A NEW SPECTRAL SEQUENCE FOR HOMOLOGY OF POSETS

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Abstract. We develop a new method to compute the homology groups of Alexandroff topological spaces (or equivalently of partially ordered sets) by means of spectral sequences giving a complete and simple description of the corresponding differentials. Applying our tools, we obtain a generalization of a result of Minian in homological Morse theory for posets.

1. Introduction

The interaction between topology and combinatorics has proved to be very fruitful. Examples of this interaction are simplicial homology, discrete Morse theory [7, 11], the celebrated proof of Kneser's conjecture given by Lovász [9], the subsequent developments in the study of graph properties by means of topological methods [8] and the theory of finite topological spaces, which has grown considerably in the last years from works by Barmak and Minian [2, 3, 4].

The theory of finite topological spaces is based in the well-known correspondence between finite posets and finite T_0–spaces given by Alexandroff [1] and in the works of Stong [12] and McCord [10] who study finite spaces from totally different perspectives. Stong studies the homotopy types of finite topological spaces by means of an elementary move which consists of removing a single point of a finite space. Surprisingly, a sequence of these simple moves is enough to determine whether two given finite spaces have the same homotopy type. On the other hand, McCord establishes a correspondence that assigns to each finite T_0–space X a simplicial complex $\mathcal{K}(X)$ together with a weak homotopy equivalence $\mathcal{K}(X) \to X$ and proves that the weak homotopy types of compact polyhedra are in one to one correspondence with the weak homotopy types of finite topological spaces.

Barmak and Minian delve deeply into this theory and obtain many interesting results, among which we mention the introduction of an elementary move in finite T_0–spaces which corresponds exactly with the elementary collapses of simple homotopy theory of compact polyhedra [4] and a generalization of McCord's result on the weak equivalence between a compact polyhedron and its order complex [3]. Moreover, they use the theory of finite spaces to study Quillen's conjecture on the poset of non-trivial p–subgroups of a group and Andrews–Curtis' conjecture [2].

As it is shown by the recent works of Barmak and Minian, finite topological spaces can be used in several different situations to develop new tools and techniques to study topological and combinatorial problems. Moreover, in many of them the finite space approach is simpler, more adequate or more tractable than

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the one given by simplicial complexes and polyhedra. In a similar way, problems regarding homotopy invariants of finite topological spaces, which can be tackled by the simplicial complex approach, can be dealt with in a more direct and natural way in their own context.

It is this idea which is exploited in this article, where we develop a new method to compute the homology groups of finite topological spaces by means of spectral sequences. Not only do we give spectral sequences which converge to the homology groups of a given finite space but also we completely describe the differentials of all the pages of those spectral sequences. Our method, which can be extended to Alexandroff spaces, involves far fewer computations than the standard one of computing the simplicial homology groups of the order complex of the corresponding poset.

As an application of our tools, we generalize a result of Minian on homological Morse theory for posets [11], largely extending the class of posets for which it is valid, with a novel and conceptual proof. We also show how our results can be applied to compute the second homotopy group of a finite topological space by means of its universal cover.

2. Preliminaries

Recall that an Alexandroff space is a topological space in which every intersection of open subsets is an open subset. If X is an Alexandroff T_0 topological space and $x \in X$, U_x denotes the minimal open set which contains x, that is, the intersection of all the open sets of X which contain x. In a similar way, F_x denotes the minimal closed set which contains x, that is, $F_x = \overline{\{x\}}$.

An order relation can be defined in an Alexandroff T_0–space X as follows: $x \leq y$ if and only if $U_x \subseteq U_y$. Conversely, if P is a poset then the subsets $\{x \in P \ / \ x \leq a\}$, $a \in P$, form a basis for a topology on P. These applications are mutually inverse and give a one-to-one correspondence between Alexandroff T_0–spaces and posets [1, 2, 10]. Moreover, continuous maps between Alexandroff T_0–spaces correspond to order-preserving maps between posets.

Hence, from now on, we will see any Alexandroff T_0–space as a poset and any poset as an Alexandroff T_0–space without further notice. Note that locally finite topological spaces are Alexandroff spaces.

If X is an Alexandroff T_0–space then $U_x = \{a \in X \ / \ a \leq x\}$ and $F_x = \{a \in X \ / \ a \geq x\}$. It is standard to define $\widehat{U}_x = \{a \in X \ / \ a < x\}$, $\widehat{F}_x = \{a \in X \ / \ a > x\}$, $C_x = U_x \cup F_x$ and $\widehat{C}_x = C_x - \{x\}$. In case several topological spaces are considered at the same time, we will denote U_x by U_x^X to indicate the space in which the minimal open set is considered. We will use similar notations for F_x, C_x, $\widehat{U}_x$, $\widehat{F}_x$ and $\widehat{C}_x$.

Let X be a finite T_0–space and let $x \in X$. The point x is an *up beat point* of X if the subposet $\widehat{F}_x$ has a minimum. The point x is a *down beat point* of X if the subposet $\widehat{U}_x$ has a maximum. The point x is a *beat point* of X if it is either an up beat point or a down beat point. Stong proves in [12] that if x is a beat point of X then $X - \{x\}$ is a strong deformation retract of X. Moreover, he gives a simple criterion to decide whether two given finite topological spaces are homotopy equivalent. In addition, he proves that two comparable continuous maps between finite T_0–spaces are homotopic. Using the results of Stong it is easy to prove that if X is a finite T_0–space and $x \in X$ then C_x is contractible. In particular, any finite T_0–space with a unique maximal (or minimal) element is contractible.

The *order complex* of an Alexandroff T_0–space X is the simplicial complex $\mathcal{K}(X)$ of the finite non-empty chains of X. McCord proves in [10] that there exists a weak homotopy equivalence $|\mathcal{K}(X)| \to X$ ($|\mathcal{K}(X)|$ denotes the geometric realization of

$\mathcal{K}(X)$). He also proves in [10] that there exists a correspondence that assigns to each Alexandroff space Z an Alexandroff T_0–space $\widehat{Z}$ which is a quotient of Z and which satisfies that the quotient map $Z \to \widehat{Z}$ is a homotopy equivalence.

The *face poset* of a simplicial complex K is the poset $\mathcal{X}(K)$ of simplices of K ordered by inclusion. Clearly, $\mathcal{K}(\mathcal{X}(K))$ is the barycentric subdivision of K.

The *non-Hausdorff suspension* of a topological space X is the space $\mathbb{S}(X)$ whose underlying set is $X \cup \{+, -\}$ and whose open sets are those of X together with $X \cup \{+\}$, $X \cup \{-\}$ and $X \cup \{+, -\}$. This definition was introduced by McCord in [10], where he proves that for every space X there exists a weak homotopy equivalence between the suspension of X and $\mathbb{S}(X)$.

A *finite model* of a topological space Z is a finite space which is weak homotopy equivalent to Z. For example, if D_2 is the discrete space of two points and $n \in \mathbb{N}$ then $\mathbb{S}^n D_2$ is a finite model of the n–sphere S^n.

If X is a poset X^{op} will denote the poset X with the inverse order.

Recall that a poset is *homogeneous* of dimension n if all its maximal chains have cardinality $n+1$. A poset X is *graded* if U_x is homogeneous for all $x \in X$. In this case, the degree of x is the dimension of U_x and is denoted by $\deg(x)$.

The following definitions were introduced by Minian in [11].

- A finite poset X is called *h–regular* if for every $x \in X$, the order complex of $\widehat{U}_x$ is homotopy equivalent to S^{n-1} where n is the maximum of the cardinality of the chains in $\widehat{U}_x$.
- A *cellular* poset is a graded poset X such that for every $x \in X$, $\widehat{U}_x$ has the homology of a $(p-1)$–sphere, where $p = \deg(x)$.

In a similar way we will say that a finite T_0–space is *cellular* if its associated poset is cellular.

If $n \in \mathbb{N}_0$, an n–chain of P is a chain of P of cardinality $n+1$. The empty chain will be regarded as a (-1)–chain. We will use the notation $[v_0, \dots, v_n]$ for an n–chain $\{v_0, \dots, v_n\}$ of P with $v_{j-1} < v_j$ for all $j \in \{1, 2, \dots, n\}$. Also, if $n \in \mathbb{N}_0$ and $s = [v_0, \dots, v_n]$ is an n–chain and $k \in \{0, \dots, n\}$ then $s_{\widehat{k}}$ will denote the $(n-1)$–chain $[v_0, \dots, \widehat{v_k}, \dots, v_n]$.

Notation 2.1. If X is a poset, $\mathrm{Ch}(X)$ will denote the set of chains of X and, for $n \in \mathbb{N}_0$, $\mathrm{Ch}_n(X)$ will denote the set of n–chains of X.

From McCord's theorem it is clear that $H_n(X) = H_n(|\mathcal{K}(X)|)$. Hence, the homology groups of a finite topological space can be computed from the simplicial chain complex associated to $\mathcal{K}(X)$. This fact can be expressed entirely in terms of the finite space X since the simplices of $\mathcal{K}(X)$ are the chains of X. Thus, making the translation to the context of posets, we can introduce the following definition which will be useful for our work.

Definition 2.2. Let X be an Alexandroff T_0–space. The *$\mathcal{P}$–chain complex associated to* X is the chain complex $C^{\mathcal{P}}(X) = (C^{\mathcal{P}}_n(X), d^{\mathcal{P}}_n)_{n\in\mathbb{Z}}$ defined by

$$C^{\mathcal{P}}_n(X) = \begin{cases} \bigoplus\limits_{\mathrm{Ch}_n(X)} \mathbb{Z} & \text{if } n \geq 0 \\ 0 & \text{if } n < 0 \end{cases}$$

and where, for $n \in \mathbb{N}$, the morphisms $d^{\mathcal{P}}_n : C^{\mathcal{P}}_n(X) \longrightarrow C^{\mathcal{P}}_{n-1}(X)$ are defined by

$$d^{\mathcal{P}}_n([v_0, \dots, v_n]) = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_n]$$

for all $[v_0, \dots, v_n] \in \mathrm{Ch}_n(X)$.

It is clear that the chain complex $C^{\mathcal{P}}(X)$ is isomorphic to the simplicial chain complex of $\mathcal{K}(X)$.

Definition 2.3. Let X be an Alexandroff T_0–space. For $n \in \mathbb{Z}$ we define $H_n^{\mathcal{P}}(X)$ as the n–th homology group of the chain complex $C^{\mathcal{P}}(X)$. The group $H_n^{\mathcal{P}}(X)$ will be called the *n–th $\mathcal{P}$–homology group of X*.

It follows that if X is an Alexandroff T_0–space then $H_n(X) = H_n^{\mathcal{P}}(X)$ for all $n \in \mathbb{Z}$.

A similar translation can be made for the relative case. We include below the notations and definitions for this case so as to set the terminology that we will use.

Definition 2.4. Let X be an Alexandroff T_0–space and let $A \subseteq X$. An *n–chain of* (X, A) is an n–chain of X which is not included in A. The set of n–chains of (X, A) will be denoted by $\mathrm{Ch}_n(X, A)$.

Definition 2.5. Let X be an Alexandroff T_0–space and let $A \subseteq X$. We define the *$\mathcal{P}$–relative chain complex associated to* (X, A) as the chain complex $C^{\mathcal{P}}(X, A) = C^{\mathcal{P}}(X)/C^{\mathcal{P}}(A)$. We also define, for $n \in \mathbb{Z}$, $H_n^{\mathcal{P}}(X, A)$ as the n–th homology group of the chain complex $C^{\mathcal{P}}(X, A)$.

Note that, for $n \in \mathbb{N}_0$, $C_n^{\mathcal{P}}(X, A) \cong \bigoplus\limits_{\mathrm{Ch}_n(X,A)} \mathbb{Z}$.

Clearly, the chain complex $C^{\mathcal{P}}(X, A)$ is isomorphic to the relative simplicial chain complex of $(\mathcal{K}(X), \mathcal{K}(A))$ and hence $H_n^{\mathcal{P}}(X, A) = H_n(X, A)$ for all $n \in \mathbb{Z}$.

In a similar way, one can define reduced $\mathcal{P}$–homology groups. In this case, the group in degree -1 of the corresponding chain complex will be the free abelian group generated by the empty chain.

3. Construction of the spectral sequence

In this section we will develop a spectral sequence which converges to the homology groups of a given finite space and we will provide an explicit description of the differentials of all the pages of this spectral sequence. In addition, we will prove that our result generalizes a cellular-type method of Minian [11] and we will give examples of application.

Notation 3.1. Let X be an Alexandroff T_0–space, let $A \subseteq X$ and let $n \in \mathbb{N}_0$. If $\sigma \in C_n^{\mathcal{P}}(X)$ then we will write $\overline{\sigma}^A$ (or simply $\overline{\sigma}$) for the class of σ in $C_n^{\mathcal{P}}(X, A)$.

If $x \in X$ and $\sigma \in C_n^{\mathcal{P}}(C_x)$, we will write $\overline{\sigma}^x$ for the class of σ in $C_n^{\mathcal{P}}(C_x, \widehat{C}_x)$.

The class of the n–cycle σ in $H_n^{\mathcal{P}}(X)$ will be denoted by $[\sigma]$, and the class of the n–relative cycle $\overline{\sigma}^A$ in $H_n^{\mathcal{P}}(X, A)$ will be denoted by $[\overline{\sigma}^A]$.

If I is a set, H is an abelian group, $\{G_i\}_{i\in I}$ is a collection of abelian groups and $\{f_i : G_i \longrightarrow H\}_{i\in I}$ is a collection of group homomorphisms, we define $\biguplus\limits_{i\in I} f_i : \bigoplus\limits_{i\in I} G_i \longrightarrow H$ by

$$\left(\biguplus_{i\in I} f_i\right)((x_i)_{i\in I}) = \sum_{i\in I} f_i(x_i) \ .$$

The following lemma, which will be used to prove proposition 3.3, contains a simple idea which will be very important in our method.

Lemma 3.2. *Let X be an Alexandroff T_0–space and let $D \subseteq X$ be an antichain. For $x \in D$ and $n \in \mathbb{N}_0$ we define:*

- $i_n^x : C_n^{\mathcal{P}}(C_x, \widehat{C}_x) \longrightarrow C_n^{\mathcal{P}}(X, X - D)$ *by* $i_n^x(\overline{\sigma}^x) = \overline{\sigma}^{X-D}$ *for every* $\sigma \in C_n^{\mathcal{P}}(C_x)$.

- $\rho_n^x : C_n^{\mathcal{P}}(X, X-D) \longrightarrow C_n^{\mathcal{P}}(C_x, \widehat{C}_x)$ *as the group homomorphism that satisfies*
$$\rho_n^x(\overline{s}^{X-D}) = \begin{cases} \overline{s}^x & \text{if } x \in s \\ 0 & \text{if } x \notin s \end{cases}$$
for every n–chain s in X.
- $\phi_n : C_n^{\mathcal{P}}(X, X-D) \longrightarrow \bigoplus_{x\in D} C_n^{\mathcal{P}}(C_x, \widehat{C}_x)$ *as the group homomorphism that satisfies*
$$\phi_n(\overline{s}^{X-D}) = (\rho_n^x(\overline{s}^{X-D}))_{x\in D}$$
for every n–chain s in X.

Then ϕ_n is a group isomorphism and $\phi_n^{-1} = \biguplus_{x\in D} i_n^x$ for every $n \in \mathbb{N}_0$.

Proof. It is easy to check that the morphisms above are well-defined.

For each $n \in \mathbb{N}_0$ and for each $x_0 \in D$, let $\mathrm{in}_n^{x_0} : C_n^{\mathcal{P}}(C_{x_0}, \widehat{C}_{x_0}) \longrightarrow \bigoplus_{x\in D} C_n^{\mathcal{P}}(C_x, \widehat{C}_x)$ be the canonical inclusion.

Let $n \in \mathbb{N}_0$. It is clear that $\biguplus_{x\in D} i_n^x$ is an epimorphism since every n–chain of $(X, X-D)$ is an n–chain of $(C_x, \widehat{C}_x)$ for some $x \in D$. Besides, $\phi_n i_n^x(s) = \mathrm{in}_n^x(s)$ for every $x \in D$ and for every n–chain s in $(C_x, \widehat{C}_x)$ since every n–chain of $(X, X-D)$ must have exactly one element of D, as D is an antichain. Therefore, $\phi_n i_n^x = \mathrm{in}_n^x$.

Then, $\phi_n \circ \biguplus_{x\in D} i_n^x = \biguplus_{x\in D} (\phi_n i_n^x) = \biguplus_{x\in D} \mathrm{in}_n^x = \mathrm{Id}$. Hence, $\biguplus_{x\in D} i_n^x$ is a monomorphism. Thus, $\biguplus_{x\in D} i_n^x$ is an isomorphism with inverse ϕ_n. □

Proposition 3.3. *Let X be an Alexandroff T_0–space and let D be an antichain in X. Then $H_n(X, X-D) \cong \bigoplus_{x\in D} \widetilde{H}_{n-1}(\widehat{C}_x)$ for every $n \in \mathbb{Z}$.*

Proof. For each $n \in \mathbb{Z}$ we have a commutative diagram

$$\begin{tikzcd}
C_n^{\mathcal{P}}(X, X-D) \arrow[r, "\overline{d}_n^{\mathcal{P}}"] & C_{n-1}^{\mathcal{P}}(X, X-D) \\
\bigoplus_{x\in D} C_n^{\mathcal{P}}(C_x, \widehat{C}_x) \arrow[u, "\biguplus_{x\in D} i_n^x"] \arrow[r, "\bigoplus_{x\in D} (\overline{d}_n^{\mathcal{P}})_x"'] & \bigoplus_{x\in D} C_{n-1}^{\mathcal{P}}(C_x, \widehat{C}_x) \arrow[u, "\biguplus_{x\in D} i_{n-1}^x"']
\end{tikzcd}$$

where for every $x \in D$ the morphism $(\overline{d}_n^{\mathcal{P}})_x$ is the restriction of $\overline{d}_n^{\mathcal{P}}$ to $C_n^{\mathcal{P}}(C_x, \widehat{C}_x)$.

Then, the chain complexes $C^{\mathcal{P}}(X, X-D)$ and $\bigoplus_{x\in D} C^{\mathcal{P}}(C_x, \widehat{C}_x)$ are isomorphic and therefore
$$H_n(X, X-D) \cong H_n^{\mathcal{P}}(X, X-D) \cong \bigoplus_{x\in D} H_n^{\mathcal{P}}(C_x, \widehat{C}_x) \cong \bigoplus_{x\in D} H_n(C_x, \widehat{C}_x)$$
for every $n \in \mathbb{Z}$.

Now, since C_x is contractible, $H_n(C_x, \widehat{C}_x) \cong \widetilde{H}_{n-1}(\widehat{C}_x)$. The result follows. □

Remark 3.4. In the previous proposition we allow $\widehat{C}_x$ to be empty, in which case $\widetilde{H}_{-1}(\widehat{C}_x) = \mathbb{Z}$ and $\widetilde{H}_n(\widehat{C}_x) = 0$ for $n \neq -1$.

The following proposition[1] is the simplicial complex version of 3.3.

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[1]We thank an anonymous referee for pointing out proposition 3.5.

Proposition 3.5. *Let K be a simplicial complex and let D be a subset of the set of vertices of K such that for every pair of distinct vertices $v_1, v_2 \in D$ the subset $\{v_1, v_2\}$ is not a simplex of K. Let L be the full subcomplex of K spanned by the vertices of K which do not belong to D. Then $H_n(K,L) \cong \bigoplus_{v\in D} \widetilde{H}_{n-1}(\mathrm{lk}_K(v))$ for every $n \in \mathbb{Z}$.*

Proof. Follows from the fact that K/L is homotopy equivalent to $\bigvee_{v\in D} \Sigma\, \mathrm{lk}_K(v)$. □

Clearly, we could have applied 3.5 to prove 3.3, but we preferred to keep the T_0–space approach which will be used to give a simple description of the differentials of the spectral sequence of 3.10.

On the other hand, 3.5 can also be obtained from 3.3. Indeed, with the notations and under the assumptions of 3.5 we have that

$$\begin{aligned} H_n(K,L) &\cong H_n(\mathcal{X}(K), \mathcal{X}(L)) \cong H_n(\mathcal{X}(K), \mathcal{X}(K) - \mathcal{X}(K[D])) \cong \\ &\cong \bigoplus_{v\in D} \widetilde{H}_{n-1}(\widehat{C}_v) \cong \bigoplus_{v\in D} \widetilde{H}_{n-1}(\mathrm{lk}_K(v)) \end{aligned}$$

where $K[D]$ is the full subcomplex of K spanned by D. Note that the third isomorphism holds by 3.3 and that the last isomorphism holds since the posets $\widehat{C}_v$ and $\mathcal{X}(\mathrm{lk}_K(v))$ are isomorphic. Also, the second isomorphism holds by the following lemma.

Lemma 3.6. *Let K be a simplicial complex and let L be a full subcomplex of K. Let L^c be the full subcomplex of K spanned by the vertices of K which are not vertices of L. Then $\mathcal{X}(L)$ is a strong deformation retract of $\mathcal{X}(K) - \mathcal{X}(L^c)$.*

Proof. Let $i : \mathcal{X}(L) \to \mathcal{X}(K) - \mathcal{X}(L^c)$ be the inclusion map. Let V_L be the set of vertices of L and let $r : \mathcal{X}(K) - \mathcal{X}(L^c) \to \mathcal{X}(L)$ be defined by $r(s) = s \cap V_L$. Then $ri = \mathrm{Id}_{\mathcal{X}(L)}$ and $ir \leq \mathrm{Id}_{\mathcal{X}(K)-\mathcal{X}(L^c)}$. The result now follows from [2, Corollary 1.2.6]. □

Therefore, proposition 3.3 is equivalent to 3.5.

Let X be an Alexandroff T_0–space. Now we will describe the connection homomorphism $\partial : H_n(C_x, \widehat{C}_x) \longrightarrow \widetilde{H}_{n-1}(\widehat{C}_x)$ of the long exact sequence associated to the $\mathcal{P}$–chain complex of $(C_x, \widehat{C}_x)$, showing that it is induced by removing the point x of the corresponding chains. We will use the following definition.

Definition 3.7. Let X be an Alexandroff T_0–space. Let s be a chain of X and let $x \in s$. We define the *index* of x in s by $\eta_x^s = \#(s \cap \widehat{U}_x)$ and the *sign* of x in s by $\mathrm{sgn}_s(x) = (-1)^{\eta_x^s}$.

Lemma 3.8. *Let X be an Alexandroff T_0–space, let $x \in X$ and let $n \in \mathbb{N}_0$.*

Let $\partial : H_n(C_x, \widehat{C}_x) \longrightarrow \widetilde{H}_{n-1}(\widehat{C}_x)$ be the connection homomorphism of the long exact sequence associated to the $\mathcal{P}$–chain complex of $(C_x, \widehat{C}_x)$.

Let $\sigma = \sum_{i=1}^{l} \alpha_i s_i + \sum_{j=1}^{m} \beta_j t_j \in C_n^{\mathcal{P}}(C_x)$, where $l, m \in \mathbb{N}$, $\alpha_i \in \mathbb{Z}$ for every $i = 1, \dots, l$, $\beta_i \in \mathbb{Z}$ for every $j = 1, \dots, m$ and where for every $i = 1, \dots, l$, s_i is an n–chain in C_x such that $x \in s_i$, and for every $j = 1, \dots, m$, t_j is an n–chain of C_x such that $x \notin t_j$.

If $\overline{\sigma} \in \ker \overline{d}_n^{\mathcal{P}}$, then $\partial([\overline{\sigma}]) = \left[\sum_{i=1}^{l} \alpha_i \,\mathrm{sgn}_{s_i}(x)(s_i - \{x\})\right]$.

Proof. Let $\tau = \sum_{i=1}^{l} \alpha_i s_i$. Note that

$$\overline{\sigma} = \overline{\sum_{i=1}^{l} \alpha_i s_i} + \overline{\sum_{j=1}^{m} \beta_j t_j} = \overline{\sum_{i=1}^{l} \alpha_i s_i} = \overline{\tau}$$

in $C_n^{\mathcal{P}}(C_x, \widehat{C}_x)$ since $\sum_{j=1}^{m} \beta_j t_j \in C_n^{\mathcal{P}}(\widehat{C}_x)$.

By the proof of the Snake Lemma, $d_n^{\mathcal{P}}(\tau) \in C_{n-1}^{\mathcal{P}}(\widehat{C}_x)$ and $\partial([\overline{\sigma}]) = \partial([\overline{\tau}])$ is the class of $d_n^{\mathcal{P}}(\tau)$ in $H_{n-1}^{\mathcal{P}}(\widehat{C}_x)$. On the other hand

$$d_n^{\mathcal{P}}(\tau) = \sum_{i=1}^{l} \alpha_i \sum_{k=0}^{n} (-1)^k (s_i)_{\widehat{k}} = \sum_{i=1}^{l} \alpha_i \operatorname{sgn}_{s_i}(x)(s_i - \{x\}) + \sum_{i=1}^{l} \alpha_i \sum_{k \neq \eta_x^{s_i}} (-1)^k (s_i)_{\widehat{k}} \,.$$

Now, note that $\sum_{i=1}^{l} \alpha_i \operatorname{sgn}_{s_i}(x)(s_i - \{x\}) \in C_{n-1}^{\mathcal{P}}(\widehat{C}_x)$, since it is a sum of $(n-1)$–chains in C_x that do not contain x. Since $d_n^{\mathcal{P}}(\tau) \in C_{n-1}^{\mathcal{P}}(\widehat{C}_x)$, then

$$\sum_{i=1}^{l} \alpha_i \left(\sum_{k \neq \eta_x^{s_i}} (-1)^k (s_i)_{\widehat{k}} \right) \in C_{n-1}^{\mathcal{P}}(\widehat{C}_x)$$

But for every $i = 1, \dots, l$, $k \neq \eta_x^{s_i}$ implies that $x \in (s_i)_{\widehat{k}}$. Then, $\sum_{k \neq \eta_x^{s_i}} (-1)^k (s_i)_{\widehat{k}}$ is a sum of chains that contain x. Since $C_{n-1}^{\mathcal{P}}(C_x)$ is a free abelian group we obtain that $\sum_{i=1}^{l} \alpha_i \left(\sum_{k \neq \eta_x^{s_i}} (-1)^k (s_i)_{\widehat{k}} \right) = 0$. Hence,

$$d_n^{\mathcal{P}}(\tau) = \sum_{i=1}^{l} \alpha_i \operatorname{sgn}_{s_i}(x)(s_i - \{x\}),$$

and therefore,

$$\partial([\overline{\sigma}]) = [d_n^{\mathcal{P}}(\tau)] = \left[\sum_{i=1}^{l} \alpha_i \operatorname{sgn}_{s_i}(x)(s_i - \{x\}) \right].$$

□

Definition 3.9. Let X be an Alexandroff T_0–space and let $\mathscr{F} = \{X_p : p \in \mathbb{Z}\}$ be a filtration of X. We say that the filtration $\mathscr{F}$ is *induced by antichains* if $X_{-1} = \varnothing$ and $X_n - X_{n-1}$ is an antichain for every $n \in \mathbb{N}$.

Note that the subposet X_0 need not be an antichain.

The following is one of the main theorems of this article.

Theorem 3.10. *Let X be a finite T_0–space and let $\{X_p : p \in \mathbb{Z}\}$ be a filtration of X which is induced by antichains. For each $p \in \mathbb{N}$, let $D_p = X_p - X_{p-1}$.*

Then there is a spectral sequence $\{(E_{p,q}^r)_{p,q \in \mathbb{Z}}, (d_{p,q}^r)_{p,q \in \mathbb{Z}}\}_{r \in \mathbb{N}}$ that converges to $H_*(X)$ such that:

- $E_{p,q}^1 = 0$ *for every* $p \leq -1$.
- $E_{0,q}^1 = H_q(X_0)$.
- $E_{p,q}^1 = \bigoplus_{x \in D_p} \widetilde{H}_{p+q-1}(\widehat{C}_x^{X_p})$ *for* $p \geq 1$.
- *The morphisms $d_{p,q}^1 : E_{p,q}^1 \longrightarrow E_{p-1,q}^1$ are defined in the following way:*
  - *If $p \leq 0$ and $q \in \mathbb{Z}$, then $d_{p,q}^1$ is the trivial homomorphism.*

– *If* $p=1$ *and* $q\in\mathbb{N}_0$, *then* $d^1_{p,q}:\bigoplus\limits_{x\in D_1}\widetilde{H}_q(\widehat{C}_x^{X_1})\longrightarrow H_q(X_0)$ *is defined by*

$$d^1_{1,q}(([\sigma_x])_{x\in D_1})=\sum_{x\in D_1}[\sigma_x]\ .$$

– *If* $p\geq 1$ *and* $q\leq -p$, *then* $d^1_{p,q}$ *is the trivial homomorphism.*

– *If* $p\geq 2$ *and* $q\geq 1-p$, *then* $d^1_{p,q}:\bigoplus\limits_{x\in D_p}\widetilde{H}_{p+q-1}(\widehat{C}_x^{X_p})\longrightarrow\bigoplus\limits_{y\in D_{p-1}}\widetilde{H}_{p+q-2}(\widehat{C}_y^{X_{p-1}})$ *is defined by*

$$d^1_{p,q}\left(\left(\left[\sum_{i=1}^{l_x}a_i^x s_i^x\right]\right)_{x\in D_p}\right)=\left(\left[\sum_{x\in D_p}\sum_{s_i^x\ni y}a_i^x\operatorname{sgn}_{s_i^x}(y)(s_i^x-\{y\})\right]\right)_{y\in D_{p-1}}$$

where for every $x\in D_p$, $l_x\in\mathbb{N}$, *and for every* $i=\{1,\dots,l_x\}$, $a_i^x\in\mathbb{Z}$ *and* $s_i^x\in C^{\mathcal{P}}_{p+q-1}(\widehat{C}_x^{X_p})$.

Proof. Let $\{(\widetilde{E}^r_{p,q})_{p,q\in\mathbb{Z}},(\widetilde{d}^r_{p,q})_{p,q\in\mathbb{Z}}\}_{r\in\mathbb{N}}$ be the bigraded spectral sequence associated to the filtration $\{X_p\}_{p\in\mathbb{Z}}$ of X, that is,

- $\widetilde{E}^1_{p,q}=H_{p+q}(X_p,X_{p-1})$ for every $p,q\in\mathbb{Z}$
- $\widetilde{d}^1_{p,q}=j_*\partial$

where j_* is the homomorphism induced in the homology groups by the projection $j:C^{\mathcal{P}}_{p+q-1}(X_{p-1})\longrightarrow C^{\mathcal{P}}_{p+q-1}(X_{p-1},X_{p-2})$ and where $\partial:H_{p+q}(X_p,X_{p-1})\longrightarrow\widetilde{H}_{p+q-1}(X_{p-1})$ is the connection homomorphism of the long exact sequence associated to the pair (X_p,X_{p-1}).

Since X is finite and $X_p=\varnothing$ for every $p\leq -1$, it follows that the spectral sequence $\{(\widetilde{E}^r_{p,q})_{p,q\in\mathbb{Z}},(\widetilde{d}^r_{p,q})_{p,q\in\mathbb{Z}}\}_{r\in\mathbb{N}}$ converges to $H_*(X)$.

For $p,q\in\mathbb{Z}$ we define:

- $E^1_{p,q}=0$ if $p\leq -1$.
- $E^1_{0,q}=H_q(X_0)$.
- $E^1_{p,q}=\bigoplus\limits_{x\in D_p}\widetilde{H}_{p+q-1}(\widehat{C}_x^{X_p})$ if $p\geq 1$.

Since the filtration $\{X_p:p\in\mathbb{Z}\}$ is induced by antichains, D_p is an antichain for every $p\in\mathbb{N}$. Thus, by the proof of 3.3, for each $p\geq 1$ and for all $q\in\mathbb{Z}$ we have isomorphisms

$$\left(\biguplus_{x\in D_p}i_n^x\right)_*\circ\left(\bigoplus_{x\in D_p}\partial^x\right)^{-1}:\bigoplus_{x\in D_p}\widetilde{H}_{p+q-1}(\widehat{C}_x^{X_p})\longrightarrow H_{p+q}(X_p,X_{p-1})$$

where $\partial^x:H_{p+q}(C_x^{X_p},\widehat{C}_x^{X_p})\to\widetilde{H}_{p+q-1}(\widehat{C}_x^{X_p})$ is the connection homomorphism of the corresponding long exact sequence.

On the other hand, we have that $E^1_{p,q}=\widetilde{E}^1_{p,q}$ for $p\leq 0$. So we have group isomorphisms $\theta_{p,q}:E^1_{p,q}\longrightarrow\widetilde{E}^1_{p,q}$ for every $p,q\in\mathbb{Z}$. We define $d^1_{p,q}=\theta^{-1}_{p-1,q}\circ\widetilde{d}^1_{p,q}\circ\theta_{p,q}$ for all $p,q\in\mathbb{Z}$.

Let $n \in \mathbb{Z}$. We have a diagram

$$\begin{tikzcd}
H_n(X_1, X_0) \arrow[r, "\partial"] \arrow[rr, bend left, "\widetilde{d}^1_{1,n-1}"] & H_{n-1}(X_0) \arrow[r, "j_*"] & H_{n-1}(X_0, X_{-1}) \arrow[d, "(j_*)^{-1}"] \\
\bigoplus\limits_{x \in D_1} H_n(C_x^{X_1}, \widehat{C}_x^{X_1}) \arrow[u, "\left(\biguplus\limits_{x \in D_1} i_n^x\right)_*"] & \bigoplus\limits_{x \in D_1} \widetilde{H}_{n-1}(\widehat{C}_x^{X_1}) \arrow[l, "\left(\bigoplus\limits_{x \in D_1} \partial^x\right)^{-1}"] \arrow[r, "d^1_{1,n-1}"'] \arrow[u, "\left(\biguplus\limits_{x \in D_1} \tau_{n-1}^x\right)_*"] & H_{n-1}(X_0)
\end{tikzcd}$$

where the maps $(i_n^x)_*$ are defined as in 3.2 and where $\tau_{n-1}^x : C_{n-1}^{\mathcal{P}}(\widehat{C}_x^{X_1}) \to C_{n-1}^{\mathcal{P}}(X_0)$ are the inclusion homomorphisms.

It is clear that $\partial (i_n^x)_* = (\tau_{n-1}^x)_* \partial^x$ for every $x \in D_1$ whenever $n \neq 1$ by the naturality of the long exact sequence. Moreover, since $\partial : H_1(X_1, X_0) \longrightarrow \widetilde{H}_0(X_0)$ is the (range) restriction of $\partial : H_1(X_1, X_0) \longrightarrow H_0(X_0)$, then we have that $\partial (i_1^x)_* = (\tau_0^x)_* \partial^x$ for every $x \in D_1$ as well. Thus the left square in the last diagram commutes and therefore $d^1_{1,n-1} = \left(\biguplus\limits_{x \in D_1} \tau_{n-1}^x\right)_*$.

Now, let $[\sigma] \in \bigoplus\limits_{x \in D_1} \widetilde{H}_{n-1}(\widehat{C}_x^{X_1})$. Then $[\sigma] = ([\sigma_x])_{x \in D_1}$ with $[\sigma_x] \in \widetilde{H}_{n-1}(\widehat{C}_x^{X_1})$ for each $x \in D_1$, and therefore $d^1_{1,n-1}([\sigma]) = \sum\limits_{x \in D_1} [\sigma_x]$.

Let $p \geq 2$ and let $n \in \mathbb{N}$. Consider the following diagram

$$\begin{tikzcd}
H_n(X_p, X_{p-1}) \arrow[r, "\partial"] \arrow[rr, bend left, "\widetilde{d}^1_{p,n-p}"] & H_{n-1}(X_{p-1}) \arrow[r, "j_*"] & H_{n-1}(X_{p-1}, X_{p-2}) \arrow[d, "(\phi_{n-1})_*"] \\
\bigoplus\limits_{x \in D_p} H_n(C_x^{X_p}, \widehat{C}_x^{X_p}) \arrow[u, "\left(\biguplus\limits_{x \in D_p} i_n^x\right)_*"] & \bigoplus\limits_{x \in D_p} \widetilde{H}_{n-1}(\widehat{C}_x^{X_p}) \arrow[l, "\left(\bigoplus\limits_{x \in D_p} \partial^x\right)^{-1}"] \arrow[u, "\left(\biguplus\limits_{x \in D_p} \tau_{n-1}^x\right)_*"] \arrow[dr, "d^1_{p,n-p}"'] & \bigoplus\limits_{y \in D_{p-1}} H_{n-1}(C_y^{X_{p-1}}, \widehat{C}_y^{X_{p-1}}) \arrow[d, "\bigoplus\limits_{y \in D_{p-1}} \partial^y"] \\
 & & \bigoplus\limits_{y \in D_{p-1}} \widetilde{H}_{n-2}(\widehat{C}_y^{X_{p-1}})
\end{tikzcd}$$

where, as above, the maps $(i_n^x)_*$ are defined as in 3.2 and where $\tau_{n-1}^x : C_{n-1}^{\mathcal{P}}(\widehat{C}_x^{X_p}) \to C_{n-1}^{\mathcal{P}}(X_{p-1})$ are the inclusion homomorphisms.

We have that

$$d^1_{p,n-p} = \left(\bigoplus_{y \in D_{p-1}} \partial^y\right) (\phi_{n-1})_* j_* \partial \left(\biguplus_{x \in D_p} i_n^x\right)_* \left(\bigoplus_{x \in D_p} \partial^x\right)^{-1}$$

As before, the left square of the diagram commutes for every $n \in \mathbb{N}$. Therefore,

$$d^1_{p,n-p} = \left(\bigoplus_{y\in D_{p-1}} \partial^y\right)(\phi_{n-1})_* j_* \left(\biguplus_{x\in D_p} \tau^x_{n-1}\right)_* = \left(\bigoplus_{y\in D_{p-1}} \partial^y\right)\left(\biguplus_{x\in D_p} \phi_{n-1} j \tau^x_{n-1}\right)_*$$

Now we will find an explicit formula for $d^1_{p,n-p}$. Let $\sigma = (\sigma_x)_{x\in D_p}$ with $\sigma_x \in C^{\mathcal{P}}_{n-1}(\widehat{C}^{X_p}_x)$ for each $x \in D_p$. Then

$$\begin{aligned}\left(\biguplus_{x\in D_p} \phi_{n-1} j \tau^x_{n-1}\right)(\sigma) &= \sum_{x\in D_p} \phi_{n-1} j \sigma_x = \sum_{x\in D_p} \phi_{n-1}(\overline{\sigma_x}^{X_{p-2}}) = \\ &= \left(\sum_{x\in D_p} \rho^y_{n-1}(\overline{\sigma_x}^{X_{p-2}})\right)_{y\in D_{p-1}}.\end{aligned}$$

Now, for each $x \in D_p$ we write $\sigma_x = \sum\limits_{i=1}^{l_x} a^x_i s^x_i$ with $l_x \in \mathbb{N}$, $a^x_i \in \mathbb{Z}$ and s^x_i an $(n-1)$–chain in $\widehat{C}^{X_p}_x$ for every $i \in \{1, \dots, l_x\}$. Then we have that

$$\rho^y_{n-1}(\overline{\sigma}^{X_{p-2}}_x) = \sum_{i=1}^{l_x} a^x_i \rho^y_{n-1}(\overline{s^x_i}^{X_{p-2}}) = \sum_{s^x_i \ni y} a^x_i \overline{s^x_i}^y$$

for every $x \in D_p$.

Then,

$$\left(\biguplus_{x\in D_p} \phi_{n-1} j \tau^x_{n-1}\right)(\sigma) = \left(\sum_{x\in D_p} \sum_{s^x_i \ni y} a^x_i \overline{s^x_i}^y\right)_{y\in D_{p-1}}$$

Finally, by lemma 3.8, we see that

$$\begin{aligned} d^1_{p,n-p}([\sigma]) &= \left(\bigoplus_{y\in D_{p-1}} \partial^y\right)\left(\left[\left(\sum_{x\in D_p} \sum_{s^x_i \ni y} a^x_i \overline{s^x_i}^y\right)_{y\in D_{p-1}}\right]\right) = \\ &= \left(\partial^y\left(\left[\sum_{x\in D_p} \sum_{s^x_i \ni y} a^x_i \overline{s^x_i}^y\right]\right)\right)_{y\in D_{p-1}} = \\ &= \left(\left[\sum_{x\in D_p} \sum_{s^x_i \ni y} a^x_i \operatorname{sgn}_{s^x_i}(y)(s^x_i - \{y\})\right]\right)_{y\in D_{p-1}}\end{aligned}$$

□

Remark 3.11. It is not difficult to prove that the morphisms of all the pages of the spectral sequence of the previous theorem can be computed in the same way as those of the first page, since for all $p, q \in \mathbb{Z}$ the groups $E^r_{p,q}$, $r \in \mathbb{N}$, are subquotients of the group $E^1_{p,q}$ and the morphisms of the r–th page of the spectral sequence are induced by the exact couple obtained from the long exact sequences in homology associated to the topological pairs (X_p, X_{p-1}).

Just as a first and simple example consider the following:

Example 3.12 (Non-Hausdorff suspension)**.** Let X be an Alexandroff T_0–space.

Consider the filtration $\{X_p\}_{p\in\mathbb{Z}}$ of $\mathbb{S}X$ given by $X_0 = U_+$ and $X_1 = \mathbb{S}X$. Note that this filtration is induced by antichains. Since U_+ is contractible and $\widehat{U}_+ = X$,

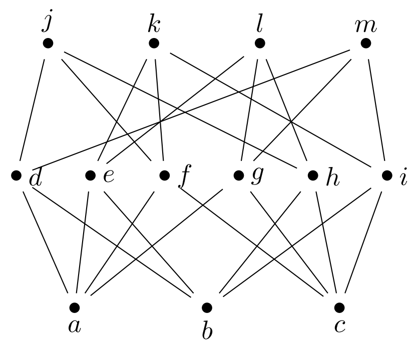
by theorem 3.10 we have a bigraded spectral sequence (E, d) that converges to $H_*(\mathbb{S}X)$ whose first page is

$$\begin{array}{c|ccc}
q & & & \\
 & \vdots & \vdots & \vdots \\
 & 0 & \widetilde{H}_2(X) & 0 \\
 & 0 & \widetilde{H}_1(X) & 0 \\
 & \mathbb{Z} & \widetilde{H}_0(X) & 0 \\
\hline
 & \vdots & \vdots & \vdots \quad p
\end{array}$$

It is not hard to see that $d^1_{1,0} : \widetilde{H}_0(X) \longrightarrow \mathbb{Z}$ is trivial. Hence, $H_0(\mathbb{S}X) = \mathbb{Z}$ and $H_n(\mathbb{S}X) = \widetilde{H}_{n-1}(X)$ for all $n \in \mathbb{N}$.

In the next example we will apply theorem 3.10 to compute the homology groups of a more complicated finite space, which turns out to be a finite model of the real projective plane. This finite space was constructed by Barmak and Minian in [3].

Example 3.13 (Finite model of the real projective plane)**.** Let X be the finite T_0–space whose Hasse diagram is



Consider the filtration $\{X_p\}_{p\in\mathbb{Z}}$ of X given by $X_0 = F_a$, $X_1 = F_a \cup \{h, i\}$ and $X_2 = X$. Note that this filtration is induced by antichains. Hence, by 3.10, there exists a bigraded spectral sequence $\{E^r, d_r\}_{r\in\mathbb{N}}$ that converges to $H_*(X)$.

Let $Z_a = E^1_{0,0} = H_0(F_a) \cong \mathbb{Z}$. Note that Z_a is generated by $[a]$.

Furthermore, $E^1_{1,0} = \widetilde{H}_0(\widehat{F}_h) \oplus \widetilde{H}_0(\widehat{F}_i) = Z_h \oplus Z_i$, where $Z_h = \widetilde{H}_0(\widehat{F}_h) \cong \mathbb{Z}$ and $Z_i = \widetilde{H}_0(\widehat{F}_i) \cong \mathbb{Z}$. Note that Z_h is generated by $[l] - [j]$ and Z_i is generated by $[m] - [k]$.

Similarly, since $\widehat{F}_b$ and $\widehat{F}_c$ are finite models for S^1, we have that $E^1_{2,0} = \widetilde{H}_1(\widehat{F}_b) \oplus \widetilde{H}_1(\widehat{F}_c) = Z_b \oplus Z_c$, where $Z_b = \widetilde{H}_1(\widehat{F}_b) \cong \mathbb{Z}$ and $Z_c = \widetilde{H}_1(\widehat{F}_c) \cong \mathbb{Z}$. In this case we see that Z_b and Z_c are generated by

$$g_0 = [d, j] + [j, h] + [h, l] + [l, e] + [e, k] + [k, i] + [i, m] + [m, d]$$

and

$$g_1 = [f, j] + [j, h] + [h, l] + [l, g] + [g, m] + [m, i] + [i, k] + [k, f]$$

respectively. Now, it is easy to see that the first page of our spectral sequence is, in fact, a chain complex:

$$\cdots \longleftarrow 0 \longleftarrow Z_a \xleftarrow{d^1_{1,0}=\alpha} Z_h \oplus Z_i \xleftarrow{d^1_{2,0}=\beta} Z_b \oplus Z_c \longleftarrow 0 \longleftarrow \cdots$$

Using theorem 3.10 it is clear that $\alpha = 0$. On the other hand, a quick calculation shows that $\beta(g_0) = ([l] - [j], [m] - [k])$ and $\beta(g_1) = ([l] - [j], [k] - [m])$. It follows that

$$E^2_{1,0} = Z_h \oplus Z_i / \operatorname{Im} \beta \cong \mathbb{Z}_2.$$

Thus, $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}_2$ and $H_n(X) = 0$ for $n \geq 2$.

It is interesting to observe that the chain complex above is much simpler than the simplicial chain complex of $\mathcal{K}(X)$.

The spectral sequence of theorem 3.10 will get a much simpler form in the case that the homology of $\widehat{U}_x$ is concentrated in some degree for all $x \in X$. This leads to the following definition.

Definition 3.14. Let X be a locally finite T_0–space. We say that X is *quasicellular* if there exists an order preserving map $\rho : X \longrightarrow \mathbb{N}_0$, which will be called *quasicellular morphism for X*, such that

(1) The set $\{x \in X : \rho(x) = n\}$ is an antichain for every $n \in \mathbb{N}_0$.
(2) For every $x \in X$, the reduced homology of $\widehat{U}_x$ is concentrated in degree $\rho(x) - 1$.

Note that if X is a quasicellular locally finite T_0–space, ρ is a quasicellular morphism for X and $x, y \in X$, then $x < y$ implies that $\rho(x) < \rho(y)$.

Note also that cellular spaces are quasicellular and that h–regular posets are quasicellular. But both inclusions are strict since the non-Hausdorff suspension of the discrete space of three points is quasicellular but neither cellular nor h–regular.

Also, since an h–regular poset is not necessarily graded (see [11, example 2.4]) we conclude that a poset might be quasicellular and non-graded.

Corollary 3.15. *Let X be a quasicellular finite space and let ρ be a quasicellular morphism for X. Let $C(X) = (C_n(X), d_n)_{n \in \mathbb{Z}}$ be the chain complex defined by*

- $C_n(X) = \bigoplus\limits_{\rho(x)=n} \widetilde{H}_{n-1}(\widehat{U}_x)$ *for each $n \in \mathbb{N}_0$ and $C_n(X) = 0$ for $n < 0$.*
- *For each $n \in \mathbb{Z}$, d_n is the group homomorphism $d^1_{n+1,-1}$ of theorem 3.10 (applied for the filtration defined by $X_p = \{x \in X : \rho(x) \leq p - 1\}$ for all $p \in \mathbb{Z}$).*

Then, $H_n(X) = H_n(C(X))$ for all $n \in \mathbb{N}_0$.

Proof. Consider the filtration $\{X_p\}_{p \in \mathbb{Z}}$ given by

$$X_p = \{x \in X : \rho(x) \leq p - 1\}$$

for each $p \in \mathbb{Z}$. Clearly, the filtration $\{X_p\}_{p \in \mathbb{Z}}$ is induced by antichains, and thus theorem 3.10 applies. Since ρ is a quasicellular morphism for X, $\widehat{C}^{X_p}_x = \widehat{U}_x$ and hence it follows that $q = -1$ is the only non-trivial row of the first page of the spectral sequence, and therefore, the homology groups of X are the homology groups of the chain complex determined by the groups and group homomorphisms on this row. By theorem 3.10, this chain complex is precisely $C(X)$. $\square$

As a corollary of 3.15, we obtain theorem 3.7 of [11]:

Theorem 3.16 (Minian)**.** *Let X be a cellular space and let $C(X)$ be its cellular chain complex* [11, definition 3.6]*. Then $H_p(X) = H_p(C(X))$ for each $p \in \mathbb{Z}$.*

Some generalizations and further applications. Note that theorem 3.10 also holds for Alexandroff T_0–spaces provided that the filtration $\mathcal{F} = \{X_p : p \in \mathbb{Z}\}$ of X is induced by antichains and satisfies the following condition

($*$) For all $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $\tilde{H}_n(\hat{C}^{X_p}_x) = 0$ for all $p \geq m$ and for all $x \in D_p = X_p - X_{p-1}$.

This condition guarantees that the spectral sequence will converge and is satisfied, for example, if the filtration $\mathcal{F}$ consists of only a finite number of distinct spaces or under the hypotheses of 3.20.

Moreover, the following relative version of 3.10 for Alexandroff T_0–spaces holds.

Theorem 3.17. *Let X be an Alexandroff T_0–space and let $A \subseteq X$ be a subspace. Let $\mathcal{F} = \{X_p : p \in \mathbb{Z}\}$ be a filtration of X such that $X_n = A$ for all $n \leq -1$. For each $p \in \mathbb{N}_0$, let $D_p = X_p - X_{p-1}$. Suppose that D_p is an antichain for all $p \in \mathbb{N}_0$ and that the filtration $\mathcal{F}$ satisfies condition $(*)$ above.*

Then there is a spectral sequence $\{(E^r_{p,q})_{p,q\in\mathbb{Z}}, (d^r_{p,q})_{p,q\in\mathbb{Z}}\}_{r\in\mathbb{N}}$ that converges to $H_*(X, A)$ such that:

- $E^1_{p,q} = 0$ *for every* $p \leq -1$.
- $E^1_{p,q} = \bigoplus_{x\in D_p} \tilde{H}_{p+q-1}(\hat{C}_x^{X_p})$ *for* $p \geq 0$.
- *The morphisms $d^1_{p,q} : E^1_{p,q} \longrightarrow E^1_{p-1,q}$ are defined in the following way:*
  - *If $p \leq 0$ and $q \in \mathbb{Z}$, then $d^1_{p,q}$ is the trivial homomorphism.*
  - *If $p \geq 1$ and $q \leq -p$, then $d^1_{p,q}$ is the trivial homomorphism.*
  - *If $p \geq 1$ and $q \geq -p+1$, then $d^1_{p,q} : \bigoplus_{x\in D_p} \tilde{H}_{p+q-1}(\hat{C}_x^{X_p}) \longrightarrow \bigoplus_{y\in D_{p-1}} \tilde{H}_{p+q-2}(\hat{C}_y^{X_{p-1}})$ is defined by*

$$d^1_{p,q}\left(\left(\left[\sum_{i=1}^{l_x} a_i^x s_i^x\right]\right)_{x\in D_p}\right) = \left(\left[\sum_{x\in D_p}\sum_{s_i^x \ni y} a_i^x \operatorname{sgn}_{s_i^x}(y)(s_i^x - \{y\})\right]\right)_{y\in D_{p-1}}$$

where for every $x \in D_p$, $l_x \in \mathbb{N}$, and for every $i = \{1, \dots, l_x\}$, $a_i^x \in \mathbb{Z}$ and $s_i^x \in C^f_{p+q-1}(\hat{C}_x^{X_p})$.

The proof of this theorem is similar to that of 3.10 and will be omitted.

Remark 3.18. Note that there exists a formulation of the previous theorem for simplicial complexes following 3.5 which, by the argument below 3.5 is equivalent to theorem 3.17. We emphasize that theorems 3.10 and 3.17 give a explicit and simple description of the differentials of the spectral sequence.

Also, the definition of quasicellular space (3.14) can be generalized as follows.

Definition 3.19. Let X be a locally finite T_0–space and let $A \subseteq X$ be a subspace. We say that (X, A) is a *relative quasicellular pair* if A is open in X and there exists an order preserving map $\rho : X - A \longrightarrow \mathbb{N}_0$, which will be called *quasicellular morphism for* (X, A), such that

(1) The set $\{x \in X - A : \rho(x) = n\}$ is an antichain for every $n \in \mathbb{N}_0$.
(2) For every $x \in X - A$, the reduced homology of $\hat{U}_x^X$ is concentrated in degree $\rho(x) - 1$.

Corollary 3.15 can be generalized accordingly.

Corollary 3.20. *Let (X, A) be a relative quasicellular pair and let ρ be a quasicellular morphism for (X, A). For each $n \in \mathbb{N}_0$, let $J_n = \{x \in X - A : \rho(x) = n\}$.*

Let $C(X, A) = (C_n(X, A), d_n)_{n\in\mathbb{Z}}$ be the chain complex defined by

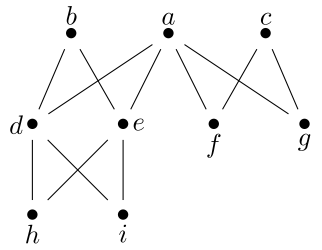
- $C_n(X, A) = \bigoplus_{x\in J_n} \tilde{H}_{n-1}(\hat{U}_x)$ *for each* $n \in \mathbb{N}_0$ *and* $C_n(X, A) = 0$ *for* $n < 0$.
- *For each $n \in \mathbb{Z}$, d_n is the group homomorphism $d^1_{n+1,-1}$ of theorem 3.17 (applied for the filtration defined by $X_0 = A$ and $X_n = X_{n-1} \cup J_{n-1}$ for all $n \in \mathbb{N}$).*

Then, $H_n(X, A) = H_n(C(X, A))$ for all $n \in \mathbb{N}_0$.

If, in addition, A is contractible, then $\widetilde{H}_n(X) = H_n(C(X, A))$ for all $n \in \mathbb{N}_0$.

The following is an example of application of the generalizations above.

Example 3.21. Consider the poset $\mathbf{S}$ defined by the following Hasse diagram.



Note that $\mathbf{S}$ is not a cellular poset since $\hat{U}_a$ does not have the homology of any sphere. Hence, theorem 3.7 of [11] does not apply. Indeed, $\mathbf{S}$ is not even a quasicellular poset since $\hat{U}_a$ does not have homology concentrated in any degree.

However, $(\mathbf{S}, U_a)$ is a relative quasicellular pair with quasicellular morphism $\rho : \mathbf{S} - U_a \to \mathbb{N}_0$ defined by $\rho(b) = 2$ and $\rho(c) = 1$.

Hence, by 3.20, the homology groups of $(\mathbf{S}, U_a)$ can be computed with the following chain complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \longrightarrow 0 \longrightarrow \cdots$$

Applying the formulas for the differentials stated in 3.10, it is easy to check that $d_2 = 0$. Thus, $H_2(\mathbf{S}, U_a) \cong H_1(\mathbf{S}, U_a) \cong \mathbb{Z}$ and $H_n(\mathbf{S}, U_a) = 0$ for all $n \in \mathbb{Z} - \{1, 2\}$. And since U_a is contractible we obtain that $H_2(\mathbf{S}) \cong H_1(\mathbf{S}) \cong \mathbb{Z}$ and $\widetilde{H}_n(\mathbf{S}) = 0$ for all $n \in \mathbb{Z} - \{1, 2\}$.

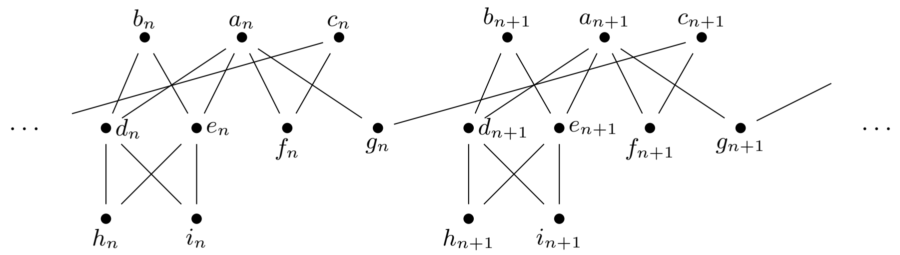
Indeed, $|\mathcal{K}(\mathbf{S})|$ is homeomorphic to $S^2 \vee S^1$.

Now, we will apply the results of this section to compute the second homotopy group of the poset of the previous example.

Example 3.22. Let $\mathbf{S}$ be the poset of example 3.21 and let $\widetilde{\mathbf{S}}$ be the Alexandroff T_0–space whose underlying set is $\mathbf{S} \times \mathbb{Z}$ and whose minimal open sets are defined by

$$U^{\widetilde{\mathbf{S}}}_{(x,n)} = \begin{cases} U^{\mathbf{S}}_x \times \{n\} & \text{if } x \neq c \\ \{(c, n), (f, n), (g, n-1)\} & \text{if } x = c \end{cases}$$

The poset $\widetilde{\mathbf{S}}$ can be represented by the following diagram in which the element (x, j) is denoted by x_j:



Let $p : \widetilde{\mathbf{S}} \to \mathbf{S}$ be the projection onto the first coordinate. Note that p is a covering map since the open subsets $U^{\mathbf{S}}_a$, $U^{\mathbf{S}}_b$ and $U^{\mathbf{S}}_c$ are evenly covered by p.

For each $n \in \mathbb{Z}$ let $V_n = \mathbf{S} \times \{n\} \cup \{(g, n-1)\}$ and $T_n = \{a, b, d, e, h, i\} \times \{n\}$. Clearly, V_n is homotopy equivalent to T_n for all $n \in \mathbb{Z}$ since the former can be obtained from the latter by successively removing beat points. And since T_n is a finite model of the sphere S^2, we obtain that V_n is simply-connected for all $n \in \mathbb{Z}$. Now, $V_n \cap V_{n+1} = \{(g, n)\}$. Thus, from van Kampen's theorem we obtain that

$V_n \cup V_{n+1}$ is simply-connected for all $n \in \mathbb{Z}$. Proceeding inductively we obtain that $\bigcup_{j=n}^{m} V_j$ is simply-connected for all $n, m \in \mathbb{Z}$ with $n \leq m$. With a compactness argument it follows that $\widetilde{\mathbf{S}} = \bigcup_{j\in\mathbb{Z}} V_j$ is simply-connected.

Therefore, $\widetilde{\mathbf{S}}$ is the universal cover of $\mathbf{S}$ and hence $\pi_2(\mathbf{S}) = H_2(\widetilde{\mathbf{S}})$. We will apply our results to compute $H_2(\widetilde{\mathbf{S}})$.

Let $A = (\mathbf{S} - \{b\}) \times \mathbb{Z}$ and let $B = (\mathbf{S} - U_b) \times \mathbb{Z}$. Let $r_0 : \mathbf{S} - \{b\} \to \mathbf{S} - U_b$ be defined by

$$r_0(x) = \begin{cases} a & \text{if } x \in \{a, d, e, h, i\} \\ x & \text{if } x \in U_c \end{cases}$$

Note that the map r_0 is continuous since it is order-preserving. Let $i : B \to A$ be the inclusion map and let $r = r_0 \times \mathrm{Id}_{\mathbb{Z}} : A \to B$. Observe that $r \circ i = \mathrm{Id}_B$ and $i \circ r \geq \mathrm{Id}_A$. Thus, r is a strong deformation retraction. Now, since $|\mathcal{K}(B)|$ is homeomorphic to $\mathbb{R}$ we obtain that A is homotopically trivial.

On the other hand, from 3.3 we obtain that

$$H_n(\widetilde{\mathbf{S}}, A) \cong \bigoplus_{j\in\mathbb{Z}} \widetilde{H}_{n-1}(\hat{U}^{\widetilde{\mathbf{S}}}_{(b,j)}) \cong \bigoplus_{j\in\mathbb{Z}} \widetilde{H}_{n-1}(\hat{U}^{\mathbf{S}}_b) = \begin{cases} \bigoplus_{j\in\mathbb{Z}} \mathbb{Z} & \text{if } n = 2 \\ 0 & \text{if } n \neq 2 \end{cases}$$

Therefore, $H_2(\widetilde{\mathbf{S}}) \cong \bigoplus_{j\in\mathbb{Z}} \mathbb{Z}$. Hence, $\pi_2(\mathbf{S}) \cong \bigoplus_{j\in\mathbb{Z}} \mathbb{Z}$.

It is interesting to mention that coverings of small categories have been studied for example in [6] and [13]. The particular case of coverings of posets is analyzed in [5]. Using the results of these works it is not difficult to construct the universal cover of a given poset.

4. Homological Morse theory for posets

G. Minian introduced in [11] a discrete version of Morse theory for posets and developed a homological variant of this theory showing that given a homologically admissible Morse matching on a cellular poset X, the homology groups of X can be computed from a chain complex which in degree p consists of the free abelian group generated by the critical points of X of degree p. Using our techniques we will give here a generalization of his result with a completely different and more conceptual proof.

We begin by recalling some definitions from [11].

Let X be a finite poset and let $\mathcal{H}(X)$ be its Hasse diagram.

Let M be a matching on $\mathcal{H}(X)$ and let $\mathcal{H}_M(X)$ be the graph obtained from $\mathcal{H}(X)$ by reversing the orientations of the edges which are not in M. We say that M is a *Morse matching* if the graph $\mathcal{H}_M(X)$ is acyclic.

Given a Morse matching M on $\mathcal{H}(X)$ we define the *critical points* of X as the points of X which are not incident to any edge of M.

We say that an edge (a, b) of the Hasse diagram of X is *homologically admissible* if the space $\widehat{U}_b - \{a\}$ is acyclic.

We say that a matching M on $\mathcal{H}(X)$ is *homologically admissible* if every edge of M is homologically admissible.

For our proof we need some lemmas. The first of them is one of the keys of our proof.

Lemma 4.1. *Let X be a poset. Let b be a maximal point of X. Suppose there exists $a \in X$ such that $\widehat{F}_a = \{x \in X \mid a < x\} = \{b\}$ and such that the edge (a, b) of the Hasse diagram of X is homologically admissible. Then $H_n(X, X - \{a, b\}) = 0$ for all $n \in \mathbb{Z}$.*

Proof. Clearly, a is upbeat point of X an hence $X-\{a\}$ is a strong deformation retract of X. Thus, $H_n(X, X-\{a\}) = 0$ for all $n \in \mathbb{Z}$.

On the other hand, applying 3.3 we obtain that

$$H_n(X-\{a\}, X-\{a,b\}) = H_{n-1}(\widehat{C}_b^{X-\{a\}}) = \widetilde{H}_{n-1}(\widehat{U}_b - \{a\}) = 0$$

for all $n \in \mathbb{Z}$ since the edge (a, b) is homologically admissible.

Then the result follows from the long exact sequence in homology of the triple $(X, X-\{a\}, X-\{a,b\})$. □

Lemma 4.2. *Let X be a quasicellular poset and let $\rho : X \to \mathbb{N}_0$ be its quasicellular morphism. Let (a, b) be an homologically admissible edge of the Hasse diagram of X. Then $\rho(b) = \rho(a) + 1$.*

Proof. We have that $\widetilde{H}_n(\widehat{U}_b - \{a\}) = 0$ for all $n \in \mathbb{Z}$ since the edge (a, b) is homologically admissible and $H_n(\widehat{U}_b, \widehat{U}_b - \{a\}) = \widetilde{H}_{n-1}(\widehat{U}_a)$ by 3.3. Then the result follows from the long exact sequence in homology of the pair $(\widehat{U}_b, \widehat{U}_b - \{a\})$. □

Now, we will prove that the homology groups of a quasicellular poset X can be computed from a chain complex constructed from a homologically admissible Morse matching, generalizing theorem 3.14 of [11]. The idea of the proof is that, by lemma 4.1, under certain hypotheses a homologically admissible edge of a poset can be removed without altering the homology groups of the poset. Thus, we will prove that the edges of a homologically admissible Morse matching can be arranged in such a way that the hypotheses of lemma 4.1 are satisfied and hence we will be able to remove all the edges of the matching to obtain that the homology groups of X can be computed from the set of critical points.

To put this idea into work, we consider a suitable filtration of the poset X and construct a spectral sequence which converges to the homology groups of X.

Theorem 4.3. *Let X be a quasicellular poset and let $p : X \to \mathbb{N}_0$ be its quasicellular morphism. Let M be a homologically admissible Morse matching on $\mathcal{H}(X)$. For $n \in \mathbb{N}_0$ let $A_n = \{x \in X \;/\; x \text{ is a critical point of } X \text{ and } \rho(x) = n\}$. Let $(C_n, d_n)_{n\in\mathbb{N}_0}$ be the chain complex defined by $C_n = \displaystyle\bigoplus_{x\in A_n} \widetilde{H}_{n-1}(\widehat{U}_x)$ and where the differentials d_n are defined as in theorem 3.10. Then the homology of X coincides with the homology of $(C_n, d_n)_{n\in\mathbb{N}_0}$.*

Proof. For $n \in \mathbb{N}_0$ let

$$X_n = \{x \in X \;/\; \rho(x) \leq n\} \cup \{z \;/\; (y, z) \in M \text{ and } \rho(y) = n\}.$$

By 4.2, $X_{n-1} \subseteq X_n$ for all $n \in \mathbb{N}$ and since X is a finite space it follows that $(X_n)_{n\in\mathbb{N}_0}$ is a filtration of X.

Let $X_{-1} = \varnothing$. We will compute now $H_i(X_n, X_{n-1})$ for all $i \in \mathbb{Z}$ and for all $n \in \mathbb{N}_0$. Fix $n \in \mathbb{N}_0$. Let $\mathcal{S} = \{(y, z) \in M \;/\; \rho(y) = n\}$. Suppose $\mathcal{S} \neq \varnothing$. We define a relation $\preceq$ in $\mathcal{S}$ as follows: $(a, b) \preceq (a', b')$ if and only if there exist $l \in \mathbb{N}$ and $(a_0, b_0), \dots, (a_l, b_l) \in \mathcal{S}$ such that $(a_0, b_0) = (a, b)$, $(a_l, b_l) = (a', b')$ and $a_j \in \widehat{U}_{b_{j+1}}$ for all $j \in \{0, \dots, l-1\}$.

This relation is clearly reflexive and transitive. We will prove now that it is also antisymmetric. Suppose that (a, b) and (a', b') are distinct elements of $\mathcal{S}$ such that $(a, b) \preceq (a', b')$ and $(a', b') \preceq (a, b)$. Then there exist $l, m \in \mathbb{N}$, $(a_0, b_0), \dots, (a_l, b_l) \in \mathcal{S}$ and $(a'_0, b'_0), \dots, (a'_m, b'_m) \in \mathcal{S}$ such that $(a_0, b_0) = (a, b) = (a'_m, b'_m)$, $(a_l, b_l) = (a', b') = (a'_0, b'_0)$, $a_j \in \widehat{U}_{b_{j+1}}$ for all $j \in \{0, \dots, l-1\}$ and $a'_k \in \widehat{U}_{b'_{k+1}}$ for all $k \in \{0, \dots, m-1\}$. In addition, we may suppose that $(a_j, b_j) \neq (a_{j+1}, b_{j+1})$ for all $j \in \{0, \dots, l-1\}$ and that $(a'_k, b'_k) \neq (a'_{k+1}, b'_{k+1})$ for all $k \in \{0, \dots, m-1\}$. And since M is a matching, we obtain that $a_j \neq a_{j+1}$ for all $j \in \{0, \dots, l-1\}$ and

$a'_k \neq a'_{k+1}$ for all $k \in \{0, \dots, m-1\}$. Now, note that $(a_j, b_{j+1}) \in \mathcal{H}(X) - M$ for all $j \in \{0, \dots, l-1\}$ since $a_j < b_{j+1}$, X is quasicellular, $\rho(a_j) = n = \rho(b_{j+1}) - 1$ (by the previous lemma) and M is a matching. In a similar way $(a'_k, b'_{k+1}) \in \mathcal{H}(X) - M$ for all $k \in \{0, \dots, m-1\}$. Thus,

$$(a'_m, b'_m), (b'_m, a'_{m-1}), (a'_{m-1}, b'_{m-1}), \dots, (a'_0, b'_0), (b_l, a_{l-1}), (a_{l-1}, b_{l-1}), \dots, (a_1, b_1), (b_1, a_0)$$

is a cycle in $\mathcal{H}_M(X)$ which entails a contradiction since M is a Morse matching.

Then the relation $\preceq$ is antisymmetric and hence it is a partial order.

Let $N = \#\mathcal{S}$. Extending the partial order in $\mathcal{S}$ to a linear order we obtain that we may label the elements of $\mathcal{S}$ as $(y_1, z_1), \dots, (y_N, z_N)$ in such a way that if $(y_j, z_j) \preceq (y_k, z_k)$ then $j \leq k$.

Note that

$$X_n = X_{n-1} \cup A_n \cup \bigcup_{j=1}^{N} \{y_j, z_j\}$$

(recall that A_n was defined as $A_n = \{x \in X \ / \ x$ is a critical point of X and $\rho(x) = n$ $\}$).

For $k \in \{0, 1, \dots, N\}$ let

$$B_k = X_{n-1} \cup A_n \cup \bigcup_{j=1}^{k} \{y_j, z_j\} \ .$$

Hence, $B_0 = X_{n-1} \cup A_n$ and $B_N = X_n$.

Let $r \in \{1, \dots, N\}$. We claim that $\widehat{U}^X_{z_r} = \widehat{U}^{B_r}_{z_r}$. Indeed, let $x \in \widehat{U}^X_{z_r}$. Then $x < z_r$ and thus $\rho(x) < \rho(z_r) = n + 1$ by the previous lemma. If $x \in X_{n-1} \cup A_n$ then $x \in B_r$. If $x \notin X_{n-1} \cup A_n$ then there exists $s \in \{1, \dots, N\}$ such that $x = y_s$. Thus, $(y_s, z_s) \preceq (y_r, z_r)$ and hence $s \leq r$. Then, $x = y_s \in B_r$. Thus, $\widehat{U}^X_{z_r} \subseteq \widehat{U}^{B_r}_{z_r}$. The other inclusion is trivial.

Hence, the edge (y_r, z_r) is homologically admissible in B_r.

On the other hand, we claim that $\widehat{F}^{B_r}_{y_r} = \{z_r\}$. Indeed, suppose that $x \in B_r$ satisfies that $x > y_r$. Hence, $\rho(x) > \rho(y_r) = n$ and thus $x = z_s$ for some $s \in \{1, \dots, r\}$. But this implies that $y_r \in \widehat{U}_{z_s}$ and hence $(y_r, z_r) \preceq (y_s, z_s)$. Then $r \leq s$ and thus $s = r$ and $x = z_r$. Hence, $\widehat{F}^{B_r}_{y_r} \subseteq \{z_r\}$ while the other inclusion is trivial.

Now, since $\rho(z_r) = n + 1$ and $B_r \subseteq X_n \subseteq \{x \in X \ / \ \rho(x) \leq n + 1\}$ (by 4.2) it follows that z_r is a maximal point of B_r. Hence, we are under the hypotheses of lemma 4.1 and thus we obtain that $H_i(B_r, B_{r-1}) = 0$ for all $i \in \mathbb{Z}$.

Hence, we have proved that $H_i(B_r, B_{r-1}) = 0$ for all $i \in \mathbb{Z}$ and for all $r \in \{1, \dots, N\}$. It follows that $H_i(B_N, B_0) = 0$ for all $i \in \mathbb{Z}$. Note that this was done under the hypothesis $\mathcal{S} \neq \varnothing$, but holds trivially if $\mathcal{S} = \varnothing$.

Thus, by 3.3,

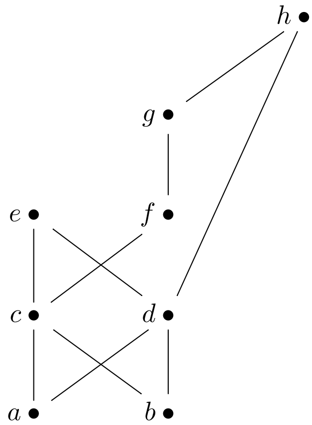
$$H_i(X_n, X_{n-1}) = H_i(B_N, X_{n-1}) \cong H_i(B_0, X_{n-1}) \cong \begin{cases} \displaystyle\bigoplus_{x \in A_n} \widetilde{H}_{n-1}(\widehat{U}_x) & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}$$

since X is quasicellular.

From the filtration $(X_n)_{n \in \mathbb{N}_0}$ of X we can contruct a spectral sequence in a similar way to the one in the proof of theorem 3.10, which will have in its first page a single nontrivial row and whose differentials can be computed in the same way as in 3.10 by naturality of the long exact sequences since the isomorphisms $H_i(X_n, X_{n-1}) \cong H_i(B_0, X_{n-1})$ are given by the inclusion maps. Thus, the result follows. $\square$

Remark 4.4. The previous theorem might not hold if the space X is not quasicellular even if for all $x \in X$ the homology of $\widehat{U}_x$ is concentrated in some degree. For

example, let X be defined by the following Hasse diagram



and let $M = \{(c,e),(d,h),(f,g)\}$. It is easy to verify that M is a homologically admissible Morse matching. On the other hand f and g are beat points of X and hence X is homotopy equivalent to $X - \{f,g\}$ which is a finite model for S^2. But the set of critical points is $\{a,b\}$ an thus if $(C_n, d_n)_{n\in\mathbb{N}_0}$ is the chain complex of the previous theorem we obtain that $C_0 = \mathbb{Z}\oplus\mathbb{Z}$ and $C_n = 0$ for all $n\in\mathbb{N}$. Clearly, the homology groups of $(C_n, d_n)_{n\in\mathbb{N}_0}$ do not coincide with those of X.

References

[1] Alexandroff, P. Diskrete Räume. *Mathematiceskii Sbornik (N.S.) 2*, 3 (1937), 501–518.

[2] Barmak, J. *Algebraic topology of finite topological spaces and applications*, vol. 2032 of *Lecture Notes in Mathematics*. Springer, Berlin, 2011.

[3] Barmak, J., and Minian, G. One-point reductions of finite spaces, h–regular CW–complexes and collapsibility. *Algebraic & Geometric Topology 8*, 3 (2008), 1763–1780.

[4] Barmak, J., and Minian, G. Simple homotopy types and finite spaces. *Advances in Mathematics 218*, 1 (2008), 87–104.

[5] Barmak, J., and Minian, G. A note on coverings of posets, A–spaces and polyhedra. *Homology, Homotopy and Applications 18*, 1 (2016), 143–150.

[6] Cibils, C., and MacQuarrie, J. Gradings, smash products and Galois coverings of a small category. *arXiv preprint arXiv:1203.5905* (2012).

[7] Forman, R. Morse theory for cell complexes. *Advances in mathematics 134*, 1 (1998), 90–145.

[8] Kozlov, D. *Combinatorial Algebraic Topology*. Springer, Berlin, 2007.

[9] Lovász, L. Kneser's conjecture, chromatic number, and homotopy. *Journal of Combinatorial Theory, Series A 25*, 3 (1978), 319–324.

[10] McCord, M. Singular homology groups and homotopy groups of finite topological spaces. *Duke Mathematical Journal 33*, 3 (1966), 465–474.

[11] Minian, G. Some remarks on Morse theory for posets, homological Morse theory and finite manifolds. *Topology and its Applications 159*, 12 (2012), 2860–2869.

[12] Stong, R. Finite topological spaces. *Transactions of the American Mathematical Society 123*, 2 (1966), 325–340.

[13] Tanaka, K. A model structure on the category of small categories for coverings. *Mathematical Journal of Okayama University 55*, 1 (2013), 95–116.

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