ABELIAN BALANCED HERMITIAN STRUCTURES ON UNIMODULAR LIE ALGEBRAS

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ABSTRACT. Let $\mathfrak{g}$ a $2n$-dimensional unimodular Lie algebra equipped with a Hermitian structure (J, F) such that the complex structure J is abelian and the fundamental form F is balanced. We prove that the holonomy group of the associated Bismut connection reduces to a subgroup of $SU(n-k)$, being $2k$ the dimension of the center of $\mathfrak{g}$. We determine conditions that allow a unimodular Lie algebra to admit this particular type of structures. Moreover, we give methods to construct them in arbitrary dimensions and classify them if the Lie algebra is 8-dimensional and nilpotent.

1. Introduction

Bismut proved in [8] that given any Hermitian structure (J, F) on a $2n$-dimensional manifold M there is a unique Hermitian connection with totally skew-symmetric torsion T given by $g(X, T(Y, Z)) = JdF(X, Y, Z) = -dF(JX, JY, JZ)$, g being the associated metric. This torsion connection, denoted by ∇, is known as the *Bismut connection* of (J, F) and it can be derived from the Levi-Civita connection ∇^g of the Riemannian metric g by $\nabla = \nabla^g + \frac{1}{2}T$, where T is identified with the 3-form JdF. Since ∇ is Hermitian, its restricted holonomy group is contained in the unitary group $U(n)$. However, stronger reductions of the holonomy group are interesting. For instance, *Calabi-Yau with torsion* structures, i.e, structures satisfying that $Hol(\nabla)$ is contained in $SU(n)$, appear in heterotic string theory, related to the Strominger system in six dimensions.

If M is a nilmanifold, i.e. a compact quotient of a simply connected nilpotent Lie group G by a lattice, Fino, Parton and Salamon proved in [16] that for invariant Hermitian structures on M, that is, for Hermitian structures arising from ones on the Lie algebra $\mathfrak{g}$ of G, the Calabi-Yau with torsion condition is equivalent to the balanced condition. Balanced metrics belong to the class $\mathcal{W}_3$ in the well-known Gray-Hervella classification [19] and they are characterized by $d^*F = 0$, where d^* is the co-differential. Equivalently, $dF^{n-1} = 0$, where $2n$ is the dimension of the Hermitian manifold.

In the particular case of dimension six, nilmanifolds admitting balanced Hermitian structures are classified in [24]. In [25] a complete study of the balanced geometry on such manifolds is carried out and solutions to the Strominger system are also found. Moreover in [25, Theorem 4.7] the associated holonomy groups for the Bismut connection are described. It turns out that the holonomy group of the Bismut connection reduces to a proper subgroup of $SU(3)$ if and only if the complex structure is abelian. We recall that a complex structure on a Lie algebra $\mathfrak{g}$ is *abelian* if $[JX, JY] = [X, Y]$ for all $X, Y \in \mathfrak{g}$. These complex structures have interesting applications in differential geometry. For instance, a pair of anticommuting abelian complex structures on $\mathfrak{g}$ gives rise to an invariant weak HKT structure on G (see [14] and [18]). It has been shown in [9] that the Dolbeault cohomology of a nilmanifold with an abelian complex structure can be computed algebraically. Also, deformations of abelian complex structures on nilmanifolds have been studied in [10, 21].

The main goal of this paper is to prove reductions of the holonomy group of the Bismut connection to $SU(n)$ in the case of Hermitian structures (J, F), with J abelian and F balanced, on unimodular, not necessarily nilpotent, $2n$-dimensional Lie algebras $\mathfrak{g}$. Moreover, if the center of $\mathfrak{g}$ is non-trivial, then the holonomy reduces to a proper subgroup of $SU(n)$.

The paper is structured as follows. Section 2 is devoted to study *abelian balanced Hermitian structures* on unimodular Lie algebras, that is Hermitian structures (J, F) where J is abelian and F is balanced. We obtain equivalent conditions for a Hermitian structure to be abelian balanced in terms of the structure constants of the Lie algebra.

In Section 3 we focus our attention on the reduction of the holonomy group of the Bismut connection. The main result is Theorem 3.12 stating that if $\mathfrak{g}$ is a unimodular Lie algebra of dimension $2n$ endowed with an abelian balanced Hermitian structure (J, F) and $\dim \mathfrak{z} = 2k$, where $\mathfrak{z}$ stands for the center of $\mathfrak{g}$, then the holonomy group of the associated Bismut connection is contained in $SU(n-k)$.

In Section 4 we study the particular case of nilpotent Lie algebras. In particular we prove that the product $\mathfrak{h}_{2n+1} \times \mathbb{R}^{2k+1}$ of a real Heisenberg Lie algebra by an odd number of copies of $\mathbb{R}$ admits abelian balanced Hermitian structures if and only if $n \geq 2$. Moreover, the holonomy group of the Bismut connection reduces to $U(1)$, which provides the strongest reduction of the holonomy. We also classify nilpotent Lie algebras in dimension 8 admitting abelian balanced Hermitian structures.

Finally, Section 5 is devoted to construct new examples of abelian balanced Hermitian structures on unimodular Lie algebras of any even dimension.

2. Abelian balanced Hermitian structures on unimodular Lie algebras

In this paper we are interested in abelian complex structures on Lie algebras endowed with balanced Hermitian metrics. We start by reviewing some known aspects about these topics.

A *complex structure* on a Lie algebra $\mathfrak{g}$ is an endomorphism J of $\mathfrak{g}$ satisfying $J^2 = -Id$ together with the vanishing of the Nijenhuis bilinear form with values in $\mathfrak{g}$,

$$N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y],$$

where $X, Y \in \mathfrak{g}$. This is equivalent to saying that the i-eigenspace $\mathfrak{g}_{1,0}$ of J in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a complex subalgebra of $\mathfrak{g}_{\mathbb{C}}$. The complex structure J is said of *abelian type* if $[JX, JY] = [X, Y]$ for any $X, Y \in \mathfrak{g}$, or equivalently $\mathfrak{g}_{1,0}$ is abelian.

In terms of the dual of $\mathfrak{g}_{\mathbb{C}}$, if we denote by $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ the eigenspaces corresponding to the eigenvalues $\pm i$ of J as an endomorphism of $\mathfrak{g}^*_{\mathbb{C}}$, respectively, then the decomposition $\mathfrak{g}^*_{\mathbb{C}} = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$ induces a natural bigraduation on the complexified exterior algebra $\bigwedge^* \mathfrak{g}^*_{\mathbb{C}} = \oplus_{p,q} \bigwedge^{p,q}(\mathfrak{g}^*) = \oplus_{p,q} \bigwedge^p(\mathfrak{g}^{1,0}) \otimes \bigwedge^q(\mathfrak{g}^{0,1})$. If d denotes the usual Chevalley-Eilenberg differential of the Lie algebra, we shall also denote by d its extension to the complexified exterior algebra, i.e. $d\colon \bigwedge^* \mathfrak{g}^*_{\mathbb{C}} \longrightarrow \bigwedge^{*+1} \mathfrak{g}^*_{\mathbb{C}}$. It is well known that the endomorphism J is a complex structure if and only if $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{2,0}(\mathfrak{g}^*) \oplus \bigwedge^{1,1}(\mathfrak{g}^*)$. Abelian complex structures are characterized by the condition $d(\mathfrak{g}^{1,0}) \subset \bigwedge^{1,1}(\mathfrak{g}^*)$. Some well-known properties of this type of structures are the following: the center $\mathfrak{z}$ of $\mathfrak{g}$ is J-invariant, the commutator ideal $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ is abelian (and therefore $J\mathfrak{g}'$ is also abelian), and $\operatorname{codim} \mathfrak{g}' \geq 2$ whenever $\dim \mathfrak{g} \geq 4$ (see for instance [4, 6]).

A *Hermitian structure* on $\mathfrak{g}$ is a pair (J, g), where J is a complex structure on $\mathfrak{g}$ and g is an inner product on $\mathfrak{g}$ compatible with J in the usual sense, i.e. $g(\cdot, \cdot) = g(J\cdot, J\cdot)$. The associated *fundamental form* $F \in \bigwedge^2 \mathfrak{g}^*$ is defined by $F(X, Y) = g(X, JY)$. Fixed J, since g and F are mutually determined by each other, we shall also denote the Hermitian structure (J, g) by the pair (J, F).

A basis $\{e_1, \ldots, e_{2n}\}$ of $\mathfrak{g}$ is said to be a *J-adapted basis* if $Je_{2k-1} = -e_{2k}$ for all $k = 1, \ldots, n$. Using the convention:

$$(J\eta)(X_1, \ldots, X_k) = (-1)^k \eta(JX_1, \ldots, JX_k), \quad \text{where } \eta \in \textstyle\bigwedge^k \mathfrak{g}^*,\ X_j \in \mathfrak{g},$$

it turns out that the dual basis $\{e^1, \ldots, e^{2n}\}$ also satisfies $Je^{2k-1} = -e^{2k}$. Moreover, if we define $F = -\sum_{k=1}^n e^{2k-1} \wedge Je^{2k-1}$, then $\{e_1, \ldots, e_{2n}\}$ is orthonormal with respect to the Hermitian metric $g(\cdot, \cdot) = F(J\cdot, \cdot)$.

A basis $\{e_1, \ldots, e_{2n}\}$ of $\mathfrak{g}$ is said to be an *adapted basis* for the Hermitian structure (J, F) if in terms of the dual basis

$$Je^{2k-1} = -e^{2k}, \quad F = -\sum_{k=1}^n e^{2k-1} \wedge Je^{2k-1} = \sum_{k=1}^n e^{2k-1} \wedge e^{2k}. \tag{1}$$

The Hermitian structure (J, F) is said to be *balanced* if F^{n-1} is a closed form or, equivalently, $d^*F = 0$, where d^* is the co-differential associated to the Riemannian metric g (see for instance [20]).

In this section we will consider unimodular Lie algebras $\mathfrak{g}$ of dimension $2n$ with balanced Hermitian structures (J, F). Recall the following expression for d^*F:

$$d^*F(X) = -\sum_{i=1}^{2n} (\nabla^g_{e_i} F)(e_i, X),$$

where $X \in \mathfrak{g}$, ∇^g is the Levi-Civita connection associated to the Hermitian metric g and $\{e_1, \ldots, e_{2n}\}$ is any orthonormal basis of $\mathfrak{g}$. Taking into account the Koszul formula for the Levi-Civita connection in the invariant setting, we obtain:

$$\begin{aligned}(\nabla^g_{e_i} F)(e_i, X) &= -F(\nabla^g_{e_i} e_i, X) - F(e_i, \nabla^g_{e_i} X)\\ &= -g(\nabla^g_{e_i} e_i, JX) + g(Je_i, \nabla^g_{e_i} X)\\ &= -\frac{1}{2}\left(2g([JX, e_i], e_i) + g([X, e_i], Je_i) + g([X, Je_i], e_i) - g([Je_i, e_i], X)\right),\end{aligned}$$

and therefore

$$d^*F(X) = \operatorname{tr}\operatorname{ad}_{JX} + \frac{1}{2}\left(-\operatorname{tr}(J\operatorname{ad}_X) + \operatorname{tr}(\operatorname{ad}_X J) - \sum_{i=1}^{2n} g([Je_i, e_i], X)\right),$$

or

$$d^*F(X) = \operatorname{tr}\operatorname{ad}_{JX} - \frac{1}{2} g\left(\sum_{i=1}^{2n} [Je_i, e_i], X\right).$$

Hence,

$$d^*F = 0 \Leftrightarrow \operatorname{tr}\operatorname{ad}_{JX} = \frac{1}{2} g\left(\sum_{i=1}^{2n} [Je_i, e_i], X\right)$$

for all $X \in \mathfrak{g}$. If, moreover, $\mathfrak{g}$ is unimodular, we have

$$d^*F = 0 \Leftrightarrow \sum_{i=1}^{2n} [Je_i, e_i] = 0.$$

Observe that this condition is independent of the choice of the orthonormal basis.

We have proved the following

Lemma 2.1. *Let $\mathfrak{g}$ be a $2n$-dimensional unimodular Lie algebra and let (J, F) be a Hermitian structure on $\mathfrak{g}$. Then (J, F) is balanced if and only if there exists an orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of $\mathfrak{g}$ such that*

$$\sum_{i=1}^{2n} [Je_i, e_i] = 0. \tag{2}$$

In particular, if $\{e_1, \ldots, e_{2n}\}$ is a J-adapted basis satisfying (2), then F given by (1) is balanced.

Remark 2.2. It follows immediately from Lemma 2.1 that if $\mathfrak{g}$ is unimodular and J is a bi-invariant complex structure on $\mathfrak{g}$ (i.e., $J[X, Y] = [JX, Y]$ for all $X, Y \in \mathfrak{g}$), then any J-Hermitian metric g on $\mathfrak{g}$ is balanced. This result was first proved in [2].

Let us consider a J-adapted basis $\{e_1, \ldots, e_{2n}\}$ of $\mathfrak{g}$ and let us express $de^k = \sum_{i<j} c^k_{ij} e^{ij}$, where e^{ij} stands for $e^i \wedge e^j$, or equivalently, $[e_i, e_j] = -\sum_{k=1}^{2n} c^k_{ij} e_k$, where c^k_{ij} are the structure constants of $\mathfrak{g}$. Here we are using the fact that $de^k(e_i, e_j) = -e^k([e_i, e_j])$. In this basis, condition (2) is equivalent to the following system of equations in terms of the structure constants of the Lie algebra:

$$\sum_{i=1}^{n} c^k_{2i-1,2i} = 0, \quad k = 1, \ldots, 2n. \tag{3}$$

As a consequence, we have the following equivalent condition for a Hermitian metric on a unimodular Lie algebra to be balanced, which does not depend on any particular basis.

Corollary 2.3. *Let $\mathfrak{g}$ be a $2n$-dimensional unimodular Lie algebra and let (J, F) be a Hermitian structure on $\mathfrak{g}$. Then (J, F) is balanced if and only if*

$$F^{n-1} \wedge d\alpha = 0, \tag{4}$$

for any 1-form $\alpha \in \mathfrak{g}^*$.

Proof. Let us consider a basis $\{e_1, \ldots, e_{2n}\}$ of $\mathfrak{g}$ adapted to the Hermitian structure. It is enough to prove the statement for $\alpha = e^k$, $k = 1, \ldots, 2n$. In term of this basis,

$$F^{n-1} = \lambda \sum_{r=1}^{n} e^1 \wedge \cdots \wedge \widehat{e^{2r-1}} \wedge \widehat{e^{2r}} \wedge \cdots \wedge e^{2n},$$

where λ is a positive real number and the symbol $\widehat{e^s}$ means that e^s does not appear in the corresponding term. Now, for any $k = 1, \ldots, 2n$,

$$F^{n-1} \wedge de^k = F^{n-1} \wedge \sum_{i<j} c_{ij}^k e^{ij} = \lambda \sum_{r=1}^{n} c_{2r-1,2r}^k \, e^{12\cdots n},$$

and it follows from (3) that F is balanced if and only if the expression above vanishes. □

Remark 2.4. Condition (4) is equivalent to the fact that F is orthogonal to the image of $d : \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$, as stated in [1, 15].

The image of $d : \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*$ provides also a way to characterize abelian complex structures. Indeed, J is abelian if and only if $J(d\alpha) = d\alpha$, for any $\alpha \in \mathfrak{g}^*$, i.e., $d\alpha$ is J-invariant.

From now on we will be interested in Hermitian structures combining abelian complex structures and balanced metrics, which motivates the following

Definition 2.5. *Let (J, F) be a Hermitian structure on a Lie algebra $\mathfrak{g}$. If J is abelian and F is balanced, we will refer to the pair (J, F) as an* abelian balanced Hermitian structure*.*

If we denote

$$\Gamma := \{\eta \in \textstyle\bigwedge^2 \mathfrak{g}^* \mid J\eta = \eta \text{ and } F^{n-1} \wedge \eta = 0\}, \tag{5}$$

we obtain from previous results the following characterization of abelian balanced Hermitian structures.

Corollary 2.6. *Let $\mathfrak{g}$ be a $2n$-dimensional unimodular Lie algebra and let (J, F) be a Hermitian structure on $\mathfrak{g}$. Then (J, F) is abelian balanced if and only if im $(d : \mathfrak{g}^* \to \bigwedge^2 \mathfrak{g}^*) \subseteq \Gamma$.*

Remark 2.7. In terms of an adapted basis $\{e_1, \ldots, e_{2n}\}$, a 2-form $\eta \in \bigwedge^2 \mathfrak{g}^*$ belongs to Γ if and only if

i) $J\eta = \eta$: $\quad \eta(e_{2r}, e_{2s}) = \eta(e_{2r-1}, e_{2s-1}), \quad \eta(e_{2r}, e_{2s-1}) = -\eta(e_{2r-1}, e_{2s})$.

ii) $F^{n-1} \wedge \eta = 0$: $\quad \sum_{r=1}^{n} \eta(e_{2r-1}, e_{2r}) = 0$.

It will be very useful to have an explicit description of the vector subspace $\Gamma \subseteq \bigwedge^2 \mathfrak{g}^*$ in terms of a basis adapted to the Hermitian structure (J, F).

Lemma 2.8. *Let (J, F) be an abelian balanced Hermitian structure on a $2n$-dimensional unimodular Lie algebra $\mathfrak{g}$ and let $\{e_1, \ldots, e_{2n}\}$ be an adapted basis. Then, a basis for Γ is given by $\mathcal{B} = \{\gamma_{ij} \mid 2i < j,\ i = 1, \ldots, n-1\} \cup \{\beta_r \mid r = 1, \ldots, n-1\}$, where:*

$$\begin{cases} \gamma_{ij} = e^{2i-1,j} - e^{2i} \wedge Je^j, \\ \beta_r = e^{2r-1,2r} - e^{2r+1,2r+2}. \end{cases} \tag{6}$$

In particular, dim $\Gamma = n^2 - 1$.

Proof. It is easy to see that the space I of J-invariant 2-forms is generated, in terms of the dual of the adapted basis, by

$$\begin{aligned} I &= \langle \{e^{ij} + J(e^{ij}) \,|\, i < j\} \rangle \\ &= \langle \{e^{ij} + J(e^{ij}) \,|\, i < j-1\} \cup \{e^{2r-1,2r} \,|\, r = 1, \dots, n\} \rangle \\ &= \langle \{e^{2i-1,j} - e^{2i} \wedge Je^{j} \,|\, 2i < j,\ i = 1, \dots, n-1\} \cup \{e^{2r-1,2r} \,|\, r = 1, \dots, n\} \rangle. \end{aligned}$$

Therefore, we can express any $\eta \in \Gamma$ as $\eta = \sum_{2i<j} a_{ij}\gamma_{ij} + \sum_{r=1}^{n} b_r e^{2r-1,2r}$. Since $F^{n-1} \wedge \eta = 0$, Remark 2.7 implies that $\sum_{r=1}^{n} b_r = 0$. This last condition allows us to express $\sum_{r=1}^{n} b_r e^{2r-1,2r}$ as $\sum_{j=1}^{n-1} m_j \beta_j$, where $m_j = \sum_{r=1}^{j} b_r$. This shows that $\mathcal{B}$ spans Γ and since all the elements of $\mathcal{B}$ are linearly independent, the proof is complete. □

3. Curvature and holonomy of the Bismut connection

Bismut proved in [8] that given any Hermitian structure (J, F) on a $2n$-dimensional manifold M there is a unique connection preserving the Hermitian structure with totally skew-symmetric torsion T given by $g(X, T(Y, Z)) = JdF(X, Y, Z) = -dF(JX, JY, JZ)$, g being the associated metric. This torsion connection, denoted by ∇, is known as the *Bismut connection* of (J, F) and it can be derived from the Levi-Civita connection ∇^g of the Riemannian metric g by $\nabla = \nabla^g + \frac{1}{2}T$, where T is identified with the 3-form JdF.

According to [16], the holonomy group of the Bismut connection associated to any invariant balanced J-Hermitian structure on a nilmanifold M^{2n} is contained in $SU(n)$.

The aim of this section is to prove such a reduction of the holonomy of the Bismut connection in the case of abelian balanced Hermitian structures on a unimodular, non necessarily nilpotent, Lie algebra $\mathfrak{g}$. Moreover, if the center of $\mathfrak{g}$ is non-trivial, then the holonomy reduces to a proper subgroup of $SU(n)$. First examples of this reduction can be found in [25].

By the well-known Ambrose-Singer theorem, the Lie algebra $\mathfrak{hol}(\nabla)$ of the holonomy group of any linear connection ∇ is generated by the curvature endomorphisms of ∇ together with their covariant derivatives. We recall now the following objects that will help us to compute $\mathfrak{hol}(\nabla)$.

Given any linear connection ∇ on an m-dimensional manifold, the connection 1-forms σ^i_j are given with respect to a local orthonormal frame $\{e_1, \dots, e_m\}$ by

$$\sigma^i_j(e_k) = g(\nabla_{e_k} e_j, e_i), \tag{7}$$

i.e. $\nabla_X e_j = \sigma^1_j(X)\, e_1 + \cdots + \sigma^m_j(X)\, e_m$. The curvature 2-forms Ω^i_j of ∇ can be expressed in terms of the connection 1-forms σ^i_j by

$$\Omega^i_j = d\sigma^i_j + \sum_{k=1}^{m} \sigma^i_k \wedge \sigma^k_j, \tag{8}$$

and the curvature endomorphisms $R(e_r, e_s)$ of the connection ∇ are given in terms of the curvature forms Ω^i_j by $g(R(e_r, e_s)e_i, e_j) = -\Omega^i_j(e_r, e_s)$. We will identify the curvature endomorphisms $R(e_r, e_s)$ with the 2-forms given by

$$R^{rs}(e_i, e_j) = -\Omega^i_j(e_r, e_s). \tag{9}$$

Finally, the covariant derivative $\nabla_{e_j}\gamma$ of any 2-form γ is given by

$$(\nabla_{e_j}\gamma)(e_r, e_s) = \sum_{k=1}^{m} \left(\sigma^k_s(e_j)\, \gamma(e_k, e_r) - \sigma^k_r(e_j)\, \gamma(e_k, e_s) \right), \qquad j = 1, \dots, m. \tag{10}$$

In the particular case of a left-invariant Hermitian structure (J, F) on a $2n$-dimensional Lie group, we can work at the Lie algebra level. Choosing an adapted basis $\{e_1, \dots, e_{2n}\}$ of $\mathfrak{g}$, the Levi-Civita connection

1-forms $(\sigma^g)^i_j$ can be expressed in terms of the structure constants c^k_{ij} by

$$(\sigma^g)^i_j(e_k) = -\frac{1}{2}\left(g(e_i,[e_j,e_k]) - g(e_k,[e_i,e_j]) + g(e_j,[e_k,e_i])\right) = \frac{1}{2}(c^i_{jk} - c^k_{ij} + c^j_{ki}).$$

Since the Bismut connection ∇ is given by $\nabla = \nabla^g + \frac{1}{2}T$, with torsion $T = JdF$, the Bismut connection 1-forms σ^i_j are determined by

$$\sigma^i_j(e_k) = (\sigma^g)^i_j(e_k) - \frac{1}{2}T(e_i,e_j,e_k) = \frac{1}{2}(c^i_{jk} - c^k_{ij} + c^j_{ki}) - \frac{1}{2}JdF(e_i,e_j,e_k). \tag{11}$$

Next we show that if the complex structure J is abelian, then it is possible to simplify some of the expressions above.

Lemma 3.1. *Let $\mathfrak{g}$ be a $2n$-dimensional Lie algebra and let (J,F) be a Hermitian structure on $\mathfrak{g}$ where J is abelian. If $\{e^1,\dots,e^{2n}\}$ is an adapted basis, then the torsion 3-form of the Bismut connection is given by*

$$T = \sum_{i=1}^{2n} e^i \wedge de^i.$$

Proof. According to (1), F can be expressed as $F = -\frac{1}{2}\sum_{i=1}^{2n} e^i \wedge Je^i$. Then:

$$-JdF = \frac{1}{2}Jd\left(\sum_{i=1}^{2n} e^i \wedge Je^i\right) = \frac{1}{2}J\left(\sum_{i=1}^{2n} de^i \wedge Je^i - e^i \wedge d(Je^i)\right).$$

Since de^k are J-invariant,

$$-JdF = -\frac{1}{2}\left(\sum_{i=1}^{2n} e^i \wedge de^i + Je^i \wedge d(Je^i)\right) = -\sum_{i=1}^{2n} e^i \wedge de^i.$$

□

Using this lemma, the Bismut connection 1-forms can be expressed in a simpler way. Observe that

$$\begin{aligned} JdF(e_i,e_j,e_k) &= \sum_{r=1}^{2n} e^r \wedge de^r(e_i,e_j,e_k) \\ &= \sum_{r=1}^{2n} \left(e^r(e_i)de^r(e_j,e_k) - e^r(e_j)de^r(e_i,e_k) + e^r(e_k)de^r(e_i,e_j)\right) \\ &= (c^i_{jk} - c^j_{ik} + c^k_{ij}). \end{aligned}$$

Lemma 3.2. *Under the conditions of Lemma 3.1, the Bismut connection 1-forms are given by*

$$\sigma^i_j = -\sum_{k=1}^{2n} c^k_{ij} e^k, \quad 1 \le i < j \le 2n. \tag{12}$$

Moreover, the following conditions hold for $i,j = 1,\dots,n$:

$$\sigma^{2i}_{2j} = \sigma^{2i-1}_{2j-1}, \quad \sigma^{2i-1}_{2j} = -\sigma^{2i}_{2j-1}. \tag{13}$$

Proof. The first equality follows directly from (11):

$$2\sigma^i_j(e_k) = (c^i_{jk} - c^k_{ij} + c^j_{ki}) - (c^i_{jk} - c^j_{ik} + c^k_{ij}) = -2\,c^k_{ij}.$$

For the second statement, it is enough to observe that since de^k are J-invariant, the structure constants satisfy

$$c^k_{2i-1,2j-1} = c^k_{2i,2j} \text{ and } c^k_{2i-1,2j} = -c^k_{2i,2j-1}, \tag{14}$$

for any $i,j = 1,\dots,n$ and $k = 1,\dots,2n$. □

Corollary 3.3. *Under the conditions of Lemma 3.1, the Bismut curvature 2-forms Ω^i_j satisfy*

$$\Omega^{2i}_{2j} = \Omega^{2i-1}_{2j-1}, \quad \Omega^{2i-1}_{2j} = -\Omega^{2i}_{2j-1}.$$

Proof. Let us prove the first equality. For the second one similar computations can be applied. From the general expression of the curvature 2-forms given in (8) we have

$$\Omega^{2i}_{2j} = d\sigma^{2i}_{2j} + \sum_{k=1}^{2n} \sigma^{2i}_k \wedge \sigma^k_{2j}.$$

Using (13) we get $d\sigma^{2i}_{2j} = d\sigma^{2i-1}_{2j-1}$. We split the second summand according to the parity of k:

$$\begin{aligned}\sum_{k=1}^{2n} \sigma^{2i}_k \wedge \sigma^k_{2j} &= \sum_{r=1}^{n} \sigma^{2i}_{2r} \wedge \sigma^{2r}_{2j} + \sum_{s=1}^{n} \sigma^{2i}_{2s-1} \wedge \sigma^{2s-1}_{2j} \\ &= \sum_{r=1}^{n} \sigma^{2i-1}_{2r-1} \wedge \sigma^{2r-1}_{2j-1} + \sum_{s=1}^{n} \sigma^{2i-1}_{2s} \wedge \sigma^{2s}_{2j-1} \\ &= \sum_{k=1}^{2n} \sigma^{2i-1}_k \wedge \sigma^k_{2j-1},\end{aligned}$$

where in the second equality we have used (13) again. Therefore $\Omega^{2i}_{2j} = \Omega^{2i-1}_{2j-1}$. □

The previous results hold for any Hermitian structure (J, F) on a Lie algebra $\mathfrak{g}$ such that J is abelian. In what follows we will apply these results for the particular case of an abelian balanced Hermitian structure on unimodular Lie algebras.

Lemma 3.4. *If $\eta \in \Gamma$, then $\nabla_X \eta \in \Gamma$, for any $X \in \mathfrak{g}$.*

Proof. According to Remark 2.7, we have to verify that (i) $J\nabla_X\eta = \nabla_X\eta$, and (ii) $F^{n-1} \wedge \nabla_X\eta = 0$, for any $X \in \mathfrak{g}$. Consider an adapted basis $\{e^1, \ldots, e^{2n}\}$. For case (i):

$$\begin{aligned}(J(\nabla_X\eta))(e_{2r}, e_{2s}) &= (\nabla_X\eta)(e_{2r-1}, e_{2s-1}) \\ &= \sum_{k=1}^{2n} \left(\sigma^k_{2s-1}(X)\,\gamma(e_k, e_{2r-1}) - \sigma^k_{2r-1}(X)\,\gamma(e_k, e_{2s-1})\right) \\ &= \sum_{p=1}^{n} \Big(\sigma^{2p}_{2s-1}(X)\,\gamma(e_{2p}, e_{2r-1}) - \sigma^{2p}_{2r-1}(X)\,\gamma(e_{2p}, e_{2s-1})\Big) + \\ &\quad \sum_{p=1}^{n} \Big(\sigma^{2p-1}_{2s-1}(X)\,\gamma(e_{2p-1}, e_{2r-1}) - \sigma^{2p-1}_{2r-1}(X)\,\gamma(e_{2p-1}, e_{2s-1})\Big) \\ &= \sum_{p=1}^{n} \Big(\sigma^{2p-1}_{2s}(X)\,\gamma(e_{2p-1}, e_{2r}) - \sigma^{2p-1}_{2r}(X)\,\gamma(e_{2p-1}, e_{2s})\Big) + \\ &\quad \sum_{p=1}^{n} \Big(\sigma^{2p}_{2s}(X)\,\gamma(e_{2p}, e_{2r}) - \sigma^{2p}_{2r}(X)\,\gamma(e_{2p}, e_{2s})\Big) \\ &= (\nabla_X\eta)(e_{2r}, e_{2s}),\end{aligned}$$

where we have used (13) and Remark 2.7. The remaining cases can be proved in a similar way.

For case (ii), it is enough to prove the statement for $\eta \in \mathcal{B}$, the basis of Γ given by (6). It is easy to see that $(\nabla_X\beta_j)(e_{2r-1}, e_{2r}) = 0$ for any $r = 1, \ldots, n$ and $j = 1, \ldots, n-1$. On the other hand, we consider

$\gamma_{i,2j-1} = e^{2i-1,2j-1} + e^{2i,2j}$ for $i < j$:

$$\begin{aligned}
\sum_{r=1}^{n} (\nabla_X \gamma_{i,2j-1})(e_{2r-1}, e_{2r}) &= \sum_{r=1}^{n}\sum_{k=1}^{2n} \left(\sigma_{2r}^k(X)\,\gamma_{i,2j-1}(e_k, e_{2r-1}) - \sigma_{2r-1}^k(X)\,\gamma_{i,2j-1}(e_k, e_{2r})\right) \\
&= \sigma_{2j}^{2i-1}(X) - \sigma_{2i}^{2j-1}(X) = \sigma_{2j}^{2i-1}(X) + \sigma_{2j-1}^{2i}(X) \\
&= 0,
\end{aligned}$$

where the last equality holds by (13). The case for $\gamma_{i,2j}$ can be treated in the same way. □

As a consequence of Lemma 3.4 we obtain the following

Proposition 3.5. *Let (J, F) be an abelian balanced Hermitian structure on a unimodular Lie algebra $\mathfrak{g}$. Then, the curvature endomorphisms of the Bismut connection R^{rs} and their covariant derivatives of any order belong to Γ.*

Proof. According to Lemma 3.4 it suffices to prove that $R^{rs} \in \Gamma$. Let us see first that R^{rs} is J-invariant for any $r, s = 1, \ldots, 2n$, i.e. $J(R^{rs}) = R^{rs}$. We compute

$$\begin{aligned}
J(R^{rs})(e_{2i}, e_{2j}) &= R^{rs}(Je_{2i}, Je_{2j}) = R^{rs}(e_{2i-1}, e_{2j-1}) \\
&= -\Omega_{2j-1}^{2i-1}(e_r, e_s) = -\Omega_{2j}^{2i}(e_r, e_s) \\
&= R^{rs}(e_{2i}, e_{2j}),
\end{aligned}$$

where we have used (9) and Corollary 3.3. The remaining cases can be checked analogously.

Next we show that $F^{n-1} \wedge R^{rs} = 0$ for any $r, s = 1, \ldots, 2n$. According to Remark 2.7, it is enough to show that $\sum_{k=1}^n R^{rs}(e_{2k-1}, e_{2k}) = 0$. This condition is equivalent to $\sum_{k=1}^n \Omega_{2k}^{2k-1}(e_r, e_s) = 0$, for any r, s, that is to say $\sum_{k=1}^n \Omega_{2k}^{2k-1} = 0$. By definition,

$$\sum_{k=1}^{n} \Omega_{2k}^{2k-1} = \sum_{k=1}^{n} \left(d\sigma_{2k}^{2k-1} + \sum_{r=1}^{2n} \sigma_r^{2k-1} \wedge \sigma_{2k}^r \right) = d\left(\sum_{k=1}^{n} \sigma_{2k}^{2k-1}\right) + \sum_{k,r}\sum_{s,t} c_{2k-1,r}^s c_{r,2k}^t e^{st}.$$

Due to (12) and (3), $\sum_{k=1}^n \sigma_{2k}^{2k-1} = 0$ and therefore:

$$\begin{aligned}
\sum_{k=1}^{n} \Omega_{2k}^{2k-1} &= \sum_{k,r}\sum_{s,t} c_{2k-1,r}^s c_{r,2k}^t e^{st} = \sum_{k,r}\sum_{s<t} (c_{2k-1,r}^s c_{r,2k}^t - c_{2k-1,r}^t c_{r,2k}^s) e^{st} \\
&= \sum_{s<t}\sum_{k,r} (c_{2k-1,r}^s c_{r,2k}^t - c_{2k-1,r}^t c_{r,2k}^s) e^{st}.
\end{aligned}$$

The expression above is zero if and only if

$$\sum_{k,r} (c_{2k-1,r}^s c_{r,2k}^t - c_{2k-1,r}^t c_{r,2k}^s) = 0$$

for any $s < t$. Now, depending on the parity of r, we can express the summand as

$$\sum_{k,m} (c_{2k-1,2m}^s c_{2m,2k}^t - c_{2k-1,2m}^t c_{2m,2k}^s) + \sum_{k,p} (c_{2k-1,2p-1}^s c_{2p-1,2k}^t - c_{2k-1,2p-1}^t c_{2p-1,2k}^s).$$

Taking into account (14), the previous sum transforms into

$$\sum_{k,m} (c_{2k-1,2m}^s c_{2m,2k}^t - c_{2k-1,2m}^t c_{2m,2k}^s) + \sum_{k,p} (c_{2k,2p}^s c_{2p-1,2k}^t - c_{2k,2p}^t c_{2p-1,2k}^s),$$

which is zero after relabelling subindices in the second summand. □

3.1. **Role of the center of $\mathfrak{g}$.** The center $\mathfrak{z}$ of $\mathfrak{g}$ will play a key role in order to determine the holonomy group of the Bismut connection ∇ for abelian balanced Hermitian structures. So, let us assume that $\mathfrak{z} \neq 0$ and consider $\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{z}^{\perp}$. We recall that if J is an abelian complex structure, then $\mathfrak{z}$ is J-invariant and therefore also $\mathfrak{z}^{\perp}$.

Proposition 3.6. *Let (J, F) be a Hermitian structure on a Lie algebra $\mathfrak{g}$ with J abelian. Then a 1-form $\eta \in \mathfrak{g}^*$ satisfies $\nabla\eta = 0$ if and only if η is dual to an element of the center $\mathfrak{z}$ of $\mathfrak{g}$.*

Proof. Any $\eta \in \mathfrak{g}^*$ can be written as $\eta(\cdot) = g(\cdot, Z_0)$ for a unique $Z_0 \in \mathfrak{g}$. For any $X, Y \in \mathfrak{g}$ we compute

$$\begin{aligned}(\nabla_X\eta)(Y) &= -\eta(\nabla_X Y) = -g(\nabla_X Y, Z_0)\\ &= -g\left(\nabla^g_X Y + \frac{1}{2}T(X,Y), Z_0\right)\\ &= -\frac{1}{2}(g([X,Y],Z_0) - g([Y,Z_0],X) + g([Z_0,X],Y)) + \frac{1}{2}dF(JZ_0, JX, JY)\\ &= -\frac{1}{2}(g([X,Y],Z_0) - g([Y,Z_0],X) + g([Z_0,X],Y))\\ &\quad + \frac{1}{2}(g([JZ_0,JX],Y) + g([JX,JY],Z_0) + g([JY,JZ_0],X)\\ &= g([Y,Z_0],X) \quad \text{since } J \text{ is abelian.}\end{aligned}$$

It follows immediately from this equation that $Z_0 \in \mathfrak{z}$ if and only if $\nabla\eta = 0$. $\square$

In order to study the curvature of the Bismut connection, we choose an adapted basis $\{e_1, \ldots, e_{2n}\}$ of $\mathfrak{g}$ such that $\{e_1, \ldots, e_{2m}\}$ is a basis of $\mathfrak{z}^{\perp}$ and $\{e_{2m+1}, \ldots, e_{2n}\}$ is a basis of $\mathfrak{z}$. Combining the previous result with equations (7) and (8) we obtain

Corollary 3.7. *Let (J, F) be a Hermitian structure on a Lie algebra $\mathfrak{g}$ with J abelian. Then the Bismut connection 1-forms and the curvature 2-forms satisfy:*

$$\sigma^i_j = 0 \quad \text{and} \quad \Omega^i_j = 0, \quad \text{if } \; i > 2m \text{ or } j > 2m. \tag{15}$$

In order to prove the reduction of the holonomy, let us define, for any $q = 1, \ldots, n$,

$$\Gamma_q = \langle \mathcal{B}_q \rangle = \langle \{\gamma_{ij} \,|\, 2i < j < 2q\} \cup \{\beta_r \mid r = 1, \ldots, q-1\} \rangle. \tag{16}$$

For the particular case of abelian balanced Hermitian structures, we obtain the following properties for the curvature endomorphisms and their covariant derivatives:

Lemma 3.8. *Let (J, F) be an abelian balanced Hermitian structure on a unimodular Lie algebra $\mathfrak{g}$ of dimension $2n$. If $\gamma \in \Gamma_m$, then $\nabla_X\gamma \in \Gamma_m$, for any $X \in \mathfrak{g}$, where $2m = \dim\mathfrak{g} - \dim\mathfrak{z}$ as above.*

Proof. By definition

$$(\nabla_X\gamma)(e_r, e_s) = \sum_{k=1}^{2n} \left(\sigma^k_s(X)\,\gamma(e_k, e_r) - \sigma^k_r(X)\,\gamma(e_k, e_s)\right).$$

Using (15) it is possible to restrict the sum up to $k = 2m$. Observe that if $j > 2m$, then $\sigma^k_j(X) = \gamma(e_k, e_j) = 0$ for any $k \leq 2m$, since $\gamma \in \Gamma_m$. We conclude that $(\nabla_X\gamma)(e_r, e_s) = 0$ if $\max\{r, s\} > 2m$, that is to say, $\nabla_X\gamma \in \Gamma_m$ for any $X \in \mathfrak{g}$. $\square$

Proposition 3.9. *Let (J, F) be an abelian balanced Hermitian structure on a unimodular Lie algebra $\mathfrak{g}$ of dimension $2n$. The curvature endomorphisms R^{rs} and their covariant derivatives of any order belong to Γ_m, where $2m = \dim\mathfrak{g} - \dim\mathfrak{z}$.*

Proof. Taking into account Lemma 3.8, it suffices to prove that $R^{rs} \in \Gamma_m$ for any r, s. Observe first that $R^{rs}(e_i, e_j) = -\Omega^i_j(e_r, e_s) = 0$ if $\max\{i, j\} > 2m$. Since $R^{rs} \in \Gamma$, let us express

$$R^{rs} = \sum_{2i<j} a^{rs}_{ij}\gamma_{ij} + \sum_{p=1}^{n-1} b^{rs}_p \beta_p.$$

If $2i < j$ and $j > 2m$ then $0 = R^{rs}(e_i, e_j) = a_{ij}^{rs}$. On the other hand,

$$R^{rs}(e_{2k-1}, e_{2k}) = \begin{cases} b_1^{rs}, & \text{if } k = 1, \\ b_k^{rs} - b_{k-1}^{rs}, & \text{if } 2 \leq k \leq n-1, \\ -b_{n-1}^{rs}, & \text{if } k = n. \end{cases}$$

Since $R^{rs}(e_{2k-1}, e_{2k}) = 0$ if $k \geq m+1$, we obtain that $b_k^{rs} = 0$ for $m+1 \leq k \leq n-1$. Therefore, $R^{rs} \in \Gamma_m$. □

3.2. **Holonomy of the Bismut connection.** According to [16], the holonomy group of the Bismut connection associated to any invariant balanced J-Hermitian structure on a nilmanifold M^{2n} is contained in $SU(n)$.

Our aim in what follows is to prove such a reduction of the holonomy of the Bismut connection in the case of abelian balanced Hermitian structures on a unimodular, non necessarily nilpotent, Lie algebra $\mathfrak{g}$. Moreover, if the center of $\mathfrak{g}$ is non-trivial, then the holonomy reduces to a proper subgroup of $SU(n)$.

To prove the reduction of the holonomy, we recall the natural identification ϕ of the space of 2-forms $\bigwedge^2 \mathfrak{g}^*$ with the Lie algebra $\mathfrak{so}(m)$, $\phi : \bigwedge^2 \mathfrak{g}^* \to \mathfrak{so}(m)$, where $\dim \mathfrak{g} = m$. Indeed, fixing a basis $\{e_1, \dots, e_m\}$ of $\mathfrak{g}$, the 2-form e^{ij} is identified with the skew-symmetric matrix $E_{ij} - E_{ji}$, where E_{ij} is the matrix whose entries are all zero except for the element (i, j) which is equal to 1, i.e. $\phi(e^{ij}) = E_{ij} - E_{ji}$.

Lemma 3.10. *Let (J, F) be an abelian balanced Hermitian structure on a unimodular $2n$-dimensional Lie algebra $\mathfrak{g}$ and $\Gamma \subseteq \bigwedge^2 \mathfrak{g}^*$ given by* (5). *Then, $\phi(\Gamma) \subseteq \mathfrak{so}(2n)$ is a Lie subalgebra isomorphic to $\mathfrak{su}(n)$. Moreover, $\phi(\Gamma_q) \subseteq \phi(\Gamma)$ is a Lie subalgebra isomorphic to $\mathfrak{su}(q) \subseteq \mathfrak{su}(n)$, where Γ_q is defined by* (16).

Proof. First we describe a particular embedding $\mathfrak{su}(n) \hookrightarrow \mathfrak{so}(2n)$. The Lie algebra $\mathfrak{su}(n)$ consists of $n \times n$ complex matrices that are traceless and antihermitian. Fix the standard complex structure J_0 on $\mathbb{R}^{2n}$ given by $J_0 e_{2k-1} = -e_{2k}$, $k = 1, \dots, n$. Then $\mathfrak{su}(n)$ can be identified with a subalgebra of $\mathfrak{gl}(n, \mathbb{C}) := \{M \in \mathfrak{gl}(2n, \mathbb{R}) \mid MJ_0 = J_0 M\}$. More precisely, a matrix in $\mathfrak{su}(n)$ can be considered as a $2n \times 2n$ real skew-symmetric matrix of the form

$$M = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix},$$

where each A_{ij} is a 2×2 block of the form

$$A_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ -b_{ij} & a_{ij} \end{pmatrix} \text{ and } A_{ji} = -A_{ij}^t \text{ if } i < j, \quad A_{ii} = \begin{pmatrix} 0 & c_i \\ -c_i & 0 \end{pmatrix}, \text{ with } \sum_{i=1}^n c_i = 0.$$

If $M = (m_{ij})$, for $i, j = 1, \dots, 2n$, then $a_{ij} = m_{2i-1,2j-1} = m_{2i,2j}$ and $b_{ij} = m_{2i-1,2j} = -m_{2i,2j-1}$, while $c_i = m_{2i-1,2i} = -m_{2i,2i-1}$. Now, taking into account that $\sum_{i=1}^n c_i = 0$, it is immediate to see that $\{\phi(\gamma_{ij}) \mid 2i < j\} \cup \{\phi(\beta_r) \mid r = 1, \dots, n-1\}$ is a basis of $\mathfrak{su}(n)$, where γ_{ij} and β_r are defined by (6). Therefore $\phi(\Gamma) = \mathfrak{su}(n)$ as described above.

For the last statement observe that $\phi(\Gamma_q)$ is the subset of $\mathfrak{su}(n)$ consisting of matrices whose last $2n - 2q$ rows and last $2n - 2q$ columns are zero, which is canonically isomorphic to $\mathfrak{su}(q)$. □

Example 3.11. For instance, if $n = 4$, a basis for Γ, or equivalently for $\mathfrak{su}(4)$ via ϕ, is given by:

$$\begin{aligned} &\langle e^{12} - e^{34}, e^{13} + e^{24}, e^{14} - e^{23}, \\ &\ e^{34} - e^{56}, e^{15} + e^{26}, e^{16} - e^{25}, e^{35} + e^{46}, e^{36} - e^{45}, \\ &\ e^{56} - e^{78}, e^{17} + e^{28}, e^{18} - e^{27}, e^{37} + e^{48}, e^{38} - e^{47}, e^{57} + e^{68}, e^{58} - e^{67} \rangle, \end{aligned}$$

where elements in the first row generate $\mathfrak{su}(2)$ and the first and the second ones generate $\mathfrak{su}(3)$.

The next theorem states the reduction of the holonomy for abelian balanced Hermitian structures:

Theorem 3.12. *Let $\mathfrak{g}$ be a unimodular Lie algebra of dimension $2n$ endowed with an abelian balanced Hermitian structure (J,F) and consider the associated Bismut connection ∇. If $\dim\mathfrak{z}=2k$, then $\mathfrak{hol}(\nabla)\subseteq\mathfrak{su}(n-k)$.*

Proof. According to the well-known Ambrose-Singer theorem, the Lie algebra of the (restricted) holonomy group of ∇, $\mathfrak{hol}(\nabla)$ is generated as a vector space by the curvature endomorphisms R^{rs} and their covariant derivatives of any order. By Proposition 3.9 all these endomorphisms belong to Γ_m which can be identified with $\mathfrak{su}(m)$, where $m=n-k$ (see Lemma 3.10). □

4. Abelian balanced Hermitian structures on nilpotent Lie algebras

In this section we consider the particular case of abelian balanced Hermitian structures (J,F) on $2n$-dimensional nilpotent Lie algebras.

We consider first the more general case of a Hermitian structure (J,F) with abelian J on a Lie algebra $\mathfrak{g}$ with $\dim\mathfrak{g}=2n$. According to [23], due to the fact that J is abelian, there exists a basis of $(1,0)$-forms $\{\alpha^1,\dots,\alpha^n\}$ such that

$$d\alpha^j\in\bigwedge\nolimits^2\langle\alpha^1,\dots,\alpha^{j-1},\bar\alpha^1,\dots\bar\alpha^{j-1}\rangle,\quad j=1,\dots,n. \tag{17}$$

We extend naturally the inner product $g(\cdot,\cdot)=F(J\cdot,\cdot)$ to $\mathfrak{g}^{1,0}$. Applying the Gram-Schmidt process to the basis $\{\alpha^1,\dots,\alpha^n\}$, we obtain an orthogonal basis $\{\omega^1,\dots,\omega^n\}$ of $\mathfrak{g}^{1,0}$ such that $\mathrm{span}_{\mathbb{C}}\{\omega^1,\dots,\omega^j\}=\mathrm{span}_{\mathbb{C}}\{\alpha^1,\dots,\alpha^j\}$ for any j. Therefore the new orthogonal basis also satisfies the structural equations (17). By rescaling appropriately and setting $\omega^j=e^{2j-1}+i\,e^{2j}$, we obtain a real orthonormal basis $\{e^1,\dots,e^{2n}\}$ of $\mathfrak{g}$ such that $Je^{2j-1}=-e^{2j}$, that is, an adapted basis for the abelian Hermitian structure.

A consequence of (17) is that the first Betti number of $\mathfrak{g}$ is greater or equal than 2. Next we show that this lower bound increases for the case of nilpotent Lie algebras endowed with abelian balanced Hermitian structures.

Proposition 4.1. *Let (J,F) be an abelian balanced Hermitian structure on a nilpotent Lie algebra $\mathfrak{g}$ of dimension $2n$. Then $b_1(\mathfrak{g})\geq 4$.*

Proof. According to (17), there exists a basis of $(1,0)$-forms $\{\omega^1,\dots,\omega^n\}$ such that $d\omega^1=0$ and $d\omega^2=a\,\omega^1\wedge\bar\omega^1$. Taking real and imaginary parts, we obtain that $de^1=de^2=0$ and $de^3=2se^{12}$, $de^4=-2re^{12}$, where $a=r+is$, $r,s\in\mathbb{R}$. Since the Hermitian structure (J,F) is abelian balanced and $\mathfrak{g}$ is unimodular, we have that de^3, $de^4\in\Gamma=\langle e^{12}-e^{34},\,e^{13}+e^{24},\,e^{14}-e^{23}\rangle$ but this happens if and only if $r=s=0$ and therefore $b_1(\mathfrak{g})\geq 4$. □

If $\dim\mathfrak{g}=4$, the previous proposition says that the only nilpotent Lie algebra admitting an abelian balanced Hermitian structure is the abelian one (observe that in dimension four, balanced metrics are Kähler, and therefore, the Lie algebra must be abelian [7]). For $\dim\mathfrak{g}=6$, these structures are studied in [25].

In what follows we denote by $\mathfrak{h}_{2n+1}$ the Lie algebra of the Heisenberg group of dimension $2n+1$.

Proposition 4.2. *The Lie algebras $\mathfrak{g}\cong\mathfrak{h}_{2n+1}\times\mathbb{R}^{2k+1}$, where $\mathfrak{h}_{2n+1}$ is a real Heisenberg group, admit abelian balanced Hermitian structures (J,F) if and only if $n\geq 2$.*

Proof. According to [22], any complex structure in $\mathfrak{h}_{2n+1}\times\mathbb{R}^{2k+1}$ is abelian and it is equivalent to one of the form

$$Je_{2i-1}=\pm e_{2i},\quad 1\leq i\leq n,\quad Jz_{2j}=-z_{2j+1},\quad 0\leq j\leq k, \tag{18}$$

$\{e_1,\dots,e_{2n},\,z_0,\dots,z_{2k+1}\}$ is a basis of $\mathfrak{h}_{2n+1}\times\mathbb{R}^{2k+1}$ satisfying $[e_{2i-1},e_{2i}]=z_0$ for any $i=1,\dots,n$. Moreover, according to [3], there exist $[n/2]+1$ equivalence classes of complex structures, depending on the number of minus signs in the complex structure (18). Concretely, consider the following complex structures:

$$J_0e_{2i-1}=e_{2i},\quad i=1,\dots,n,$$

$$\begin{cases} J_r e_{2i-1} = -e_{2i}, & i = 1, \dots, r, \\ J_r e_{2i-1} = e_{2i}, & i = r+1, \dots, n, \end{cases} \qquad r = 1, \dots, n-1,$$
$$J_n e_{2i-1} = -e_{2i}, \quad i = 1, \dots, n.$$

Now, it is clear that J_k is equivalent to J_{n-k} and therefore, we can consider $0 \le r \le [n/2]$.

If $n = 1$, then the only complex structure up to equivalence is J_0 and (2) cannot be satisfied, which means that there are no balanced metrics. On the other hand, if $n \ge 2$, let us consider structures J_r with $r \ge 1$. Clearly, the basis $\{e_1, \dots, e_{2n}, z_0, \dots, z_{2k+1}\}$ is not adapted to the complex structure J_r. However, we can define a new adapted basis $\{f_1, \dots, f_{2n+1}, z_0, \dots, z_{2k+1}\}$ in the following way:

$$f_i = \sqrt{n+1-2r}\; e_i, \text{ for } i = 1, 2, \quad f_{2i} = -e_{2i}, \text{ for } i = r+1, \dots, n,$$

and $f_i = e_i$ for the remaining cases. Moreover,

$$\begin{aligned} [f_1, f_2] &= (n+1-2r) z_0, \\ [f_{2i-1}, f_{2i}] &= z_0, \text{ for } i = 1, \dots, r, \\ [f_{2i-1}, f_{2i}] &= -z_0, \text{ for } i = r+1, \dots, n. \end{aligned}$$

Now,

$$\begin{aligned} \sum_{i=1}^{n} [J_r f_{2i-1}, f_{2i-1}] &= [f_1, f_2] + \sum_{i=2}^{r} [f_{2i-1}, f_{2i}] + \sum_{i=r+1}^{n} [f_{2i-1}, f_{2i}] \\ &= [(n+1-2r) + (r-1) - (n-r)]\, z_0 = 0. \end{aligned}$$

By Lemma 2.1, the metric that makes $\{f_1, \dots, f_{2n+1}, z_0, \dots, z_{2k+1}\}$ orthonormal is balanced. □

Moreover, we obtain the following result:

Proposition 4.3. *The Lie algebra $\mathfrak{h}_{2n+1} \times \mathbb{R}^{2k+1}$ possesses $[\frac{n}{2}] + 1$ equivalence classes of abelian complex structures. If $n \ge 2$, among them, only $[\frac{n}{2}]$ admit balanced metrics.*

Proof. In the proof of the previous proposition we have seen that J_r, $r \neq 0$, admit balanced metrics. Let us see now that in fact J_0 does not admit this type of metrics. The complex structure J_0 is equivalent to J_n, therefore the basis $\{e_1, \dots, e_{2n}, z_0, \dots, z_{2k+1}\}$ is J_n-adapted and $[e_{2i-1}, e_{2i}] = z_0$ for $i = 1, \dots, n$. In particular, this basis does not satisfy equation (2). Let us see that no J_n-adapted basis may satisfy that equation.

Consider a generic basis of the form $\{f_1, \dots, f_{2n}, z_0, \dots, z_{2k+1}\}$ where $f_i = \sum_{j=1}^{2n} \lambda_j^i e_j$, for $i = 1, \dots, 2n$, and $(\lambda_j^i) \in GL(2n, \mathbb{R})$. Imposing the condition that this basis is J_n-adapted, that is $J_n f_{2i-1} = -f_{2i}$, we obtain the conditions

$$\lambda_{2j-1}^{2i-1} = \lambda_{2j}^{2i}, \quad \lambda_{2j}^{2i-1} = -\lambda_{2j-1}^{2i}.$$

Now,

$$\begin{aligned} [f_{2i-1}, f_{2i}] &= \left[\sum_{j=1}^{n} \lambda_{2j-1}^{2i-1} e_{2j-1} + \sum_{j=1}^{n} \lambda_{2j}^{2i-1} e_{2j},\ \sum_{j=1}^{n} \lambda_{2j-1}^{2j} e_{2j-1} + \sum_{j=1}^{n} \lambda_{2j}^{2i} e_{2j} \right] \\ &= \left[\sum_{j=1}^{n} \lambda_{2j-1}^{2i-1} e_{2j-1},\ \sum_{j=1}^{n} \lambda_{2j}^{2i} e_{2j} \right] + \left[\sum_{j=1}^{n} \lambda_{2j}^{2i-1} e_{2j},\ \sum_{j=1}^{n} \lambda_{2j-1}^{2j} e_{2j-1} \right] \\ &= \sum_{j=1}^{n} \left(\lambda_{2j-1}^{2i-1} \lambda_{2j}^{2i} - \lambda_{2j-1}^{2i} \lambda_{2j}^{2i-1} \right) [e_{2j-1}, e_{2j}] \\ &= \left(\sum_{j=1}^{n} \left(\lambda_{2j-1}^{2i-1} \right)^2 + \left(\lambda_{2j}^{2i-1} \right)^2 \right) z_0, \end{aligned}$$

i.e., $[f_{2i-1}, f_{2i}] = \mu_i z_0$ where $\mu_i > 0$. Now it is clear that (2) cannot be satisfied. □

Remark 4.4. According to [11], the Lie algebra $\mathfrak{h}_{2n+1} \times \mathbb{R}$, $n \geq 1$, admits a J_0-Hermitian metric g which is locally conformally Kähler, i.e. the fundamental form F satisfies $dF = \theta \wedge F$ for some closed 1-form θ. In this case, we have $d^*F \neq 0$. Moreover, it was proved in [5] that these are the only Lie algebras admitting locally conformally Kähler structures with abelian complex structure.

As a consequence of previous results we can conclude that abelian balanced Hemitian structures also impose conditions on the center of the nilpotent Lie algebra and therefore on the holonomy group of the associated Bismut connection.

Proposition 4.5. *Let (J, F) be an abelian balanced Hermitian structure on a nilpotent Lie algebra $\mathfrak{g}$ of dimension $2n \geq 6$. Then $2 \leq \dim \mathfrak{z} \leq 2n - 4$.*

Proof. It is known that any nilpotent Lie algebra has non-trivial center. Moreover, since J is abelian, then the center is J-invariant and therefore $\dim \mathfrak{z} \geq 2$. Let us suppose now that $\dim \mathfrak{z} = 2n - 2$. This means that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_3 \times \mathbb{R}^{2n-3}$. By Proposition 4.2, this algebra does not admit balanced Hermitian metrics. □

Corollary 4.6. *Let (J, F) be an abelian balanced Hermitian structure on a nilpotent Lie algebra $\mathfrak{g}$ of dimension $2n \geq 6$. Then, the holonomy group of the Bismut connection reduces to a proper subgroup of $SU(n)$.*

Proof. The result follows directly from Theorem 3.12 and the fact that $\dim \mathfrak{z} \geq 2$. □

The abelian balanced Hermitian structures on $\mathfrak{h}_{2n+1} \times \mathbb{R}^{2k+1}$ provide the strongest reduction of the holonomy of the Bismut connection.

Proposition 4.7. *Let ∇ be the Bismut connection associated to any abelian balanced Hermitian structure (J, F) on $\mathfrak{h}_{2n+1} \times \mathbb{R}^{2k+1}$, $n \geq 2$. Then, $\mathfrak{hol}(\nabla) = \mathfrak{u}(1)$.*

Proof. Consider an adapted basis $\{e_1, \ldots, e_{2n}, z_1, \ldots, z_{2k+1}, z_0\}$ of $\mathfrak{h}_{2n+1} \times \mathbb{R}^{2k+1}$ and its dual $\{e^1, \ldots, e^{2n+2k+2}\}$. The structure equations for any abelian balanced Hermitian structure are given by

$$de^i = 0, \quad i = 1, \ldots, 2n + 2k + 1, \quad de^{2n+2k+2} = \sum_{r=1}^{n-1} b_r \beta_r,$$

with β_r defined in (6). Since the only non-zero structure constants are $c_{2i-1,2i}^{2n+2k+2}$ for $i = 1, \ldots, n$, the only non-zero Bismut connection 1-forms are σ_{2i}^{2i-1}, which are multiples of $e^{2n+2k+2}$ by (12). Moreover, the curvature 2-forms Ω_{2i}^{2i-1} are multiples of $de^{2n+2k+2}$ and the curvature endomorphisms $R^{2i-1,2i}$ can be identified also with multiples of $de^{2n+2k+2}$ via (9). In this way, the space generated by the curvature endomorphisms is 1-dimensional.

For covariant derivatives, observe that if X is not a multiple of z_0, then $\sigma_q^p(X) = 0$ and therefore $\nabla_X \gamma = 0$. Now, if $X = z_0$, consider $\gamma = e^{2i-1,2i}$. The terms $\sigma_s^k(z_0)\gamma(e_k, e_r)$ in (10) are non-zero if and only if $k = 2i - 1$, $r = s = 2i$ or $k = 2i$, $r = s = 2i - 1$ but in both cases, $\nabla_{z_0}\gamma(e_r, e_r) = 0$. Therefore, $\nabla_X R = 0$ for any curvature endomorphism R, which means that $\mathfrak{hol}(\nabla) = \langle de^{2n+2k+2} \rangle$ and in particular, $\dim \mathfrak{hol}(\nabla) = 1$. □

Remark 4.8. The result of Corollary 4.6 does not hold in general for solvable Lie algebras. For instance, as it was shown in [25], the 6-dimensional Lie algebra with structure equations

$$de^1 = de^2 = 0, \quad de^3 = -e^{13} - e^{24}, \quad de^4 = -e^{14} + e^{23}, \quad de^5 = e^{15} + e^{26}, \quad de^6 = e^{16} - e^{25},$$

endowed with the abelian balanced Hermitian structure (J, F), where $Je^{2i-1} = -e^{2i}$ and $F = e^{12} + e^{34} + e^{56}$, has Bismut holonomy group equal to $SU(3)$. According to [3], this is the only 6-dimensional unimodular non-nilpotent Lie algebra that admits abelian complex structures.

4.1. **Abelian balanced Hermitian structures on 8-dimensional nilpotent Lie algebras.** Let $\mathfrak{g}$ be an 8-dimensional nilpotent Lie algebra endowed with an abelian balanced Hermitian structure (J, F). Using Corollary 2.6, Propositions 4.1 and 4.5, we can consider an adapted basis $\{e_1, \ldots, e_8\}$ of $\mathfrak{g}$ such that the Lie bracket is generically given by:

$$
\begin{aligned}
&[e_1, e_3] = [e_2, e_4] = -c_{13}^5 e_5 - c_{13}^6 e_6 - c_{13}^7 e_7 - c_{13}^8 e_8, && [e_1, e_2] = -c_{12}^5 e_5 - c_{12}^6 e_6 - c_{12}^7 e_7 - c_{12}^8 e_8,\\
&[e_1, e_4] = -[e_2, e_3] = -c_{14}^5 e_5 - c_{14}^6 e_6 - c_{14}^7 e_7 - c_{14}^8 e_8, && [e_3, e_4] = c_{12}^5 e_5 + c_{12}^6 e_6 - c_{34}^7 e_7 - c_{34}^8 e_8,\\
&[e_1, e_5] = [e_2, e_6] = -c_{15}^7 e_7 - c_{15}^8 e_8, && [e_5, e_6] = (c_{12}^7 + c_{34}^7) e_7 + (c_{12}^8 + c_{34}^8) e_8,\\
&[e_1, e_6] = -[e_2, e_5] = -c_{16}^7 e_7 - c_{16}^8 e_8, &&\\
&[e_3, e_5] = [e_4, e_6] = -c_{35}^7 e_7 - c_{35}^8 e_8, &&\\
&[e_3, e_6] = -[e_4, e_5] = -c_{36}^7 e_7 - c_{36}^8 e_8, &&
\end{aligned}
$$

where coefficients $c_{ij}^k \in \mathbb{R}$ must satisfy relations that ensure the Jacobi identity. Analogously, the corresponding structure equations are:

$$
\begin{cases}
de^i = 0, \quad i = 1, 2, 3, 4,\\
de^5 = c_{12}^5 \beta_1 + c_{13}^5 \gamma_{13} + c_{14}^5 \gamma_{14},\\
de^6 = c_{12}^6 \beta_1 + c_{13}^6 \gamma_{13} + c_{14}^6 \gamma_{14},\\
de^7 = c_{12}^7 \beta_1 + (c_{12}^7 + c_{34}^7)\beta_2 + c_{13}^7 \gamma_{13} + c_{14}^7 \gamma_{14} + c_{15}^7 \gamma_{15} + c_{16}^7 \gamma_{16} + c_{35}^7 \gamma_{25} + c_{36}^7 \gamma_{26},\\
de^8 = c_{12}^8 \beta_1 + (c_{12}^8 + c_{34}^8)\beta_2 + c_{13}^8 \gamma_{13} + c_{14}^8 \gamma_{14} + c_{15}^8 \gamma_{15} + c_{16}^8 \gamma_{16} + c_{35}^8 \gamma_{25} + c_{36}^8 \gamma_{26},
\end{cases} \tag{19}
$$

where β_r and γ_{ij} are defined in (6). Observe that e_7 and e_8 are always central elements and e_5 and e_6 are central if and only if $c_{12}^i + c_{34}^i = c_{15}^i = c_{16}^i = c_{35}^i = c_{36}^i = 0$ for $i = 7$ and 8.

Proposition 4.9. *Let $\mathfrak{g}$ be an 8-dimensional nilpotent Lie algebra with 4-dimensional center endowed with an abelian balanced Hermitian structure (J, F). Then, $\mathfrak{g}$ is isomorphic to one and only one algebra in the following list:*

- I.1) $\mathfrak{g}_1$: $de^i = 0, \quad i = 1, 2, 3, 4, 5, \quad de^6 = e^{12} - e^{34}, \quad de^7 = e^{13} + e^{24}, \quad de^8 = e^{14} - e^{23}$;
- I.2) $\mathfrak{g}_2$: $de^i = 0, \quad i = 1, 2, 3, 4, 5, 6, \quad de^7 = e^{13} + e^{24}, \quad de^8 = e^{14} - e^{23}$;
- I.3) $\mathfrak{g}_3$: $de^i = 0, \quad i = 1, 2, 3, 4, 5, 6, 7, \quad de^8 = e^{12} - e^{34}$.

Proof. Let $\{e_1, \ldots, e_8\}$ be an adapted basis such that the center of $\mathfrak{g}$ is given by $\mathfrak{z} = \langle e_5, e_6, e_7, e_8 \rangle$, i.e. the structure equations are of type:

$$de^i = 0, \quad i = 1, 2, 3, 4, \quad de^i = c_{12}^i \beta_1 + c_{13}^i \gamma_{13} + c_{14}^i \gamma_{14}, \quad i = 5, 6, 7, 8.$$

Clearly, the Jacobi identity is satisfied for any value of c_{ij}^k. Since $de^i \in \langle \beta_1, \gamma_{13}, \gamma_{14} \rangle$ for $i = 5, 6, 7, 8$, we can assume that $de^5 = 0$. Now, depending on the dimension of $\langle de^6, de^7, de^8 \rangle$, it is not hard to show that the Lie algebra must be isomorphic to one of these:

$$
\begin{array}{lllll}
de^i = 0, & i = 1, 2, 3, 4, 5, & de^6 = \beta_1, & de^7 = \gamma_{13}, & de^8 = \gamma_{14},\\
de^i = 0, & i = 1, 2, 3, 4, 5, & de^6 = 0, & de^7 = \gamma_{13}, & de^8 = \gamma_{14},\\
de^i = 0, & i = 1, 2, 3, 4, 5, & de^6 = 0, & de^7 = 0, & de^8 = \beta_1.
\end{array}
$$

□

Remark 4.10. The Lie algebras that appear in the previous proposition are algebras associated to groups of type H: $\mathfrak{g}_1$ is the product of the real Lie algebra of the quaternionic Heisenberg group and $\mathbb{R}$; $\mathfrak{g}_2$ is the product of the real Lie algebra of the complex Heisenberg group and $\mathbb{R}^2$ and $\mathfrak{g}_3$ is $\mathfrak{h}_5 \times \mathbb{R}^3$. These algebras are the only nilpotent ones that admit abelian hypercomplex structures in dimension 8, [13].

Proposition 4.11. *Let $\mathfrak{g}$ be an 8-dimensional nilpotent Lie algebra with 2-dimensional center endowed with an abelian balanced Hermitian structure (J, F). Then, $\mathfrak{g}$ satisfies one of the following conditions:*

- II.1) $de^5 = de^6 = 0$ *and $\mathfrak{g}$ is 2-step.*
- II.2) de^5 *and* de^6 *are linearly independent and $\mathfrak{g}$ is 3-step.*

Proof. For case II.1) observe that $de^5 = de^6 = 0$ is equivalent to say that $e_5, e_6 \notin [\mathfrak{g}, \mathfrak{g}]$. In fact, $[\mathfrak{g}, \mathfrak{g}] \subseteq \langle e_7, e_8 \rangle$ and therefore $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] = 0$.

For case II.2), let $\{e'_1, \dots, e'_8\}$ be an adapted basis such that $\mathfrak{z} = \langle e'_7, e'_8 \rangle$, i.e. the structure equations are given by (19), where some of the coefficients $c^i_{12} + c^i_{34}$, c^i_{15}, c^i_{16}, c^i_{35}, c^i_{36} for $i = 7$ or $i = 8$ are non-zero and suppose that de'^5 and de'^6 are linearly dependent, i.e., there exists $\lambda \in \mathbb{R}$ such that $de'^5 = \lambda de'^6$. We can define a new basis in the following way:

$$e^i = e'^i,\ i \neq 5, 6, \quad e^5 = e'^5 - \lambda e'^6, \quad e^6 = \lambda e'^5 + e'^6.$$

This basis preserves equations (19) and since $Je^5 = -e^6$, it is an adapted basis for the Hermitian structure. Moreover, $de^5 = 0$ and we can suppose that $de^6 \neq 0$.

Imposing $d^2 e^7 = 0$ in (19) we obtain that $c^7_{12} + c^7_{34} = 0$ and the following system of equations:

$$\begin{aligned} c^6_{13}c^7_{15} + c^6_{14}c^7_{16} - c^6_{12}c^7_{36} &= 0, \\ c^6_{14}c^7_{15} - c^6_{13}c^7_{16} - c^6_{12}c^7_{35} &= 0, \\ c^6_{12}c^7_{16} - c^6_{13}c^7_{35} + c^6_{14}c^7_{36} &= 0, \\ c^6_{12}c^7_{15} + c^6_{14}c^7_{35} - c^6_{13}c^7_{36} &= 0. \end{aligned}$$

If we consider the previous system as homogeneous in variables c^7_{ij}, the determinant of the associated matrix is simply $((c^6_{12})^2 + (c^6_{13})^2 + (c^6_{14})^2)^2 \neq 0$ since $de^6 \neq 0$. Therefore, $c^7_{15} = c^7_{16} = c^7_{35} = c^7_{36} = 0$.

Similarly, we obtain that $c^8_{12} + c^8_{34} = c^8_{15} = c^8_{16} = c^8_{35} = c^8_{36} = 0$. But in this case, the center should be 4-dimensional, which is a contradiction.

Finally, observe that $de^5, de^6 \neq 0$ and therefore $e^5, e^6 \in [\mathfrak{g}, \mathfrak{g}]$. Since $e_5 \notin \mathfrak{z}$, $[e_i, e_5] \neq 0$ for some $i = 1, \dots, 4$, which means that $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] \neq 0$. However $[[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}], \mathfrak{g}] = 0$. □

Remark 4.12. A particular case of II.1) in the previous proposition is $\mathfrak{h}_7 \times \mathbb{R}$ which appear if de^7 and de^8 are linearly dependent but not zero simultaneously.

5. Examples

In this section we will provide two ways of constructing unimodular Lie algebras endowed with abelian balanced Hermitian structures. In the first one the examples arise from nilpotent commutative associative algebras, whereas the second method uses semidirect products of abelian complex Lie algebras.

5.1. **Nilpotent commutative associative algebras.** Let us recall the following construction, introduced in [6]. If A is a finite-dimensional commutative associative algebra, then $\mathfrak{aff}(A) := A \oplus A$ is a Lie algebra with the following bracket:

$$[(a, b), (a', b')] = (0, ab' - ba'). \tag{20}$$

Moreover, the endomorphism $J : \mathfrak{aff}(A) \to \mathfrak{aff}(A)$ defined by $J(a, b) = (b, -a)$ is an abelian complex structure, called the *standard* abelian complex structure. According to [5], the Lie algebra $\mathfrak{aff}(A)$ is unimodular if and only if A is nilpotent, and in this case, it follows that $\mathfrak{aff}(A)$ is actually a nilpotent Lie algebra.

So, let us assume from now on that A is a nilpotent commutative associative algebra of dimension n. Let us fix an inner product $\langle \cdot, \cdot \rangle$ on A and extend it to an inner product on $\mathfrak{aff}(A)$ by considering $\langle \cdot, \cdot \rangle \oplus \langle \cdot, \cdot \rangle$. It is readily seen that $\langle J\cdot, J\cdot \rangle = \langle \cdot, \cdot \rangle$, so that (J, F) is a Hermitian structure on $\mathfrak{aff}(A)$, where $F(\cdot, \cdot) := \langle \cdot, J\cdot \rangle$, as usual. Let $\{e_1, \dots, e_n\}$ denote an orthonormal basis of A, and set $u_i := (e_i, 0)$, $v_i := (0, e_i)$, $i = 1, \dots, n$, so that $Ju_i = -v_i$, thus $\{u_1, v_1, \dots, u_n, v_n\}$ is an adapted basis of $\mathfrak{aff}(A)$.

According to Lemma 2.1, the Hermitian structure is balanced if and only if $\sum_{i=1}^n ([Ju_i, u_i] + [Jv_i, v_i]) = 0$, but due to (20), this condition is equivalent to $\sum_{i=1}^n e_i^2 = 0$. This proves the following result:

Proposition 5.1. *Let A be a nilpotent commutative associative algebra of dimension n with an inner product, and consider the Hermitian structure (J, F) on $\mathfrak{aff}(A)$ as above, where J is the standard abelian complex structure. Then (J, F) is balanced if and only if there exists an orthonormal basis $\{e_1, \dots, e_n\}$ of A such that $\sum_{i=1}^n e_i^2 = 0$.*

Example 5.2. According to [12] (see also [17]), there are 4 isomorphism classes of 3-dimensional nilpotent commutative associative algebras. However, it is easy to verify that only one of them satisfies the conditions of Proposition 5.1. We will denote this algebra by A_1; it has a basis $\{e_1, e_2, e_3\}$ such that $e_1^2 = -e_3$, $e_2^2 = e_3$, and we will define an inner product on A_1 by declaring this basis orthonormal. The corresponding nilpotent Lie algebra $\mathfrak{aff}(A_1)$ is isomorphic to $\mathfrak{h}_5 \times \mathbb{R}$, and its abelian balanced Hermitian structure (J, F) corresponds to one of the balanced structures on this Lie algebra appearing in [25], with the complex structure $\tilde{J}^-$.

Example 5.3. In dimension 4, there are 3 (up to isomorphism) nilpotent commutative associative algebras that satisfy the conditions of Proposition 5.1 (see [12]), which we denote B_1, B_2 and B_3.

(i) The algebra B_1 is isomorphic to the direct product $B_1 = A_1 \times \mathbb{R}$, and accordingly $\mathfrak{aff}(B_1) = \mathfrak{aff}(A_1) \times \mathbb{R}^2 \cong (\mathfrak{h}_5 \times \mathbb{R}) \times \mathbb{R}^2$, a direct product of Hermitian Lie algebras.

(ii) The algebra B_2 has a basis $\{e_1, e_2, e_3, e_4\}$ such that $e_1^2 = e_4$, $e_2^2 = e_4$, $e_3^2 = -2e_4$. For any $\lambda > 0$, we will define an inner product $\langle \cdot, \cdot \rangle_\lambda$ on B_2 by declaring the basis $\{e_1, e_2, e_3, \lambda^{-1} e_4\}$ orthonormal. Equivalently, we can consider a new orthonormal basis $\{e_1, e_2, e_3, e_4\}$ where now the product is given by $e_1^2 = \lambda e_4$, $e_2^2 = \lambda e_4$, $e_3^2 = -2\lambda e_4$, $\lambda > 0$. The corresponding nilpotent Lie algebra $\mathfrak{aff}(B_2)_\lambda$ has an adapted basis $\{u_1, v_1, \dots, u_4, v_4\}$ such that
$$[u_1, v_1] = \lambda v_4, \quad [u_2, v_2] = \lambda v_4, \quad [u_3, v_3] = -2\lambda v_4,$$
and the fundamental 2-form is given in terms of the dual basis $\{u^i, v^i\}$ by $F = \sum_{i=1}^4 u^i \wedge v^i$. Clearly, this Lie algebra is isomorphic to $\mathfrak{h}_7 \times \mathbb{R}$ for any $\lambda > 0$. Moreover, the associated Hermitian metrics $g_\lambda = F(J\cdot, \cdot)$ are pairwise non-isometric, since the corresponding scalar curvature is $-3\lambda^2$. Therefore we obtain a curve (J_2, F_λ), $\lambda > 0$, of non-equivalent abelian balanced Hermitian structures on $\mathfrak{h}_7 \times \mathbb{R}$, where J_2 is the abelian complex structure given in the proof of Corollary 4.3.

(iii) The algebra B_3 has a basis $\{e_1, e_2, e_3, e_4\}$ such that $e_1^2 = -e_3$, $e_2^2 = e_3$, $e_1 e_2 = e_2 e_1 = e_4$, and we will define an inner product on B_3 by declaring this basis orthonormal. The corresponding nilpotent Lie algebra $\mathfrak{aff}(B_3)$ has an adapted basis $\{u_1, v_1, \dots, u_4, v_4\}$ such that
$$[u_1, v_1] = -v_3, \quad [u_2, v_2] = v_3, \quad [u_1, v_2] = [u_2, v_1] = v_4,$$
and the fundamental 2-form is given in terms of the dual basis $\{u^i, v^i\}$ by $F = \sum_{i=1}^4 u^i \wedge v^i$. Consider the new basis given by:
$$h^1 = u^1,\ h^2 = -u^2,\ h^3 = v^1,\ h^4 = v^2,\ h^5 = u^3,\ h^6 = u^4,\ h^7 = v^3,\ h^8 = -v^4.$$
Now, it is clear that $\mathfrak{aff}(B_3)$ is isomorphic to $\mathfrak{g}_2$ in Proposition 4.9.

5.2. **Semidirect products of abelian complex Lie algebras.** Let $m, n \in \mathbb{N}$ and consider a representation $\pi : \mathbb{C}^m \to \mathrm{End}(\mathbb{C}^n) \cong \mathfrak{gl}(n, \mathbb{C})$ such that $\mathrm{tr}(\pi(x)) = 0$ for any $x \in \mathbb{C}^m$, i.e., $\pi(\mathbb{C}^m) \subset \mathfrak{sl}(n, \mathbb{C})$. We can therefore build a semidirect product and obtain in this way the complex Lie algebra $\mathfrak{g} := \mathbb{C}^m \ltimes_\pi \mathbb{C}^n$, which is 2-step solvable and unimodular. We will denote by $\tilde{\mathfrak{g}}$ its realification, $\tilde{\mathfrak{g}} := \mathfrak{g}_\mathbb{R}$, which is also 2-step solvable and unimodular. Choosing some identifications $\mathbb{C}^m \equiv \mathbb{R}^{2m}$ and $\mathbb{C}^n \equiv \mathbb{R}^{2n}$, the representation π induces a representation $\tilde{\pi} : \mathbb{R}^{2m} \to \mathfrak{sl}(n, \mathbb{C}) \hookrightarrow \mathfrak{sl}(2n, \mathbb{R})$, and therefore we have the decomposition $\tilde{\mathfrak{g}} = \mathbb{R}^{2m} \ltimes_{\tilde{\pi}} \mathbb{R}^{2n}$. Note that the Lie algebra $\tilde{\mathfrak{g}}$ does not depend on the chosen identifications, it depends only on the representation π.

Multiplication by i on $\mathfrak{g}$ induces a canonical bi-invariant complex structure I on $\tilde{\mathfrak{g}}$, that is $I^2 = -\,\mathrm{id}$ and $[Ix, y] = [x, Iy] = I[x, y]$ for any $x, y \in \tilde{\mathfrak{g}}$. By suitably modifying the bi-invariant complex structure I, we will obtain an abelian complex structure on $\tilde{\mathfrak{g}}$. Indeed, let us define $J : \tilde{\mathfrak{g}} \to \tilde{\mathfrak{g}}$ by
$$J|_{\mathbb{R}^{2m}} = -I, \qquad J|_{\mathbb{R}^{2n}} = I,$$
and compute, for $x_j \in \mathbb{R}^{2m}$, $y_j \in \mathbb{R}^{2n}$, $j = 1, 2$:
$$[J(x_1 + y_1), J(x_2 + y_2)] = [-Ix_1 + Iy_1, -Ix_2 + Iy_2] = [x_1, y_2] + [y_1, x_2] = [x_1 + y_1, x_2 + y_2],$$

thus J is abelian. Note that both $\mathbb{R}^{2m}$ and $\mathbb{R}^{2n}$ are J-invariant.

Now let us choose an inner product $\langle\cdot,\cdot\rangle$ such that it is Hermitian with respect to J and the two factors are orthogonal. There exists an adapted basis $\{e_1,\dots,e_{2m+2n}\}$ of $\tilde{\mathfrak{g}}$ such that $\{e_1,\dots,e_{2m}\}$ is an adapted basis of $\mathbb{R}^{2m}$ and $\{e_{2m+1},\dots,e_{2m+2n}\}$ is an adapted basis of $\mathbb{R}^{2n}$. Clearly, $\sum_{i=1}^{2m+2n}[Je_i,e_i]=0$, and therefore, according to Lemma 2.1, (J,F) is an abelian balanced Hermitian structure on $\tilde{\mathfrak{g}}$, where F is the associated fundamental 2-form.

To sum up, we state the following

Proposition 5.4. *The realification of the unimodular complex Lie algebra $\mathbb{C}^m\ltimes_\pi\mathbb{C}^n$, where $\pi:\mathbb{C}^m\to\mathfrak{sl}(n,\mathbb{C})$ is a representation, admits abelian balanced Hermitian structures.*

Let us consider in more detail the case $m=1$. In this situation, $\mathfrak{g}=\mathbb{C}\ltimes_\pi\mathbb{C}^n$ is called an almost abelian complex Lie algebra, and the action of $\mathbb{C}$ on $\mathbb{C}^n$ is determined by a single traceless operator $M:=\pi(1)\in\mathfrak{sl}(n,\mathbb{C})$. Let us consider now its realification $\tilde{\mathfrak{g}}=\mathbb{R}^2\ltimes_{\tilde\pi}\mathbb{R}^{2n}$. If $\{A,B\}$ is a basis of $\mathbb{R}^2$ with $IA=B$, then the associated real representation $\tilde\pi:\mathbb{R}^2\to\mathfrak{sl}(2n,\mathbb{R})$ is characterized by $\tilde\pi(A)$ and $\tilde\pi(B)$. If $\tilde M$ denotes the image of M under the embedding $\mathfrak{sl}(n,\mathbb{C})\hookrightarrow\mathfrak{sl}(2n,\mathbb{R})$ determined by the identification $\mathbb{C}^n\equiv\mathbb{R}^{2n}$, then clearly $\tilde\pi(A)=\tilde M$ and $\tilde\pi(B)=I\tilde M=\tilde M I$. Therefore the Lie bracket on the Lie algebra $\tilde{\mathfrak{g}}=\mathbb{R}^2\ltimes_{\tilde\pi}\mathbb{R}^{2n}$ is described by $[A,X]=\tilde MX$, $[B,X]=I\tilde MX$ for any $X\in\mathbb{R}^{2m}$; its abelian complex structure is given by $JA=-B$, $JX=IX$ for any $X\in\mathbb{R}^{2m}$.

Remark 5.5. It is well known that two almost abelian Lie algebras $\mathbb{C}\ltimes_M\mathbb{C}^n$ and $\mathbb{C}\ltimes_{M'}\mathbb{C}^n$ are isomorphic if and only if $M'=cPMP^{-1}$ for some non-zero $c\in\mathbb{C}$ and $P\in GL(n,\mathbb{C})$. Clearly, the realification of two isomorphic complex Lie algebras are isomorphic as real Lie algebras.

Example 5.6. (1) If we consider $\mathfrak{g}=\mathbb{C}\ltimes_M\mathbb{C}^2$, then taking into account the canonical Jordan form of a 2×2 traceless complex matrix and Remark 5.5, we only have to consider two possibilities for M, either

$$(i)\,M=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad\text{or}\quad(ii)\,M=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

In case (i), $\mathfrak{g}$ is a 2-step nilpotent Lie algebra, and its realification $\tilde{\mathfrak{g}}$ is isomorphic to the Lie algebra with structure equations

$$de^i=0,\ i=1,2,3,4,\quad de^5=e^{13}+e^{42},\quad de^6=e^{14}+e^{23}, \tag{21}$$

i.e., it is the realification of the complex Heisenberg Lie algebra. It was shown in [25] that any abelian complex structure on this Lie algebra admits balanced metrics. In case (ii), we reobtain the unimodular solvable Lie algebra from Remark 4.8, with the same abelian balanced Hermitian structure.

(2) We consider now $\mathfrak{g}=\mathbb{C}\ltimes_M\mathbb{C}^3$, where M is a 3×3 traceless complex matrix. The possible canonical Jordan forms for such a matrix are

$$(i)\,M=\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix},\quad(ii)\,M=\begin{pmatrix}\lambda&0&0\\1&\lambda&0\\0&0&-2\lambda\end{pmatrix},\quad(iii)\,M=\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix},$$

where $\lambda,\lambda_j\in\mathbb{C}$ with $\sum_{j=1}^3\lambda_j=0$. From this, we easily see that the only nilpotent examples are obtained in case (ii), for $\lambda=0$, and in case (iii). In the former case, the Lie algebra $\mathfrak{g}$ is 2-step nilpotent, and $\tilde{\mathfrak{g}}$ is isomorphic to the direct product of the Lie algebra underlying equations (21) with $\mathbb{R}^2$. In the latter case, $\mathfrak{g}$ is the only 4-dimensional 3-step nilpotent complex Lie algebra, and its realification $\tilde{\mathfrak{g}}$ has a basis $\{e_1,\dots,e_8\}$ adapted to an abelian balanced Hermitian structure satisfying:

$$\begin{cases}de^i=0,\quad i=1,2,3,4,\\de^5=-e^{13}-e^{24},\\de^6=-e^{14}+e^{23},\\de^7=-e^{15}-e^{26},\\de^8=-e^{16}+e^{25}.\end{cases}$$

This is a particular case of Proposition 4.11, case II.2). According to Theorem 3.12, the holonomy algebra of the Bismut connection ∇ is contained in $\mathfrak{su}(3)$, since the center $\mathfrak{z} = \text{span}\{e_7, e_8\}$ is 2-dimensional. An example of a non-nilpotent example where this holonomy reduction occurs is in case (i), when $\lambda_3 = 0$ and $\lambda_2 = -\lambda_1$. According to Remark 5.5, we may assume $\lambda_1 = -\lambda_2 = 1$, and therefore $\tilde{\mathfrak{g}}$ is isomorphic to the direct product of $\mathbb{R}^2$ and the unimodular solvable Lie algebra from Remark 4.8.

(3) If we consider now almost abelian complex Lie algebras of the form $\mathfrak{g} = \mathbb{C} \ltimes_M \mathbb{C}^4$ with M a 4×4 traceless complex matrix, we will be able to provide examples of 10-dimensional indecomposable solvable non-nilpotent Lie algebras admitting an abelian balanced Hermitian structure whose holonomy algebra is contained properly in $\mathfrak{su}(5)$. Indeed, for any $\lambda = a + bi \in \mathbb{C} - \{0\}$, consider the matrix

$$M_\lambda = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}.$$

In this case $\mathfrak{g}_\lambda = \mathbb{C} \ltimes_{M_\lambda} \mathbb{C}^4$ is a 5-dimensional solvable non-nilpotent complex Lie algebra, which can be seen to be indecomposable, and its realification $\tilde{\mathfrak{g}}_\lambda$ has a basis $\{e_1, \dots, e_{10}\}$ adapted to an abelian balanced Hermitian structure satisfying:

$$\begin{cases} de^i = 0, \quad i = 1,2,3,4, \\ de^5 = -e^{13} - e^{24}, \\ de^6 = -e^{14} + e^{23}, \\ de^7 = -a(e^{17} + e^{28}) - b(e^{18} - e^{27}), \\ de^8 = b(e^{17} + e^{28}) - a(e^{18} - e^{27}), \\ de^9 = a(e^{19} + e^{2,10}) + b(e^{1,10} - e^{29}), \\ de^{10} = -b(e^{19} + e^{2,10}) + a(e^{1,10} - e^{29}). \end{cases}$$

According to Theorem 3.12, the holonomy algebra of the Bismut connection ∇ is contained in $\mathfrak{su}(4)$, since the center $\mathfrak{z} = \text{span}\{e_5, e_6\}$ is 2-dimensional.

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