indicators of final products that are unable to capture the progressive, non-linear and contextual nature of learning processes (Clark, 1997; Rogoff, 2012). An example of this kind of applications are the massive assessments performed to children at clear-cut times in their schooling. Children's responses to these individual written tests are marked according to fixed external parameters. This kind of assessment that seeks to determine what and how much individual students know, as if knowledge were a stable and fixed mental content, allows to measure the distance between the results achieved by students and a set of expectations (devised by experts who usually have no or scarce contact with local school communities), and situate the students, their teachers and school systems into global rankings. Students whose results fall far away from the expectations are considered to be deficient. This verification demarks a border between such 'defective' students on the one hand, and the body of knowledge and behaviours valued by school culture on the other (Edwards, 1991), hindering the 'meetings of mind' that are necessary so that the process of education can take place 'across the age and experience gaps that separate teachers from children' (Olson & Bruner, 1996, p. 10). For these reasons, it is unlikely that an approach to quality of education that privileges learners' adequacy to standards as its measuring stick can contribute to an education capable of offering students with different backgrounds and needs genuine opportunities to expand their physical, cognitive and social capabilities (Marchesi & Martín, 2014).

In the domain of number, adequacy-oriented approaches look to measure children's knowledge of conventional number forms and determine the extent to which children use such forms correctly to achieve numerical goals established by others (teacher, psychologist, etc.). The focus is to measure how close the child's knowledge is with respect to culturally established uses. The underlying notion of a learner is that of a potential knower who, if exposed to adequate instruction, will become capable of applying and transferring a given set of cultural knowledge, independently of contextual factors.

A resources-based approach does not seek to compare learners' performance, but to capture the ways and extent to which they make use of what they know or are in the way to learn. The focus is on learners' active exploration, generation, adaptation and reflection about number problems, forms and procedures as they approach meaningful purposes, with a special interest on how they articulate different bodies of knowledge and venture into innovations. According to Hammer et al. (2005), students' resources are not correct or wrong per se. Rather, the ways in which students activate and coordinate their resources may be more or less useful in different contexts. The underlying notion of a learner is that of a thinker and experienced agent, even in the early years. Quality of education is envisaged in terms of acknowledging, appraising and fostering learners' experience and meaning-making processes. This approach is more sensitive to a repertoire of learning trajectories, and thus may offer an alternative to allocating a deficit to those learners who do not meet expected standards.

## *Children's numerical thinking from a resources-based approach*

Children's knowledge of numbers in the oral and notational semiotic modes (Kress, 2010) is relevant for their conceptual number development, because oral and notational numeration systems condense basic principles organizing the number domain (Martí & Scheuer, 2015; Nunes & Bryant, 1996). Children's uptake of conventional symbolic number forms is evident from approximately 18 months of age in the oral mode, and from around 30 months in the notational mode. Children begin to take up number words *one*, *two* and *three* in order to designate small collections (Mix, Huttenlocher, & Cohen Levine, 2002). They progressively extend their repertoire of number words, in connection with different number functions (Bideaud, Meljac, & Fischer, 1992; Saxe et al., 1987). By the age of four, children can acknowledge many of the functions numerals (or written numbers) accomplish in daily life (Sinclair & Sinclair, 1984).

The repertoire of notational strategies three- to eight-year olds carry out have been extensively explored (Alvarado & Brizuela, 2005; Broitman, 2013; Hughes, 1986; Scheuer, Sinclair, Merlo de Rivas, & Tièche Christinat, 2000; Sinclair, Siegrist, & Sinclar, 1983; Teubal, Dockrell, & Tolchinsky, 2007), by requesting them to note down physically present collections of objects (usually ≥30) or numbers on dictation (up to the thousands). One of the earliest strategies children implement is to produce a pictogram or a scheme for each object in the collection. Grasping the ideographic principle establishing that a single graphic sign stands for a cardinal value drives children's understanding of numbers further. However, these strategies may coexist, so that children implement one or the other according to task conditions. Logogrammic strategies are frequent numbers in the tens, hundreds and beyond, by decomposing the oral number word into smaller numerical parts for which the notation is known, and noting down each of them in a string (e.g. 100701 for *one hundred and seventy-one*).

These different studies have shown that in order to investigate children's numerical thinking, it is necessary to confront them with meaningful situations involving a variety of semiotic modes and of 'cognitive zones', ranging from zones where the child feels comfortable and knowledgeable, to spaces he/she has seldom gone through, yet may be willing to explore. Based on research on playfulness in childhood, we propose that a playful setting is particularly suitable to this end. Playfulness encourages children to imagine and try out alternatives that make sense to them even if they depart from conventions (Sarlé, 2006). According to Huizinga (1955), in play activity takes place on the basis of established rules that are nevertheless freely accepted and is accompanied by a mix of tension and happiness. Playful situations are perceived as fictional. This awareness, together with the permission to agree upon or create rules, allows children to explore and rehearse roles, rules and procedures that they cannot adequately implement in ordinary life (Cekaite, Blum-Kulka, Grøver, & Teubal, 2014). Engaging in a playful attitude fosters awareness of rules and meanings (Baquero, 1996). 'In play a child always behaves beyond his average age, above his daily behavior; in play it is as though he were a head taller than himself' (Vygotsky, 1978, p. 102).

## Aims

The main aim of this study is to achieve a deeper understanding of the repertoire of resources children in the first years of compulsory education display, explore and transform in relation to a range of numerical tasks in the oral and notational modes. We are interested in going beyond solely recording the extent of children's conventional number knowledge, in order to consider the dimensions that indicate children's exploratory thinking, agency and learning potential in this field. Specific aims are to:

(1) Study the relationships among Kindergarten and Year-One children's resources regarding the afore-mentioned dimensions in terms of response patterns that capture the tension between using what is known and exploring novelties in order to represent numerical meanings.
(2) Describe and interpret main response patterns and reflect about their relevance for early mathematics education.

## Methods

### *Participants*

Participants were 42 children attending Age 4 Kindergarten Class (K4) or Year-One (Y1) in two public schools in Río Negro, Argentina. School supervisors and teachers were informed about the aims of the study and asked for permission to host the study. Children's parents were requested to sign a form in order to allow their children to be interviewed at school and to use their productions in scientific communications. Children were informed about the study in class and their volunteer participation was requested orally. About 10 children were randomly selected from each class in each school, from those without any reported learning difficulties and who had provided individual and family consent to participate. Twenty children attended K4 (10 boys and 10 girls; mean age = 54.5 months) and 22 attended Y1 (13 boys and 9 girls; mean age = 76.5 months).

In Argentina, compulsory education starts at four years of age. According to curricular guidelines, numerical instruction in K4 focuses in formalizing the use of numbers up to 10, mostly in the oral mode. In Y1 teaching focuses on solving diverse problems with numbers up to 100 in the oral and notational modes.

### *Procedure, tasks and materials*

We individually interviewed children in dedicated rooms at their schools. The script (see Appendix) included 20 tasks that required to represent definite numbers/quantities and indefinite large quantities in two semiotic modes (oral and notational), in relation to a variety of referents (see Table 1). The interviewer (INT) showed her interest in the child (CH)'s views. She introduced herself by saying: *I know children have very good ideas about numbers and I would like to know more about them. Would you help me by sharing your ideas with me?* In the context of the different tasks, she pointed

**Table 1.** Summary of tasks.

| Numerical status | Tasks (in oral and notational modes) | Specifications |
|---|---|---|
| Definite | Expression of own age | |
| | Number series | Starting from 1 |
| | Quantification of manipulable objects (chips) | 3 |
| | | 10 |
| | | 30 |
| | Expression of absence of quantity | Manipulable objects (chips) |
| | | Visual distant objects (stars) |
| Indefinite | Imagination and expression of large numbers | Of manipulable objects (chips) |
| | | Of a temporal entity (an elderly age) |
| | | Of visual distant objects (stars) |

out: *I am sure you can think of a way to do it with your own ideas, even if you have not been taught this yet.* In order to facilitate children's playful attitude, tasks allowed for various procedures and no time limitations were set.

Interviews lasted approximately 40 minutes and were audio-taped and transcribed verbatim.

## *Qualitative analysis*

For each child, the transcript of the interview and his/her notational productions were analysed together. A multidimensional system of analysis was elaborated, with the following dimensions: Scope of conventional number knowledge, Non-conventional responses, Playfulness and Progress.

The coding of data occurred in conjunction with the adjustment or development of categories, in an iterative process that continued until new data did not change the categories that we were developing. Each author was randomly assigned to be the lead coder of a set of interviews and second coder of another one. Once the initial coding had been completed for each child, the second coder assessed the categories established by the first coder and eventually added new categories. In instances in which coding was not aligned, coding assignments were discussed in order to resolve the discrepancy and/or review the coding scheme. Next, we present the dimensions and their categories. Examples for most of them are provided in the Results section.

### *Scope of conventional knowledge*

This dimension seeks to grasp the extent of children's conventional knowledge of numbers. Responses were compared to the norm by means of mutually excluding categories.

*Oral/notational number series.* The highest number stated/noted down by the child without skipping or repeating any number was coded according to ranges: none, 1–9, 10–19, 20–29, 30–49, ≥50.

*Oral/notational quantification of collections of chips.* The largest collection the child cardinalized correctly was recorded: none, 3, 10, 30 or a number of chips between 20 and 29.

*Notation of indefinite large quantities.* We coded the number of referents for which the CH was able to say a number and note it down conventionally.

*Writing own name, oral/notational expression of own age and of absence of quantity.* We coded whether the CH offered a conventional response or not.

In the following dimensions, categories are not mutually excluding, since in the context of the same task (or group of tasks) children displayed activity corresponding to more than one kind of category.

### *Non-conventional responses*

For the different tasks, this dimension distinguishes among categories of responses that do not match convention.

*Oral quantification of collections of chips (3, 10 and 30).* In agreement with Saxe et al. (1987), categories are Enumeration without cardinalization (*one, two, three*, and when asked: *How many are there?* CH enumerates again or further), Global quantification (*many*, *a lot*, *thousands*), Distant number (with a difference ≥2 for 3 chips, and ≥3 for 10 and 30 chips) and Close number (with differences of 1 or 2, respectively).

*All the tasks requiring to note quantities down.* We distinguished the following categories (based on studies referred to in the Introduction):

- Any numeral, pseudo-numeral, letter, pseudo-letter.
- Multiple notation, using:
  - (i) non-ordered digits, pseudo-numerals, letters, pseudo-letters, schemes or pictograms (e.g. XXX, 14E or AIO for 3 chips), or
  - (ii) ordered digits (either fully outspread, as 12345678910 for 10 chips, or compressed, as 1 10), repeating the numeral for the whole collection as many times as objects in the collection

(e.g. 333 for 3 chips), eventually including schemes, pictograms (123 XXX) or the name of the objects (e.g. 123 CHIPS).

We analysed whether the correspondence among graphic signs forming multiple notations and the quantity of objects in the collection was precise or global.

- Partially correct multidigit numeral (e.g. 31 or 13 for 30, 17 for 27, intentional variation of number of digits in order to represent the magnitude of the number to be noted down).
- Intentional variation of size of the notation according to magnitude.
- Graphic connection among the signs that form the notation, indicating that individual objects make part of an encompassing major unit.

*Notation of absence of quantity.* Some of the following categories have been reported by Hughes (1986), Teubal et al. (2007) and Tolchinsky (2003):

- Any numeral, pseudo-numeral, letter, pseudo-letter.
- Deliberate refusal to produce a notation, based on the principle that each object is to be represented by one graphic sign, so that if there are none, no sign is traced.
- Sign of negation (X, -), crossing a previous notation out, drawing an opposite referent (e.g. drawing the sun for no visible stars).
- Reductive notation, as compared to the notation for 'a few' (e.g. a numeral or a multiple notation standing for a minimal quantity as 1 or 2, a noticeably small shape, crossing part of a previous notation out).
- Circle, or writing 'there is none' or similar expression.



### *Playfulness*

Children related to the tasks in ways that reminded us of some of the indicators of playfulness (Huizinga, 1955; Sarlé, 2006). We established the following categories:

- Accepts/gets into the 'rule' to respond based on his/her own ideas (e.g. Maia (K), after having refused to note down 30 chips *because I don't know it*, was reminded that she could try to do it with her own ideas, *perhaps inventing*. Maia listened attentively and then enthusiastically said: *Fine, I'll go ahead!* and produced YP).
- Establishes his/her own rules to shape a situation.
- Expresses that he/she is operating in a fictional space (e.g. Nahuel (K4), dismissing a query by INT: *May be you forgot I'm just playing?*).
- Proposes alternatives for a same situation, or adapts a solution to different situations, in such a way that she/he opens up the relations among means and ends.
- Visibly enjoys (e.g. laughs, smiles and makes jokes).

### *Progress*

Based on Vygotsky's formulation of the zone of proximal development (1978) and later contributions (Baquero, 1996; Corral Ruso, 2001), we devised the following categories of intra- and inter-task progress:

- From non-conventional response to conventional response. CH provides a conventional response after having offered a non-conventional response for the same task (e.g. when noting the number series produces 12345689, rereads it and inserts the missing 7) or for a very similar problem further on in the interview (e.g. counts up to 15 when requested to say the number series aloud, and up to 23 when looking to quantify 30 chips).

- More elaborate non-conventional response. After providing a non-conventional response, CH provides another, more elaborate response for the same task (e.g. a Close quantification response after a Distant one for 10 chips, or a Multiple notation with ordered digits [as 123 for 3 chips] after a Multiple notation with non-specific digits [as 276]), or for a comparable situation further on.
- Expresses new awareness of a regularity or use, or expresses a new understanding.
- Sets new learning goals, identifies new problems.

### *Statistical analysis*

In order to achieve our specific aims, we applied Hierarchical Ascendant Classification (HAC) (Ward, 1963). This is a tool to study a group of individuals as described by a set of qualitative variables. HAC classifies the individuals on the basis of their coordinates on the main factorial axes provided by a previous Multiple Correspondence Analysis (Greenacre, 1984). HAC starts from the population of participants so that each one is the only element of a class. In each successive iteration, two classes which are least distant are grouped into a new class, until a single class encompasses all the individuals. This iterative process is represented in a tree-structured graph called a dendrogram. For each class of individuals, HAC informs over- and under-represented categories ($p < .05$). Thus, different classes are characterized by particular combinations which may be interpreted as response patterns.

We performed a HAC for each dimension of analysis (variables are informed in Online Supplemental File) and a fifth one in order to study the relationships among the four dimensions. Children's School grade was considered as another variable. SPAD 5.5 was used.

## Results

Results of the HACs studying each dimension of analysis separately showed that all the children displayed some kind of activity in each of the four dimensions, in one or more occasions in the course of the interview. A baseline was found for Scope of conventional knowledge (stating the number series between 3 and 9 and quantifying 3 chips in the oral mode) and for Playfulness (Accepting the rules proposed by INT).

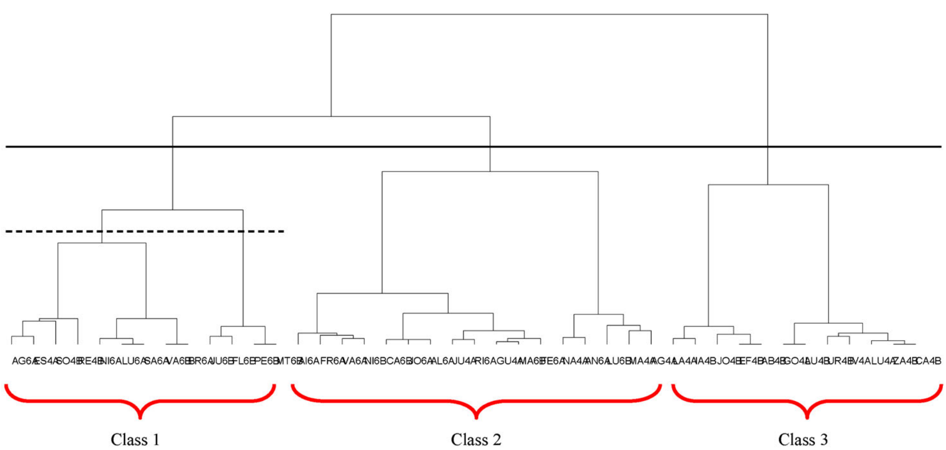


**Figure 1.** Dendrogram resulting from the HAC analysis of the interplay among dimensions of analysis.

For the sake of concision, we only present the results of the fifth HAC, which studies the interplay among the four dimensions.

### *HAC analysis of the interplay among dimensions*

By analysing the dendrogram, as well as the three best partitions provided by the programme, we determined three main classes (see full line in Figure 1). In the case of Class 1, information provided by the two subclasses distinguished at a lower hierarchy (indicated by the dashed line) was relevant.

For each class, we report the number of children and their school grade, as well as over- and under-represented categories in each dimension, and provide illustrative examples.

#### Class 1 *(13 children; 10 from Y1 and 3 from K4)*

This class is characterized by kinds of Non-conventional responses and of Playfulness.

*Non-conventional responses.* The range of non-conventional responses offered by these children evidence certain knowledge of the principles guiding the uses of numbers. When these children quantified collections of chips in the oral mode in a non-standard way, they proposed Distant numbers – never a Global quantification. Most of their non-conventional notations were based on digits and corresponded to the more sophisticated categories (Partially correct multidigit numeral and eventually Multiple notation with ordered digits).

*Playfulness.* In notational tasks with definite quantities, playfulness was revealed by explicitly Accepting the 'rule' to respond in function of the own ideas, Proposing alternatives and in a lesser extent (for instance, none of these children repeated a same solution for the three large quantities), Establishing rules.

Finer-grained analysis within Class 1 reveals two subclasses.

*Class 1a (4 children from Y1).* This subclass is characterized by a particular interplay of Scope of conventional knowledge and Non-conventional responses – on the grounds of playfulness that, as we have seen, unites the whole class. These four children show an extensive number series (both oral and notational). This does not prevent them from recurrently proposing non-conventional notations.

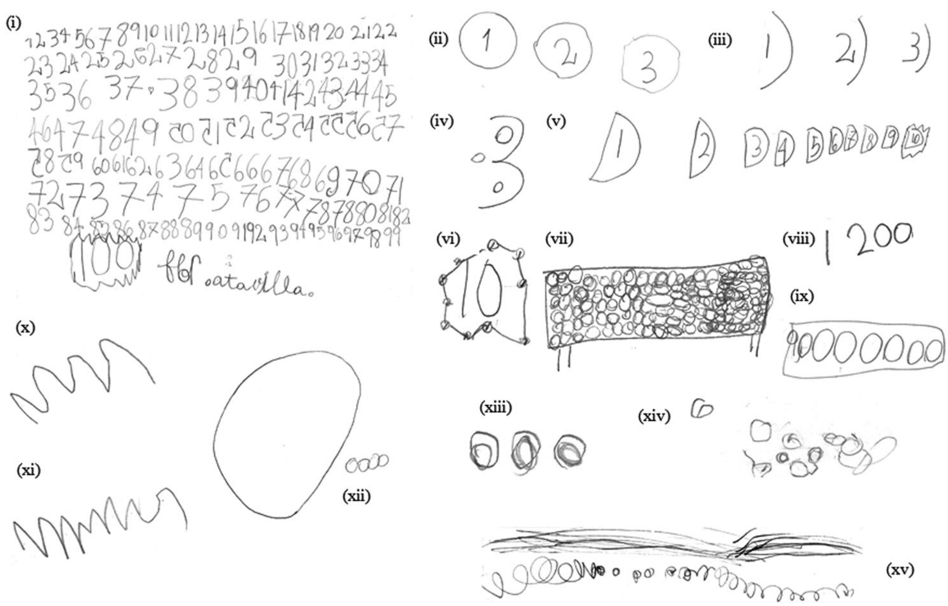


**Figure 2.** Examples of notational production in Classes 1 and 3.

For instance, Florencia (Y1) recited numbers up to 100 perfectly and succeeded in noting down all of them (see Figure 2, i). However, in the notational quantification tasks with chips, her behaviour evidenced tension between some of the information she was trying to record on paper (qualitative aspects of the referent and each object in the collection) and the ideographic principle of conventional number notations. In order to note 3 chips down, Florencia began by drawing three circles (she placed one of the chips on the paper, outlined its contour and drew two other circles freehand) and then traced the digits 1, 2 and 3 inside the circles (ii). When INT asked her if she could note down 3 chips in *another way*, Florencia produced another Multiple notation with ordered digits, this time using quasi-schemes instead of pictograms. She drew three semi-circles, as if the chips were overlapping, and placed the digits 1, 2 and 3 in sequence, one inside each quasi-scheme (iii). Upon INT's query to express three chips with *one number only*, Florencia produced a single 3, but did not leave it 'alone', as she completed it with three small-sized pictographic characters (iv). She based her first attempt for 10 chips on the notational strategy she had devised in her second trial for 3 chips; that is, Multiple notation with the complete series of digits, each one inserted in a semi-circle (v). Once again, INT asked her if it was possible to express such a quantity by means of only one number. Florencia responded by producing the numeral '10' enclosed by a string of 10 small circles standing for chips (vi). In order to represent 30 chips on paper, she broke away from the costly notational strategy directed at keeping graphic track of each of the objects that make up the collection.

For the request to imagine and note down an indefinite, large number of chips, Florencia asked (partly as to herself, partly as to INT): *Two hundred?* and drew a table covered with chips (vii). She said she could not note that quantity down *because they are so many*. Upon INT's renewed query about the possibility to note down *two hundred* by means of one numeral, Florencia replied: *But with two hundred my hand gets tired*. This comment shows she had become aware of the effort involved in noting down all the numbers from 1 up to the chosen cardinal. She solved this conflict by noting down only the first and the last number of the complete series (viii). With this new kind of notation, she broke away with representing qualitative aspects of the objects, but retained a minimal part of the whole series of numbers that she would have noted down one by one, had it not been so tiresome. Instead of compacting the graphic representation for the chips (as she had done when she used semi-circles instead of circles for 3 and 10 chips), this time Florencia shortened the series of numerals. In this way, we had the opportunity to witness the unfolding of a sequence of compromise solutions, making original deals among the following three parties: (i) the will to keep graphic track of qualitative aspects of the referent as well as of each object in the collection, (ii) attending to INT's queries to use *one number only* and (iii) avoiding excessive work (as she had already experienced when she had traced one or even two signs [a pictogram or quasi-scheme and a digit] for each object). The analysis of the whole interview shows that even if Florencia moved comfortably through extensive number intervals and even ventured into noting down very high numerals (probably for the first time, see ix), she did not ever abandon her commitment to record at least some extra-cardinal aspect of the referents motivating the notation.

*Class 1 b (9 children; 6 from Y1 and 3 from K4)*. The feature that distinguishes these children is intertask Progress, consisting mainly of Setting new learning goals in close connection with previous tasks where some kind of conflict or doubt had arisen. An example of this is given by Valentín (Y1), who mastered the oral and notational number series up to 1000. In order to represent large numbers of chips and of visible stars, he said there would be *infinite*, and noted ∞ down. His repeating the same answer for both referents suggests he was using *infinite*/∞ as a number label, suitable for any very large quantity. However, when at the end of the interview he was asked if and what he would like to go on learning about numbers, he stated that he wanted to learn large numbers better (*up to one million* [ … ] *It is very high numbers that I find hardest, because some times I get them wrong*).

### Class 2 *(17 children, 12 from Y1 and 5 from K4)*

These children share characteristics regarding Scope of conventional knowledge, Non-conventional responses and Progress.

*Scope of conventional knowledge.* Children in Class 2 usually noted their age and name conventionally. They mastered the number series further on in the oral mode than in the notational one (up to 19, 29 or 49 orally; up to 9 or to 19 notationally). They were able to quantify at least 20 chips correctly (many of them had to count twice or thrice to succeed). These children provided conventional notations for 3 and 10, but no one was able to do so for 30 chips.

*Non-conventional responses.* Their quantification for 30 chips was very close to correct. The slight difference was due to missing the word *thirty* or to insufficient stability in the application of one-to-one correspondence between number tags and pointing. Some children spontaneously counted the chips twice as they realized they had made a mistake and attempted to resolve it. These children noted down 30 chips with Partially correct multidigit numerals and varied kinds of Multiple notations. In the latter, correspondence was not well kept (in contrast with the multiple notations they proposed for 3 or 10).

*Progress.* All the children in Class 2 revealed progress across tasks and many of them also within tasks, by proposing more Elaborate non-conventional responses (or ones that were closer to conventions), achieving a Conventional solution and/or Expressing new awareness about the ways numbers work. The process carried out by Candela (Y1) illustrates the last two categories.

INT: What if there are no chips left? How would you show it?
Candela: It's not a number!
INT: Isn't it a number?
Candela: (as she notes down 0 and looks at it). It is a number! But tiny. But it is almost nothing. It cannot be counted.
INT: I see. Fine, and what have you noted down here?
Candela: Zero.

Candela's appropriation of *zero*/0 was evident in the task referring to stars, when she noted down 0 straightforward and reread it conventionally.

### Class 3 *(12 children from K4)*

Children in Class 3 share characteristics regarding Scope of conventional knowledge, Non-conventional responses and Progress.

*Scope of conventional knowledge.* Their conventional knowledge was restricted to the oral mode, with a number series usually up to a number between 2 and 9 but never beyond 25. Most of the children could only quantify up to 3 chips, while no one could go beyond 10 chips. The very few children who did not state their age correctly belonged to this class. Very few children wrote down their name and none offered any conventional response for any of the following tasks: noting down the own age, the number series, absence of quantity, definite quantities of chips or large indefinite quantities.

*Non-conventional responses.* These children's non-conventional quantification were Global, Distant or Enumerating – never a Close number. Although they did not demonstrate to know any digit, most of the time they managed to note quantities by using Multiple notation with different signs (pictograms, pseudo-numerals, letters and pseudo-letters). Concern for one-to-one correspondence between number of characters and of objects in the represented collection was evident only for the smallest collection (3 chips). Some children made endeavour to graphically show the fact that collections form soldered entities. For example, Abril (K4) attempted to draw a hand with four fingers (see Figure 2, x), just as she had shown in order to express her own age. She used the same type of drawing again in order to note down the elderly age (xi). Other children augmented the size of the notations they produced for indefinite large quantities. For instance, Lucía (K4) represented large numbers of chips by drawing only *a* [chip] *but a very big one,* while for representing few chips she drew 4 chips, much smaller than the previous one (xii). Children in Class 3 displayed a variety of resources to represent the absence of quantity: that is, Reductive notation, use Sign of negation (-) and Deliberate refusal to produce a notation. A few children wrote down Any numeral or letter, seemingly not acknowledging the specific restriction involved in this request.

*Playfulness.* Children from this class are characterized by Accepting the rules established by INT in the oral tasks with definite quantities – and not only in notational tasks, as most children.

*Progress.* This class is characterized by inter-task progress. Regarding notational level, children generally progressed in the complexity and adjustment of some non-conventional answers; for example, when going from many pictographic characters to schemes, or when using a curl to represent the idea of many more characters after producing multiple notations with unconnected signs. Efraín (K4) drew the correct quantity of chips for representing 3 and 10 chips (see Figure 2, xiii and xiv), but he proposed curls and afterwards lines for 30 (xv). In the oral mode, these children accomplished Conventional responses after having provided non-conventional ones. Some children also Set new learning goals. New awareness is under-represented in this class.

## Discussion

Classes resulting from the series of HACs of children's activity along Conventional knowledge, Non-conventional responses, Playfulness and Progress may be interpreted as response patterns. These response patterns offer a picture of children's number knowledge in positive terms, offering hints about the width of children's zone of comfort, and also about the kinds of canonical and original solutions they reached, the challenges they engaged with and even the emotions they experienced as they dealt with numbers.

Non-conventional responses revealed children's capacity to generate meaning and use their resources as they approached number problems through pathways that involved alternatives to the conventional wisdom they were in the way to learn. All the kinds of non-conventional responses devised by these K4 and Y1 children to express quantities have been documented in the literature. A difference is that in this corpus multiple notations are more pervasive and varied, at the time that logogrammic notations (Scheuer et al., 2000) are absent. This may be due to the very nature of the tasks. Logogrammic notations have been abundantly reported for requests to note down specific quantities conveyed through words. In this interview, we purposely avoided placing the child in the position of having to decode an oral number string and transpose it into a (particular) notation. Rather, we intended to highlight the quantitative aspect of the referents to be noted down, whether presented physically or through imagination. Another contribution of our findings with respect to previous studies is that the multidimensional intra-individual analyses carried out allowed us to understand the role of non-conventional numerical production in children's meaning-making processes better.

Non-conventional production contributed to shape the three response patterns. Even those children who demonstrated extensive mastery of conventional knowledge exhibited extensive non-conventional production – frequently with as much dedication as enjoyment (as illustrated by Florencia, Subclass 1a). Coexistence of conventional and creative procedures points at the gap existing between the uptake of conventional number forms on the one hand, and deliberate and appropriate use of such forms in context, on the other (Broitman, 2013; Nunes & Bryant, 1996). For children at the other end of mastery of conventional knowledge (in Class 3, whose conventional responses were restricted to the first nine numbers conveyed in the oral mode), non-conventional productions were the strands to share their emerging numerical understandings and efforts with INT, and also to push their ideas and achievements a bit further. Had these children been assessed solely in terms of adequacy, only their lack of knowledge would have been detected. In contrast, Class 3 provides a picture of active and creative thinkers, capable of making up notations for quantity by using forms coming from other semiotic systems, or of exploiting their analogical resources to convey magnitude differences between two quantities (e.g. Efraín's drawing of four to represent his current age and many fingers for an elderly age).

Analysing the knowledge contained in the repertoire of non-conventional productions (versus considering them monolithically as incorrect) enables us to recognize subtle gradations within these kinds of meaningful responses. No wonder, then, that certain kinds of non-conventional responses coexisted, while others did not (as shown by the interplay among Close, Distant, Global and Enumerative quantifications in Classes 1 and 3). Moreover, analysing the situations in which children activated their non-conventional responses, or achieved convention, also gives important hints about the transitions among their zone of comfort, uneasiness or challenge. This is particularly evident in Class 2, drawing together children who were comfortable when they dealt with definite quantities in the first and part of the second decades, and were working out the third one.

In sum, in the particular socio-cognitive context, we co-constructed with the participating children, non-conventional productions operated as cognitive resources (Hammer et al., 2005) and so did conventional productions. Far from presenting culturally established solutions as prepackaged pieces of knowledge, most of these children showed that such responses arose from non-linear exploratory and constructive processes, which introduced adjustments and fostered awareness of regularities or of new shades of meaning (as illustrated by Candela, Class 2).

The specific HAC performed for Playfulness showed that all the children accepted the rules proposed by INT. This enabled them to participate in the interview as thinkers who enjoyed the permission to generate and try out ideas and procedures. Children who demonstrated the major Scope of conventional knowledge (in Class 1) went markedly beyond this baseline, as they related to the activities in other playful modes as well, as Proposing a broad age of alternatives and Setting rules of their own.

Progress was evidenced by Classes 2, 3 and a minor subclass within Class 1. This result indicates that children demonstrating very different scopes of conventional knowledge, attending Kindergarten or Year-One, found opportunities to push their solutions and/or ideas forward as they responded to the same tasks. Detecting progress was possible because of the broad conception of zone of proximal development (Baquero, 1996; Corral Ruso, 2001) and of learning (Hammer et al., 2005) endorsed. Based on such views, at the level of final products progress can be detected not only when convention is accomplished following other kinds of production, but also when refining or reshaping non-conventional solutions occurs. Moreover, progress is recognized also at representational levels, in regard to changes in the ways of conceiving the learning object (i.e. numbers and the problems in which they take part) and also in regard to the learning-self, as happens when awareness of ignorance or uncertainty leads to set new learning goals (Pozo, 2014).

Besides, the finding that all the children showed progress (in one way or another) might also be explained by the constellation of inter-related features shaping the particular socio-cognitive environment where children's thinking with and about numbers was explored. We highlight some of them:

- *Meaningful, embedded and open-ended tasks* have probably facilitated that children made sense of requests, generated goals of their own, and situated themselves in a range of cognitive zones so that they calibrated the level of difficulty and challenge.
- *A playful frame* might have allowed children to deploy exploratory thought and activity, with no temporal or achievement pressures.
- *Multimodality* seems to have also contributed to such deployment of thought and activity. Even if the focus was on the numeration system in the oral and notational modes, tasks bring other semiotic instruments about, as oral speech, drawing, writing and gesture. All the language systems children use to express themselves were welcome.
- *Dialogical interaction, acknowledging children as creative and conscientious thinkers.* Open-ended, genuine questions together with interventions requesting children to go further in their attempts

might have also played a role in children's generation and transformation of productions, concerns and understandings.

## Conclusions

We propose that one of the ways to foster the quality of education in the first years of compulsory school is to integrate the kind of features we have just pointed out in pedagogic interaction with children. This is so because a central aspect of quality of education is the extent to which it succeeds in empowering learners – even when they are very young – in their skills to actively participate in a range of problem spaces; to make use of different sources of knowledge and of semiotic modes to formulate their own voice; to explore alternative solutions and to become aware of their progress and challenges, strategies and resources.

## Acknowledgements

We thank Alfredo Bautista, Cecilia Bordoli and Montserrat de la Cruz for collaboration in the initial phases of this study; Lucía Bugallo, Silvina Marquez and Ana Clara Ventura for their helpful suggestions.

## Funding

This work was supported by Universidad Nacional del Comahue [C-107]; Agencia Nacional de Promoción Científica y Tecnológica [PICT 2014-1016]; Ministerio de Economía y Competitividad de España [EDU2010-21995-C02-02].

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## Appendix 1. Script used in the interview

Tasks are described following the order of presentation. INT stands for Interviewer and CH for Child.

*Oral expression of the own age*

In the context of establishing social contact with CH: *How old are you?*

### *Oral number series*

*Do you know the numbers? Do you know how to count? Would you show me?* If CH remained silent: *One, two … and what comes next?* If CH stopped counting or incorrectly said one or several numbers, INT repeated the last numbers CH had correctly stated and asked: *Which one comes next? And after that?*

### *Notational number series*

*Could you write down all the numbers you just said?* (a blank sheet of paper and a pencil were provided).

### *Quantification of 3, 10 and 30 chips in the oral and notational mode*

Chips were 2.5 cm diameter circles made of green EVA rubber. *Do you like collectable chips? Do you collect any? Let's imagine a child who collects chips, who is given these* (INT placed 3 chips in a casual array on the table). *How many chips are there? Can you note down how many chips the child has got?* (a blank sheet of paper was provided). INT added 7 chips and requested CH to quantify the whole amount and note it down on the same sheet of paper. Finally, INT added another 20 chips and repeated the previous requests.

***Imagination and expression of large quantities and absence of quantity***

*We'll play the game of enlarging and shrinking numbers. Do you feel like playing?* INT explained that this game was about thinking of and noting large quantities and then doing so for every time smaller ones. For each referent (chips, age or stars), a new sheet of paper was provided.

*Large number of chips*

*Let's imagine that the child now has got many, many chips. How would you note down, in any way you can, a very large number of chips? As if the table were full of them.*

*No chips*

After asking to imagine the child had less and less chips: *How would you note down that she/has none, not a single chip?*

*Notation of own age and elderly age*

*How would you note down your age? How would you note down the age you will be when you grow into an elderly person?*

*Large number of stars*

*Let's imagine the sky on a clear night, full with stars. Can you note down the number of all the stars you can see?*

*No visible stars.*

After asking to imagine an increasingly cloudy sky: *And what if it's so cloudy that not even a single star can be seen?*

In all the notational tasks, if children did not produce any notation, they were encouraged to do so in any way they could. They were always asked to reread their notations at the end of each task.

At the end of the interview, INT asked about CH's learning goals: *Do you think that you can go on learning about numbers? What would you like to learn?*