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# On the complexity of the minimum domination problem restricted by forbidden induced subgraphs of small size

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ABSTRACT

We study the computational complexity of the minimum dominating set problem on graphs restricted by forbidden induced subgraphs. We give some dichotomies results for the problem on graphs defined by any combination of forbidden induced subgraphs with at most four vertices, implying either an NP-Hardness proof or a polynomial time algorithm. We also extend the results by showing that dominating set problem remains NP-hard even when the graph has maximum degree three, it is planar and has no induced claw, induced diamond, induced $K_4$ nor induced cycle of length 4, 5, 7, 8, 9, 10 and 11.



## 1. Introduction

Dominating set is a fundamental problem of algorithmic graph theory [14] with many applications [7,13,21]. It arises naturally in many areas including mathematics, operations research, logistics, economics, and computer science. A typical application related with the problem is the following: Assume you have a representation of a city given by a grid, where intersection points are the corners, and you want to put cameras on it to observe the entire grid (city). The goal is to choose a set of points in the plane in order to *observe* the remaining points. The grid can be represented as a graph where corners are vertices and adjacent corners are joined by edges. Since one can be interested in the observation of certain regions of the city, it makes sense to use as representation an induced graph of the grid. These graphs are a subclass of $K_3$-free, which is one of the classes we analyze.

The problem remains NP-hard in many restricted graph families such as planar [18], chordal [3], split and bipartite [1], planar graphs of maximum degree 3 [11], among others. On the other hand, the problem has efficient algorithms for classes such as interval and circular-arc graphs [5], AT-free graphs [16], and strongly chordal graphs [10]. Many graph classes can be defined by forbidding certain induced subgraphs. For instance, split graphs, interval graphs, bipartite graphs, permutation graphs, among others can be defined as $\mathcal{F}$-free graphs, for some graph family $\mathcal{F}$. Several of them are interesting from the geometric point of view. In this paper we present a systematic analysis of the complexity for the minimum dominating set problem on $\mathcal{F}$-free graphs, where $\mathcal{F}$ is a family of graphs of order at most four. For unary family $\mathcal{F} = \{F\}$, usually we denote $F$-Free instead of $\mathcal{F}$-free.

The next section gives the notations used throughout this paper. In the third section we show previous results that will be used for simplifying the analysis, along with some remarks that were not mentioned previously but are needed for our work. At the end of this section we present the current known information, and also give a graph representation of the results as a help for visualization. In the following section we prove the complexity of the domination problem for several graph classes in order to obtain the desired results. Finally we give the conclusions of this paper.

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## 2. Notations

Let $G = (V, E)$ be an undirected graph where $V(G)$ and $E(G)$ denote the vertex set and the edge set respectively. Throughout the paper, $n_G = |V(G)|$ and $m_G = |E(G)|$ denote the numbers of vertices and edges of the graph $G$, respectively. Denote by $N_G(v)$ the subset of vertices adjacent to $v$, and let $N_G[v] = N_G(v) \cup \{v\}$. Set $N_G(v)$ is called the *neighborhood* of $v$, while $N_G[v]$ is the *closed neighborhood* of $v$. Let $S \subseteq V(G)$, we denote the *neighborhood* of $S$ as $N_G(S) = \bigcup_{v\in S} N_G(v) \setminus S$ and the *closed neighborhood* of $S$ as $N_G[S] = N_G(S) \cup S$. The *degree* of $v$ is $d_G(v) = |N_G(v)|$. By $\Delta(G)$ we denote the maximum vertex degree in $G$. When there is no ambiguity, we may omit the subscripts from $n$, $m$, $N$ and $d$. We say that $u$ is *universal* when $N[u] = V(G)$. Say that $w$ is *dominated* by vertex $v$ if $N[w] \subseteq N[v]$. A subset $S$ of $V$ dominates another subset $T$ of $V$ if $T \subseteq N[S]$.

As usual, $C_n$ and $P_n$ denote the chordless cycle and the chordless path on $n$ vertices, respectively. A *dominating set* of $G$ is a set $W \subseteq V(G)$ such that every vertex in $V(G) \setminus W$ is adjacent to some vertex of $W$. The size of a minimum dominating set in a graph $G$ is called the domination number of $G$ and is denoted as $\gamma(G)$. An induced subgraph $H$ of $G$ is said *dominating H* if $V(H)$ is a dominating set of $G$. Clearly, if there is a dominating $H$ then $\gamma(G) \le |V(H)|$.

The graph $K_q$ is the complete graph of $q$ vertices, and the graph $tK_q$ consists of the disjoint union of $t$ copies of the graph $K_q$.

Subdivision of an edge is the operation of creation of a new vertex on the edge. When a polynomial-time algorithm has been shown for a problem, we say that the problem is in $P$. Whenever the problem is in the complexity class NP-Complete we say that the problem is *NPC*.

We assume $G$ is connected, since domination problem can be solved independently in each connected component.

## 3. Previous results

The set of graphs of order three are: $P_3$, $K_3$, $3K_1$ and co-$P_3$. Since $P_3$-free is a subclass of $P_4$-free, which is a subclass of permutation graphs, then by applying [6], we deduce that domination problem restricted to $P_3$-free graphs is in $P$. In addition, $P_3$-free graphs are also known as cluster graphs which are disjoint unions of complete graphs and there is a trivial linear-time algorithm to solve the minimum dominating set problem in this class of graphs. From [1] it is known that domination problem restricted to bipartite graphs is in *NPC*, hence for $K_3$-free graphs the problem is also in *NPC* and from [16] we can conclude that domination restricted to co-paw-free graphs or $3K_1$-free graphs is in $P$ since both are subclasses of AT-free graphs.

**Lemma 3.1** ([1]). *Domination problem restricted to split graphs is in NPC.*

**Corollary 3.2.** *Since $(2K_2, C_4)$-free graphs is a superclass of split graphs, then domination problem restricted to this class is also in NPC.*

**Lemma 3.3** ([23]). *Domination Problem restricted to $(K_p, P_5)$-free graphs for fixed p is in P.*

**Corollary 3.4.** *Domination problem restricted to $(2K_2, K_4)$-free graphs is in P.*

**Lemma 3.5** ([2]). *The clique-width of (claw, co-claw)-free graphs and (claw, paw)-free graphs is bounded.*

**Lemma 3.6** ([9]). *The clique-width of $(K_2 \cup \text{claw}, K_3)$-free graphs is bounded.*

**Lemma 3.7** ([8]). *Domination problem is in P for graph classes with bounded clique-width.*

**Corollary 3.8.** *Domination problem restricted to (claw, co-claw)-free graphs or (claw, paw)-free or $(K_2 \cup \text{claw}, K_3)$-free graphs is in P.*

We add some remarks for easy results for which we could not find any references:

**Remark 3.9.** Note that (claw, $K_3$)-free $\subseteq$ $(K_2 \cup \text{claw}, K_3)$-free graphs. Hence the domination problem on (claw, $K_3$)-free graph class is in $P$.

**Remark 3.10.** The minimum dominating set problem restricted to $4K_1$-free graphs is in $P$.

**Proof.** Given a $4K_1$-free graph $G$, any maximal independent set $I$ of $G$ has at most 3 vertices. It is well-known that every maximal independent set is a dominating set, then there is at least one dominating set of size at most 3. Hence, the minimum dominating set for $G$ can be solved by checking for each possible subset of at most three vertices if it is a dominating set. This can be done in $O(n^3)$. □

**Remark 3.11.** The minimum dominating set problem restricted to co-diamond-free graphs is in $P$.

**Proof.** Given a co-diamond-free graph $G$, if $G$ has no edge then $\gamma(G) = n$. Otherwise, let $e = v_1v_2$ be an arbitrary edge. If $\{v_1, v_2\}$ is not a dominating set then $\exists\ w$ such that $\{v_1, v_2\} \cap N(w) = \emptyset$. Clearly, $\{v_1, v_2, w\}$ is a dominating set because if exists some vertex $x \in V(G)$ such that is not adjacent to any of them, then $\{v_1, v_2, w, x\}$ induces a co-diamond which is a contradiction. □

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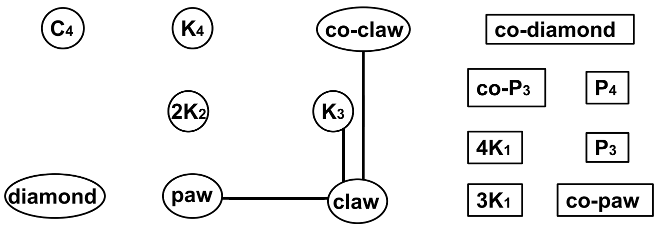


**Fig. 1.** Scheme of graph classes complexity for the dominating set problem.

### 3.0.1. Summary

We enumerate existent results for the complexity of the problem restricted to $\mathcal{F}$-free graphs where $\mathcal{F}$ is a family of graphs with 3 or 4 vertices:

| Forbidden induced subgraphs | Complexity | References |
|---|---|---|
| $P_3$ | P | [6] |
| $3K_1$ | P | [16] |
| co-$P_3$ | P | [16] |
| $P_4$ | P | [6] |
| $4K_1$ | P | Remark 3.10 |
| co-diamond | P | Remark 3.11 |
| paw, claw | P | [2,8] |
| claw, co-claw | P | [2,8] |
| co-paw | P | [16] |
| claw, $K_3$ | P | [9,8] |
| $K_3$ | NPC | [22] |
| claw | NPC | [12] |
| paw | NPC | [19] |
| diamond | NPC | [19] |
| co-claw | NPC | [19] |
| $2K_2$, $C_4$ | NPC | [1] |

We represent certain information contained in above table as a meta-graph where each vertex corresponds to an $F$-free graph (a vertex gets rectangle shape if the domination problem restricted to its corresponding class is in $P$; otherwise, it gets circular shape which means the corresponding class is in $NPC$) and each edge $vw$ corresponds to an intersection of $NPC$-classes corresponding to $v$ and $w$ such that the intersection is in $P$. In next section, we will complete the meta-graph with new edges. This allows to determine the complexity of domination problem restricted to any $\mathcal{F}$-free graphs where $\mathcal{F}$ is a family of graphs with at most 4 vertices.

## 4. Completing hierarchy

In order to complete Fig. 1, we need to prove the complexity of the domination problem for the intersection of classes that belongs to $NPC$. Note that any connected graph which is $K_3$-free and has at least four vertices is also ($K_4$, paw, diamond)-free graphs. Since the problem is in NPC for $K_3$-free graphs then it is NPC for ($K_4$, paw, diamond)-free graphs, therefore the vertices corresponding to $K_4$-free, paw-free and diamond-free graphs form an independent set.

**Theorem 4.1.** *Domination problem restricted to ($2K_2$, claw)-free graphs class is in $P$.*

**Proof.** Let $G = (V, E)$ be a connected ($2K_2$, claw)-free graph. Since the domination problem for $P_4$-free graphs is in $P$, then we consider the case where $G$ has an induced $P_4 = \{p_1, p_2, p_3, p_4\}$ ($p_1p_2$, $p_2p_3$ and $p_3p_4$ are edges of $P_4$).

Let $U = V(G) \setminus N[P_4]$. If $U = \emptyset$, then $\gamma(G)$ is at most 4. Hence, the problem is in $P$. Now, we suppose that $U \neq \emptyset$. Clearly, $U$ is an independent set because $G$ is $2K_2$-free. Hence, $N(U) \subseteq N(P_4)$. As $G$ is a connected graph, $N(u) \neq \emptyset$ for every $u \in U$. Moreover, if $u, u'$ are two different vertices of $U$ then $N(u) \cap N(u') = \emptyset$ by claw-freeness. Let $v$ be any vertex in $N(U)$ which means $v$ is neighbor of some $u \in U$. Clearly, $N(v) \cap P_4 = \{p_2, p_3\}$ because otherwise there is a $2K_2$ or a claw as induced subgraph which is a contradiction. Now, we will prove that $N(U)$ induces a complete subgraph. Suppose there are two vertices $v, v' \in N(U)$, $(v, v') \notin E(G)$. Recall that $N(v) \cap \{p_1, p_2, p_3, p_4\} = \{p_2, p_3\}$ $\{p_1, p_2, v, v'\}$ induces a

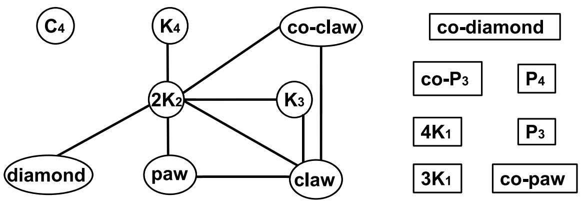


**Fig. 2.** Scheme of graph classes complexity for the dominating set problem.

claw which is a contradiction. Hence, $N(U)$ induces a complete subgraph. We show that $p_1$ and $p_4$ cannot have a common neighbor. Suppose $w$ is a common neighbor of $p_1$ and $p_4$. In this case, $w$ belongs to $N(P_4) \setminus N[U]$. If $w$ is adjacent to some vertex $v \in N(U)$ (Recall that $N(v) \cap \{p_1, p_2, p_3, p_4\} = \{p_2, p_3\}$), then $\{w, v, p_1, p_4\}$ induces a claw, a contradiction. Hence, $w$ is not adjacent to any $v \in N(U)$. Choose any edge $uv$ where $u \in U$. Clearly, $\{u, v, w, p_1\}$ induces a $2K_2$. Again, this is a contraction. In Consequence, $N[p_1] \cap N[p_4] = \emptyset$ which implies the closed neighborhoods of vertices in $U \cup \{p_1, p_4\}$ are all disjoint. Then, $\gamma(G) \geq |U| + 2$. Now, we prove that $D = U \cup \{p_2, p_3\}$ is a dominating set of $G$ and it must be minimum. Suppose there is some vertex $w$ not dominated by $D$. Clearly, $w \in N(\{p_1, p_4\}) \setminus N[\{p_2, p_3\}]$. If $w$ is not adjacent to $p_1$ ($p_4$), then $\{w, p_4, p_1, p_2\}$ ($\{w, p_1, p_3, p_4\}$) induces a $2K_2$ which is a contradiction. Hence, $w$ is adjacent to $p_1$ and $p_4$. Again, this is a contradiction. Consequently, $D$ is a (minimum) dominating set and it can be obtained in polynomial time. □

**Theorem 4.2.** *Domination problem restricted to ($2K_2$, diamond)-free graphs class is in P.*

**Proof.** Let $G$ be a ($2K_2$, diamond)-free graph and $K_p = \{u_1, \dots, u_p\}$ be a maximum clique in $G$, which can be obtained in polynomial time (the number of maximal cliques is polynomial). We separate the proof in two cases according to the size of $K_p$

- $p \leq 3$: Then $G$ is $(2K_2, K_4)$-free. By Corollary 3.4 the problem is in $P$
- $p \geq 4$: If $G = K_p$ then $\gamma(G) = 1$. Otherwise, let $v \in N(K_p)$. It is easy to see that $|N(v) \cap K_p| = 1$, otherwise $G$ is not diamond-free. Suppose w.l.o.g. $u_1 \in N(v)$ and let $w \in N(v) \setminus K_p$. Since $G$ must be $2K_2$-free then $w$ must be connected to at least $p-2$ vertices of $\{u_2, \dots, u_p\}$ but then again $G$ is not diamond-free, hence $w$ cannot exist. Thus the graph is a split graph $(K = K_p, S)$ where every vertex of $S$ has degree one. Clearly, $N(S)$ is the minimum dominating set of G, which can be obtained in polynomial time. □

Now we proceed to prove the complexity for the domination problem restricted to ($2K_2$, co-claw)-free is in $P$. We begin showing a lemma that turns out to be useful for proving the desired property.

**Lemma 4.3.** *Let $G$ be a connected $(2K_2, K_3)$-free graph with an induced $C_5 = \{v_1, v_2, v_3, v_4, v_5\}$. Then $C_5$ is a dominating set.*

**Proof.** Suppose that $C_5$ is not a dominating set and let $w$ be a vertex from $G$ which is not in $N[C_5]$. Since $w$ cannot be an isolated vertex it must be connected with some other vertex $x \notin C_5$. Then $\{w, x\}$ forms a $K_2$. Since $G$ is $2K_2$-free then $(w, x)$ cannot be disjoint with any $K_2$ from $C_5$. In this case $x$ must be connected to at least three vertices from $C_5$ which implies there is some $K_3$ formed by $x$ and two consecutive vertices of $C_5$. Absurd since $G$ is $K_3$-free. □

**Theorem 4.4.** *Domination problem restricted to ($2K_2$, co-claw)-free graphs class is in P.*

**Proof.** Let $G$ be a connected ($2K_2$, co-claw)-free graph. Suppose $G$ has a cycle, since if it has no cycle it is a tree where domination problem is in $P$ [20]. Suppose also that $G$ is $K_3$-free, since if it has $K_3 = U \subseteq V(G)$ then any vertex from $V(G)$ must be in $N[U]$, hence there is a dominating set of cardinality 3 and minimum dominating set can be obtained in $O(n^3)$.

It is clear that $G$ is $C_{n+5}$-free ($n > 0$) because it is $2K_2$-free. If $G$ is $C_5$-free, then from [16] it is known that domination problem restricted to $(2K_2, C_5, K_3)$-free graphs is in $P$. Now, we consider that $G$ contains a $C_5$, then we apply Lemma 4.3 described above and we affirm there is a dominating set of $G$ with size at most 5. Thus the minimum dominating set can be found in $O(n^5)$. □

**Theorem 4.5.** *Domination problem restricted to ($2K_2$, paw)-free graphs class is in P.*

**Proof.** Let $T = \{t_1, t_2, t_3\}$ be a $K_3 \in G$. Let $v$ be a vertex not adjacent to $T$ and $P = \{u_1, \dots, u_k = v\}$ a shortest path from $T$ to $v$, w.l.o.g. $u_1 = t_1$. Then $u_2$ must be adjacent to $t_1$ and some other vertex of $T$, otherwise $\{u_2, t_1, t_2, t_3\}$ induces a paw. Without loss of generality, $u_2$ is adjacent to $t_2$. Let $u_i$ be the vertex from $P$ such that $u_i$ is not adjacent to $t_2$ with smallest index $i$. Clearly, $\{t_2, u_{i-2}, u_{i-1}, u_i\}$ induces a paw which is a contradiction. Since we take $v$ as an arbitrary vertex from $G$, then we showed that any vertex is adjacent to $T$ and the minimum dominating set has at most three vertices. □

Fig. 2 shows the final meta-graph, which means the complexity of any $\mathcal{F}$-free graphs, where $\mathcal{F}$ consists of graphs with at most 4 vertices, can be determined from this picture.

Let $W$ be the subset of corresponding vertices of meta-graph to each member of $\mathcal{F}$: (a) if the submeta-graph induced by $W$ is an independent set of circular-shaped vertices, then the domination problem restricted to $\mathcal{F}$-free graphs is in $NPC$;

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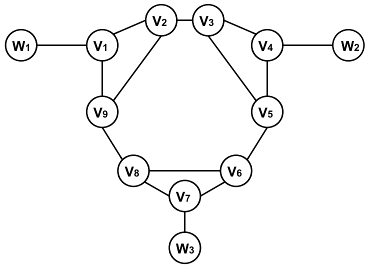


**Fig. 3.** Replace each claw-vertex in the graph with this new structure. $H_9$.

(b) otherwise, the problem is in $P$. By described results, Let $W$ be an induced subgraph which is an independent set with only circular-shaped vertices. We can extend $W$ to a maximal circular-shaped vertex-independent set $W'$. Clearly, $W'$ corresponds to $\mathcal{F}'$-free graphs which is a subclass of $\mathcal{F}$-free graphs. If we prove that the problem restricted to $\mathcal{F}'$-free graphs is in $NPC$ then the problem restricted to $\mathcal{F}$-free graphs is also in $NPC$. It is easy to see there are exactly 3 maximal circular-shaped vertex-independent sets: $W_1 = \{C_4, 2K_2\}$, $W_2 = \{C_4, K_3, K_4, \text{diamond, paw, co-claw}\}$ and $W_3 = \{C_4, K_4, \text{diamond, claw}\}$. By [1], the problem restricted to $(C_4, 2K_2)$-free graphs is in $NPC$. Theorem 4.7 proves the problem restricted to $(C_4, K_3, K_4$, diamond, paw, co-claw)-free graphs is in $NPC$ and Corollary 4.10 proves the problem restricted to $(C_4, K_4$, diamond, claw)-free graphs is in $NPC$.

**Lemma 4.6** ([17]). *If a graph $G'$ is obtained from a graph $G$ by triple subdivision of an edge, then $\gamma(G') = \gamma(G) + 1$.*

**Theorem 4.7.** *The domination problem restricted to $(K_3, C_4)$-free graphs is in NPC.*

**Proof.** It is trivial to check that after applying a triple subdivision to every edge of an arbitrary graph $G = (V, E)$, the result graph $G'$ is $(K_3, C_4)$-free. Applying Lemma 4.6, if we can solve in polynomial time the domination problem restricted to $(K_3, C_4)$-free graphs, then we can solve the problem on an arbitrary graph $(\gamma(G) = \gamma(G') + |E(G)|)$. Consequently domination problem restricted to $(K_3, C_4)$-free graphs is in $NPC$. $\square$

**Definition.** Say a vertex $v$ is a *claw-vertex* if $d(v) = 3$ and its neighbors form an independent set.

**Definition.** Let $G$ be an arbitrary graph that has a claw-vertex $v$ and its neighbors are $w_1$, $w_2$ and $w_3$. Say a *magnification* for $v$ is the replacement of $v$ by a cycle $C_9 = \{v_1, v_2, \ldots, v_9\}$ with three additional edges $v_2v_9$, $v_3v_5$ and $v_6v_8$, where $v_1$ connects to $w_1$, $v_4$ connects to $w_2$ and $v_7$ connects to $w_3$ (See Fig. 3). We call the $C_9$ with three additional arcs $H_9$.

**Lemma 4.8.** *Given a graph $G$ with a claw-vertex $v$. If $G'$ is the resulting graph after magnification of $v$, then $\gamma(G') = \gamma(G) + 2$.*

**Proof.** Let $D$ be a minimum dominating set of $G$, $|D| = \gamma(G)$. We construct a dominating set $D'$ of $G'$ such that $|D'| \leq |D| + 2$ for the following cases, then $\gamma(G') \leq |D'| \leq |D| + 2 = \gamma(G) + 2$.

1. $v \in D$: Thus $D' = (D \setminus v) \cup \{v_1, v_4, v_7\}$ is a dominating set from $G'$ and $|D'| = |D| + 2$.
2. $v \notin D$: Suppose w.l.o.g. $w_1 \in D$. Then $D' = D \cup \{v_3, v_8\}$ is a dominating set from $G'$ and $|D'| = |D| + 2$.

Next, we will show that $\gamma(G') \geq \gamma(G) + 2$. Let $D'$ a minimum dominating set of $G'$, $|D'| = \gamma(G')$. As $N_{G'}[v_3] \cap N_{G'}[v_8] = \emptyset$ then $|D' \cap H_9| \geq 2$. We analyze each possible value of $|\{w_1, w_2, w_3\} \cap D'|$. For each case, we construct a dominating set $D$ of $G$ such that $|D| \leq |D'| - 2$ which implies $\gamma(G') \geq \gamma(G) + 2$.

1. $|w_2, w_2, w_3 \cap D'| = 0$: $N[v_1], N[v_4], N[v_7]$ are disjoint sets. Each neighborhood must contain a vertex from $D' \cap H_9$. Hence, $H_9$ has at least 3 vertices from $D'$. We remove those vertices from $D'$ and add $v$. The result set $D$ is a dominating set from $G$.
2. $|\{w_2, w_2, w_3\} \cap D'| = 1$: Suppose w.l.o.g. $w_1 \in D'$ and $w_2, w_3 \notin D'$. In case $H_9$ contains at least three vertices from $D'$, then we repeat the reasoning of the previous item. Otherwise, there are exactly two vertices from $H_9$ belong to $D'$. Those should be $v_3$ and $v_8$, hence $w_2, w_3$ are dominated with vertices outside $H_9$. Therefore, $D$ can be obtained by removing $\{v_3, v_8\}$ from $D'$. Clearly, $D$ is a dominating set from $G$.
3. $|\{w_2, w_2, w_3\} \cap D'| = 2$: Suppose w.l.o.g. $\{w_1, w_2\} \subseteq D'$. If in $|H_9 \cap D'| \geq 3$, we can apply the same reasoning as in the first item and obtain the same result. Otherwise, there are exactly two vertices from $H_9$ belong to $D'$. If $v_7 \in D'$, then no vertex from $H_9$ can dominate the set $\{v_9, v_2, v_3, v_5\}$. Thus, $v_7 \notin D'$, and $w_3$ is dominated by a vertex outside $H_9$. Therefore, the set $D = D' \setminus H_9$ is a dominating set of $G$.
4. $|\{w_2, w_2, w_3\} \cap D'| = 3$: $\{w_1, w_2, w_3\} \in D'$. Thus $D = D' \setminus H_9$ is a dominating set of $G$. $\square$

**Theorem 4.9.** *The domination problem restricted to maximum degree 3 planar $(K_4, C_4, C_5, C_7, C_8, C_9, C_{10}, C_{11}$, diamond, claw)-free graphs is in NPC.*

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**Proof.** The domination problem restricted to planar graph $G$ of maximum degree 3 is in *NPC* [11]. Given any planar graph $G$ of maximum degree 3, we apply a triple subdivision for each edge of $G$ and obtain a graph $G'$. It is easy to see that $G'$ is a planar graph of maximum degree 3 and it's ($K_4$, $C_4$, $C_5$, $C_7$, $C_8$, $C_9$, $C_{10}$, $C_{11}$, diamond, claw)-free. Clearly, $G$ and $G'$ have the same number of claw-vertices. By Lemma 4.6, we know that $\gamma(G') = \gamma(G) + |E(G)|$. We can apply magnification for each claw-vertex of $G'$ in order to remove claws, obtaining a graph $G''$ which is maximum degree 3 planar ($K_4$, $C_4$, diamond, claw)-free graph. Applying Lemma 4.8 we know that $\gamma(G'') = \gamma(G') + 2|U'| = \gamma(G) + |E(G)| + 2|U|$, where $U'$ is the set of claw-vertices of $G'$ and $U$ is the set of claw-vertices of $G$. Therefore the problem remains in *NPC* restricted to maximum degree 3 planar ($K_4$, $C_4$, diamond, claw)-free graphs. □

**Corollary 4.10.** *The domination problem restricted to ($K_4$, $C_4$, claw, diamond)-free graph $G$ is in NPC.*

## 5. Conclusions

We close the gaps of missing information about the complexity of the minimum dominating set problem for graph classes given by forbidden induced subgraphs of order at most four. The study of induced subgraphs of size at most four had been already studied [4,15]. Many of these classes defines the first barrier where problems become *P* instead of *NPC*. On the other hand, it seems that the same technique can be applied to many of the classic problems in order to close the similar gaps.

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