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Sharp regularity estimates for second order fully nonlinear parabolic equations

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Abstract We prove sharp regularity estimates for viscosity solutions of fully nonlinear parabolic equations of the form

$$u_t - F\left(D^2u, Du, X, t\right) = f(X,t) \quad \text{in} \quad Q_1, \tag{Eq}$$

where F is elliptic with respect to the Hessian argument and $f \in L^{p,q}(Q_1)$. The quantity $\Xi(n,p,q) := \frac{n}{p} + \frac{2}{q}$ determines to which regularity regime a solution of (Eq) belongs. We prove that when $1 < \Xi(n,p,q) < 2 - \epsilon_F$, solutions are parabolically α-Hölder continuous for a sharp, quantitative exponent $0 < \alpha(n,p,q) < 1$. Precisely at the critical borderline case, $\Xi(n,p,q) = 1$, we obtain sharp parabolic Log-Lipschitz regularity estimates. When $0 < \Xi(n,p,q) < 1$, solutions are locally of class $C^{1+\sigma, \frac{1+\sigma}{2}}$ and in the limiting case $\Xi(n,p,q) = 0$, we show parabolic $C^{1,\text{Log-Lip}}$ regularity estimates provided F has "better" a priori estimates.

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1 Introduction

The study of second order parabolic equations plays a fundamental role in the development of several fields in pure and applied mathematics, such as differential geometry, functional and harmonic analysis, infinite dimensional dynamical systems, probability, as well as in mechanics, thermodynamics, electromagnetism, among others. The non-homogeneous heat equation,

$$u_t - \Delta u = f \quad \text{in} \quad Q_1 = B_1 \times (-1, 0], \tag{1.1}$$

where $f \in L^p(Q_1)$, $p > \frac{n+2}{2}$, represents the simplest linear prototype. Its mathematical analysis goes back to nineteenth century and the regularity theory for such an equation is nowadays fairly complete. The fully nonlinear parabolic theory is quite more recent. The fundamental works of Krylov and Safonov [15,16] on linear, non-divergence form elliptic equations set the beginning of the development of the regularity theory for viscosity solutions to fully nonlinear parabolic equations. Since then this has been a central subject of research. Wang in [20,21] proves Harnack inequality and $C^{1+\alpha, \frac{1+\alpha}{2}}$ estimates for fully nonlinear parabolic equations, and Crandall et al. in [4] develop an L^p-viscosity theory, see also Imbert–Silvestre's survey in [8] as regards to existence, comparison and Hölder regularity of viscosity solutions. Krylov in [11,12] obtains $C^{2+\alpha, \frac{2+\alpha}{2}}$ estimates for solutions to $u_t - F(D^2u) = 0$, under convexity assumptions (see also [21, Section 4.3] for similar results), and Caffarelli and Stefanelli in [1] exhibit solutions to uniform parabolic equations that are not $C^{2,1}$.

Non-divergence form parabolic equations involving sources with mixed integrability conditions $f \in L^{p,q}(Q_1)$, as in (Eq) have also been fairly well studied in the literature. Existence in suitable parabolic Sobolev spaces has been proven by Krylov, see [13,14], see also the sequence of works by Kim [9,10]. Insofar as regularity estimates are concerned, only qualitative results are available when p and q are sufficient large. Nonetheless, as in a number of physical, geometric and free boundary problems, obtaining a quantitative sharp regularity estimate for solutions is decisive for a finer analysis. Hence, the purpose of this paper is to obtain sharp moduli of continuity of solutions for second order parabolic equation (Eq), involving sources with mixed norms, which depend only on dimension, p, q and universal parameters.

Hereafter we denote by

$$\Xi(n, p, q) := \frac{n}{p} + \frac{2}{q}.$$

The first quantitative regularity result we show states that if $1 < \Xi(n, p, q) < \frac{n+2}{\mathfrak{p}_0}$, where $\frac{n+2}{2} \leq \mathfrak{p}_0 < n+1$ is a constant,[1] then solutions are parabolically α-Hölder con-

[1] The universal constant $\mathfrak{p}_0$ is one which gives the minimal range for which the Aleksandrov–Bakelman–Pucci–Krylov–Tso maximum principle holds for L^p-viscosity solutions provided $p > \mathfrak{p}_0$ (cf. [4, Section 2] and [5] for more details).

tinuous for the sharp exponent $\alpha := 2 - \Xi(n, p, q)$ (see Sect. 3 for the treatment of this case).

Intuitively, as $\Xi(n, p, q)$ decreases, one should expect that regularity estimates of solutions improve . The borderline is $\Xi(n, p, q) = 1$, where we prove that solutions are parabolically Log-Lipschitz continuous (see Sect. 4 for this analysis). This result is a further quantitative improvement to the fact that $u \in C^{\alpha, \frac{\alpha}{2}}_{loc}(Q_1)$ for any $0 < \alpha < 1$.

When $0 < \Xi(n, p, q) < 1$, we show that solutions are $C^{1+\beta, \frac{1+\beta}{2}}$, for $\beta \leq 1 - \Xi(n, p, q)$ (see Sect. 5 for this case). Qualitative results, when $p = q > n + 1$, were previously obtained by Crandall et al. [4, Section 7] and Wang [21, Section 1.2].

Finally, we deal with the upper borderline case, $f \in \mathrm{BMO}(Q_1)$. Under appropriate higher a priori estimates on F, we show that solutions are parabolically $C^{1,\text{Log-Lip}}_{loc}(Q_1)$ (see Sect. 6 for this approach). Particularly, $u \in C^{1+\alpha, \frac{1+\alpha}{2}}_{loc}(Q_1)$ for any $0 < \alpha < 1$.

The table below provides a global picture of the parabolic regularity theory for equations with anisotropic sources, in comparison with the sharp elliptic estimate from [18]: where $\varsigma := 2 - \Xi(n, p, q)$ and $\mu := \min\{\alpha^-, 1 - \Xi(n, p, q)\}$. Here α^- means $\alpha - \zeta$ for every $0 < \zeta < \alpha$ and $\epsilon \in (0, \frac{n}{2})$ is a universal constant.[2]

$f \in L^p(B_1)$	Regularity of u	$f \in L^{p,q}(Q_1)$	Regularity of u
$n - \varepsilon \leq p < n$	$C^{0,2-\frac{n}{p}}_{loc}(B_1)$	$1 < \Xi(n, p, q) < \frac{n+2}{p_0}$	$C^{\varsigma, \frac{\varsigma}{2}}_{loc}(Q_1)$
$p = n$	$C^{0,\text{Log-Lip}}_{loc}(B_1)$	$\Xi(n, p, q) = 1$	$\text{par} - C^{0,\text{Log-Lip}}_{loc}(Q_1)$
$p > n$	$C^{1,\min\left\{\alpha^-, 1-\frac{n}{p}\right\}}_{loc}(B_1)$	$0 < \Xi(n, p, q) < 1$	$C^{1+\mu, \frac{1+\mu}{2}}_{loc}(Q_1)$
$\mathrm{BMO} \supsetneq L^\infty$	$C^{1,\text{Log-Lip}}_{loc}(B_1)$	$\mathrm{BMO} \supsetneq L^\infty$	$\text{par} - C^{1,\text{Log-Lip}}_{loc}(Q_1)$
Elliptic Theory		**Parabolic Theory**	

It is interesting to note that the parabolic regularity estimates agree with its elliptic counterpart provided $f \in L^{p,\infty}(Q_1)$.

Next picture (Fig. 1) shows the critical surfaces and the regions they define for the optimal regularity estimates available for solutions to (Eq).

[2] Here ε is the Escauriaza's universal constant which provides the minimal range which the Harnack inequality (resp. Hölder regularity) holds for viscosity solutions to fully nonlinear elliptic equations, since $p \geq n - \varepsilon$ (see [6] for more details).

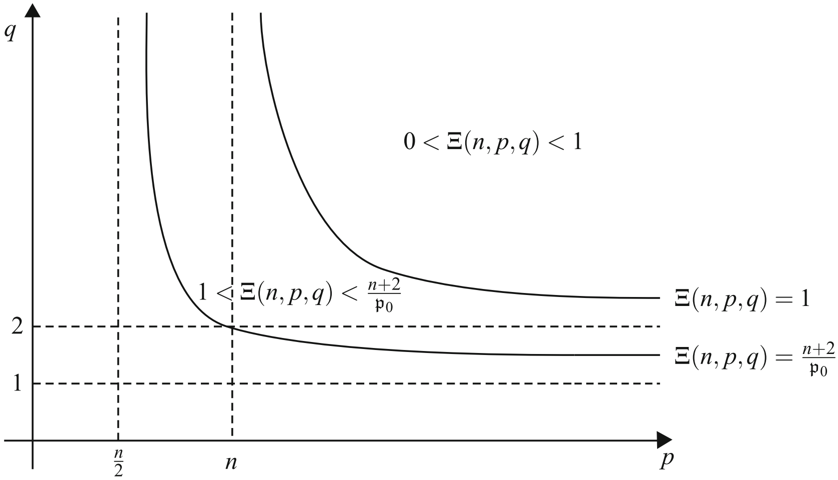

Fig. 1 Critical surfaces for optimal regularity estimates

2 Definitions and preliminary results

Throughout this paper $F\colon Sym(n) \times \mathbb{R}^n \times B_1(0) \times (-1, 0] \longrightarrow \mathbb{R}$ is a fully nonlinear uniformly elliptic operator with respect to the Hessian argument and Lipschitz with respect to gradient dependence. That is, there are constants $\Lambda \geq \lambda > 0$ and $\Gamma \geq 0$ such that for all $Z, W \in R^n$ and $M, N \in \mathrm{Sym}(n)$, space of $n \times n$ symmetric matrices, with $M \geq N$, there holds

$$\mathscr{P}^{-}_{\lambda,\Lambda}(M-N)-\Gamma|Z-W| \leq F(M, Z, X, t)-F(N, W, X, t) \leq \mathscr{P}^{+}_{\lambda,\Lambda}(M-N)+\Gamma|Z-W|. \tag{2.1}$$

Hereafter, $\mathscr{P}^{\pm}_{\lambda,\Lambda}$ denote the Pucci's extremal operators:

$$\mathscr{P}^{+}_{\lambda,\Lambda}(M) := \lambda \cdot \sum_{e_i<0} e_i + \Lambda \cdot \sum_{e_i>0} e_i \quad \text{and} \quad \mathscr{P}^{-}_{\lambda,\Lambda}(M) := \lambda \cdot \sum_{e_i>0} e_i + \Lambda \cdot \sum_{e_i<0} e_i$$

where $\{e_i : 1 \leq i \leq n\}$ are the eigenvalues of M. Any operator F which satisfies the condition (2.1) will be referred in this article as a $(\lambda, \Lambda, \Gamma)$-*parabolic operator*. Following classical terminology, any constant or mathematical term which depends only on dimension and of the parameters λ, Λ and Γ will be called *universal*.

We can (and will) always assume that F is normalized in the sense

$$F(0, 0, X, t) = 0, \tag{2.2}$$

and, unless otherwise stated, conditions (2.1) and (2.2) are always assumed throughout the text; sometimes we will refer F as a normalized $(\lambda, \Lambda, \Gamma)$ operator.

Equations and problems studied here are designed in the $(n + 1)$-dimensional Euclidean space, $\mathbb{R}^{n+1}$. The semi-open cylinder is denoted by

$Q_r(X_0,\tau) = B_r(X_0) \times (\tau - r^2, \tau]$. For simplicity we refer $Q_1(0,0) = Q_1$. The *parabolic distance* between the points $P_1 = (X_1, t_1)$ and $P_2 = (X_2, t_2) \in Q_1$ is defined by

$$d_{\text{par}}(P_1, P_2) := \sqrt{|X_1 - X_2|^2 + |t_1 - t_2|}.$$

For a function $u\colon Q_1 \to \mathbb{R}$ the semi-norm and norm for the *parabolic Hölder space* are defined respectively by

$$[u]_{C^{\alpha,\frac{\alpha}{2}}(Q_1)} := \sup_{\substack{(X,t),(Y,s)\in Q_1\\(X,t)\neq(Y,s)}} \frac{|u(X,t)-u(Y,s)|}{d_{\text{par}}((X,t),(Y,s))^\alpha} \quad \text{and} \quad \|u\|_{C^{\alpha,\frac{\alpha}{2}}(Q_1)} := \|u\|_{C^0(Q_1)} + [u]_{C^{\alpha,\frac{\alpha}{2}}(Q_1)}.$$

Under finiteness of such a norm one concludes that u is α-Hölder continuous with respect to the spatial variables and $\frac{\alpha}{2}$-Hölder with respect to the temporal variable.

We say that u is locally (parabolically) Log-Lipschitz continuous if the following quantity

$$[u]_{\text{par}-C^{0,\text{Log-Lip}}(Q_r(x_0,t_0))} := \sup_{(X,t),(Y,s)\in Q_r(x_0,t_0)} \frac{|u(X,t)-u(Y,s)|}{-r\log r} \quad \forall\, r \ll 1.$$

is finite for $(x_0, t_0) \in Q_1$. Moreover, the corresponding parabolic Log-Lipschitz norm is given by

$$\|u\|_{\text{par}-C^{0,\text{Log-Lip}}(Q_r(x_0,t_0))} := \|u\|_{C^0(Q_r(x_0,t_0))} + [u]_{\text{par}-C^{0,\text{Log-Lip}}(Q_r(x_0,t_0))}.$$

In what follows, $C^{1+\alpha,\frac{1+\alpha}{2}}(Q_1)$ denotes the space of u whose spacial gradient $Du(X,t)$ there exists in the classical sense for every $(X,t) \in Q_1$ and such that

$$\|u\|_{C^{1+\alpha,\frac{1+\alpha}{2}}(Q_1)} := \|u\|_{L^\infty(Q_1)} + \|Du\|_{L^\infty(Q_1)} + \sup_{\substack{(X,t),(Y,s)\in Q_1\\(X,t)\neq(Y,s)}} \frac{|u(X,t)-[u(Y,\tau)-Du(Y,s)\cdot(X-Y)]|}{d_{\text{par}}^{1+\alpha}((X,t),(Y,s))}$$

is finite. It is easy to verify that $u \in C^{1+\alpha,\frac{1+\alpha}{2}}(Q_1)$ implies every component of Du is $C^{0,\alpha}(Q_1)$, and u is $\frac{1+\alpha}{2}$-Hölder continuous in the variable t, see for instance [4, Section 1].

Now, we say that u is locally (parabolically) $C^{1,\text{Log-Lip}}$ continuous if the quantity

$$[u]_{\text{par}-C^{1,\text{Log-Lip}}(Q_r(x_0,t_0))} := \sup_{(X,t),(Y,s)\in Q_r(x_0,t_0)} \frac{|u(X,t)-[u(Y,s)+Du(Y,s).(X-Y)]|}{-r^2\log r}$$

is finite. Moreover, its parabolic $C^{1,\text{Log-Lip}}$-norm is given by

$$\begin{aligned}&\|u\|_{\text{par}-C^{1,\text{Log-Lip}}(Q_r(x_0,t_0))}\\&\quad := \|u\|_{C^0(Q_r(x_0,t_0))} + \|Du\|_{L^\infty(Q_r(x_0,t_0))} + [u]_{\text{par}-C^{1,\text{Log-Lip}}(Q_r(x_0,t_0))}.\end{aligned}$$

A function u belongs to the Sobolev space $W^{2,1,p}(Q_1)$ if it satisfies $u, Du, D^2u, u_t \in L^p(Q_1)$. The corresponding norm is given by

$$\|u\|_{W^{2,1,p}(Q_1)} = \left[\|u\|^p_{L^p(Q_1)} + \|u_t\|^p_{L^p(Q_1)} + \|Du\|^p_{L^p(Q_1)} + \|D^2u\|^p_{L^p(Q_1)}\right]^{\frac{1}{p}}$$

It follows by Sobolev embedding that if $p > \frac{n+2}{2}$ then $W^{2,1,p}(Q_1)$ is continuously embedded in $C^0(Q_1)$. Also, $u \in W^{2,1,p}_{loc}(Q_1)$ implies that u is twice parabolically differentiable a.e., see for more details [2].

Definition 2.1 (L^{P}*-viscosity solutions*) Let $\mathscr{G}: Sym(n) \times \mathbb{R}^n \times B_1(0) \times (-1, 0] \to \mathbb{R}$ be a uniformly elliptic operator, $\mathrm{P} > \frac{n+2}{2}$ and $f \in L^{\mathrm{P}}_{loc}(Q_1)$. We say that a function $u \in C^0(Q_1)$ is an L^{P}-viscosity subsolution (respectively supersolution) to

$$u_t - \mathscr{G}\left(D^2u(X,t), Du(x,t), X, t\right) = f(X,t) \quad in \quad Q_1 \tag{2.3}$$

if for all $\varphi \in W^{2,1,\mathrm{P}}_{loc}(Q_1)$ whenever $\varepsilon > 0$ and $\mathscr{O} \subset Q_1$ is an open set and

$$\varphi_t - \mathscr{G}\left(D^2\varphi(X,t), D\varphi(x,t), X, t\right) - f(X,t) \geq \varepsilon \quad (resp. \leq -\varepsilon) \quad a.e. \quad in \quad \mathscr{O}$$

then $u - \varphi$ cannot attains a local maximum (resp. minimum) in $\mathscr{O}$. In an equivalent manner, u is an L^{P}-viscosity subsolution (resp. supersolution) if for all test function $\varphi \in W^{1,2,\mathrm{P}}_{loc}(Q_1)$ and $(X_0, t_0) \in Q_1$ at which $u - \varphi$ attain a local maximum (resp. minimum) one has

$$\begin{cases} \underset{(X,t)\to(X_0,t_0)}{\text{essliminf}} \left[\varphi_t - \mathscr{G}\left(D^2\varphi(X,t), D\varphi(x,t), X, t\right) - f(X,t)\right] \leq 0 \\ \underset{(X,t)\to(X_0,t_0)}{\text{esslimsup}} \left[\varphi_t - \mathscr{G}\left(D^2\varphi(X,t), D\varphi(x,t), X, t\right) - f(X,t)\right] \geq 0 \end{cases} \tag{2.4}$$

Finally we say that u is an L^{P}-viscosity solution to (2.3) if it is both an L^{P}-viscosity supersolution and an L^{P}-viscosity subsolution.

Remark 2.2 We say that a function $u \in C^0(Q_1)$ is a C^0-viscosity solution to (2.3) when the sentences in (2.4) are evaluated pointwisely for all "test function" $\varphi \in C^{2,1}_{loc}(Q_1)$. This is the notion used by Imbert–Silvestre in [8] and Wang in [20,21].

According to [4, Section 6] (see also [20, Section 5]) for a fixed $(X_0, t_0) \in Q_1$, we measure the oscillation of the coefficients of F around (X_0, t_0) by the quantity

$$\Theta_F(X_0, t_0, X, t) := \sup_{M \in Sym(n)} \frac{|F(M, 0, X, t) - F(M, 0, X_0, t_0)|}{\|M\| + 1}. \tag{2.5}$$

For notation purposes, we shall often write $\Theta_F(0,0,X,t) = \Theta_F(X,t)$.

We recall that a function f is said to belong to the *Anisotropic Lebesgue* space, $L^{p,q}(Q_1)$ if

$$\|f\|_{L^{p,q}(Q_1)} := \left(\int_{-1}^{0}\left(\int_{B_1}|f(X,t)|^p dX\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} = \|\|f(\cdot,t)\|_{L^p(B_1)}\|_{L^q((-1,0])} < +\infty.$$

This is a Banach space when endowed with the norm above. When $p = q$, this is the standard definition of L^p spaces. The definition are naturally extended when either p or q are infinity. It is plain to verify that $L^{p,q}(Q_1) \subset L^s(Q_1)$ for $s := \min\{p,q\}$.

We recall again the existence of the constant $\mathfrak{p}_0$, satisfying $\frac{n+2}{2} \le \mathfrak{p}_0 < n+1$, for which Harnack inequality (resp, Hölder regularity) holds for L^{P}-viscosity solutions, provided $\mathrm{P} > \mathfrak{p}_0$, see for instance [4, Section 5]. The following compactness result becomes then available:

Proposition 2.3 (Compactness of solutions) *Let u be an L^{P}-viscosity solution to* (Eq) *in Q_r under the assumption $\mathrm{P} \ge \min\{p,q\} > \mathfrak{p}_0$. Then u is locally of class $C^{\beta,\frac{\beta}{2}}$ for some $0 < \beta < 1$ and*

$$\|u\|_{C^{\beta,\frac{\beta}{2}}(Q_r)} \le C(n,\lambda,\Lambda,\Gamma)r^{-\beta}\left(\|u\|_{L^\infty(Q_r)} + r^{2-\Xi(n,p,q)}\|f\|_{L^{p,q}(Q_r)}\right).$$

Upon appropriate regularity assumption on the boundary data, solutions are pre-compact in the C^0-topology up-to-the-boundary. We state such a result for future references and refer to [3, Proposition 4.6] and [4, Lemma 6.3] for a proof.

Proposition 2.4 (Pre-compactness up to the boundary) *Let Ω satisfy a uniform exterior cone condition, $Q := \Omega \times ((-T,0])$ and $\mathscr{C} \subset C^0(\partial_p Q)$ be compact, $R > 0$ and $\mathfrak{B}_R := \{f \in L^{\mathrm{P}}(Q) : \|f\|_{L^{\mathrm{P}}(Q)} \le R\}$. Then the set all functions $u \in C^0(\overline{Q})$ such that there exists $\psi \in \mathscr{C}$ and $f \in \mathfrak{B}_R$ for which u is an L^{P}-viscosity solution to*

$$u_t - \mathscr{P}^-_{\lambda,\Lambda}(D^2u) - \Gamma|Du| - f \le 0 \le u_t - \mathscr{P}^+_{\lambda,\Lambda}(D^2u) + \Gamma|Du| + f \quad in \quad Q$$

and $u = \psi$ on $\partial_p Q$ is pre-compact in $C^0(\overline{Q})$.

Another piece of information we need in our approach concerns the stability of the notion of viscosity solutions; that is the limit of a sequence of viscosity solutions turns out to be a viscosity solution of the limiting equation. More precisely, we refer to the following Lemma, whose proof can be found, for instance, in [4, Theorem 6.1].

Lemma 2.5 (Continuity with respect to equation) *Let F_j, F be normalized (λ,Λ,Γ) operators, $\mathrm{P} > \mathfrak{p}_0$, $f, f_j \in L^{\mathrm{P}}(Q_1)$ and u_j be L^{P}-viscosity solutions to*

$$(u_j)_t - F_j(D^2u_j, Du_j, x, t) = f_j \quad in \quad Q_1$$

for all $j \in \mathbb{N}$*. Assume that* $u_j \to u$ *locally uniformly as* $j \to \infty$*. Moreover, for all* $Q_r(x_0, t_0) \subset Q_1$ *and all* $\varphi \in W^{2,1,\mathrm{P}}(Q_r(x_0, t_0))$ *(test function), assume that*

$$g_j(x,t) := F_j(D^2\varphi(x,t), D\varphi(x,t), x, t) - f_j(x,t)$$

and

$$g(x,t) := F(D^2\varphi(x,t), D\varphi(x,t), x, t) - f(x,t)$$

satisfy

$$\|g - g_j\|_{L^{\mathrm{P}}(Q_r(x_0,t_0))} \to 0 \quad as \quad j \to \infty. \tag{2.6}$$

Then, u *is an* L^{P}*-viscosity solution to*

$$u_t - F(D^2u, Du, x, t) = f \quad in \quad Q_r(x_0, t_0).$$

Furthermore, if F *and* f *are continuous functions, then* u *is a* C^0*-viscosity solution if* (2.6) *holds for all* $\varphi \in C^{2,1}(Q_1)$ *(test function).*

In the sequel, we obtain a Lemma which provides a tangential path toward the regularity theory available for constant coefficient, homogeneous $\mathfrak{F}$-caloric functions.

Lemma 2.6 ($\mathfrak{F}$-Caloric approximation Lemma) *Let* u *be an* L^{P}*-viscosity solution to* (Eq) *with* $|u| \le 1$ *and* $f \in L^{p,q}(Q_1)$ *with* $\mathrm{P} := \min\{p,q\} > \mathfrak{p}_0$*. Define* $\mathfrak{h}\colon Q_{1/2} \to \mathbb{R}$ *to be the* L^{P}*-viscosity solution of*

$$\begin{cases} \mathfrak{h}_t - F(D^2\mathfrak{h}, 0, 0, 0) = 0 & in \ Q_{\frac{1}{2}} \\ \mathfrak{h} = u & on \ \partial_p Q_{\frac{1}{2}} \end{cases} \tag{2.7}$$

Given $\delta > 0$*, there exists* $\eta = \eta(\delta, n, \lambda, \Lambda, \mathrm{P}) > 0$ *such that if*

$$\max\left\{ \left(⨍_{Q_1} \Theta_F^{\mathrm{P}}(X,t) \right)^{\frac{1}{\mathrm{P}}}, \ \|f\|_{L^{p,q}(Q_1)}, \ \Gamma \right\} \le \eta,$$

then

$$\sup_{Q_{\frac{1}{2}}} |u - \mathfrak{h}| \le \delta. \tag{2.8}$$

Proof The proof is based on a contradiction argument. Suppose there exists a $\delta_0 > 0$ for which the thesis of the Lemma, namely sentence (2.8), does not hold, regardless how small we make η. That means we could find sequences of functions $(u_j)_{j\ge1}$, $(\mathfrak{h}_j)_{j\ge1}$ with $|u_j| \le 1$, a sequence of normalized $(\lambda, \Lambda, \Gamma_j)$-operators $F_j\colon Sym(n) \times \mathbb{R}^n \times Q_1 \to \mathbb{R}$ and a sequence of functions $(f_j)_{j\ge1}$ all linked through the set of equations

$$\begin{cases} (u_j)_t - F_j(D^2u_j, Du_j, X, t) = f_j & \text{in } Q_1 \\ (\mathfrak{h}_j)_t - F_j(D^2\mathfrak{h}_j, 0, 0, 0) = 0 & \text{in } Q_{\frac{1}{2}} \\ \mathfrak{h}_j = u_j & \text{on } \partial_p Q_{\frac{1}{2}} \end{cases} \tag{2.9}$$

in the L^{P}-viscosity sense, with

$$\max\left\{\left(\fint_{Q_1}\Theta^{\mathrm{P}}_{F_j}(X,t)\right)^{\frac{1}{\mathrm{P}}},\ \|f_j\|_{L^{p,q}(Q_1)},\ \Gamma_j\right\} = \mathrm{o}(1) \quad as \quad j \to \infty; \tag{2.10}$$

however

$$\sup_{Q_{\frac{1}{2}}} |u_j - \mathfrak{h}_j| > \delta_0 \quad \text{for all} \quad j \in \mathbb{N}. \tag{2.11}$$

By compactness of the sequences $(u_j)_{j\geq 1}$ and $(\mathfrak{h}_j)_{j\geq 1}$, namely Propositions 2.3 and 2.4, we may assume, passing to a subsequence if necessary, that $u_j \to u_0$ and $\mathfrak{h}_j \to \mathfrak{h}_0$ uniformly in $\overline{Q_{\frac{1}{2}}}$.

Next we will prove that $u_0 = \mathfrak{h}_0$. The idea is to conclude that both functions solve the same PDE, for which uniqueness is available, and hence this would contradict (2.11) for $j \gg 1$ large enough.

Initially we note that it follows from structural condition imposed on the operators $(F_j)_{j\geq 1}$, namely, (2.1), that, up to a subsequence, can may assume

$$F_j(M, 0, 0, 0) \to \mathfrak{F}_0(M) \tag{2.12}$$

locally uniformly in the space $Sym(n)$, where $\mathfrak{F}_0$ is a $(\lambda, \Lambda, 0)$ operator with constant coefficients—see for instance [4, Sections 1 and 6] and [21, Lemma 1.4]. Also, applying Lemma 2.5 along with uniqueness result, for instance [4, Lemma 6.2], we know $\mathfrak{h}_0$ is the unique C^0-viscosity solution to

$$\begin{cases} (\mathfrak{h}_0)_t - \mathfrak{F}_0(D^2\mathfrak{h}_0) = 0 & \text{in } Q_{\frac{1}{2}} \\ \mathfrak{h}_0 = u_0 & \text{on } \partial_p Q_{\frac{1}{2}}. \end{cases} \tag{2.13}$$

To conclude the proof, we will show that u_0 also solves (2.13) in the viscosity sense. For that end, let $\varphi \in C^{2,1}(Q_{\frac{1}{2}})$ be a test function and define

$$\mathfrak{K}_j(\varphi) := \left|F_j\left(D^2\varphi(X,t), D\varphi(X,t), X, t\right) - f_j(X,t) - \mathfrak{F}_0\left(D^2\varphi(X,t)\right)\right|.$$

We estimate

$$\begin{aligned} \mathfrak{K}_j(\varphi) \leq \Gamma_j|D\varphi(X,t)| + \Theta_{F_j}(X,t)\left(|D^2\varphi(X,t)| + 1\right) + |f_j(X,t)| \\ + \left|F_j\left(D^2\varphi(X,t), 0, 0, 0\right) - \mathfrak{F}_0\left(D^2\varphi(X,t)\right)\right|. \end{aligned} \tag{2.14}$$

Finally, since from (2.10) and (2.12) one has that the L^{P}-norm of RHS of (2.14) goes to zero as $j \to \infty$, we can apply once more Lemma 2.5, which assures u_0 is too a solution of (2.13), and by uniqueness, $u_0 = \mathfrak{h}_0$, which yields a contradiction as indicated before. □

We conclude this section by commenting on reduction processes to be used throughout the proof.

Remark 2.7 (Preserving ellipticity) If F is a $(\lambda, \Lambda, \Gamma)$-parabolic operator then

$$\mathscr{G}\left(M, \overrightarrow{Z}, X, t\right) = \kappa_0^2 \cdot F\left(\frac{M}{\kappa_0^2}, \frac{\overrightarrow{Z}}{\kappa_0}, X, t\right)$$

is a $(\lambda, \Lambda, \kappa_0 \cdot \Gamma)$-parabolic operator for any $\kappa_0 > 0$.

Remark 2.8 (Normalization and scaling) We can always suppose, without loss of generality, that viscosity solutions of

$$u_t - F(D^2u, Du, X, t) = f(X, t) \quad in \quad Q_1$$

satisfy $\|u\|_{L^\infty(Q_1)} \leq 1$. Also given a small number $\varepsilon_0 > 0$, we can also suppose that $\Gamma + \|f\|_{L^{p,q}(Q_1)} < 2\varepsilon_0$. Indeed, for

$$\kappa := \frac{\varepsilon_0}{\varepsilon_0(\|u\|_{L^\infty(Q_1)} + 1) + \|f\|_{L^{p,q}(Q_1)}} \quad \text{and} \quad R > \max\left\{1, \frac{\Gamma}{\varepsilon_0}, \sqrt{\kappa}\right\},$$

we define

$$v(X, t) := \kappa u\left(\frac{1}{R}X, \frac{1}{R^2}t\right).$$

It is easy to verify that

1. $\|v\|_{L^\infty(Q_1)} \leq 1$;
2. $v_t - \mathscr{G}(D^2v, Dv, X, t) = g(X, t)$ in Q_1, in the L^{P}-viscosity sense, where

$$\mathscr{G}(M, \overrightarrow{Z}, X, t) = \frac{\kappa}{R^2}F\left(\frac{R^2}{\kappa}M, \frac{R}{\kappa}\overrightarrow{Z}, \frac{1}{R}X, \frac{1}{R^2}t\right) \quad and$$
$$g(X, t) = \frac{\kappa}{R^2}f\left(\frac{1}{R}X, \frac{1}{R^2}t\right);$$

3. $\mathscr{G}$ is a $\left(\lambda, \Lambda, \Gamma^\sharp\right)$-parabolic operator, with $\Gamma^\sharp < \varepsilon_0$;
4. $\|g\|_{L^{p,q}(Q_1)} \leq \varepsilon_0$;
5. $\left(⨍_{Q_r} \Theta^{\mathrm{P}}_{\mathscr{G}}(X, t)\right)^{\frac{1}{\mathrm{P}}} \leq \max\left\{1, \frac{\kappa}{R^2}\right\}\left(⨍_{Q_{rR^{-1}}} \Theta^{\mathrm{P}}_{F}(X, t)\right)^{\frac{1}{\mathrm{P}}}$ (cf. [4, Remark 6.4]).

Once a universal estimate is proven for v, a corresponding one becomes available for the general solution u, properly adjusted by the choices of κ and R.

3 Optimal $C^{\alpha,\frac{\alpha}{2}}$ regularity

Our strategy for proving optimal $C^{\alpha,\frac{\alpha}{2}}$ regularity estimates is based on a refined compactness method as in [4,18,20,21]. It relies on a control of oscillation decay obtained from the regularity theory available for a "better" limiting equation; the realm of the so-called *geometric tangential analysis*. Next lemma is the key access point for the approach, as it provides the first step in the iteration process to be implemented.

Lemma 3.1 *Let u be a normalized L^{P}-viscosity solution for* (Eq)*, that is,* $|u| \leq 1$ *in* Q_1*. Given* $0 < \gamma < 1$*, there exist* $\eta(n,\lambda,\Lambda,\gamma) > 0$ *and* $0 < \rho(n,\lambda,\Lambda,\gamma) \ll \frac{1}{2}$*, such that if*

$$\max\left\{\left(\fint_{Q_1} \Theta_F^{\mathrm{P}}(X,t)\right)^{\frac{1}{\mathrm{P}}},\ \|f\|_{L^{p,q}(Q_1)},\ \Gamma\right\} \leq \eta \quad \textit{with} \quad 1 < \Xi(n,p,q) < \frac{n+2}{\mathfrak{p}_0}$$

then, for some $\varsigma \in \mathbb{R}$*, with* $|\varsigma| \leq C(n,\lambda,\Lambda)$ *there holds*

$$\sup_{Q_\rho} |u - \varsigma| \leq \rho^\gamma. \tag{3.1}$$

Proof For a $\delta > 0$ to be chosen a posteriori, let $\mathfrak{h}$ be a solution to a homogeneous uniformly parabolic equation with constant coefficients, that is δ-close to u in the L^∞-norm, i.e.,

$$\mathfrak{h}_t - F(D^2\mathfrak{h}) = 0 \quad \text{in} \quad Q_1 \quad \text{and} \quad \sup_{Q_{\frac{1}{2}}} |u - \mathfrak{h}| \leq \delta. \tag{3.2}$$

Lemma 2.6 assures the existence of such a function. Once our choice for δ, to be set of the end of this proof, is universal, then the choice of $\eta(n,\lambda,\Lambda,\delta)$ is universal too. From the regularity theory available for $\mathfrak{h}$, see for instance [4, Section 7] or [21, Section 1.2], we can estimate

$$|\mathfrak{h}(X,t) - \mathfrak{h}(0,0)| \leq C(n,\lambda,\Lambda) d_{\text{par}}((X,t),(0,0)) \ \text{ for } \sqrt{|X|^2 + |t|} < \frac{1}{3}, \tag{3.3}$$

and also,

$$|\mathfrak{h}(0,0)| \leq C(n,\lambda,\Lambda). \tag{3.4}$$

For $\varsigma = \mathfrak{h}(0,0)$ it follows from Eqs. (3.2) and (3.3) via triangular inequality that

$$\sup_{Q_\rho} |u - \varsigma| \leq \delta + C(n,\lambda,\Lambda)\rho. \tag{3.5}$$

We make the following universal selections:

$$\rho := \min\left\{r_0, \left(\frac{1}{2C}\right)^{\frac{1}{1-\gamma}}\right\} \quad \textit{and} \quad \delta := \frac{1}{2}\rho^\gamma \tag{3.6}$$

where $C > 0$ is a universal constant from Eq. (3.3) and $0 < r_0 \leq 1$ is a universal constant to appear in the Theorem 3.2. Let us stress that the choices above depend only upon dimension, ellipticity parameters and the fixed exponent. From the above choices we obtain

$$\sup_{Q_\rho} |u - \varsigma| \leq \rho^\gamma .$$

and the Lemma is concluded. □

Theorem 3.2 *Let u be an L^{P}-viscosity solution of (Eq) with $f \in L^{p,q}(Q_1)$ and*

$$1 < \Xi(n, p, q) < \frac{n+2}{\mathfrak{p}_0}.$$

There exist universal, positive constants r_0 and θ_0 such that if

$$\sup_{0<r\leq r_0} \sup_{(Y,\tau)\in Q_{\frac{1}{2}}} \left(\fint_{Q_r(Y,\tau)} \Theta_F^{\mathrm{P}}(Y, \tau, X, t) \right)^{\frac{1}{\mathrm{P}}} \leq \theta_0,$$

then, for a constant $C > 0$ and $\gamma := 2 - \Xi(n, p, q)$, there holds

$$\|u\|_{C^{\gamma, \frac{\gamma}{2}}\left(Q_{\frac{1}{2}}\right)} \leq C(n, \lambda, \Lambda, \gamma) \left[\|u\|_{L^\infty(Q_1)} + \|f\|_{L^{p,q}(Q_1)} + 1 \right].$$

Proof Through normalization and scaling processes, see Remark 2.8, we can suppose without losing generality that $|u| \leq 1$ and $\|f\|_{L^{p,q}(Q_1)} \leq \eta$, where η is the universal constant from Lemma 3.1 when we set $\gamma = \gamma(n, p, q) = 2 - \Xi(n, p, q)$. Once selected $\theta_0 = \eta$ the goal will be to iterate the Lemma 3.1. For a fixed $(Y, \tau) \in Q_{\frac{1}{2}}$ we claim that there exists a convergent sequence of real numbers $\{\varsigma_k\}_{k\geq 1}$, such that

$$\sup_{Q_{\rho^k}(Y,\tau)} |u - \varsigma_k| \leq \rho^{k.\gamma} \tag{3.7}$$

where the radius $0 < \rho \ll \frac{1}{2}$ is given by Lemma 3.1, upon the selection of γ as above.

The proof of (3.7) will follow by induction process. Lemma 3.1 gives the first step of induction, $k = 1$. Now suppose verified the k^{th} step in (3.7). We define

$$v_k(X, t) = \frac{u(Y + \rho^k X, \tau + \rho^{2k} t) - \varsigma_k}{\rho^{k.\gamma}}$$

and

$$F_k(M, Z, X, t) := \rho^{k[2-\gamma]} F \left(\frac{1}{\rho^{k[2-\gamma]}} M, \frac{1}{\rho^{k[1-\gamma]}} Z, Y + \rho^k X, \tau + \rho^{2k} t \right).$$

As commented before, see Remark 2.7, F_k is $(\lambda, \Lambda, \Gamma)$-parabolic operator, moreover by the induction hypothesis, $|v_k| \leq 1$ and

$$(v_k)_t - F_k(D^2 v_k, Dv_k, X, t) = \rho^{k.[2-\gamma]} f(Y + \rho^k X, \tau + \rho^{2k} t) =: f_k(X, t),$$

in the L^{P}-viscosity sense. One easily computes,

$$\|f_k\|_{L^{p,q}(Q_1)} = \rho^{k(2-\gamma)} \rho^{-k.\Xi(n,p,q)} \left(\int_{\tau-\rho^{2k}}^{\tau} \left(\int_{B_{\rho^k}(Y)} |f(Z, s)|^p dZ \right)^{\frac{q}{p}} ds \right)^{\frac{1}{q}}.$$

Due to the sharp choice of $\gamma(n, p, q) = 2 - \Xi(n, p, q)$, we have that

$$\|f_k\|_{L^{p,q}(Q_1)} = \|f\|_{L^{p,q}\left(B_{\rho^k}(Y)\times(\tau-\rho^{2k},\tau]\right)} \le \|f\|_{L^{p,q}(Q_1)} \le \eta,$$

as well as,

$$\left(⨍_{Q_1} \Theta^{\mathrm{P}}_{F_k}(X, t) \right)^{\frac{1}{\mathrm{P}}} \le \max\left\{1, \rho^{k(2-\gamma)}\right\} \left(⨍_{Q_{\rho^k}(Y,\tau)} \Theta^{\mathrm{P}}_{F}(Y, \tau, X, t) \right)^{\frac{1}{\mathrm{P}}} \le \eta.$$

In conclusion, we are allowed to employ Lemma 3.1 to v_k, which provides the existence of a universally bounded real number $\overline{\varsigma}_k$ with $|\overline{\varsigma}_k| \le C$, such that

$$\sup_{Q_\rho} |v_k - \overline{\varsigma}_k| \le \rho^{\gamma}. \tag{3.8}$$

Finally, if we select

$$\varsigma_{k+1} := \varsigma_k + \rho^{k.\gamma} \overline{\varsigma}_k \tag{3.9}$$

and rescale (3.8) back to the unit domain, we obtain the $(k+1)$th step in the induction process (3.7). In addition, we have that

$$|\varsigma_{k+1} - \varsigma_k| \le C\rho^{k.\gamma}, \tag{3.10}$$

and hence the sequence $\{\varsigma_k\}_{k\ge 1}$ is Cauchy, and so it converges. From (3.7) $\varsigma_k \to u(Y, \tau)$. As well as from (3.10) it follows that

$$|u(Y, \tau) - \varsigma_k| \le \frac{C}{1-\rho^{\gamma}} \rho^{k.\gamma}, \tag{3.11}$$

Finally, for $0 < r < \rho$, let k the smallest integer such that $(X, t) \in Q_{\rho^k}(Y, \tau) \setminus Q_{\rho^{k+1}}(Y, \tau)$. It follows from (3.7) and (3.11) that

$$\begin{aligned}
\sup_{Q_r(Y,\tau)} \tfrac{|u(X,t)-u(Y,\tau)|}{d_{\mathrm{par}}((X,t),(Y,\tau))^{\gamma}} &\le \sup_{Q_r(Y,\tau)} \tfrac{|u(X,t)-\varsigma_k|+|u(Y,\tau)-\varsigma_k|}{d_{\mathrm{par}}((X,t),(Y,\tau))^{\gamma}} \\
&\le \left(1 + \tfrac{C}{1-\rho^{\gamma}}\right) \sup_{Q_r(Y,\tau)} \tfrac{\rho^{k.\gamma}}{d_{\mathrm{par}}((X,t),(Y,\tau))^{\gamma}} \\
&\le \left(1 + \tfrac{C}{1-\rho^{\gamma}}\right) \tfrac{1}{\rho^{\gamma}}.
\end{aligned}$$

Last estimate, combined with Remark 2.8 and a standard covering argument provide

$$\|u\|_{C^{\gamma,\frac{\gamma}{2}}\left(Q_{\frac{1}{2}}\right)} \leq C(n,\lambda,\Lambda,\gamma)\left[\|u\|_{L^{\infty}(Q_1)} + \|f\|_{L^{p,q}(Q_1)} + 1\right].$$

and hence the proof of Theorem is verified. □

Remark 3.3 The exponent of Hölder regularity of our result is sharp. This can be verified through of following example from [19]: Let $u \in C_{loc}((-1,0]; L^2_{loc}(B_1)) \cap L^2_{loc}((-1,0]; W^{1,2}_{loc}(B_1))$ be a weak solution to

$$u_t - \Delta u = f \quad in \quad Q_1$$

Suppose that $1 < \Xi(n,p,q) < 2$ then for $\gamma := 2 - \Xi(n,p,q)$ we have that $u \in C^{\gamma,\frac{\gamma}{2}}_{loc}(Q_1)$. Remark that in this case $\mathfrak{p}_0 = \frac{n+2}{2}$.

Remark 3.4 Under VMO assumption of the coefficients of the operator F:

$$\fint_{Q_r(Y,\tau)} \Theta^{\mathrm{P}}_F(Y,\tau,X,t) = \mathrm{o}(1),$$

as $r \to 0$, Theorem 3.2 holds without the smallness oscillation condition, as it can always be assumed upon an appropriate scaling.

Remark 3.5 Under no assumptions on the coefficients, other than ellipticity, adjustments in the proof of previous Theorem yields $C^{\gamma,\frac{\gamma}{2}}_{loc}(Q_1)$ where $\gamma := \min\left\{\beta^-, 2 - \Xi(n,p,q)\right\}$ where $0 < \beta < 1$ is the maximal exponent from Proposition 2.3.

4 Parabolic Log-Lipschitz type estimates

In this section we address the question of finding the optimal and universal modulus of continuity for solutions of uniformly parabolic equations of the form (Eq) whose right hand side lies in the borderline space $L^{p,q}(Q_1)$, when p and q lie on the critical surface:

$$\Xi(n,p,q) = 1.$$

Such estimate is particularly important to the general theory of fully nonlinear parabolic equations. Through a simple analysis one verifies that solutions of (Eq), with sources under the above borderline integrability condition should be asymptotically Lipschitz continuous. Indeed, as $\Xi(n,p,q) \to 1^+$, solutions are parabolically Hölder continuous for every exponent $0 < \alpha < 1$. The key goal in this section is to obtain the sharp, quantitative modulus of continuity for u.

Lemma 4.1 *Let u be a normalized L^{P}-viscosity solution to* (Eq). *There exist $\eta(n,\lambda,\Lambda)>0$ and $0<\rho(n,\lambda,\Lambda)\ll\frac{1}{2}$, such that if*

$$\max\left\{\left(\fint_{Q_1}\Theta_F^{\mathrm{P}}(X,t)\right)^{\frac{1}{\mathrm{P}}},\ \|f\|_{L^{p,q}(Q_1)},\ \Gamma\right\}\le\eta \tag{4.1}$$

under the condition $\Xi(n,p,q)=1$, then, we can find an affine function $\mathfrak{L}(X):=\mathfrak{A}+\langle\mathfrak{B},X\rangle$, with universally bounded coefficients, $|\mathfrak{A}|+|\mathfrak{B}|\le C(n,\lambda,\Lambda)$, such that

$$\sup_{Q_\rho}|u-\mathfrak{L}|\le\rho. \tag{4.2}$$

Proof For a $\delta>0$ which will be chosen a posteriori, we apply Lemma 2.6 and find a function $\mathfrak{h}\colon Q_{\frac{1}{2}}\to\mathbb{R}$ satisfying

$$\mathfrak{h}_t-F(D^2\mathfrak{h})=0\quad\text{in}\quad Q_{\frac{1}{2}},$$

in the L^{P}-viscosity sense such that

$$\sup_{Q_{\frac{1}{2}}}|u-\mathfrak{h}|\le\delta. \tag{4.3}$$

We now define

$$\mathfrak{L}(X)=\mathfrak{h}(0,0)+\langle D\mathfrak{h}(0,0),X\rangle, \tag{4.4}$$

and apply the regularity theory available for $\mathfrak{h}$, see for instance [4, Section 7] or [21], as to assure the existence of a universal constants $0<\alpha_F<1$ and $C>0$ such that

$$|\mathfrak{h}(X,t)-\mathfrak{L}(X)|\le Cd_{\mathrm{par}}((X,t),(0,0))^{1+\alpha_F},\quad\text{for}\quad\sqrt{|X|^2+|t|}<\frac{1}{3}. \tag{4.5}$$

It is time to make universal choices: we set

$$\rho:=\min\left\{r_0,\left(\frac{1}{2C}\right)^{\frac{1}{\alpha_F}}\right\}<\frac{1}{2}\quad\text{and}\quad\delta:=\frac{1}{2}\rho, \tag{4.6}$$

which decides the value of $\eta(n,\lambda,\Lambda)>0$ through the approximation Lemma 2.6. It the sequel we estimate

$$\sup_{Q_\rho}|u-\mathfrak{L}|\le\sup_{Q_\rho}|u-\mathfrak{h}|+\sup_{Q_\rho}|\mathfrak{h}-\mathfrak{L}|\le\rho,$$

and the proof is complete. □

Theorem 4.2 *Let u be an L^{P}-viscosity solution to (Eq). There exists universal constants, $r_0 > 0$ and $\theta_0 > 0$, such that if*

$$\sup_{0<r\le r_0}\ \sup_{(Y,\tau)\in Q_{\frac{1}{2}}}\left(\fint_{Q_r(Y,\tau)} \Theta_F^{\mathrm{P}}(Y,\tau,X,t)\right)^{\frac{1}{\mathrm{P}}} \le \theta_0,$$

then, for a universal constant $C > 0$ and any $(X,t), (Y,\tau) \in Q_{\frac{1}{2}}$, there holds

$$|u(X,t)-u(Y,\tau)| \le C\left[\|u\|_{L^\infty(Q_1)} + \|f\|_{L^{p,q}(Q_1)} + 1\right]\cdot \omega(d_{par}((X,t),(Y,\tau))),$$

where $\omega(s) := s\log\frac{1}{s}$ is the Lipschitz logarithmic modulus of continuity.

Proof We start off the proof by assuming, with no loss of generality, that $|u| \le 1$ and

$$\|f\|_{L^{p,q}(Q_1)} < \frac{\eta}{4} \quad \text{and} \quad \Gamma < \frac{\eta}{8\max\{1, \mathscr{L}^n(B_1(0))\}},$$

where $\eta = \eta(n,\lambda,\Lambda)$ the largest positive number such that the Lemma 4.1 holds. Choose $\theta_0 = \frac{\eta}{8}$. For a fixed $(Y,\tau) \in Q_{\frac{1}{2}}$ we will prove the existence of a sequence of affine functions

$$\mathfrak{L}_k(X) = \mathfrak{A}_k + \langle \mathfrak{B}_k, X - Y\rangle$$

such that

$$\sup_{B_{\rho^k}(Y)\times(\tau-\rho^{2k},\tau]} |u - \mathfrak{L}_k| \le \rho^k \tag{4.7}$$

and

$$|\mathfrak{A}_{k+1} - \mathfrak{A}_k| \le C\rho^k \quad \text{and} \quad |\mathfrak{B}_{k+1} - \mathfrak{B}_k| \le C, \tag{4.8}$$

where $0 < \rho \ll \frac{1}{2}$ is the radius given by Lemma 4.1. Notice that the second estimate in (4.8) gives the growing estimate on the linear coefficients of order

$$|\mathfrak{B}_k| \le Ck. \tag{4.9}$$

We now argue by induction. Lemma 4.1 provides the first step and now we suppose that we have already verified the kth step of (4.7). Define

$$v_k(X,t) := \frac{u(Y+\rho^k X, \tau+\rho^{2k}t) - \mathfrak{L}_k(Y+\rho^k X)}{\rho^k},$$

which verifies $|v_k| \le 1$ in Q_1, by the induction condition. Define

$$F_k(M, \overrightarrow{p}, X, t) := \rho^k F\left(\frac{M}{\rho^k}, \overrightarrow{p}, Y+\rho^k X, \tau+\rho^{2k}t\right).$$

It is plain to check that F_k is a $(\lambda, \Lambda, \Gamma)$-parabolic operator and

$$(v_k)_t - F_k(D^2 v_k, Dv_k, X, t) = f_k(X,t) + g_k(X,t) = H_k(X,t)$$

in the L^{P}-viscosity sense, where

$$f_k(X,t) := \rho^k f(Y + \rho^k X, \tau + \rho^{2k} t)$$

and

$$g_k(X,t) := F_k(D^2 v_k, Dv_k + B_k, X, t) - F_k(D^2 v_k, Dv_k, X, t).$$

Moreover,

$$\|f_k\|_{L^{p,q}(Q_1)} = \rho^k \rho^{-k.\Xi(n,p,q)} \left(\int_{\tau-\rho^{2k}}^{\tau} \left(\int_{B_{\rho^k}(Y)} |f(Z,s)|^p dZ \right)^{\frac{q}{p}} ds \right)^{\frac{1}{q}}.$$

By the critical condition, $\Xi(n,p,q) = 1$, we verify that

$$\|f_k\|_{L^{p,q}(Q_1)} = \|f\|_{L^{p,q}\left(B_{\rho^k}(Y)\times(\tau-\rho^{2k},\tau]\right)} < \frac{\eta}{4}.$$

Moreover, given the smallest regime on Γ, assumption (2.1) and (4.9), we have

$$|g_k(X,t)| \le Ck\rho^k\Gamma < \frac{\eta}{8\max\{1, \mathscr{L}^n(B_1(0))\}}.$$

Thus,

$$\|g_k\|_{L^{p,q}(Q_1)} \le \frac{\eta}{8\max\{1, \mathscr{L}^n(B_1(0))\}} \sqrt[p]{\mathscr{L}^n(B_1(0))} \le \frac{\eta}{8}.$$

Therefore, $\|H_k\|_{L^{p,q}(Q_1)} < \frac{3\eta}{8}$. Furthermore,

$$\left(⨍_{Q_1} \Theta^{\mathrm{P}}_{F_k}(X,t) \right)^{\frac{1}{\mathrm{P}}} \le \max\left\{1, \rho^k\right\} \left(⨍_{Q_{\rho^k}(Y,\tau)} \Theta^{\mathrm{P}}_{F}(Y,\tau,X,t) \right)^{\frac{1}{\mathrm{P}}} \le \frac{\eta}{8}.$$

We have verified that we can apply Lemma 4.1 to the function v_k, assuring the existence of an affine function $\widetilde{\mathfrak{L}}_k(X) = \widetilde{\mathfrak{A}}_k + \langle \widetilde{\mathfrak{B}}_k, X\rangle$ satisfying $|\widetilde{\mathfrak{A}}_k|, |\widetilde{\mathfrak{B}}_k| \le C$, such that

$$\sup_{Q_\rho} |v_k - \widetilde{\mathfrak{L}}_k| \le \rho. \tag{4.10}$$

We now define

$$\mathfrak{A}_{k+1} := \mathfrak{A}_k + \rho^k \widetilde{\mathfrak{A}}_k \quad \text{and} \quad \mathfrak{B}_{k+1} := \mathfrak{B}_k + \widetilde{\mathfrak{B}}_k. \tag{4.11}$$

Rescaling (4.10) to the unit domain gives the $(k+1)$th induction step. The first estimate in (4.7) assures that the sequence $\{\mathfrak{A}_k\}_{k\geq 1}$ converges to $u(Y,\tau)$. Also we can estimate, by geometric series,

$$|u(Y,\tau)-\mathfrak{A}_k| \leq \frac{C\rho^k}{1-\rho}. \tag{4.12}$$

Finally, for $0<r<\rho$, let k be the lowest integer such that

$$(X,t) \in Q_{\rho^k}(Y,\tau)\setminus Q_{\rho^{k+1}}(Y,\tau).$$

It follows by (4.7), (4.9) and (4.12) that

$$\begin{aligned}\sup_{Q_r(Y,\tau)} \frac{|u(X,t)-u(Y,\tau)|}{r\log r^{-1}} &\leq \sup_{Q_{\rho^k}(Y,\tau)} \frac{|u-\mathfrak{L}_k|+|u(Y,\tau)-\mathfrak{A}_k|+|\mathfrak{B}_k|\rho^k}{r\log r^{-1}}\\ &\leq C \sup_{Q_{\rho^k}(Y,\tau)} \frac{k\rho^k}{r\log r^{-1}}\\ &\leq C(n,\lambda,\Lambda),\end{aligned}$$

and the proof of the theorem is concluded. □

Remark 4.3 As a consequence of the estimate given by Theorem 4.2, we are able to derive a precise integral behaviour of the gradient of a solution to (Eq). Indeed, one can derive the following pointwise control, say near (0, 0):

$$|Du(X,t)| \lesssim -C\log(|X|^2+|t|), \quad \text{for} \ \sqrt{|X|^2+|t|} \ll \frac{1}{2}$$

Under suitable smallness regime on $f \in L^{p,q}(Q_1)$ and on $\Theta_F \in L^{\mathrm{P}}(Q_1)$, it follows by an adjustment of our arguments, combined with $H^{1,\frac{1}{2},s}$ interior estimates from [4, Theorem 7.3] that one can approximate an L^{P}-viscosity solution of (Eq) by an $\mathfrak{F}$-caloric function

$$\mathfrak{h}_t - F(D^2\mathfrak{h}, X_0, t_0) = 0 \quad \text{in} \quad Q_{\frac{1}{2}},$$

in the $H^{1,\frac{1}{2},s}(Q_{\frac{1}{2}})$ topology. Thus, through an iterative process as indicated in the proof of Theorem 4.2, one can find affine functions $\mathfrak{L}_k$ such that

$$\fint_{Q_{\rho^k}} |D(u-\mathfrak{L}_k)|^s \leq 1.$$

Therefore, it is possible to establish s-BMO type of estimates for the gradient in terms of the $L^{p,q}(Q_1)$ norm of f, when the critical condition $\frac{n}{p}+\frac{2}{q}=1$ is verified. That is,

$$\|Du\|_{s-\mathrm{BMO}(Q_r)} \leq C\left[\|u\|_{L^\infty(Q_1)}+\|f\|_{L^{p,q}(Q_1)}+1\right], \quad \text{for} \quad 0<r\ll \frac{1}{4}.$$

Comparing such an estimate with the results from [4, Theorem 7.3], it synthesizes quantitatively the fact of

$$|Du| \in \bigcap_{m\geq 1} L^m_{loc}(Q_1),$$

since L^P-viscosity solutions have its gradient in $L^s_{loc}(Q_1)$ for all $s < \frac{n+2}{\Xi(n,p,q)-1}$.

5 Optimal $C^{1+\alpha, \frac{1+\alpha}{2}}$ regularity

In this section we obtain asymptotically sharp $C^{1+\sigma, \frac{1+\sigma}{2}}$ interior regularity estimates for solutions of (Eq). Such estimates are already available in the literature, see for instance [4] and [20]. We shall only comment on how we can deliver them by means of the arguments designed in Sect. 4.

Initially, we revisit Lemma 4.1 and observe that if $0 < \alpha_F \leq 1$ represents the optimal exponent from the $C^{1+\overline{\alpha}, \frac{1+\overline{\alpha}}{2}}$ regularity theory for solutions to homogeneous $(\lambda, \Lambda, \Gamma)$-parabolic operators with constant coefficients, then given

$$\alpha \in (0, \alpha_F) \cap (0, 1 - \Xi(n, p, q)]\,, \tag{5.1}$$

since $\sup\limits_{0<r\leq r_0} \sup\limits_{(Y,\tau)\in Q_{\frac{1}{2}}} \left(⨏_{Q_r(Y,\tau)} \Theta^{\mathrm{P}}_F(Y, \tau, X, t)\right)^{\frac{1}{\mathrm{P}}}$ and $\|f\|_{L^{p,q}}$ are under universal smallest regime assumption, we are able to choose

$$\rho := \min\left\{r_0, \left(\frac{1}{2C}\right)^{\frac{1}{\alpha_F-\alpha}}\right\} \tag{5.2}$$

such that

$$\sup_{Q_\rho} |u - \mathfrak{L}| \leq \rho^{1+\alpha}, \tag{5.3}$$

where $\mathfrak{L}$ is given by (4.4). This is the first step in our induction process. Now, suppose that has been checked the kth step in the induction process

$$\sup_{Q_{\rho^k}} |u - \mathfrak{L}_k| \leq \rho^{k(1+\alpha)} \tag{5.4}$$

with the following order of approximation for the coefficients

$$|\mathfrak{A}_{k+1} - \mathfrak{A}_k| \leq C\rho^{k(1+\alpha)} \quad \text{and} \quad |\mathfrak{B}_{k+1} - \mathfrak{B}_k| \leq C\rho^{k\alpha}. \tag{5.5}$$

We define the re-scaled function

$$v_k(X, t) := \frac{u(Y + \rho^k X, \tau + \rho^{2k} t) - \mathfrak{L}_k(Y + \rho^k X)}{\rho^{k(1+\alpha)}},$$

which verifies $|v_k| \leq 1$ in Q_1, and satisfies in the L^{P}-viscosity sense

$$(v_k)_t - G_k(D^2 v_k, Dv_k, X, t) = f_k(X,t) + g_k(X,t) = H_k(X,t) \qquad (5.6)$$

where

$$G_k(M, \overrightarrow{p}, X, t) = \rho^{k(1-\alpha)} F\left(\frac{1}{\rho^{k(1-\alpha)}} M, \rho^{k\alpha}\overrightarrow{p}, \rho^k X, \rho^{2k} t\right)$$

is a $(\lambda, \Lambda, \Gamma)$-parabolic operator and

$$f_k(X,t) := \rho^{k(1-\alpha)} f(Y + \rho^k X, \tau + \rho^{2k} t)$$

$$g_k(X,t) := G_k(D^2 v_k, Dv_k + B_k, X, t) - G_k(D^2 v_k, Dv_k, X, t).$$

Now,

$$\|f_k\|_{L^{p,q}(Q_1)} = \omega(\rho^k)\|f\|_{L^{p,q}\left(Q_{\rho^k}(Y,\tau)\right)} < \frac{\eta}{2},$$

where $\omega(\rho^k) = \rho^{k[1-\alpha-\Xi(n,p,q)]}$ is computed by change of variables. By the integrability relation and the value of α, we conclude $\omega(\rho^k) \leq 1$ for all integer $k \geq 1$. Also

$$|g_k(X,t)| \leq \mu(\rho^k)\Gamma$$

where as before μ is easily computed explicitly using (5.5). Now, we have verified $\mu(\rho^k) < 1$ and in fact $\lim\limits_{k\to\infty} \mu(\rho^k) = 0$. Thus,

$$\|g_k\|_{L^{p,q}(Q_1)} \leq \Gamma \sqrt[p]{\mathscr{L}^n(B_1(0))} < \frac{\eta}{2}.$$

Finally,

$$\left(\fint_{Q_1} \Theta^{\mathrm{P}}_{G_k}(X,t)\right)^{\frac{1}{\mathrm{P}}} \leq \max\left\{1, \rho^{k(1-\alpha)}\right\}\left(\fint_{Q_{\rho^k}(Y,\tau)} \Theta^{\mathrm{P}}_F(Y,\tau,X,t)\right)^{\frac{1}{\mathrm{P}}}$$
$$\text{and} \quad \|H_k\|_{L^{p,q}(Q_1)} \leq \eta;$$

therefore, we can apply the first induction step, which gives the existence of an affine function $\overline{\mathfrak{L}}_k(X) := \overline{\mathfrak{A}}_k + \langle \overline{\mathfrak{B}}_k, X\rangle$ with $|\overline{\mathfrak{A}}_k|, |\overline{\mathfrak{B}}_k| \leq C(n,\lambda,\Lambda)$ such that

$$\sup_{Q_\rho} |v_k - \overline{\mathfrak{L}}_k| \leq \rho^{1+\alpha}.$$

Rewriting the previous estimate in the unit domain gives

$$\sup_{Q_{\rho^{k+1}}} |u - \mathfrak{L}_{k+1}| \le \rho^{(k+1)(1+\alpha)},$$

for $\mathfrak{L}_{k+1}(X) := \mathfrak{L}_k(x) + \rho^{k(1+\alpha)}\overline{\mathfrak{L}}_k(\rho^{-k}X)$. The coefficients fulfils

$$|\mathfrak{A}_{k+1} - \mathfrak{A}_k| + \rho^k|\mathfrak{B}_{k+1} - \mathfrak{B}_k| \le C_0(n, \lambda, \Lambda)\rho^{(1+\alpha)k}, \tag{5.7}$$

hence, from (5.7), we conclude that $(\mathfrak{A}_k)_{k\ge 1} \subset \mathbb{R}$ and $(\mathfrak{B}_k)_{k\ge 1} \subset \mathbb{R}^n$ converge to $u(Y, \tau)$ and to $Du(Y, \tau)$ respectively. Moreover we have the following control

$$|u(Y, \tau) - \mathfrak{A}_k| \le C_0\frac{\rho^{k(1+\alpha)}}{1-\rho} \quad \text{and} \quad |Du(Y, \tau) - \mathfrak{B}_k| \le C_0\frac{\rho^{k\alpha}}{1-\rho} \tag{5.8}$$

Finally, given any $0 < r < \rho$, let k be an integer such that $(X, t) \in Q_{\rho^k}(Y, \tau)\setminus Q_{\rho^{k+1}}(Y, \tau)$. Therefore, we estimate from (5.8) that

$$\sup_{Q_r(Y,\tau)} |u(X, t) - [u(Y, \tau) + \langle Du(Y, \tau), X - Y\rangle]| \le C_0(n, \lambda, \Lambda, \alpha).r^{1+\alpha}$$

and the sketch is finished.

Remark 5.1 We highlight that the previous result must be interpreted in following way

$$\begin{cases} \text{If} \quad 1 - \Xi(n, p, q) < \alpha_F & \text{then } u \in C^{1+\sigma, \frac{1+\sigma}{2}}_{loc}(Q_1), \quad \text{for} \qquad \sigma = 1 - \Xi(n, p, q) \\ \text{If} \quad 1 - \Xi(n, p, q) \ge \alpha_F & \text{then } u \in C^{1+\gamma, \frac{1+\gamma}{2}}_{loc}(Q_1), \quad \text{for any } \ \gamma < \alpha_F. \end{cases}$$

Remark 5.2 The optimality of previous result can be verified by an example due to Krylov in [13, Page 209].

6 Parabolic $C^{1,\text{Log-Lip}}$ type estimates

In this last section we address the issue of finding the optimal regularity estimate for the limiting upper borderline case $f \in \text{BMO}$, which encompasses the case $f \in L^{\infty,\infty} \simeq L^{\infty}$.

In view of the almost optimal estimates given in the previous section, establishing a quantitative regularity result for solutions to (Eq) with bounded forcing term, requires that F-harmonic functions are $C^{2+\sigma,\frac{2+\sigma}{2}}$ smooth; otherwise no further information could be revealed from better hypotheses on the source function f. Evans–Krylov's regularity theory [7,11,12] assures that convex/concave equations do satisfy the $C^{2+\sigma,\frac{2+\sigma}{2}}$ smoothness assumption.

We now state and prove our sharp par $-$ $C^{1,\text{Log-Lip}}$ interior regularity theorem. For simplicity we will work on equations with constant coefficients and with no gradient dependence. Similar result can be easily obtained under continuity condition on the coefficients and Lipschitz control on the gradient dependence.

Theorem 6.1 *Let u be a C^0-viscosity solution to $u_t - F(D^2u) = f(X,t)$ in Q_1. If any solution of $v_t - F(D^2v + M) = K$, where $M \in Sym(n)$ and $K \in \mathbb{R}$ are on the surface $-F(M) = K$, has interior $C^{2+\sigma, \frac{2+\sigma}{2}}$ a priori estimates, i.e.,*

$$\|v\|_{C^{2+\sigma,\frac{2+\sigma}{2}}(Q_r)} \le \frac{\overline{\Phi}}{r^{2+\sigma}} \|v\|_{L^\infty(Q_1)} \tag{6.1}$$

for some $\overline{\Phi}(n,\lambda,\Lambda,K) > 0$ and $\sigma(n,\lambda,\Lambda) \in (0,1)$. Then, for a constant $C(n,\lambda,\Lambda,\sigma,\overline{\Phi}) > 0$, there holds

$$\begin{aligned} &|u(X,t) - [u(0,0) + \langle Du(0,0), X\rangle]| \\ &\quad \le C\left[\|u\|_{L^\infty(Q_1)} + \|f\|_{BMO(Q_1)} + 1\right] \cdot \omega(d_{par}((X,t),(0,0))) \end{aligned} \tag{6.2}$$

where $\omega(r) = r^2 \log \frac{1}{r}$ is the C^1-Log-Lipschitz modulus of continuity.

Proof By standard reduction arguments, we can assume that $\|u\|_{L^\infty(Q_1)} \le \frac{1}{2}$ and $\|f\|_{\text{BMO}(Q_1)} \le \vartheta_0$ for some $\vartheta_0 > 0$ which will be chosen a posteriori. Throughout the proof we use the notation

$$[f]_{\text{avg},Q_1} := \fint_{Q_1} f(Z,\varsigma)\,dZ\,d\varsigma.$$

The strategy is to find parabolic quadratic polynomials

$$\mathfrak{P}_k(X,t) := \frac{1}{2}\langle \mathfrak{A}_k X, X\rangle + \mathfrak{B}_k t + \langle \mathfrak{C}_k, X\rangle + \mathfrak{D}_k$$

such that $\mathfrak{P}_0 = \mathfrak{P}_{-1} = \frac{1}{2}\langle NX, X\rangle$, where $-F(N) = [f]_{\text{avg},Q_1}$ and for all $k \ge 0$,

$$\mathfrak{B}_k - F(\mathfrak{A}_k) = [f]_{\text{avg},Q_1} \quad \textit{and} \quad \sup_{Q_{\rho^k}} |u - \mathfrak{P}_k| \le \rho^{2k}, \tag{6.3}$$

with

$$\rho^{2(k-1)}(|\mathfrak{A}_k - \mathfrak{A}_{k-1}| + |\mathfrak{B}_k - \mathfrak{B}_{k-1}|) + \rho^{k-1}|\mathfrak{C}_k - \mathfrak{C}_{k-1}| + |\mathfrak{D}_k - \mathfrak{D}_{k-1}| \le C\rho^{2(k-1)} \tag{6.4}$$

where the radius $0 < \rho \ll \frac{1}{2}$ in (6.3) and (6.4) will also be determined a posteriori. We prove the existence of such polynomials by induction process in k. The first step of induction, $k = 0$, it is obviously satisfied. Suppose now that we have verified the thesis of induction for $k = 0, 1, \ldots, i$. Then, defining the re-scaled function $v := Q_1 \to \mathbb{R}$ given by

$$v_k(X,t) = \frac{(u - \mathfrak{P}_k)(\rho^k X, \rho^{2k} t)}{\rho^{2k}},$$

we have, by induction hypothesis, that $|v_k| \leq 1$ and it solves

$$(v_k)_t - F_k(D^2 v_k) = f(\rho^k X, \rho^{2k} t) := f_k(X, t)$$

in the C^0-viscosity sense, where $F_k(M) := F(M + \mathfrak{A}_k) - \mathfrak{B}_k$ which is a $(\lambda, \Lambda, 0)$-parabolic operator with

$$\begin{aligned} \|f_k\|_{\mathrm{BMO}(Q_1)} &:= \sup_{0<r\leq 1} \fint_{Q_r} \left| f_k(X,t) - [f_k]_{\mathrm{avg}, Q_r} \right| dXdt \\ &= \sup_{0<r\leq 1} \fint_{Q_{r\rho}} \left| f(Z,\varsigma) - [f]_{\mathrm{avg}, Q_{r\rho}} \right| dZd\varsigma \\ &\leq \|f\|_{\mathrm{BMO}(Q_1)} \\ &\leq \vartheta_0. \end{aligned}$$

As in Lemma 2.6, with some slight changes, and, under smallness assumption on $\|f\|_{BMO(Q_1)}$ to be set soon, we can find a C^0-viscosity solution $\mathfrak{h}$ to

$$\mathfrak{h}_t - F(D^2\mathfrak{h} + M_k) = [f]_{\mathrm{avg}, Q_1} \quad in \quad Q_1,$$

such that

$$\sup_{Q_{\frac{1}{2}}} |v_k - \mathfrak{h}| \leq \delta,$$

for some $\delta > 0$ which we will choose below. From hypothesis (6.1), $\mathfrak{h}$ is $C^{2+\sigma, \frac{2+\sigma}{2}}$ at the origin with universal bounds. Thus, if we define

$$\mathfrak{P}(X,t) := \frac{1}{2}\langle D^2\mathfrak{h}(0,0)X, X\rangle + \mathfrak{h}_t(0,0)t + \langle D\mathfrak{h}(0,0), X\rangle + \mathfrak{h}(0,0),$$

by the $C^{2+\sigma, \frac{2+\sigma}{2}}$ regularity assumption (6.1), we can estimate

$$|D^2\mathfrak{h}(0,0)| + |\mathfrak{h}_t(0,0)| + |D\mathfrak{h}(0,0)| + |\mathfrak{h}(0,0)| \leq C\overline{\Phi}$$

where

$$|(\mathfrak{h} - \mathfrak{P})(X,t)| \leq C(n)\overline{\Phi} d_{\mathrm{par}}((X,t),(0,0))^{2+\sigma}.$$

Now, we are able to select

$$\rho := \left(\frac{1}{2C\overline{\Phi}}\right)^{\frac{1}{\sigma}} \quad \text{and} \quad \delta := \frac{1}{2}\rho^2.$$

The choice above for $\rho(\overline{\Phi}, \sigma, \Lambda, \lambda, n) \ll \frac{1}{2}$ decides the value for $\delta(\overline{\Phi}, \sigma, \Lambda, \lambda, n) > 0$ which determines, by Lemma 2.6, the universal smallness regime given by the constant $\vartheta_0 > 0$. From the previous choices, we readily obtain

$$\sup_{Q_\rho} |v_k - \mathfrak{P}| \le \rho^2. \tag{6.5}$$

Rewriting (6.5) back to the unit domain yields

$$\sup_{Q_{\rho^{k+1}}} \left| u(X,t) - \left[\mathfrak{P}_k(X,t) + \rho^{2k} \mathfrak{P}\left(\frac{X}{\rho^k}, \frac{t}{\rho^{2k}} \right) \right] \right| \le \rho^{2(k+1)}. \tag{6.6}$$

Therefore, defining

$$\mathfrak{P}_{k+1}(X,t) := \mathfrak{P}_k(X,t) + \rho^{2k} \mathfrak{P}\left(\frac{X}{\rho^k}, \frac{t}{\rho^{2k}} \right),$$

we verify the $(k+1)^{\underline{th}}$ step of induction and, clearly, the required conditions (6.3) and (6.4) are satisfied. From (6.4) we conclude that $\mathfrak{D}_k \to u(0,0)$ and $\mathfrak{D}_k \to Du(0.0)$, with the following estimates

$$|u(0,0) - \mathfrak{D}_k| \le \frac{C\rho^{2k}}{1-\rho} \quad \text{and} \quad |Du(0,0) - \mathfrak{C}_k| \le \frac{C\rho^k}{1-\rho}. \tag{6.7}$$

Furthermore, Eq. (6.4) yields the growth estimates:

$$|\mathfrak{A}_k| \le \sum_{j=1}^{k} |\mathfrak{A}_j - \mathfrak{A}_{j-1}| \le Ck \quad \text{and} \quad |\mathfrak{B}_k| \le \sum_{j=1}^{k} |\mathfrak{B}_j - \mathfrak{B}_{j-1}| \le Ck. \tag{6.8}$$

Finally, given any $0 < r < \rho$, let k be an integer such that

$$(X,t) \in Q_{\rho^k}(Y,\tau) \setminus Q_{\rho^{k+1}}(Y,\tau)$$

We estimate from Eqs. (6.3), (6.7) and (6.8),

$$\begin{aligned} &\sup_{Q_r(0)} |u(X,t) - [u(0,0) + \langle Du(0,0), X\rangle]| \\ &\quad \le \rho^{2k} + |u(0,0) - \mathfrak{D}_k| + \rho^k |Du(0,0) - \mathfrak{C}_k| + \rho^{2k}(|\mathfrak{B}_k| + |\mathfrak{A}_k|) \\ &\quad \le C(n, \lambda, \Lambda, \sigma, \overline{\Phi}).r^2 \log r^{-1}, \end{aligned}$$

and the proof of Theorem is finished. □

Remark 6.2 The final estimate says that solutions to (Eq) are asymptotically $C^{2,1}$ in the parabolic sense. Furthermore, adjustments in the previous explanation yield $u_t, D^2u \in s - BMO(Q_{\frac{1}{2}})$, with appropriate a priori estimate in terms of the BMO-norm of f in Q_1. Indeed, under appropriate smallness regime on $f \in BMO(Q_1)$ we can approximate u by a solution $\mathfrak{h}$ to

$$\mathfrak{h}_t - F(D^2\mathfrak{h}, X_0, t_0) = [f]_{\text{avg}, Q_1} \quad \text{in} \quad Q_{\frac{1}{2}}$$

in the $W^{2,1,s}(Q_{\frac{1}{2}})$ topology. Thus, by an iterative process similar to the one used here one finds parabolic quadratic polynomials $\mathfrak{P}_k$ such that

$$\fint_{Q_{\rho^k}} \left(|\partial_t(u - \mathfrak{P}_k)|^s + |D^2(u - \mathfrak{P}_k)|^s\right) \leq 1$$

Therefore, the previous sentence provides the desired s-BMO estimate. In other words,

$$\|u_t\|_{s-BMO(Q_r)} + \|D^2u\|_{s-BMO(Q_r)} \leq C\{\|u\|_{L^\infty(Q_1)} + \|f\|_{BMO(Q_1)}\}, \\ \text{for} \quad 0 < r \ll 1$$

Remark 6.3 The result proven in this section can be further applied to equations of the form $u_t - F(D^2u, X, t) = f(u, X, t)$, where f is continuous. It is particularly meaningful to geometric flow problems:

$$H_t - \Delta H - H|A|^2 = 0,$$

where H is the inwards mean curvature vector of the surface at position X and time t and $|A|$ represents the norm of the second fundamental form. This equation describes the mean curvature hyper-surface in the Euclidean space $\mathbb{R}^{n+1}$, see for example [17].

Remark 6.4 As a final remark, we note that the results proven in this article can be generalized for a more general class of anisotropic Lebesgue spaces with mixed norms. Namely, consider $\overrightarrow{p} = (p_1, \ldots, p_n)$. Let $f \in L^{p_1,\ldots,p_n,q}(Q_1)$, i.e., $f \in L^{p_1}_{X_1} \ldots L^{p_n}_{X_n} L^q_t$. The quantity

$$\Xi(n, p_1, \ldots, p_n, q) := \left(\sum_{i=1}^{n} \frac{1}{p_i}\right) + \frac{2}{q}$$

sets up the following regularity regimes, with universal a priori estimates:

- $1 < \Xi(n, p_1, \ldots, p_n, q) < \frac{n+2}{\mathfrak{p}_0} < 2$ for the $C^{\alpha, \frac{\alpha}{2}}$ regularity regime;
- $\Xi(n, p_1, \ldots, p_n, q) = 1$ for the Lipschitz logarithmic type estimates;
- $0 < \Xi(n, p_1, \ldots, p_n, q) < 1$ for the $C^{1+\alpha, \frac{1+\alpha}{2}}$ regularity regime.

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