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Linearizing well quasi-orders and bounding the length of bad sequences

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ABSTRACT

We study the length functions of controlled bad sequences over some well quasi-orders (wqo's) and classify them in the Fast Growing Hierarchy. We develop a new and self-contained study of the length of bad sequences over the disjoint product in $\mathbb{N}^n$ (Dickson's Lemma), which leads to recently discovered upper bounds but through a simpler argument. We also give a tight upper bound for the length of controlled decreasing sequences of multisets of $\mathbb{N}^n$ with the underlying lexicographic ordering, and use it to give an upper bound for the length of controlled bad sequences in the majoring ordering with the underlying disjoint product ordering. We apply this last result to attain complexity upper bounds for the emptiness problem of ITCA and ATRA automata.
For the case of the product and majoring wqo's the idea is to *linearize* bad sequences, i.e. to transform a *bad* sequence over a wqo into a *decreasing* one over a well-order, for which upper bounds can be more easily handled.

1. Well quasi-orders and bad sequences

A quasi-order is a binary relation $\leq$ over a given set A that is reflexive and transitive. A sequence $\mathbf{X} = x_0, x_1, x_2, \ldots$ of elements of A is called *good* if there are $i < j$ such that $x_i \leq x_j$. A sequence is *bad* if it is not good. A *well quasi-order* (wqo) is a quasi-order where all infinite sequences are good, or, equivalently, all bad sequences are finite.

The theory of well quasi-orderings was initially developed by Higman [1] (under the name of "finite basis property") and by Erdös and Rado in an unpublished manuscript, although some early evidence of the theory had already appeared in a work by Neumann [2]. Further developments were [3–5], and more recently [6]. Wqo's have become a key ingredient in a great number of results related with decidability, finiteness, and regularity results that appear in areas such as termination proofs for rewriting systems [7,8], their extensions [9,10], complexity upper bounds [11,12], well-structured transition systems [13–16], etc.

From the analysis of a termination proof of a given algorithm $\mathcal{S}$, whose correctness is grounded in the analysis of a certain wqo, one may extract a computational complexity upper bound for $\mathcal{S}$. Roughly, the idea is that any sequence of successive configurations of $\mathcal{S}$ (with a given input) is transformed into a bad sequence in the wqo. Thus, having an upper bound for the length of the bad sequence entails an upper bound for the number of steps that the algorithm needs to terminate.

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Consider the product ordering over $\mathbb{N}^n$, one of the wqo's analyzed in this work, defined as $\langle z_1,\dots,z_n\rangle \le_{\mathrm{pr}} \langle y_1,\dots,y_n\rangle$ iff $\forall i \in \{1,\dots,n\}$ $z_i \le y_i$. A sequence $x_0,\dots,x_k$ being bad in $(\mathbb{N}^n, \le_{\mathrm{pr}})$ means that for all $i<j$, there exists m such that the m-th component of x_j is strictly smaller than the m-th component of x_i. Dickson's Lemma [17] is the statement that $(\mathbb{N}^n, \le_{\mathrm{pr}})$ is a wqo, so all bad sequences of this wqo are finite. How long can a bad sequence starting with a given tuple be? For $\mathbb{N}^2$ and any $N \in \mathbb{N}$, the sequence

$$\mathbf{x} = \langle 0,1\rangle\ ,\ \langle N,0\rangle\ ,\ \langle N-1,0\rangle\ ,\ \langle N-2,0\rangle\ ,\ \dots\ ,\ \langle 1,0\rangle\ ,\ \langle 0,0\rangle \tag{1}$$

is $\le_{\mathrm{pr}}$-bad and has length greater than N. So in general there is no bound to the length of a bad sequence starting with a given element: bad sequences in a wqo are finite but could be *arbitrarily* large.

In practice, in the analysis of termination proofs, one has two additional assumptions of a wqo $(A, \le)$. First, one has some effective way of measuring the *size* of each element $x \in A$, notated $|x|_A$ or simply $|x|$.

Definition 1. (See [18].) A *norm function* $|\cdot|_A$ over a set A is a mapping $|\cdot|_A : A \to \mathbb{N}$ that provides every element of A with a nonnegative integer, its *norm*. The norm function is said to be *proper* if $\{x \in A \mid |x|_A < n\}$ is finite for every $n \in \mathbb{N}$. In this article, whenever we consider $A = \mathbb{N}^n$ then $|\cdot|_A$ will be defined as $|\cdot|_\infty$, the infinity norm.

Second, we may restrict ourselves to bad sequences $\mathbf{x} = x_0, x_1, x_2 \dots$ with a *controlled behavior*, which means that there is an effective way of computing, given i, an upper bound for $|x_i|$.

Definition 2. Let $g : \mathbb{N} \to \mathbb{N}$ be an increasing function and let $(A, \le)$ be a wqo with a proper norm. A sequence $\mathbf{x} = x_0, x_1, x_2 \dots$ is *g,t-controlled* if for all i, $|x_i|_A < g(t+i)$. We say that g is the *control function* for $\mathbf{x}$.

Controlled bad sequences exclude arbitrary jumps as the one going from 1 to N in the sequence (1). As a consequence of König's Lemma, controlled bad sequences over wqo's cannot be *arbitrarily* large: given a control, there exist upper bounds for their lengths.

1.1. Some examples of bad sequences

Let us go back to the example of the $\le_{\mathrm{pr}}$-bad sequence in (1). If we further impose that the sequence is $g,0$-controlled, where $g(x) = x+2$, and we fix $|x|_{\mathbb{N}^2}$ to be the infinity norm of x, then the *longest* $g,0$-controlled $\le_{\mathrm{pr}}$-bad sequence has length 8, as it is shown by the following sequence:

$$\underbrace{\langle 1,1\rangle}_{x_0}\ ,\ \underbrace{\langle 2,0\rangle}_{x_1}\ ,\ \underbrace{\langle 1,0\rangle}_{x_2}\ ,\ \underbrace{\langle 0,4\rangle}_{x_3}\ ,\ \underbrace{\langle 0,3\rangle}_{x_4}\ ,\ \underbrace{\langle 0,2\rangle}_{x_5}\ ,\ \underbrace{\langle 0,1\rangle}_{x_6}\ ,\ \underbrace{\langle 0,0\rangle}_{x_7}\ . \tag{2}$$

Indeed, observe that both coordinates of x_0 take the maximum value allowed by the control, that is, $g(0)-1=1$. Since (2) is bad, for all $i>0$ it must be $x_i \not\ge_{\mathrm{pr}} \langle 1,1\rangle$, and this means that one of the coordinates of x_i must be 0. Now x_1 has the second coordinate equal to 0 while the first one takes the maximum possible value allowed by the control function, i.e. $g(1)-1=2$. One can check that if we seek the *longest* $g,0$-controlled $\le_{\mathrm{pr}}$-bad sequence then x_2 (which must be $\not\ge_{\mathrm{pr}} \langle 1,1\rangle$ and also $\not\ge_{\mathrm{pr}} \langle 2,0\rangle$) should be $\langle 1,0\rangle$—and not, for instance $\langle 0,3\rangle$. The next pair, x_3 (which must be $\not\ge_{\mathrm{pr}} \langle 1,1\rangle$, $\not\ge_{\mathrm{pr}} \langle 2,0\rangle$ and $\not\ge_{\mathrm{pr}} \langle 1,0\rangle$), must now swap the zero-valued coordinate, and take value $g(3)-1=4$ for the other coordinate. From this point on, all the subsequent pairs $x_4, x_5, \dots$ preserve the zero-valued first coordinate while the second one decreases linearly down to 0. Of course, (2) is not the *unique* longest g,t-controlled $\le_{\mathrm{pr}}$-bad sequence.

The above example might suggest that it is always fairly easy to calculate the length of the longest g,t-controlled $\le_{\mathrm{pr}}$-bad sequence of $\mathbb{N}^n$. This is not the case: just consider the same problem for $\mathbb{N}^3$ and one will shortly be captured by cumbersome combinatorial calculations. Even more, getting upper bounds for the length of such sequences is also rather complex.

Let us move to our second example of wqo, also analyzed in this work: the product ordering for disjoint union of tuples. In general, the disjoint union of s sets $A_0,\dots,A_{s-1}$ can be formalized as $\bigcup_{0\le i<s}\{i\}\times A_i$. Following this definition, the disjoint union of s copies of $\mathbb{N}^n$ is the space $[s]\times\mathbb{N}^n$, where $[s]$ denotes $\{0,\dots,s-1\}$. The *disjoint product* over $[s]\times\mathbb{N}^n$ is defined as $\langle a, z_1,\dots,z_n\rangle \le^{\mathrm{d}}_{\mathrm{pr}} \langle b, y_1,\dots,y_n\rangle$ iff $a=b$ and $\langle z_1,\dots,z_n\rangle \le_{\mathrm{pr}} \langle y_1,\dots,y_n\rangle$. It is known that $\le^{\mathrm{d}}_{\mathrm{pr}}$ over $[s]\times\mathbb{N}^n$ is a wqo. A sequence $x_0,\dots,x_k$ of $[s]\times\mathbb{N}^n$ is $\le^{\mathrm{d}}_{\mathrm{pr}}$-bad if for all $i<j$ either the first component of x_i is different from the first component of x_j, or there is m such that the m-th component of x_j is strictly smaller than the m-th component of x_i. The norm we will use for $[s]\times\mathbb{N}^n$ is the infinity norm on $\mathbb{N}^{n+1}$. For instance, the reader can verify that

$$\langle 0,1\rangle\ ,\ \langle 0,0\rangle\ ,\ \langle 1,3\rangle\ ,\ \langle 1,2\rangle\ ,\ \langle 1,1\rangle\ ,\ \langle 1,0\rangle \tag{3}$$

is a $\le^{\mathrm{d}}_{\mathrm{pr}}$-bad and $g,0$-controlled sequence over $[2]\times\mathbb{N}$, for $g(x)=x+2$. Observe that (3) is not $\le_{\mathrm{pr}}$-bad over $\mathbb{N}^2$, as for instance $\langle 0,0\rangle$ is $\le_{\mathrm{pr}}$ than any other tuple. A simple analysis also shows that (3) has maximum length, so the length of the longest $g,0$-controlled sequence over $[2]\times\mathbb{N}$ is 6.

The disjoint product wqo $\le^{\mathrm{d}}_{\mathrm{pr}}$ over $[s]\times\mathbb{N}^n$ is a generalization of the product wqo $\le_{\mathrm{pr}}$ over $\mathbb{N}^n$: For $s=1$ we have $\langle 0, z_1,\dots,z_n\rangle \le^{\mathrm{d}}_{\mathrm{pr}} \langle 0, y_1,\dots,y_n\rangle$ iff $\langle z_1,\dots,z_n\rangle \le_{\mathrm{pr}} \langle y_1,\dots,y_n\rangle$.

i	j	possible values for x
0	1	$\langle 1,4\rangle$
0	2	$\langle 1,4\rangle$ or $\langle 4,1\rangle$
0	3	$\langle 4,1\rangle$

i	j	possible values for x
1	2	$\langle 3,2\rangle$ or $\langle 5,1\rangle$
1	3	$\langle 3,2\rangle$ or $\langle 5,1\rangle$
2	3	$\langle 2,1\rangle$

Fig. 1. The sequence $\mathbf{X}$ in (4) is $\leq_{\mathrm{maj}}^{(\leq_{\mathrm{pr}})}$-bad: for any $i<j$, the tuple $x \in X_i$ is $\not\leq_{\mathrm{pr}}$ any element of X_j.

As a final example, given a wqo $(A,\leq)$, consider the *majoring ordering* over finite sets of elements of A, defined as $X \leq_{\mathrm{maj}}^{(\leq)} Y$ iff every element of X is $\leq$-majored by some element of Y. The majoring ordering depends on an underlying $(A,\leq)$—this is the reason of the superscript in $\leq_{\mathrm{maj}}$. It is known that $\leq_{\mathrm{maj}}^{(\leq)}$ is a wqo, provided $(A,\leq)$ is also a wqo. In this article we study the length of controlled $\leq_{\mathrm{maj}}^{(\leq_{\mathrm{pr}})}$-bad sequences of finite sets of $\mathbb{N}^n$ and of controlled $\leq_{\mathrm{maj}}^{(\leq_{\mathrm{pr}}^{\mathrm{d}})}$-bad sequences of finite sets of $[s]\times\mathbb{N}^n$. A sequence $X_0,\dots,X_k$ of $\mathbb{N}^n$ is $\leq_{\mathrm{maj}}^{(\leq_{\mathrm{pr}})}$-bad if for all $i<j$, there is $x\in X_i$ such that for all $y\in X_j$, we have $x\not\leq_{\mathrm{pr}} y$. For example, the following sequence

$$\mathbf{X} \;=\; \underbrace{\{\langle 1,4\rangle,\langle 4,1\rangle\}}_{X_0}\;,\;\underbrace{\{\langle 5,1\rangle,\langle 3,2\rangle\}}_{X_1}\;,\;\underbrace{\{\langle 2,1\rangle\}}_{X_2}\;,\;\underbrace{\{\langle 1,5\rangle\}}_{X_3} \tag{4}$$

is $\leq_{\mathrm{maj}}^{\leq_{\mathrm{pr}}}$-bad, as it is explained in Fig. 1. To *control* a finite set of tuples means to control both the infinity norm of its tuples and also the cardinality of the set. Thus, the sequence (4) is g,t-controlled, for $g(x)=x+5$, though clearly it does not have maximum length. Devising the longest g,t-controlled $\leq_{\mathrm{maj}}^{(\leq_{\mathrm{pr}})}$-bad sequences over finite sets of $\mathbb{N}^2$ is far from simple.

In this paper we give upper bounds for the length of g,t-controlled bad sequences, when t is a parameter. That is, given a well (quasi-) order under study $(A,\leq)$, we define $L_g^A(t)$ as the length of the longest g,t-controlled bad sequence in $(A,\leq)$, and we study upper bounds for L_g^A, which are classified in the Fast Growing Hierarchy $(\mathfrak{F}_\alpha)_{\alpha<\epsilon_0}$ of Löb and Wainer [19].

1.2. Linearizing

Our technique to obtain an upper bound for L_g^A is to *linearize* the wqo $(A,\leq_A)$ with a proper norm $|\cdot|_A$ into a suitable well linear order $(B,\leq_B)$ with a proper norm $|\cdot|_B$. This means to find a function $h: A^+\to B$ such that for every $\mathbf{a}\in A^+$ and $a\in A$, if $\mathbf{a}^\frown a$ is a bad sequence in $(A,\leq_A)$ then $h(\mathbf{a}) >_B h(\mathbf{a}^\frown a)$. So if $\mathbf{a}=a_0,\dots,a_k$ is bad in $(A,\leq_A)$ then

$$\mathbf{b}=h(a_0),h(a_0,a_1),h(a_0,a_1,a_2),\dots,h(\mathbf{a})$$

is decreasing in $(B,\leq_B)$. Furthermore, for any control function g we seek a control function $\tilde{g}$ such that if $\mathbf{a}$ is g,t-controlled then $|h(\mathbf{a})|_B<\tilde{g}(|\mathbf{a}|+t-1)$—here $|\mathbf{a}|$ denotes the length of $\mathbf{a}$. Hence if $\mathbf{a}$ is g,t-controlled then $\mathbf{b}$ is $\tilde{g},t$-controlled and therefore from a g,t-controlled bad sequence in $(A,\leq_A)$ one can get a $\tilde{g},t$-decreasing sequence in $(B,\leq_B)$ *of the same length*. Hence $L_g^A\leq L_{\tilde{g}}^B$, and the task is now to find an upper bound for $L_{\tilde{g}}^B$. In practice, these upper bounds are easier to devise for well-orders than for wqo's.

1.3. Contributions

Lexicographic ordering—Section 3 Being a well-order, *bad* is synonym of *decreasing*. Let $L_{n,g}^{\mathrm{lex}}(t)$ and $L_{s,n,g}^{\mathrm{lex}}(t)$ denote the length of the longest g,t-controlled decreasing sequence over $\mathbb{N}^n$ and over $[s]\times\mathbb{N}^n$ respectively. In Theorem 1 we show that $L_{n,g}^{\mathrm{lex}}$ is tightly upper bounded by a function at the level $\mathfrak{F}_{\gamma+n-1}$ of the Fast Growing Hierarchy, whenever g is at the level $\mathfrak{F}_\gamma$. In Theorem 2 we extend this result for $L_{s,n,g}^{\mathrm{lex}}$, showing that, irrespective of s, it is still upper bounded by a function in $\mathfrak{F}_{\gamma+n-1}$.

Product and disjoint product ordering—Section 4 Let us denote $L_{n,g}^{\mathrm{pr}}(t)$ (resp. $L_{s,n,g}^{\mathrm{pr}}(t)$) the length of the longest g,t-controlled $\leq_{\mathrm{pr}}$-bad (resp. $\leq_{\mathrm{pr}}^{\mathrm{d}}$-bad) sequence over $\mathbb{N}^n$ (resp. over $[s]\times\mathbb{N}^n$). In Theorems 3 and 5 (and their respective Corollaries 4 and 6) we give a novel and elementary proof that both $L_{n,g}^{\mathrm{pr}}$ and $L_{s,n,g}^{\mathrm{pr}}$ have an upper bound in $\mathfrak{F}_{\gamma+n-1}$. This result has been obtained in [20] but the argument presented here is markedly simpler, as it only uses a linearization of $(\mathbb{N}^n,\leq_{\mathrm{pr}})$ and $([s]\times\mathbb{N}^n,\leq_{\mathrm{pr}}^{\mathrm{d}})$ into the lexicographic ordering over $\mathbb{N}^n$ and $[s]\times\mathbb{N}^n$ respectively.

Multiset ordering—Section 5 This well-order will consume a rather large amount of development and technicalities for showing the maximizing strategy first and devising its lower and upper bound next. Let $L_{n,g}^{\mathrm{ms}}(t)$ (resp. $L_{s,n,g}^{\mathrm{ms}}(t)$) denote the length of the longest g,t-controlled multiset-decreasing sequence of finite multisets of $\mathbb{N}^n$ (resp. of $[s]\times\mathbb{N}^n$) and the underlying lexicographic ordering over $\mathbb{N}^n$ (resp. over $[s]\times\mathbb{N}^n$). In Theorems 15 and 21 we show that $L_{s,n,g}^{\mathrm{ms}}$ has a tight upper bound in $\mathfrak{F}_{\omega^n\cdot s}$ whenever g is primitive recursive, which as a particular case implies a tight upper bound in $\mathfrak{F}_{\omega^n}$ for $L_{n,g}^{\mathrm{ms}}$.

Majoring ordering—Section 6 In Theorem 22 we give a general linearization of the majoring wqo into the multiset well-order. Then we specialize it in the majoring ordering over the product and the disjoint product orderings. Let us denote $L^{\mathrm{maj}}_{n,g}(t)$ (resp. $L^{\mathrm{maj}}_{s,n,g}(t)$) the length of the longest g,t-controlled $\leq^{(\leq_{\mathrm{pr}})}_{\mathrm{maj}}$-bad (resp. $\leq^{(\leq^{\mathrm{d}}_{\mathrm{pr}})}_{\mathrm{maj}}$-bad) sequence of finite sets of $\mathbb{N}^n$ (resp. of $[s]\times\mathbb{N}^n$). In Corollary 28 we show that if g is primitive recursive then $L^{\mathrm{maj}}_{s,n,g}$ is upper bounded by a function in $\mathfrak{F}_{\omega^n\cdot s}$ and hence, in particular, $L^{\mathrm{maj}}_{n,g}$ is upper bounded by a function in $\mathfrak{F}_{\omega^n}$.

Applications—Section 7 We finally give some applications on how our upper bound for $L^{\mathrm{maj}}_{s,n,g}$ can be used in a known decision procedure for the emptiness problem of two kind of automata over data trees: ITCA [21] and ATRA [22].

1.4. Related work

McAloon [23] shows an upper bound for $L^{\mathrm{pr}}_{n,g}$ when g is linear, and places it at the level $\mathfrak{F}_{n+1}$ of the Fast Growing Hierarchy. Later Clote [24] simplifies McAloon's argument and finds an upper bound in $\mathfrak{F}_{n+6}$. Neither of these proofs are self-contained and both are quite complex.

In [20] D. and S. Figueira, Schmitz and Schnoebelen show an improved upper bound of $\mathfrak{F}_n$ with a simpler proof, relying in a mathematically more general setting of disjoint unions of powers of $\mathbb{N}$. In fact, the main result of [20] is both more general (because it is stated for $L^{\mathrm{pr}}_{s,n,g}$ instead of $L^{\mathrm{pr}}_{n,g}$) and more precise (because it refines the upper bound in the Fast Growing Hierarchy) than those of McAloon and Clote: if $g\in\mathfrak{F}_\gamma$ then $L^{\mathrm{pr}}_{s,n,g}$ is bounded by a function in $\mathfrak{F}_{\gamma+n-1}$. Although this proof is markedly simpler than those of [23] and [24], there are still some technical lemmas regarding this richer setting of sum of powers. We arrive to the same results as in [20] through an argumentation which is shorter and still fully self-contained. It consists in linearizing the disjoint product wqo into the lexicographic well-ordering, based on a constructive proof of Dickson's Lemma given by Harwood, Moller and Setzer [25].

To the best of our knowledge there are no rigorous study of the length of the controlled bad sequences for the other orderings studied in this paper, namely the lexicographic, multiset and majoring ordering.

However there are some works that address Higman's Lemma (subword ordering). Cichoń and Tahhan Bittar [26] show a method for reducing bounds for tuples of words over a finite alphabet of p letters from bounds on the case for $p-1$ letters. Weiermann [27] gives an $\mathfrak{F}_{\omega^{p-1}}$-like bound through a more general approach and a more involved analysis. More recently, based on the techniques developed in [20], Schnoebelen and Schmitz [18] exhibit a new proof of the result for finite alphabets—which is even more general than Weiermann's—using an algebraic framework for handling normed wqo's. Finally in [28] these results are extended to infinite alphabets and an upper bound in $\mathfrak{F}_{\omega^{\omega^k}}$ is given for the length of the longest controlled bad sequence in $(\mathbb{N}^k)^*$ with the subword ordering.

2. Basic definitions

If A is a set then $|A|$ denotes the cardinality of A. If $x\in A^n$ then the i-th coordinate of x is denoted $x[i]$, so $x=\langle x[1],\dots,x[n]\rangle$. Sequences are always in boldface and if $\mathbf{x}$ is a finite sequence then $|\mathbf{x}|$ denotes its length. A^* denotes the set of all sequences (including the empty sequence, notated $\emptyset$) of elements of A and A^+ denotes the set of nonempty sequences of elements of A. The concatenation of the sequence $\mathbf{x}$ and the element x at the rightmost place is denoted $\mathbf{x}^\frown x$. We fix $g:\mathbb{N}\to\mathbb{N}$ to be an increasing function. For $s\in\mathbb{N}$, $[s]$ denotes the set $\{0,\dots,s-1\}$.

Given a set X provided with a total order $\leq$, $(X,\leq)$ is called a *well-order* if every non-empty subset of X has a minimum. Recall that a *quasi-order* is a binary relation $\leq$ over a given set A that is reflexive and transitive. A sequence $\mathbf{X}=x_0,x_1,x_2,\dots$ of elements of A is called *good* if there are $i<j$ such that $x_i\leq x_j$. A sequence is *bad* if it is not good. A *well quasi-order* (wqo) is a quasi-order where all infinite sequences are good, or, equivalently, all bad sequences are finite.

We work with the following wqo's:

Lexicographic ordering If $x,y\in\mathbb{N}^n$ then it is the well-order defined as

$$x<_{\mathrm{lex}} y \overset{\mathrm{def}}{\Leftrightarrow} x[1]<y[1]\vee(x[1]=y[1]\wedge\langle x[2],\dots,x[n]\rangle<_{\mathrm{lex}}\langle y[2],\dots,y[n]\rangle)\,.$$

The length of the longest g,t-controlled decreasing sequence in $(\mathbb{N}^n,\leq_{\mathrm{lex}})$ is denoted by $L^{\mathrm{lex}}_{n,g}(t)$. We will work with $\leq_{\mathrm{lex}}$ over $[s]\times\mathbb{N}^n$, seen as a subset of $\mathbb{N}^{n+1}$.

Product ordering If $x,y\in\mathbb{N}^n$ then it is the wqo defined as

$$x\leq_{\mathrm{pr}} y \overset{\mathrm{def}}{\Leftrightarrow} (\forall i\in\{1,\dots,n\})\; x[i]\leq y[i].$$

The length of the longest g,t-controlled bad sequence in $(\mathbb{N}^n,\leq_{\mathrm{pr}})$ is denoted by $L^{\mathrm{pr}}_{n,g}(t)$.

$(\mathbb{N}^n, \leq_{\text{lex}})$	lexicographic ordering over $\mathbb{N}^n$	$L^{\text{lex}}_{n,g}(t)$
$([s] \times \mathbb{N}^n, \leq_{\text{lex}})$	lexicographic ordering over $[s] \times \mathbb{N}^n$	$L^{\text{lex}}_{s,n,g}(t)$
$(\mathbb{N}^n, \leq_{\text{pr}})$	product ordering over $\mathbb{N}^n$	$L^{\text{pr}}_{n,g}(t)$
$([s] \times \mathbb{N}^n, \leq^{\text{d}}_{\text{pr}})$	disjoint product ordering over $[s] \times \mathbb{N}^n$	$L^{\text{pr}}_{s,n,g}(t)$
$(\mathcal{M}_{<\infty}(A), \leq^{(\leq_A)}_{\text{ms}})$	multiset ordering over finite multisets of elements of A with underlying $\leq_A$ over A	–
$(\mathcal{M}_{<\infty}(\mathbb{N}^n), \leq_{\text{ms}})$	multiset ordering over finite multisets $\mathbb{N}^n$ with underlying $\leq_{\text{lex}}$	$L^{\text{ms}}_{n,g}(t)$
$(\mathcal{M}_{<\infty}([s] \times \mathbb{N}^n), \leq_{\text{ms}})$	multiset ordering over finite multisets $[s] \times \mathbb{N}^n$ with underlying $\leq_{\text{lex}}$	$L^{\text{ms}}_{s,n,g}(t)$
$(\mathcal{P}_{<\infty}(A), \leq^{(\leq_A)}_{\text{maj}})$	majoring ordering over finite sets of elements of A with underlying $\leq_A$ over A	–
$(\mathcal{P}_{<\infty}(\mathbb{N}^n), \leq_{\text{maj}})$	majoring ordering over finite sets of $\mathbb{N}^n$ with underlying $\leq_{\text{pr}}$	$L^{\text{maj}}_{n,g}(t)$
$(\mathcal{P}_{<\infty}([s] \times \mathbb{N}^n), \leq^{\text{d}}_{\text{maj}})$	majoring ordering over finite sets of $[s] \times \mathbb{N}^n$ with underlying $\leq^{\text{d}}_{\text{pr}}$	$L^{\text{maj}}_{s,n,g}(t)$

Fig. 2. Summary of notation for the studied wqo's and the length of g, t-controlled bad sequences.

Disjoint product ordering If $x, y \in [s] \times \mathbb{N}^n$ then it is the wqo defined as

$$x \leq^{\text{d}}_{\text{pr}} y \overset{\text{def}}{\Leftrightarrow} x[1] = y[1] \wedge \langle x[2], \ldots, x[n+1] \rangle \leq_{\text{pr}} \langle y[2], \ldots, y[n+1] \rangle.$$

The length of the longest g, t-controlled bad sequence in $([s] \times \mathbb{N}^n, \leq^{\text{d}}_{\text{pr}})$ is denoted by $L^{\text{pr}}_{s,n,g}(t)$.

Multiset ordering A multiset M over a set X is a function $X \to \mathbb{N}$. Intuitively a multiset is a generalization of a set, where elements may be repeated. For $x \in X$, $M(x)$ is called the *multiplicity* of x. A multiset is finite if the set of elements with positive multiplicity is finite. We notate $x \in M$ for $M(x) > 0$. Let $\mathcal{M}_{<\infty}(X)$ denote the class of finite multisets over X.

Let $(X, \leq)$ be a poset and let $M, N \in \mathcal{M}_{<\infty}(X)$. We define

$$N <^{(\leq)}_{\text{ms}} M \overset{\text{def}}{\Leftrightarrow} M \neq N \wedge (\forall x \in X)[N(x) > M(x) \Rightarrow (\exists y \in X)[y > x \wedge M(y) > N(y)]].$$

Intuitively, this says that N can be obtained from M by replacing some elements by finitely many (possibly zero) smaller (with respect to $\leq$) elements. If $(X, \leq)$ is a well-order then $(\mathcal{M}_{<\infty}(X), \leq^{(\leq)}_{\text{ms}})$ is also a well-order. See [29] for more details.

We will study $(\mathcal{M}_{<\infty}(\mathbb{N}^n), \leq^{(\leq_{\text{lex}})}_{\text{ms}})$, the multiset ordering of finite multisets of tuples with the underlying lexicographic ordering. In this context, we simply write $\leq_{\text{ms}}$ for $\leq^{(\leq_{\text{lex}})}_{\text{ms}}$. Observe that it is a well-order because $(\mathbb{N}^n, \leq_{\text{lex}})$ is so.

The length of the longest g, t-controlled decreasing sequence in $(\mathcal{M}_{<\infty}(\mathbb{N}^n), \leq_{\text{ms}})$ is denoted by $L^{\text{ms}}_{n,g}(t)$ and the length of the longest g, t-controlled decreasing sequence in $(\mathcal{M}_{<\infty}([s] \times \mathbb{N}^n), \leq_{\text{ms}})$ is denoted by $L^{\text{ms}}_{s,n,g}(t)$.

Majoring ordering Let $\mathcal{P}_{<\infty}(X)$ denote the finite and non-empty parts of X. For a wqo $(X, \leq)$ and $A, B \in \mathcal{P}_{<\infty}(X)$, the *majoring ordering* is defined as

$$A \leq^{(\leq)}_{\text{maj}} B \overset{\text{def}}{\Leftrightarrow} (\forall x \in A)(\exists y \in B)\, x \leq y.$$

We will study $(\mathcal{P}_{<\infty}(\mathbb{N}^n), \leq^{(\leq_{\text{pr}})}_{\text{maj}})$, the majoring ordering of finite sets of tuples with the underlying product ordering. In this context, we write $\leq_{\text{maj}}$ for $\leq^{(\leq_{\text{pr}})}_{\text{maj}}$. We will also study $(\mathcal{P}_{<\infty}([s] \times \mathbb{N}^n), \leq^{(\leq^{\text{d}}_{\text{pr}})}_{\text{maj}})$, the majoring ordering of finite sets of $[s] \times \mathbb{N}^n$ with the underlying $\leq^{\text{d}}_{\text{pr}}$ wqo. In this case, we write $\leq^{\text{d}}_{\text{maj}}$ for $\leq^{(\leq^{\text{d}}_{\text{pr}})}_{\text{maj}}$. Observe that these two orderings are wqo's because $(\mathbb{N}^n, \leq_{\text{pr}})$ and $([s] \times \mathbb{N}^n, \leq^{\text{d}}_{\text{pr}})$ are so (see for instance [30, Prop. 2.15]).

The length of the longest g, t-controlled bad sequence in $(\mathcal{P}_{<\infty}(\mathbb{N}^n), \leq_{\text{maj}})$ is denoted by $L^{\text{maj}}_{n,g}(t)$ and the length of the longest g, t-controlled bad sequence in $(\mathcal{P}_{<\infty}([s] \times \mathbb{N}^n), \leq^{\text{d}}_{\text{maj}})$ is denoted by $L^{\text{maj}}_{s,n,g}(t)$.

Fig. 2 summarizes the notation for the wqo's considered in this article and the notation for the length of the longest controlled bad sequences over such wqo's (whenever it applies).

The fast growing hierarchy $(F_\alpha)_{\alpha<\epsilon_0}$ Let ϵ_0 be the least infinite ordinal α such that $\omega^\alpha = \alpha$. The Fast Growing Hierarchy is defined as

$$F_0(x) \overset{\text{def}}{=} x + 1 \qquad F_{\alpha+1}(x) \overset{\text{def}}{=} F^{x+1}_\alpha(x) \qquad F_\lambda(x) \overset{\text{def}}{=} F_{\lambda_x}(x),$$

where in general g^k denotes the k-th iteration of g (i.e. $g^1 = g$ and $g^{k+1} = g \circ g^k$), $\alpha < \epsilon_0$ is an ordinal, $\lambda < \epsilon_0$ is a limit ordinal and $(\lambda_x)_{x<\omega}$ is an increasing sequence of ordinals with limit λ (a *fundamental sequence*), which we fix to be:

$$(\gamma + \omega^{\beta+1})_x \overset{\text{def}}{=} \gamma + \omega^\beta \cdot (x+1) \qquad (\gamma + \omega^\lambda)_x \overset{\text{def}}{=} \gamma + \omega^{\lambda_x}.$$

Observe that, in particular, for $1 \le s, n < \omega$, we have

$$F_{\omega^n \cdot s}(t) = F_{\omega^n \cdot (s-1)+\omega^n}(t) \stackrel{\text{def}}{=} F_{\omega^n \cdot (s-1)+\omega^{n-1}\cdot(t+1)}(t).$$

The class $\mathfrak{F}_\alpha$ of the Fast Growing Hierarchy is the closure under substitution and limited recursion of the constant, sum, projections, and the functions F_α. $\mathfrak{F}_0 = \mathfrak{F}_1$ contains all linear functions, $\mathfrak{F}_2$ contains all the elementary functions, $\mathfrak{F}_3$ contains all the tetration functions. $\bigcup_{n<\omega} \mathfrak{F}_n$ is the class of all primitive recursive functions and in general $\bigcup_{\alpha<\omega^k} \mathfrak{F}_\alpha$ is the class of k-recursive functions [31]. There are a number of important *monotonicity* results regarding the Fast Growing Hierarchy: for ordinals $\alpha < \beta < \epsilon_0$, the function F_α is strictly increasing, $F_{\alpha+1} \ge F_\alpha$, F_α is eventually majorized by F_β, and then $\mathfrak{F}_\alpha \subsetneq \mathfrak{F}_\beta$ (except for $\alpha = 0$ and $\beta = 1$), etc. For more results on the Fast Growing Hierarchy, cf. [19].

3. Lexicographic ordering

In [20, Section VI], it is shown that

$$L^{\text{lex}}_{1,g}(t) = g(t), \quad L^{\text{lex}}_{n+1,g}(t) = \sum_{j=1}^{g(t)} L^{\text{lex}}_{n,g}\left(o^{j-1}_{n,g}(t)\right), \quad o_{n,g}(t) \stackrel{\text{def}}{=} t + L^{\text{lex}}_{n,g}(t). \tag{5}$$

Theorem 1 *(Tight upper bound for $L^{\text{lex}}_{n,g}$). For any ordinal $\gamma \ge 1$, if g has an upper bound in $\mathfrak{F}_\gamma$ then $L^{\text{lex}}_{n,g}$ has an upper bound in $\mathfrak{F}_{\gamma+n-1}$. This bound is tight.*

Proof. Observe that if $g \le G \in \mathfrak{F}_\gamma$, then $L^{\text{lex}}_{n,g} \le L^{\text{lex}}_{n,G}$. Thus, for the upper bound, we can assume without loss of generality that $g \in \mathfrak{F}_\gamma$; the general result will then follow for g upper bounded in $\mathfrak{F}_\gamma$. We proceed by induction on n. If $n = 1$ then $L^{\text{lex}}_{1,g}(t) = g(t)$, and by hypothesis $g \in \mathfrak{F}_\gamma$. Now suppose $L^{\text{lex}}_{n,g} \le h \in \mathfrak{F}_{\gamma+n-1}$. We have $L^{\text{lex}}_{n+1,g}(t) \le g(t) \cdot L^{\text{lex}}_{n,g}(o^{g(t)-1}_{n,g}(t)) \le g(t) \cdot o^{g(t)}_{n,g}(t)$, where the first inequality follows from (5), since $o_{n,g}$ is growing, and the second one because $L^{\text{lex}}_{n,g} \le o_{n,g}$.

Since $L^{\text{lex}}_{n,g} \le h \in \mathfrak{F}_{\gamma+n-1}$ then $o_{n,g}(t) \le h(t) + t$ and so $o_{n,g} \in \mathfrak{F}_{\gamma+n-1}$. In [19, Thm. 2.10] it is proved that if $f \in \mathfrak{F}_\alpha$, then there is a number p such that $\forall x, f(x) < F^p_\alpha(x)$. Consequently there is p such that $F^p_{\gamma+n-1}$ majorizes $o_{n,g}$. Therefore

$$\begin{aligned} L^{\text{lex}}_{n+1,g}(t) &< g(t) \cdot F^{p\cdot g(t)}_{\gamma+n-1}(t) \\ &\le g(t) \cdot F^{p\cdot g(t)+1}_{\gamma+n-1}(p \cdot g(t)) && \text{(mononpremonicity of } F_{\gamma+n-1}) \\ &= g(t) \cdot F_{\gamma+n}(p \cdot g(t)), \end{aligned}$$

which lies in $\mathfrak{F}_{\gamma+n}$, since it is the composition and product of functions in $\mathfrak{F}_{\gamma+n}$ (and since $\gamma + n \ge 2$, $\mathfrak{F}_{\gamma+n}$ is closed by products).

In [20, Prop. VI.3] it is shown that if $g = F_\gamma$ then $L^{\text{lex}}_{n,g} \ge F_{\gamma+n-1}$. Hence our upper bound is tight if $g \ge F_\gamma$, since $L^{\text{lex}}_{n,g} \ge L^{\text{lex}}_{n,F_\gamma} \ge F_{\gamma+n-1}$. $\square$

Theorem 2 *(Tight upper bound for $L^{\text{lex}}_{s,n,g}$). For any ordinal $\gamma \ge 1$, if g has an upper bound in $\mathfrak{F}_\gamma$ then $L^{\text{lex}}_{s,n,g}$ has an upper bound in $\mathfrak{F}_{\gamma+n-1}$. This bound is tight.*

Proof. As in the proof of Theorem 1, for the upper bound we can assume that $g \in \mathfrak{F}_\gamma$. It is clear that

$$L^{\text{lex}}_{s,n,g}(t) = \sum_{j=1}^{s} L^{\text{lex}}_{n,g}\left(o^{j-1}_{n,g}(t)\right), \tag{6}$$

and then

$$\begin{aligned} L^{\text{lex}}_{s,n,g}(t) &\le s \cdot L^{\text{lex}}_{n,g}(o^{s-1}_{n,g}(t)) && \text{(by (6), since } o_{n,g} \text{ is growing)} \\ &\le s \cdot o^s_{n,g}(t) && \text{(since } L^{\text{lex}}_{n,g} \le o_{n,g}). \end{aligned}$$

As we explained in the proof of Theorem 1, $o_{n,g} \in \mathfrak{F}_{\gamma+n-1}$. Since multiplication by a constant and bounded iteration do not make us change the level of the hierarchy, we conclude that $L^{\text{lex}}_{s,n,g} \in \mathfrak{F}_{\gamma+n-1}$.

Since $L^{\text{lex}}_{n,g} \le L^{\text{lex}}_{s,n,g}$ and by [20, Prop. VI.3] $L^{\text{lex}}_{n,g} \ge F_{\gamma+n-1}$ if $g \ge F_\gamma$, we conclude that $\mathfrak{F}_{\gamma+n-1}$ is a tight bound for $L^{\text{lex}}_{s,n,g}$. $\square$

4. Product and disjoint product ordering

In Section 4.1 we linearize the wqo $(\mathbb{N}^n, \leq_{\text{pr}})$ into the well-order $(\mathbb{N}^n, \leq_{\text{lex}})$ and derive an upper bound for $L^{\text{pr}}_{n,g}(t)$. In Section 4.2 we extend this result and linearize the wqo $([s] \times \mathbb{N}^n, \leq^{\text{d}}_{\text{pr}})$ into $([s] \times \mathbb{N}^n, \leq_{\text{lex}})$ to get an upper bound for $L^{\text{pr}}_{s,n,g}(t)$.

4.1. Product ordering

The next result follows the idea of Harwood, Moller and Setzer [25] adapted to *controlled* bad sequences. For the sake of completeness we include the full proof.

First, let us mention the intuition behind the proof. For $x \in \mathbb{N}^n$, define $\uparrow x \stackrel{\text{def}}{=} \{z \in \mathbb{N}^n \mid x \leq_{\text{pr}} z\}$. Let $n = 2$, and suppose

$$\mathbf{x} = \langle x_0, y_0\rangle, \langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \dots, \langle x_k, y_k\rangle$$

is a bad sequence in $(\mathbb{N}^2, \leq_{\text{pr}})$. Let $a(\mathbf{x}) = \min_{0\leq i<|\mathbf{x}|} x_i$, $b(\mathbf{x}) = \min_{0\leq i<|\mathbf{x}|} y_i$ and $C(\mathbf{x}) = \uparrow\langle a(\mathbf{x}), b(\mathbf{x})\rangle \setminus \bigcup_{0\leq i<|\mathbf{x}|} \uparrow\langle x_i, y_i\rangle$. It is easy to see that $C(\mathbf{x})$ is finite. Here is how we can linearize $(\mathbb{N}^2, \leq_{\text{pr}})$ into $(\mathbb{N}^2, \leq_{\text{lex}})$: Define $h(\mathbf{x}) \stackrel{\text{def}}{=} \langle a(\mathbf{x}) + b(\mathbf{x}), |C(\mathbf{x})|\rangle \in \mathbb{N}^2$ and suppose that $\mathbf{x}^\frown\langle x, y\rangle$ is bad. If $x < a(\mathbf{x}) \vee y < b(\mathbf{x})$ then $h(\mathbf{x}^\frown\langle x, y\rangle)[1] < h(\mathbf{x})[1]$; in case $x \geq a(\mathbf{x}) \wedge y \geq b(\mathbf{x})$ then $C(\mathbf{x}^\frown\langle x, y\rangle) \subseteq C(\mathbf{x})$. In this last case, since $\langle x, y\rangle \in C(\mathbf{x}) \setminus C(\mathbf{x}^\frown\langle x, y\rangle)$, we have $|C(\mathbf{x}^\frown\langle x, y\rangle)| < |C(\mathbf{x})|$. Therefore $h(\mathbf{x}^\frown\langle x, y\rangle) <_{\text{lex}} h(\mathbf{x})$. Furthermore, if $\mathbf{x}$ is g, t-controlled then $C(\mathbf{x})$ has at most $g(t + |\mathbf{x}| - 1)^2$ elements, and $a(\mathbf{x}) + b(\mathbf{x}) < 2g(t + |\mathbf{x}| - 1)$. Hence if $\mathbf{x}$ is g, t-controlled, then the sequence

$$\mathbf{y} = h(\langle x_0, y_0\rangle), h(\langle x_0, y_0\rangle, \langle x_1, y_1\rangle), \dots, h(\mathbf{x}),$$

is $<_{\text{lex}}$-decreasing and $\tilde{g}, t$-controlled, where $\tilde{g}(x) = 2g(x)^2$.

The argument cannot be generalized straightforwardly for any $n > 2$ to obtain a linearization into $(\mathbb{N}^n, \leq_{\text{lex}})$. For instance, for $n=3$ and $\mathbf{x}=\langle 0, 0, 1\rangle, \langle 0, 1, 0\rangle$, we would have $C(\mathbf{x}) = \uparrow\langle 0, 0, 0\rangle \setminus (\uparrow\langle 0, 0, 1\rangle \cup \uparrow\langle 0, 1, 0\rangle)$ and this set is infinite ($(N, 0, 0) \in C(\mathbf{x})$ for any N). However, $(\mathbb{N}^n, \leq_{\text{pr}})$ can be linearized into $(\mathbb{N}^n, \leq_{\text{lex}})$ by an inductive argument.

Theorem 3. *There is a function $h_n : (\mathbb{N}^n)^+ \to \mathbb{N}^n$ such that if $\mathbf{x}^\frown x$ is bad in $(\mathbb{N}^n, \leq_{\text{pr}})$ and $\mathbf{x}$ is nonempty, then $h_n(\mathbf{x}^\frown x) <_{\text{lex}} h_n(\mathbf{x})$. Furthermore if $\mathbf{x}$ is g, t-controlled then $|h_n(\mathbf{x})|_\infty < \tilde{g}(|\mathbf{x}| - 1 + t)$, for $\tilde{g}(x) = n!\, g(2nx)^n$. That is, $h_n(\mathbf{x})$ is $\tilde{g}, t$-controlled.*

Proof. We define the functions h_n by induction on n. If $\mathbf{x} = x_0, x_1, x_2, \dots, x_k$ is a bad sequence in $\mathbb{N}$ then define $h_1(x_0, x_1, x_2, \dots, x_k) \stackrel{\text{def}}{=} x_k$. Since in $\mathbb{N}$ the product order and the lexicographic order coincide, we have $h_1(\mathbf{x}^\frown x) <_{\text{lex}} h_1(\mathbf{x})$.

For the inductive construction of h_n, let $n > 1$ and assume the truth of the statement of the theorem for dimension $n - 1$. For $1 \leq i \leq n$ and $x \in \mathbb{N}^n$ we define

$$\textsc{del}_i(x) \stackrel{\text{def}}{=} \langle x[1], \dots, x[i-1], x[i+1], \dots, x[n]\rangle,$$

i.e. $\textsc{del}_i(x)$ deletes the i-th component of the n-tuple x. Given a finite and nonempty bad sequence $\mathbf{x} = x_0, x_1, \dots, x_k$ of n-tuples, we define the set

$$\begin{aligned}\textsc{bad}_i(\mathbf{x}) \stackrel{\text{def}}{=} \{\textsc{del}_i(x_{j_0}), \dots, \textsc{del}_i(x_{j_p}) \mid \quad & p \geq 0, 0 \leq j_0 < \dots < j_p \leq k, \text{ and} \\ & \textsc{del}_i(x_{j_0}), \dots, \textsc{del}_i(x_{j_p}) \text{ is bad}\},\end{aligned}$$

i.e. $\textsc{bad}_i(\mathbf{x})$ consists of the bad subsequences of $(n-1)$-tuples of $\mathbf{x}$ in which the i-th components of the n-tuples have been deleted. Finally we define

$$\begin{aligned}&\textsc{min}_i(\mathbf{x}) \stackrel{\text{def}}{=} \min_{<_{\text{lex}}}\{h_{n-1}(\mathbf{y}) \mid \mathbf{y} \in \textsc{bad}_i(\mathbf{x})\} \quad \text{and} \\ &\textsc{ext}_n(\mathbf{x}) \stackrel{\text{def}}{=} \{x \in \mathbb{N}^n \mid (\forall i \in \{1, \dots, n\})\ \textsc{min}_i(\mathbf{x}) = \textsc{min}_i(\mathbf{x}^\frown x), \text{ and} \\ &\qquad\qquad (\forall j \in \{0, \dots, k\})\ x_j \not\leq_{\text{pr}} x\},\end{aligned}$$

which consists of the n-tuples with which the sequence $\mathbf{x}$ can be extended without altering the $\textsc{min}_i$ values and yet while maintaining badness.

Fact 1. *$|\textsc{ext}_n(\mathbf{x})| < \infty$, and if $\mathbf{x}$ is g, t-controlled, then $|\textsc{ext}_n(\mathbf{x})| < g(k + t)^n$.*

Proof. Let $\mathbf{z} = \textsc{del}_i(x_{j_0}), \dots, \textsc{del}_i(x_{j_p}) \in (\mathbb{N}^{n-1})^+$ be a bad sequence, suppose $\textsc{min}_i(\mathbf{x}) = h_{n-1}(\mathbf{z})$, and suppose that $s \in \textsc{ext}_n(\mathbf{x})$. If the sequence $\mathbf{z}^\frown\textsc{del}_i(s)$ were bad, then by the ind. hyp. we would get that $\textsc{min}_i(\mathbf{x}^\frown s) \leq_{\text{lex}} h_{n-1}(\mathbf{z}^\frown\textsc{del}_i(s)) <_{\text{lex}} h_{n-1}(\mathbf{z}) = \textsc{min}_i(\mathbf{x})$, contradicting $s \in \textsc{ext}_n(\mathbf{x})$. Therefore, since $\mathbf{z}$ is bad but $\mathbf{z}^\frown\textsc{del}_i(s)$ is not, we have $\textsc{del}_i(x_{j_m}) \leq_{\text{pr}} \textsc{del}_i(s)$ for some m. But since $s \in \textsc{ext}_n(\mathbf{x})$ we have that $x_{j_m} \not\leq_{\text{pr}} s$, and therefore $s[i] < x_{j_m}[i]$. Now, since this goes for all i, we conclude that $|\textsc{ext}_n(\mathbf{x})|$ is finite.

Now if $\mathbf{x}$ is g,t-controlled, then $x_j[i] < g(k+t)$ for all j, because g is increasing. By the above argument $|\mathrm{EXT}_n(\mathbf{x})| \leq g(k+t)^n$, but since $\mathbf{x}$ was nonempty and $x_0 \notin \mathrm{EXT}_n(\mathbf{x})$, we conclude $|\mathrm{EXT}_n(\mathbf{x})| < g(k+t)^n$. □

We finally define

$$h_n(\mathbf{x}) \stackrel{\text{def}}{=} \left\langle \sum_{i=1}^{n} \mathrm{MIN}_i(\mathbf{x}), |\mathrm{EXT}_n(\mathbf{x})| \right\rangle \in \mathbb{N}^n,$$

where the sum is taken componentwise and thus results in a tuple in $\mathbb{N}^{n-1}$. We conclude the proof with the following two facts:

Fact 2. *If $\mathbf{x}^\frown x$ is bad then $h_n(\mathbf{x}^\frown x) <_{\text{lex}} h_n(\mathbf{x})$.*

Proof. Suppose that $\mathbf{y} = \mathbf{x}^\frown x$ bad. Since for any $i \in \{1, \ldots, n\}$, $\mathrm{BAD}_i(\mathbf{x}) \subseteq \mathrm{BAD}_i(\mathbf{y})$, then $\mathrm{MIN}_i(\mathbf{y}) \leq_{\text{lex}} \mathrm{MIN}_i(\mathbf{x})$; and if $\mathrm{MIN}_i(\mathbf{y}) = \mathrm{MIN}_i(\mathbf{x})$ for all i then $\mathrm{EXT}_n(\mathbf{y}) \subsetneq \mathrm{EXT}_n(\mathbf{x})$, since $\mathrm{EXT}_n(\mathbf{y}) \subseteq \mathrm{EXT}_n(\mathbf{x})$ but $x \in \mathrm{EXT}_n(\mathbf{x}) \setminus \mathrm{EXT}_n(\mathbf{y})$. Thus $|\mathrm{EXT}_n(\mathbf{y})| < |\mathrm{EXT}_n(\mathbf{x})|$. □

Fact 3. *If $\mathbf{x}$ is g,t-controlled then $|h_n(\mathbf{x})|_\infty < \tilde{g}(|\mathbf{x}| - 1 + t)$, where $\tilde{g}(x) = n!\; g(2nx)^n$.*

Proof. By induction on $n \geq 1$. If $n = 1$ then if $\mathbf{x} = x_0, \ldots, x_k$ is g,t-controlled, then $h_1(\mathbf{x}) = x_k < g(t+k) = g(t + |\mathbf{x}| - 1) \leq g(2(t + |\mathbf{x}| - 1)) = \tilde{g}(|\mathbf{x}| - 1 + t)$.

Since any $\mathbf{y} \in \mathrm{BAD}_i(\mathbf{x})$ is a $g,(t+k)$-controlled bad sequence of $\mathbb{N}^{n-1}$, by inductive hypothesis we get

$$\begin{aligned} |h_{n-1}(\mathbf{y})|_\infty &< (n-1)!\; g((n-1)(|\mathbf{y}| - 1 + t + k))^{n-1} \\ &\leq (n-1)!\; g((n-1)(k+t+k))^{n-1} \\ &\leq (n-1)!\; g(2n(k+t))^{n-1} \end{aligned}$$

In particular, for $\mathbf{y}$ such that $\mathrm{MIN}_i(\mathbf{x}) = h_{n-1}(\mathbf{y})$, we conclude $|\mathrm{MIN}_i(\mathbf{x})|_\infty < (n-1)!\; g(2n(k+t))^{n-1}$, and so the first $n-1$ coordinates of $h_n(\mathbf{x})$ are strictly bounded by $n!\; g(2n(k+t))^{n-1}$ (the factor n comes from the n additions). By Fact 1, the last coordinate of $h_n(\mathbf{x})$ is strictly bounded by $g(k+t)^n$. Therefore,

$$\begin{aligned} |h_n(\mathbf{x})|_\infty &< \max\{n!\; g(2n(k+t))^{n-1}, g(k+t)^n\} \\ &\leq n!\; g(2n(k+t))^n \\ &= n!\; g(2n(|\mathbf{x}| - 1 + t))^n \\ &= \tilde{g}(|\mathbf{x}| - 1 + t) \end{aligned}$$

and this concludes the proof of Fact 3. □

This concludes the proof of Theorem 3. □

Recall that $L^{\text{pr}}_{n,g}(t)$ denotes the length of the longest g,t-controlled bad sequence in $(\mathbb{N}^n, \leq_{\text{pr}})$, and $L^{\text{lex}}_{n,g}(t)$ denotes the length of the longest g,t-controlled decreasing sequence in $(\mathbb{N}^n, \leq_{\text{lex}})$.

Corollary 4 *(Tight upper bound for $L^{\text{pr}}_{n,g}$). $L^{\text{pr}}_{n,g} \leq L^{\text{lex}}_{n,\tilde{g}}$, for $\tilde{g}$ as in Theorem 3. Hence if g has an upper bound in $\mathfrak{F}_\gamma$, and $\gamma \geq 2$ is an ordinal, then $L^{\text{pr}}_{n,g}$ has an upper bound in $\mathfrak{F}_{\gamma+n-1}$. This bound is tight.*

Proof. Observe that for the upper bound of $L^{\text{pr}}_{n,g}$ we can assume without loss of generality that $g \in \mathfrak{F}_\gamma$. The function $\tilde{g}$ is defined through finite substitution from g and product. Since $\mathfrak{F}_2$ and higher levels are closed under finite products, we have $\tilde{g} \in \mathfrak{F}_\gamma$. By Theorem 1, there is a function $h \in \mathfrak{F}_{\gamma+n-1}$ such that $h \geq L^{\text{lex}}_{n,\tilde{g}}$. On the other hand, since $L^{\text{pr}}_{n,g} \geq L^{\text{lex}}_{n,g}$, by Theorem 1 the bound is tight. □

4.2. Disjoint product ordering

We extend Theorem 3 from $\leq_{\text{pr}}$ over $\mathbb{N}^n$ to $\leq^{\text{d}}_{\text{pr}}$ over $[s] \times \mathbb{N}^n$:

Theorem 5. *There is a function $h_{s,n} : ([s] \times \mathbb{N}^n)^* \to [s] \times \mathbb{N}^n$ such that if $\mathbf{x}^\frown x$ is bad in $([s] \times \mathbb{N}^n, \leq^{\text{d}}_{\text{pr}})$ then $h_{s,n}(\mathbf{x}^\frown x) <_{\text{lex}} h_{s,n}(\mathbf{x})$. Furthermore if $\mathbf{x}$ is g,t-controlled then $|h_{s,n}(\mathbf{x})|_\infty < \hat{g}(|\mathbf{x}| - 1 + t)$, for $\hat{g}(x) = s \cdot \tilde{g}(2(x+1))$ and $\tilde{g}$ as in Theorem 3.*

Proof. Given a (possibly empty) sequence $\mathbf{x}$ in $[s] \times \mathbb{N}^n$ and $i \in [s]$, we define $\pi_i(\mathbf{x})$ as the (possibly empty) sequence of $\mathbb{N}^n$ which results in selecting (in the order of $\mathbf{x}$) the last n components of each element of $\mathbf{x}$ whose first component is i. More formally, define $\pi_i : ([s] \times \mathbb{N}^n)^* \to (\mathbb{N}^n)^*$ as follows:

$$\pi_i(\emptyset) \stackrel{\text{def}}{=} \emptyset$$

$$\pi_i(x_1, x_2, \ldots, x_k) \stackrel{\text{def}}{=} \begin{cases} \langle x_1[2], \ldots, x_1[n+1]\rangle ^\frown \pi_i(x_2, \ldots, x_k) & \text{if } x_1[1] = i; \\ \pi_i(x_2, \ldots, x_k) & \text{otherwise.} \end{cases}$$

Clearly, if $\mathbf{x}$ is $\leq^{\mathrm{d}}_{\mathrm{pr}}$-bad and g, t-controlled then $\pi_i(\mathbf{x})$ is $\leq_{\mathrm{pr}}$-bad and $g, t + |\mathbf{x}|$-controlled. If $\mathbf{x} = x_1, \ldots, x_k$, let $c(\mathbf{x}) \stackrel{\text{def}}{=} \{x_j[1] \mid j \in \{1, \ldots, k\}\}$. Define

$$h_{s,n}(\mathbf{x}) \stackrel{\text{def}}{=} \left\langle s - |c(\mathbf{x})|, \sum_{i \in [s], \pi_i(\mathbf{x}) \neq \emptyset} h_n(\pi_i(\mathbf{x})) \right\rangle \in [s] \times \mathbb{N}^n,$$

where $h_n : (\mathbb{N}^n)^+ \to \mathbb{N}^n$ is the one from Theorem 3 and the sum is taken componentwise.

Fact 4. *If $\mathbf{x}^\frown x$ is $\leq^{\mathrm{d}}_{\mathrm{pr}}$-bad then $h_{s,n}(\mathbf{x}^\frown x) <_{\mathrm{lex}} h_{s,n}(\mathbf{x})$.*

Proof. If $x[1] \notin c(\mathbf{x})$ then $h_{s,n}(\mathbf{x}^\frown x)[1] < h_{s,n}(\mathbf{x})[1]$, so $h_{s,n}(\mathbf{x}^\frown x) <_{\mathrm{lex}} h_{s,n}(\mathbf{x})$. If $x[1] \in c(\mathbf{x})$ then $c(\mathbf{x}) = c(\mathbf{x}^\frown x)$ and

$$\sum_{\substack{i \in [s] \\ \pi_i(\mathbf{x}^\frown x) \neq \emptyset}} h_n(\pi_i(\mathbf{x}^\frown x)) = h_n(\pi_{x[1]}(\mathbf{x}^\frown x)) + \sum_{\substack{i \in [s] \\ \pi_i(\mathbf{x}^\frown x) \neq \emptyset \\ i \neq x[1]}} h_n(\pi_i(\mathbf{x}^\frown x)). \tag{7}$$

On the one hand, for $i \neq x[1]$ we have $\pi_i(\mathbf{x}^\frown x) = \pi_i(\mathbf{x})$. On the other hand, $\pi_{x[1]}(\mathbf{x}^\frown x)$ is $\leq_{\mathrm{pr}}$-bad, $\pi_{x[1]}(\mathbf{x}^\frown x) = \pi_{x[1]}(\mathbf{x})^\frown \langle x[2], \ldots, x[n+1]\rangle$, and hence by Theorem 3, $h_n(\pi_{x[1]}(\mathbf{x}^\frown x)) <_{\mathrm{lex}} h_n(\pi_{x[1]}(\mathbf{x}))$. So from (7) we obtain

$$\sum_{\substack{i \in [s] \\ \pi_i(\mathbf{x}^\frown x) \neq \emptyset}} h_n(\pi_i(\mathbf{x}^\frown x)) <_{\mathrm{lex}} h_n(\pi_{x[1]}(\mathbf{x})) + \sum_{\substack{i \in [s] \\ \pi_i(\mathbf{x}) \neq \emptyset \\ i \neq x[1]}} h_n(\pi_i(\mathbf{x})),$$

and we conclude that $h_{s,n}(\mathbf{x}^\frown x) <_{\mathrm{lex}} h_{s,n}(\mathbf{x})$. $\square$

Fact 5. *If $\mathbf{x}$ is g, t-controlled then $|h_{s,n}(\mathbf{x})|_\infty < \hat{g}(|\mathbf{x}| - 1 + t)$, where $\hat{g}(x) = s \cdot \tilde{g}(2(x + 1))$.*

Proof. It is clear that the first component of $h_{s,n}(\mathbf{x})$ is at most s. Since $\pi_i(\mathbf{x})$ is $g, t + |\mathbf{x}|$-controlled, by Theorem 3 $|h_n(\pi_i(\mathbf{x}))|_\infty < \tilde{g}(|\mathbf{x}| - 1 + t + |\mathbf{x}|) = \tilde{g}(2|\mathbf{x}| - 1 + t)$, for $\tilde{g}$ as in Theorem 3. By the definition of $h_{s,n}$ we have $|h_{s,n}(\mathbf{x})|_\infty < s \cdot \tilde{g}(2|\mathbf{x}| - 1 + t)$. Since g is an increasing function, $\tilde{g}(2|\mathbf{x}| - 1 + t) \leq \tilde{g}(2(|\mathbf{x}| + t))$, and then $|h_{s,n}(\mathbf{x})|_\infty < \hat{g}(|\mathbf{x}| - 1 + t)$ for $\hat{g}(x) = s \cdot \tilde{g}(2(x + 1))$. $\square$

This concludes the proof of Theorem 5. $\square$

We arrive to the same result as in [20]:

Corollary 6 *(Tight upper bound for $L^{\mathrm{pr}}_{s,n,g}$). $L^{\mathrm{pr}}_{s,n,g} \leq L^{\mathrm{lex}}_{s,n,\hat{g}}$, for $\hat{g}$ as in Theorem 5. Hence if g has an upper bound in $\mathfrak{F}_\gamma$, and $\gamma \geq 2$ is an ordinal, then $L^{\mathrm{pr}}_{s,n,g}$ has an upper bound in $\mathfrak{F}_{\gamma+n-1}$. This bound is tight.*

Proof. The upper bound is straightforward from Theorem 5 and Theorem 2. Also, since $L^{\mathrm{pr}}_{s,n,g} \geq L^{\mathrm{lex}}_{s,n,g}$, by Theorem 2 this bound is tight. $\square$

5. Multiset ordering

We need a notion of g, t*-controlled sequence of (multi)sets*. By Definition 2 it suffices to give a proper norm:

Definition 3 *(A proper norm of sets and multisets of tuples).* Given a set A with a proper norm $|\cdot|_A$, and given $X \in \mathcal{M}_{<\infty}(A)$, we define $|X|$, the norm of X, as the maximum between

$$\max_{x \in A} X(x) \qquad \text{and} \qquad \max\{|x|_A \mid x \in A \wedge X(x) > 0\}.$$

For $X \in \mathcal{P}_{<\infty}(A)$, $|X|$ is defined analogously, as any set is a multiset.

In this section we give a tight upper bound for $L^{\mathrm{ms}}_{s,n,g}$ (and consequently for $L^{\mathrm{ms}}_{n,g}$) in terms of the Fast Growing Hierarchy, for $g \leq h \in \mathfrak{F}_\alpha$ and $\alpha < \omega$.

5.1. Maximizing strategy

To study the longest g, t-controlled $\leq_{\mathrm{ms}}$-decreasing sequence of multisets we define the maximizing strategy which, given a nonempty g, t-controlled multiset M, determines the greatest $g, (t+1)$-controlled multiset N which is smaller than M. The strategy says that to obtain N one should take out one of the minimum elements of M, say m, (i.e. decrement in one the multiplicity of m) and add as many elements smaller than m as the control function permits.

For the rest of this subsection, assume $(X, \leq)$ is a well-order. We write $<_{\mathrm{ms}}$ instead of $<^{(\leq)}_{\mathrm{ms}}$. Let $M \in \mathcal{M}_{<\infty}(X)$ be g, t-controlled and let $|\cdot|_X = |\cdot|$ be a proper norm for X. We define the g, t*-predecessor* of M as follows: For $x \in X$,

$$\textsc{pred}^g_t(M)(x) \stackrel{\text{def}}{=} \begin{cases} g(t+1)-1 & \text{if } x < \min M \wedge |x| < g(t+1); \\ M(x)-1 & \text{if } x = \min M; \\ M(x) & \text{otherwise.} \end{cases}$$

where $\min M \stackrel{\text{def}}{=} \min\{x \mid M(x) > 0\}$.

Lemma 7. *Let M be a g, t-controlled nonempty finite multiset over a totally ordered set P, and let $N = \textsc{pred}^g_t(M)$. Then (1) N is $g, (t+1)$-controlled; (2) $N <_{\mathrm{ms}} M$; and (3) if N' is $g, (t+1)$-controlled and $N' <_{\mathrm{ms}} M$ then $N' \leq_{\mathrm{ms}} N$.*

Proof. (1) is clear from the definition of N and the fact that g is monotone increasing. For (2), it is obvious that $M \neq N$. By definition, if $N(x) > M(x)$ then $x < m = \min M$ and $M(m) > N(m)$.

For (3), assume $N' < M$ is $g, (t+1)$-controlled. We show that if $N'(x) > N(x)$ then there is $z > x$ such that $N(z) > N'(z)$. Suppose $N'(x) > N(x)$.

1. Suppose $x < \min M$. Then $N(x) = g(t+1) - 1 \geq N'(x)$, contradicting $N'(x) > N(x)$.
2. Suppose $x > \min M$. Then $N(x) = M(x)$ and therefore $N'(x) > M(x)$. Since $N' <_{\mathrm{ms}} M$ there is $z > x$ such that $N(z) = M(z) > N'(z)$.
3. Suppose $x = \min M$. Then $N(x) = M(x) - 1$, and so $N'(x) \geq M(x)$. If $N'(x) > M(x)$ then, since $M <_{\mathrm{ms}} N'$, there is $z > x$ with $M(z) > N'(z)$. For such z, by definition of N, we have $N(z) = M(z) > N'(z)$. If $N'(x) = M(x)$ then, since $N' \neq M$, there is y such that $N'(y) \neq M(y)$. Any such y must be different from x. Suppose that all such y's were smaller than $x = \min M$. In this case $M \leq_{\mathrm{ms}} N'$ and this contradicts the hypothesis. Hence there is $y > x$ such that $N'(y) \neq M(y)$. If $N'(y) > M(y)$, there is $z > y > x$ such that $N'(z) < M(z) = N(z)$. If $N'(y) < M(y)$, since $M(y) = N(y)$, we conclude $N'(y) < N(y)$.

This concludes the proof. □

We represent a finite multiset M such that $\{x \mid M(x) > 0\} = \{x_1, \dots, x_n\}$ as $M \stackrel{\text{def}}{=} M(x_1) \cdot x_1 + \cdots + M(x_n) \cdot x_n$. For $n \in \mathbb{N}$ and S a set, we denote with $n \cdot S$ the multiset M such that $M(x) = n$ if $x \in S$ and $M(x) = 0$ otherwise. For a finite multiset M, let $L_{g,M}(t)$ denote the length minus one of the longest g, t-controlled and $<_{\mathrm{ms}}$-decreasing sequence of multisets starting with the multiset M. For $x \in X$, let $o_{g,x}(t) = t + L_{g,1\cdot\{x\}}(t)$.

Lemma 8. *If $k \geq 1$ then $L_{g,k\cdot\{x\}}(t) = \sum_{i=0}^{k-1} L_{g,1\cdot\{x\}}(o^i_{g,x}(t))$.*

Proof. We write L_k for $L_{g,k\cdot\{x\}}$ and o for $o_{g,x}$. First we show that

Fact 6. $o^i(t) = t + \sum_{j=0}^{i-1} L_1(o^j(t))$.

Proof. By induction on $i \geq 0$. If $i = 0$ it is trivial. Now

$$\begin{aligned} o^{i+1}(t) &= o(o^i(t)) \\ &= o(t + \sum_{j=0}^{i-1} L_1(o^j(t))) && \text{(ind. hyp.)} \\ &= t + \sum_{j=0}^{i-1} L_1(o^j(t))) + L_1(o^i(t)) && \text{(def. of } o \text{ and ind. hyp.)} \\ &= t + \sum_{j=0}^{i} L_1(o^j(t)). \end{aligned}$$

This concludes the proof of Fact 6. □

Now we show the statement of the lemma by induction on $k \geq 1$: If $k = 1$ it is straightforward. For the inductive step, observe that the longest g, t-controlled decreasing sequence of multisets beginning with $M_1 = (k+1) \cdot \{x\}$ is

$$M_1 >_{\text{ms}} M_2 >_{\text{ms}} \dots, >_{\text{ms}} M_{l_1} >_{\text{ms}} N_2 >_{\text{ms}} N_3 >_{\text{ms}} \dots >_{\text{ms}} N_{l_2},$$

of length $l_1 + l_2 - 1$ and where $l_1 = L_k(t) + 1$, $M_{l_1} = 1 \cdot \{x\}$, $l_2 = L_1(t + L_k(t)) + 1$ and $N_{l_2} = \emptyset$. We have:

$$\begin{aligned}
L_{k+1}(t) &= l_1 + l_2 - 2 \\
&= L_k(t) + L_1(t + L_k) \\
&= \sum_{i=0}^{k-1} L_1(o^i(t)) + L_1(t + \sum_{i=0}^{k-1} L_1(o^i(t))) && \text{(ind. hyp.)} \\
&= \sum_{i=0}^{k-1} L_1(o^i(t)) + L_1(o^k(t)) + 1 && \text{(Fact 6)} \\
&= \sum_{i=0}^{k} L_1(o^i(t)),
\end{aligned}$$

and this concludes the proof of Lemma 8. □

The following are straightforward consequences of Lemma 8.

Corollary 9. *For $k \geq 1$, $t \geq 0$, $L_{g,k\cdot\{x\}}(t) \geq L^k_{g,1\cdot\{x\}}(t)$.*

Corollary 10. *For $k \geq 1$, $t \geq 0$, $L_{g,k\cdot\{x\}}(t) \leq k \cdot L_{g,1\cdot\{x\}}(o^{k-1}_{g,x}(t))$.*

5.2. Lower bound for multisets of $[s] \times \mathbb{N}^n$

In the sequel we will initially fix $(X, \leq)$ to be $(\mathbb{N}^n, \leq_{\text{lex}})$. At the end of the subsection we will derive a lower bound for the maximum length of controlled sequences of decreasing multisets over $([s] \times \mathbb{N}^n, \leq_{\text{lex}})$, which in the case $s = 1$ is equivalent to a lower bound for multisets over $(\mathbb{N}^n, \leq_{\text{lex}})$.

If $M \in \mathcal{M}_{<\infty}(\mathbb{N}^n)$ then let $P_{n,g}(M, t)$ denote the length minus one of the longest g, t-controlled $<_{\text{ms}}$-decreasing sequence of multisets starting with M. If M consists of one copy of $(x_1, \dots, x_n)$, we simply write $P_{n,g}(x_1, \dots, x_n, t)$ instead of $P_{n,g}(1 \cdot \{(x_1, \dots, x_n)\}, t)$.

Observe that, having fixed $(X, \leq)$ as $(\mathbb{N}^n, \leq_{\text{lex}})$, we have $L_{g,M}(t) = P_{n,g}(M, t)$.

Define $G_{n,g} : \mathbb{N}^{n+1} \setminus \{(0, \dots, 0)\} \to \mathbb{N}$ by multiple recursion as:

$$G_{n,g}(0, \dots, 0, 1, t) \stackrel{\text{def}}{=} g(t+1) \tag{8}$$

$$G_{n,g}(\bar{x}, x_0 + 1, t) \stackrel{\text{def}}{=} G^{g(t+1)-1}_{n,g}(\bar{x}, x_0, t) \tag{9}$$

$$G_{n,g}(\bar{x}, x_j + 1, \bar{0}, t) \stackrel{\text{def}}{=} G_{n,g}(\bar{x}, x_j, g(t+1) - 1, \bar{0}, t). \tag{10}$$

In Eqn. (9), we let $\bar{x} = x_{n-1}, \dots, x_1$. In Eqn. (10) we let $\bar{x} = x_{n-1}, \dots, x_{j+1}$ and this equation only applies when $j > 0$. $G^k_{n,g}(\bar{a}, t)$ denotes the k-th iteration of $G_{n,g}$ in the last component, i.e. $G^1_{n,g}(\bar{a}, t) = G_{n,g}(\bar{a}, t)$ and $G^{k+1}_{n,g}(\bar{a}, t) = G_{n,g}(\bar{a}, G^k_{n,g}(\bar{a}, t))$.

Lemma 11. *If $g(t) \geq t + 1$ then $P_{n,g} \geq G_{n,g}$.*

Proof. By induction on the lexicographic order of $(x_{n-1}, \dots, x_0)$:

If $(x_1, \dots, x_{n-1}, x_n) = (0, \dots, 0, 1)$ then for (8) the longest g, t-controlled $<_{\text{ms}}$-decreasing sequence starting with $1 \cdot \{(\bar{0}, 1)\}$ is

$$1 \cdot \{(\bar{0}, 1)\} >_{\text{ms}} (g(t+1) - 1) \cdot \{(\bar{0}, 0)\} >_{\text{ms}} \dots >_{\text{ms}} 0 \cdot \{(\bar{0}, 0)\} = \emptyset,$$

which has length $g(t+1) + 1$ and then $P_{n,g}(0, \dots, 0, 1, t) = g(t+1) = G_{n,g}(0, \dots, 1, t)$.

For (9), the longest g, t-controlled $<_{\text{ms}}$-decreasing sequence of multisets starting with $1 \cdot \{(\bar{x}, x_0 + 1)\}$ contains the multiset $M = (g(t+1) - 1) \cdot \{(\bar{x}, x_0)\}$, so $P_{n,g}(\bar{x}, x_0 + 1, t) \geq P_{n,g}(M, t + 1)$. Therefore

$$
\begin{aligned}
P_{n,g}(\bar{x}, x_0+1, t) &\geq P_{n,g}((g(t+1)-1)\cdot\{(\bar{x}, x_0)\}, t+1) \\
&\geq P_{n,g}^{g(t+1)-1}(\bar{x}, x_0, t+1) && \text{(Corollary 9)} \\
&\geq G_{n,g}^{g(t+1)-1}(\bar{x}, x_0, t) && \text{(ind. hyp. and monot. of } G_{n,g}\text{)} \\
&= G_{n,g}(\bar{x}, x_0+1, t).
\end{aligned}
$$

Finally, for (10), the longest g, t-controlled $<_{\text{ms}}$-decreasing sequence of multisets starting with $1\cdot\{(\bar{x}, x_j+1, \bar{0})\}$ contains $1\cdot\{(x_1, \ldots, x_j, g(t+1)-1, \bar{0})\}$ as one of its terms, so

$$
\begin{aligned}
P_{n,g}(\bar{x}, x_j+1, \bar{0}, t) &\geq P_{n,g}(\bar{x}, x_j, g(t+1)-1, \bar{0}, t) \\
&\geq G_{n,g}(x_1, \ldots, x_j, g(t+1)-1, 0, \ldots, 0, t) && \text{(ind. hyp.)} \\
&= G_{n,g}(x_1, \ldots, x_j+1, 0, \ldots, 0, t).
\end{aligned}
$$

This concludes the proof. □

Now we prove some results regarding the Fast Growing Hierarchy that will be needed later on.

Lemma 12. *Let $c < \omega$, and $1 \leq n < \omega$. For every $t \geq c-1$ and every ordinal γ of the form $\sum_{i=n}^{m} \omega^i \cdot a_i$ (where $a_i < \omega$), we have $F_{\gamma+\omega^n}(t) \geq F_{\gamma+c}(t)$.*

Proof. By induction on n. □

Corollary 13. *For every $0 \leq n < \omega$, every $a \leq b < \omega$, and every ordinal γ of the form $\sum_{i=n}^{m} \omega^i \cdot a_i$ (where $a_i < \omega$), we have*

1. $F_{\gamma+\omega^n\cdot a+1} \leq F_{\gamma+\omega^n\cdot(a+1)}$
2. $F_{\gamma+\omega^n\cdot a} \leq F_{\gamma+\omega^n\cdot b}$

Proof. Both cases are trivial for $n=0$, and both are immediate consequences of Lemma 12 for $n>0$. □

Lemma 14. *If g is such that $g(t) \geq t+2$ for all t, then*

$$G_{n,g}(x_{n-1}, \ldots, x_0, t) \geq F_\alpha(t),$$

where $\alpha = \omega^{n-1}\cdot x_{n-1} + \cdots + \omega^0 \cdot x_0$ if $x_i > 0$ for some $i > 0$, and $\alpha = x_0 - 1$ if $x_0 > 0$ and $x_i = 0$ for all $i > 0$.

Proof. We proceed by induction on $(x_{n-1}, \ldots, x_0)$.

First we have that $G_{n,g}(0, \ldots, 0, 1, t) = g(t+1) \geq t+2 \geq F_0(t)$.

Second, for any $\bar{x} = x_{n-1}, \ldots, x_1$,

$$
\begin{aligned}
G_{n,g}(\bar{x}, x_0+1, t) &= G_{n,g}^{g(t+1)-1}(\bar{x}, x_0, t) \\
&\geq G_{n,g}^{t+1}(\bar{x}, x_0, t) && (g(t) \geq t+1) \\
&= F_\alpha^{t+1}(t) = F_{\alpha+1}(t).
\end{aligned}
$$

Third, we consider two possibilities. First:

$$
\begin{aligned}
G_{n,g}(\bar{0}, 1, 0, t) &= G_{n,g}(\bar{0}, 0, g(t+1)-1, t) \\
&\geq F_{g(t+1)-2}(t) && \text{(ind. hyp.)} \\
&\geq F_{t+1}(t) && (g(t) \geq t+2) \\
&= F_\omega(t),
\end{aligned}
$$

and lastly, if $\bar{x} = x_{n-1}, \ldots, x_{j+1}$, $\beta = \omega^{n-1}\cdot x_{n-1} + \cdots + \omega^{j+1}\cdot x_{j+1}$ and $(\bar{x}, x_j+1, 0, \ldots, 0) >_{\text{lex}} (0, \ldots, 1, 0)$:

$$
\begin{aligned}
G_{n,g}(\bar{x}, x_j+1, \bar{0}, t) &= G_{n,g}(\bar{x}, x_j, g(t+1)-1, \bar{0}, t) \\
&\geq F_{\beta+\omega^j\cdot x_j+\omega^{j-1}\cdot(g(t+1)-1)}(t) && \text{(ind. hyp.)} \\
&\geq F_{\beta+\omega^j\cdot x_j+\omega^{j-1}\cdot(t+1)}(t) && \text{(Corollary 13.2 and } g(t) \geq t+1\text{)} \\
&= F_{\beta+\omega^j\cdot(x_j+1)}(t).
\end{aligned}
$$

This concludes the proof. □

We now give a lower bound for $L^{\mathrm{ms}}_{s,n,g}$ in terms of the Fast Growing Hierarchy. Observe that since $L^{\mathrm{ms}}_{1,n,g} = L^{\mathrm{ms}}_{n,g}$ we also get a lower bound for multisets of $\mathbb{N}^n$.

Theorem 15 *(Lower bound for $L^{\mathrm{ms}}_{s,n,g}$). Let g be a function such that $g(t) \geq t+2$ for all t and $s \leq g(0)$, then for all t, $L^{\mathrm{ms}}_{s,n,g}(t) \geq F_{\omega^n \cdot s}(t-1)$ (this last function belongs to $\mathfrak{F}_{\omega^n \cdot s} \setminus \bigcup_{\alpha<\omega^n \cdot s} \mathfrak{F}_\alpha$).*

Proof. Except when we have both $s=1$ and $n=1$, we proceed as follows:

$$
\begin{aligned}
L^{\mathrm{ms}}_{s,n,g}(t) &\geq P_{n+1,g}(s-1, t+1, \bar{0}, t) \\
&\geq G_{n+1,g}(s-1, t+1, \bar{0}, t) && \text{(Lemma 11)} \\
&\geq F_{\omega^n \cdot (s-1) + \omega^{n-1} \cdot (t+1)}(t) && \text{(Lemma 14)} \\
&= F_{\omega^n \cdot s}(t).
\end{aligned}
$$

If $s=n=1$, we proceed similarly, but in the third inequality Lemma 14 yields $\geq F_t(t) \geq F_t(t-1) = F_\omega(t-1)$. □

5.3. Upper bound for multisets of $[s] \times \mathbb{N}^n$

Define $U_{n,g} : \mathbb{N}^{n+1} \setminus \{(0,\dots,0)\} \to \mathbb{N}$ by multiple recursion as:

$$U_{n,g}(0,\dots,0,1,t) \stackrel{\text{def}}{=} g(t+1) \tag{11}$$

$$U_{n,g}(\bar{x}, x_0+1, t) \stackrel{\text{def}}{=} g(t+1) \cdot U_{n,g}(\bar{x}, x_0, o^{g(t+1)-1}_{x_{n-1},\dots,x_0}(t+2)) \tag{12}$$

$$U_{n,g}(\bar{x}, x_j+1, \bar{0}, t) \stackrel{\text{def}}{=} U_{n,g}(\bar{x}, x_j, g(t+1), \bar{0}, t+2), \tag{13}$$

where $o_{x_{n-1},\dots,x_0}(t) = t + U_{n,g}(x_{n-1},\dots,x_1,x_0,t)$; Eqn. (13) applies when $j>0$ and $\bar{x}=x_{n-1},\dots,x_{j+1}$.

Lemma 16. $P_{n,g} \leq U_{n,g}$.

Proof. By induction on the lexicographic order of $(x_{n-1},\dots,x_0)$. For (11), as in the proof of Lemma 11, the longest g,t-controlled $<_{\mathrm{ms}}$-decreasing sequence starting with $1 \cdot \{(\bar{0},1)\}$ has length $g(t+1)+1$ and then $P_{n,g}(\bar{0},1,t) = g(t+1) = U_{n,g}(\bar{0},1,t)$. For (12) the longest g,t-controlled $<_{\mathrm{ms}}$-decreasing sequence starting with $M_0 = 1 \cdot \{(\bar{x}, x_0+1)\}$ continues with a multiset M_1 whose $<_{\mathrm{lex}}$-maximum element is $(\bar{x}, x_0)$, of multiplicity $g(t+1)-1$. Therefore if $N = g(t+1) \cdot \{(\bar{x}, x_0)\}$ then $M_0 >_{\mathrm{ms}} N >_{\mathrm{ms}} M_1$ and N is $g,(t+2)$-controlled since g is increasing. Hence

$$
\begin{aligned}
P_{n,g}(\bar{x}, x_0+1, t) &\leq P_{n,g}(g(t+1) \cdot \{(\bar{x}, x_0)\}, t+2) \\
&\leq g(t+1) \cdot P_{n,g}(\bar{x}, x_0, \tilde{o}^{g(t+1)-1}_{x_1,\dots,x_0}(t+2)) \\
&\leq g(t+1) \cdot U_{n,g}(\bar{x}, x_0, o^{g(t+1)-1}_{x_{n-1},\dots,x_0}(t+2)) \\
&= U_{n,g}(\bar{x}, x_0+1, t)
\end{aligned}
$$

where $\tilde{o}_{x_{n-1},\dots,x_0}(t) = t + P_{n,g}(x_{n-1},\dots,x_0,t)$, the second inequality follows from Corollary 10, and the third one from ind. hyp. and monotonicity of $U_{n,g}$. For (13) the longest g,t-controlled $<_{\mathrm{ms}}$-decreasing sequence of multisets starting with $M'_0 = 1 \cdot \{(\bar{x}, x_j+1, \bar{0})\}$ continues with a multiset M'_1 whose $<_{\mathrm{lex}}$-maximum element is $(\bar{x}, x_j, g(t+1)-1, \dots, g(t+1)-1)$, of multiplicity $g(t+1)-1$. Then $M'_0 >_{\mathrm{ms}} N' >_{\mathrm{ms}} M'_1$, where $N' = 1 \cdot \{(\bar{x}, x_j, g(t+1), \bar{0})\}$, and hence N' is $g,(t+2)$-controlled. Therefore by inductive hypothesis we have

$$
\begin{aligned}
P_{n,g}(\bar{x}, x_j+1, \bar{0}, t) &\leq P_{n,g}(\bar{x}, x_j, g(t+1), \bar{0}, t+2) \\
&\leq U_{n,g}(\bar{x}, x_j, g(t+1), \bar{0}, t+2) \\
&= U_{n,g}(\bar{x}, x_j+1, \bar{0}, t),
\end{aligned}
$$

and this concludes the proof. □

Proposition 17. *Let $\gamma < \omega^\omega$ and let g and k be such that $t \geq k$, $g(t+1) \leq F_\gamma(t)$ and $g(t) \geq t+1$. Then for all $t \geq \max\{2,k\}$, all $\bar{x} = x_{n-1},\dots,x_1$ and all x_0: if $U_{n,g}(\bar{x}, x_0, t) \leq F_\gamma(t)$ then $U_{n,g}(\bar{x}, x_0+c, t) \leq F_{\gamma+3c}(t)$. In particular, for all $t \geq \max\{2,k\}$, $U_{n,g}(\bar{0}, x_0, t) \leq F_{\gamma+3(x_0-1)}(t)$, since $U_{n,g}(\bar{0},1,t) = g(t+1)$.*

Proof. We proceed by induction on c. For $c = 1$, let $t \geq \max\{2, k\}$:

$$
\begin{aligned}
U_{n,g}(x_{n-1}, \ldots, x_0 + 1, t) &= g(t+1) \cdot U_{n,g}(\bar{x}, x_0, o^{g(t+1)-1}_{x_{n-1},\ldots,x_0}(t+2)) \\
&\leq g(t+1) \cdot F_\gamma(o^{g(t+1)-1}_{x_{n-1},\ldots,x_0}(t+2)) && \text{(hyp.)} \\
&\leq g(t+1) \cdot F_\gamma(F^{g(t+1)-1}_{\gamma+1}(t+2)) \\
&\leq g(t+1) \cdot (F^{g(t+1)+1}_{\gamma+1}(t+2) \\
&\leq g(t+1) \cdot (F^{g(t+1)+1}_{\gamma+1}(g(t+1)) && (g(t) \geq t+1) \\
&= g(t+1) \cdot (F_{\gamma+2}(g(t+1)) \\
&\leq F_\gamma(t) \cdot (F_{\gamma+2}(F_{\gamma+2}(t)) && (g(t+1) \leq F_\gamma(t)) \\
&\leq F_{\gamma+2}(t) \cdot (F_{\gamma+2}(F_{\gamma+2}(t)) \\
&\leq F_{\gamma+2}(F_{\gamma+2}(F_{\gamma+2}(t)) \\
&= F^3_{\gamma+2}(t) \\
&\leq F_{\gamma+3}(t) && (t \geq 2).
\end{aligned}
$$

Where for the inequality in the third line we are using that $o_{x_{n-1},\ldots,x_0}(t) = t + U_{n,g}(x_{n-1}, \ldots, x_0, t) \leq t + F_\gamma(t) \leq F_{\gamma+1}(t)$.

Now, for the inductive step, we prove the result for $c+1$ assuming it is true for $1, \ldots, c$. For $t \geq \max\{2, k\}$, by ind. hyp. $U_{n,g}(\bar{x}, x_0 + c, t) \leq F_{\gamma+3c}(t)$. Therefore, if $\tilde{x_0} = x_0 + c$ and using the ind. hyp. we have $U_{n,g}(\bar{x}, x_0 + c + 1, t) = U_{n,g}(\bar{x}, \tilde{x_0} + 1, t) \leq F_{\gamma+3c+3}(t)$. □

It is known that if $\alpha \leq \beta$ then there is t_0 such that for all $t \geq t_0$ we have $F_\alpha(t) \leq F_\beta(t)$. However, in principle, t_0 may depend on β in a non-uniform way. The following lemma makes a uniform statement that will be needed later.

Lemma 18. *Let $\alpha \in \mathbb{N}$. Then for all $t \geq \alpha - 1$ and for all $\beta \geq \alpha$ we have $F_\alpha(t) \leq F_\beta(t)$.*

Proof. We proceed by transfinite induction on β. Let $t \geq \max\{\alpha - 1\}$. For $\beta = \alpha$ the result is trivial. Next, $F_{\beta+1}(t) \geq F_\beta(t)$ and $F_\beta(t) \geq F_\alpha(t)$ by inductive hypothesis. Finally, for β a limit ordinal we have $F_\beta(t) = F_{\beta_t}(t) \geq F_\alpha(t)$, as we can use ind. hyp. because $t \geq \alpha - 1$ and $\beta > \beta_t \geq \alpha$ □

Corollary 19. *Let g be upper bounded in $\mathfrak{F}_\alpha$, for $\alpha < \omega$. Then there exists k such that for all $\beta \geq \omega$ and all $t \geq k$ we have $F_\beta(g(t)) \leq F_{\beta+1}(t)$.*

Proof. Since g is upper bounded in $\mathfrak{F}_\alpha$, there exist a k' such that if $t \geq k'$ then $g(t) \leq F_{\alpha+1}(t)$. Therefore

$$
\begin{aligned}
F_\beta(g(t)) &\leq F_\beta(F_\beta(t)) && (\text{for } t \geq \max\{\alpha, k'\} \text{ by Lemma 18}) \\
&= F^2_\beta(t) \\
&\leq F_{\beta+1}(t). && (\text{for } t \geq 1)
\end{aligned}
$$

To conclude, take $k = \max\{1, \alpha, k'\}$. □

Lemma 20. *If g has a primitive recursive upper bound and $\forall^\infty t$ (that is, for all t save for a finite number), $g(t) \geq t + 1$, then there is k such that for all $t \geq k$ and all $\bar{x} = x_{n-1}, \ldots, x_1 \neq \bar{0}$ we have $U_{n,g}(\bar{x}, 0, t) \leq F_\alpha(t)$, where $\alpha = \omega^{n-1} \cdot x_{n-1} + \omega^{n-2} \cdot x_{n-2} + \cdots + \omega^2 \cdot x_2 + \omega \cdot x_1 + 1$.*

Proof. First, let $e < \omega$ be an ordinal such that $\forall^\infty t$, $g(t+1) \leq F_e(t)$. Now, let $k'_0 \geq 2$ be a constant such that for all $t \geq k'_0$ we have $g(t+1) \leq F_e(t)$ and $g(t) \geq t + 1$. Also, let k''_0 be the constant given by Lemma 18 so that, in particular, for all $t \geq k''_0$ and for all $\beta \geq \omega$, $g(t) \leq F_\beta(t)$. We now take $k_0 = \max\{k'_0, k''_0\}$, restrict ourselves to $t \geq k_0$ and proceed by induction on $\bar{x} \neq \bar{0}$.

$$
\begin{aligned}
U_{n,g}(\bar{0}, 1, 0, t) &= U_{n,g}(\bar{0}, g(t+1), t+2) \\
&\leq F_{d(t)}(t+2) && (\text{Proposition 17}) \\
&\leq F_{d(t)+1}(d(t)) \\
&= F_\omega(d(t)) \\
&\leq F_{\omega+1}(t), && (\forall t \geq k_1)
\end{aligned}
$$

where $d(t) \stackrel{\text{def}}{=} 3(g(t+1)-1)+e$.

Next,

$$\begin{aligned} U_{n,g}(\bar{0}, x_1+1, 0, t) &= U_{n,g}(\bar{0}, x_1, g(t+1), t+2) \\ &\leq F_{\omega\cdot x_1+1+d(t)}(t+2) && \text{(ind. hyp. and Proposition 17)} \\ &\leq F_{\omega\cdot x_1+d(t)+1}(d(t)) \\ &= F_{\omega\cdot(x_1+1)}(d(t)) \\ &\leq F_{\omega\cdot(x_1+1)+1}(t). && (\forall t \geq k_2 \text{ by Corollary 19}) \end{aligned}$$

Observe that k_2 does not depend on x_1.

Finally, let $\bar{x} = x_{n-1}, \ldots, x_{j+1}$ and let $\beta = \omega^{n-1} \cdot x_{n-1} + \cdots + \omega^{j+1} \cdot x_{j+1}$.

$$\begin{aligned} U_{n,g}(\bar{x}, x_j+1, \bar{0}, t) &= U_{n,g}(\bar{x}, x_j, g(t+1), \bar{0}, t+2) \\ &\leq F_{\beta+\omega^j\cdot x_j+\omega^{j-1}\cdot g(t+1)+1}(t+2) && \text{(ind. hyp.)} \\ &\leq F_{\beta+\omega^j\cdot x_j+\omega^{j-1}\cdot(g(t+1)+1)}(t+2) \\ &\leq F_{\beta+\omega^j\cdot x_j+\omega^{j-1}\cdot(g(t+1)+1)}(g(t+1)) && (g(t)\geq t+1) \\ &= F_{\beta+\omega^j\cdot(x_j+1)}(g(t+1)) \\ &\leq F_{\beta+\omega^j\cdot(x_j+1)+1}(t). && (\forall t \geq k_3, \text{ by Corollary 19}) \end{aligned}$$

Observe that k_3 does not depend on $\bar{x}, x_j$. To finish, take $k = \max\{k_0, k_1, k_2, k_3\}$, which is clearly independent of $\bar{x}, x_j$. $\square$

We now give a tight upper bound for $L^{\text{ms}}_{s,n,g}$ in terms of the Fast Growing Hierarchy. Observe that since $L^{\text{ms}}_{1,n,g} = L^{\text{ms}}_{n,g}$ we also get a tight upper bound for multisets of $\mathbb{N}^n$.

Theorem 21 *(Tight upper bound for $L^{\text{ms}}_{s,n,g}$). If g has a primitive recursive upper bound then $L^{\text{ms}}_{s,n,g}$ has an upper bound in $\mathfrak{F}_{\omega^n\cdot s}$. If $s \leq g(0)$ and $\forall^\infty t,\ g(t) \geq t+2$ this bound is tight.*

Proof. Without loss of generality, we can assume that g is primitive recursive and that $\forall t,\ g(t) \geq t+2$ (if the upper bound holds for functions g with this last condition, it will also hold for potentially smaller functions by the monotonicity of $L^{\text{ms}}_{s,n,g}$ respect to g). Observe that the tightness will follow from Theorem 15. Suppose $g \in \mathfrak{F}_{e-1}$ for $1 \leq e < \omega$. Observe that $\forall^\infty t,\ g(t+1) \leq F_e(t)$. We first show for $n > 1$ that

$$\forall^\infty t,\ L^{\text{ms}}_{s,n,g}(t) \leq F_{\omega^n\cdot s}(g(t)), \tag{14}$$

and, if $n = 1$, that

$$\forall^\infty t,\ L^{\text{ms}}_{s,n,g}(t) \leq F_{\omega\cdot s}(3(g(t)-1)+e). \tag{15}$$

Let $M_1, M_2, \ldots$ be a g,t-controlled bad sequence of multisets over $([s] \times \mathbb{N}^n, \leq_{\text{lex}})$. Since the $\leq_{\text{lex}}$-greatest element of M_1 is not greater than $(s-1, g(t)-1, \ldots, g(t)-1)$, we have that $M_1 <_{\text{ms}} 1 \cdot \{(s-1, g(t), 0, \ldots, 0)\}$.

Suppose now $n > 1$. Recall the definitions of $P_{n,g}$ from Section 5.2 and of $U_{n,g}$ from Section 5.3. We have:

$$\begin{aligned} L^{\text{ms}}_{s,n,g}(t) &\leq P_{n+1,g}(s-1, g(t), \bar{0}, t+1) \\ &\leq U_{n+1,g}(s-1, g(t), \bar{0}, t+1) && \text{(Lemma 16)} \\ &\leq U_{n+1,g}(s-1, g(t), \bar{0}, g(t)) && (g(t) \geq t+1 \text{ and monot. of } U_{n+1,g}) \\ &\leq F_{\omega^n\cdot(s-1)+\omega^{n-1}\cdot g(t)+1}(g(t)) && (\forall t \geq k \text{ by Lemma 20}) \\ &\leq F_{\omega^n\cdot(s-1)+\omega^{n-1}\cdot(g(t)+1)}(g(t)) && (\forall t \text{ by Corollary 13.1}) \\ &= F_{\omega^n\cdot s}(g(t)), \end{aligned}$$

and we conclude (14). Notice that $F_{\omega^n\cdot s} \circ g \in \mathfrak{F}_{\omega^n\cdot s}$, since g is primitive recursive.

If $n = 1$, we use Proposition 17 at the third inequality above and obtain:

$$\begin{aligned} L^{\text{ms}}_{s,n,g}(t) &\leq F_{\omega\cdot(s-1)+1+3(g(t)-1)+e}(t+1) \\ &\leq F_{\omega\cdot(s-1)+1+3(g(t)-1)+e}(3(g(t)-1)+e) && (g(t) \geq t+2) \\ &= F_{\omega\cdot s}(3(g(t)-1)+e), \end{aligned}$$

and we conclude (15). Notice that this last function is in $\mathfrak{F}_{\omega^n\cdot s}$. $\square$

6. Majoring ordering

In this section we state a general result for linearizing the majoring ordering into the multiset ordering. We apply it to linearize the majoring ordering over finite sets of $\mathbb{N}^n$ with $\leq_{\text{pr}}$ as the underlying ordering into the multiset ordering over finite multisets of $\mathbb{N}^n$ with $\leq_{\text{lex}}$ as the underlying ordering, and to linearize the majoring ordering over finite sets of $[s] \times \mathbb{N}^n$ with $\leq^{\text{d}}_{\text{pr}}$ as the underlying ordering into the multiset ordering over finite multisets of $[s] \times \mathbb{N}^n$ and $\leq_{\text{lex}}$ as the underlying ordering. From these results we derive an upper bound for the length of the longest controlled bad sequences of the respective wqo's.

For a given wqo $(A, \leq_A)$ with a proper norm $|\cdot|_A$ and $x \in \mathbb{N}$, we define

$$[x]_A \stackrel{\text{def}}{=} |\{y \in A \mid |y|_A < x\}|.$$

Theorem 22. *Let $(A, \leq_A)$ be a wqo with a proper norm $|\cdot|_A$ and let $(B, \leq_B)$ be well-order with a proper norm $|\cdot|_B$ such that $(A, \leq_A)$ is linearized into $(B, \leq_B)$ via $h : A^+ \to B$. Then there is a function $f : (\mathcal{P}_{<\infty}(A))^+ \to \mathcal{M}_{<\infty}(B)$ such that if $\mathbf{X}^\frown X$ is a bad sequence in $(\mathcal{P}_{<\infty}(A), \leq^{(\leq_A)}_{\text{maj}})$, $\mathbf{X}$ is a nonempty sequence and X is a nonempty set, then $f(\mathbf{X}^\frown X) <^{(\leq_B)}_{\text{ms}} f(\mathbf{X})$.*

Furthermore, suppose $\mathbf{X}$ is g, t-controlled, and let $\tilde{g}$ be a control function independent of $\mathbf{X}$ such that if $\mathbf{a}$ is a g, t-controlled and nonempty bad sequence in $(A, \leq_A)$ then $|h(\mathbf{a})|_B < \tilde{g}(|\mathbf{a}| + t - 1)$. Then $|f(\mathbf{X})| < \bar{g}(|\mathbf{X}| + t - 1)$, for any $\bar{g}(x) > \max\{([g(x)]_A)^{x+1}, \tilde{g}(2x)\}$.

Proof. Our linearization will be done in two steps. First, from a $\leq^{(\leq_A)}_{\text{maj}}$-bad sequence $\mathbf{X} = X_0, X_1, \ldots, X_k$ of finite and nonempty sets of A, we define a sequence $T_0, T_1, \ldots, T_k$ of trees whose nodes (except the root) are labeled with elements of A. If w is a node of T_i, we denote as $v(w)$ the value of its label. The trees T_i will satisfy the following conditions:

1. For any $i \in \{0, \ldots, k\}$, if $w_0, \ldots, w_m$ is a path in T_i then $v(w_1), \ldots, v(w_m)$ is a $\leq_A$-bad sequence of elements of A. Furthermore, if $\mathbf{X}$ is g, t-controlled then $v(w_1), \ldots, v(w_m)$ is $g, t + i$-controlled.
2. T_{i+1} is a strict extension of T_i.

The second step of our construction uses the hypothesis that the wqo $(A, \leq_A)$ is linearized into the well-order $(B, \leq_B)$ and transforms the trees T_i into finite multisets M_i of elements of B in such a way that $\mathbf{M} = M_0, M_1, \ldots, M_k$ is $<^{(\leq_B)}_{\text{ms}}$-decreasing. Observe that since $\leq_B$ is a well-order then $\leq^{(\leq_B)}_{\text{ms}}$ also is one. Furthermore, given a control for $\mathbf{X}$, we find a control for $\mathbf{M}$. By taking $f(\mathbf{X}) = M_{|\mathbf{X}|-1}$ we obtain the desired result.

Here are the details of the construction. Let $X \subseteq A$. We say X *avoids* x if for all $y \in X$ we have $x \not\leq_A y$. Since $\mathbf{X} = X_0, X_1, \ldots, X_k$ is bad, then for any $i < j$, X_j avoids some tuple of X_i. In particular for all $j \in \{1, \ldots, k\}$, X_j avoids some tuple of X_0.

Construction of the trees T_i Define the following sequence of finite trees, whose nodes, except the root, are labeled with elements of A. By a *path* we always refer to a simple path (i.e. a path without backtracking) from the root to a leaf.

- If $X_0 = \{a_1, \ldots, a_p\}$ then T_0 is the tree formed by a root r (the value of $v(r)$ is irrelevant) and r has exactly p children, say $w_1, \ldots, w_p$, such that $v(w_i) = a_i$ for all $i \in \{1, \ldots, p\}$.
- T_{i+1} is formed by extending T_i as follows. Suppose $X_{i+1} = \{a_1, \ldots, a_p\}$. For any path $w_0, \ldots, w_m$ in T_i do the following: if for all $j = 1, \ldots, m$, X_{i+1} avoids $v(w_j)$ then add exactly p new children of w_m, say $w'_1, \ldots, w'_p$, such that $v(w'_i) = a_i$ for all $i \in \{1, \ldots, p\}$.

See Fig. 3 for an example of this construction when $(A, \leq_A) = (\mathbb{N}^2, \leq_{\text{pr}})$ and the sequence (4) of page 5.

Proposition 23. *At least one path of T_i is strictly extended in T_{i+1}.*

Proof. Recall that $X_j \neq \emptyset$ for all j. It is clear that if all internal nodes (i.e. nodes that are not leaves) of T_i have a child whose label is avoided by X_{i+1} then there is a path $w_0, \ldots, w_m$ in T_i such that X_{i+1} avoids $v(w_j)$ for all $j \in \{1, \ldots, m\}$.

If $T_{i+1} = T_i$ then, by construction, there is no path $w_0, \ldots, w_m$ in T_i such that $v(w_j)$ is avoided by X_{i+1} for all $j \in \{1, \ldots, m\}$. Then there is an internal node of T_i, say w, with none of its children containing labels avoided by X_{i+1}. But this contradicts the badness of $\mathbf{X}$ since by construction the set of children's values of w is X_j for some $j \leq i$. □

As the example in Fig. 3 shows, the height of T_{i+1} is not necessarily greater than the height of T_i. The following easily follows by construction:

Proposition 24. *If $w_0, \ldots, w_m$ is a path in T_i then $v(w_1), \ldots, v(w_m)$ is a $\leq_A$-bad sequence of elements of A. Furthermore if $\mathbf{X}$ is g, t-controlled then $v(w_1), \ldots, v(w_m)$ is $g, (t + i)$-controlled.*

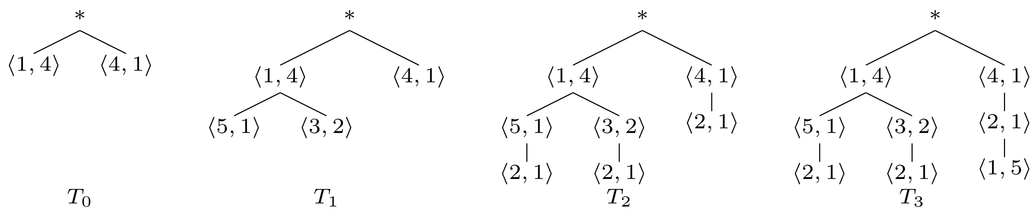

Fig. 3. Construction of the trees for the $\leq_{\mathrm{maj}}$-bad sequence $\mathbf{X} = X_0, X_1, X_2, X_3$ defined in (4) of page 5.

Construction of the multisets M_i By hypothesis, $h : A^+ \to B$ satisfies that for every $\leq_A$-bad sequence $\mathbf{a}^\frown a$ of elements of A we have $h(\mathbf{a}^\frown a) <_B h(\mathbf{a})$.

Let $M_i \in \mathcal{M}_{<\infty}(B)$ be defined as: $M_i(y) \overset{\text{def}}{=} d$ iff there are exactly d paths in T_i, say

$$\begin{aligned} &w_0^1, \ldots, w_{m_1}^1 \\ &\qquad \vdots \\ &w_0^d, \ldots, w_{m_d}^d, \end{aligned}$$

such that $h(v(w_1^j), \ldots, v(w_{m_j}^j)) = y$ for all $j \in \{1, \ldots, d\}$. In other words, M_i is the multiset where we put $h(v(w_1), \ldots, v(w_m))$ for every path $w_0, \ldots, w_m$ in T_i.

Proposition 25. $M_{i+1} <_{\mathrm{ms}}^{(\leq_B)} M_i$.

Proof. If the path $w_0, \ldots, w_m$ in T_i is extended to $w_0, \ldots, w_m, w$ in T_{i+1} then $h(v(w_1), \ldots, v(w_m), v(w)) <_B h(v(w_1), \ldots, v(w_m))$. By the construction of the multisets M_i, Proposition 23, and the definition of $\leq_{\mathrm{ms}}^{(\leq_B)}$, it follows straightforwardly that $M_{i+1} <_{\mathrm{ms}}^{(\leq_B)} M_i$. $\square$

Observe that the need for working with multisets and not simply with sets resides in the fact that h may not be injective.

Proposition 26. *Let $\bar{g}(x) > \max\{([g(x)]_A)^{x+1}, \tilde{g}(2x)\}$. If $\mathbf{X} = X_0, \ldots, X_k$ is g, t-controlled then $|M_k| < \bar{g}(|\mathbf{X}| + t - 1)$. Therefore, $M_0, M_1, \ldots, M_k$ is $\bar{g}, t$-controlled.*

Proof. The maximum multiplicity of an element in M_k is bounded by

$$\begin{aligned} \prod_{j=0}^{k} [g(t+j)]_A &\leq ([g(t+k)]_A)^{k+1} \\ &\leq ([g(t+k)]_A)^{t+k+1} \\ &< \bar{g}(t+k) \\ &= \bar{g}(|\mathbf{X}| + t - 1). \end{aligned}$$

On the other hand, suppose $w_0, \ldots, w_m$ is a path in T_k. By Proposition 24, $v(w_1), \ldots, v(w_m)$ is a $\leq_A$-bad sequence of elements of A which is $g, (t+k)$-controlled. Then

$$\begin{aligned} |h(v(w_1), \ldots, v(w_m))|_B &\leq \tilde{g}(m+t+k-1) && \text{(hypothesis)} \\ &\leq \tilde{g}(t+2k-1) && (m \leq k) \\ &\leq \tilde{g}(2(t+k)) \\ &< \bar{g}(t+k) \\ &= \bar{g}(|\mathbf{X}| + t - 1). \end{aligned}$$

By Definition 3 we conclude that $|M_k| < \bar{g}(|\mathbf{X}| + t - 1)$. $\square$

By taking $f(\mathbf{X}) = M_{|\mathbf{X}|-1}$ we conclude the proof of Theorem 22. $\square$

We are now ready to derive the upper bounds for the length of bad sequences over the majoring ordering over finite sets of $\mathbb{N}^n$ and $[s] \times \mathbb{N}^n$.

Corollary 27. *There is a function $f_{s,n} : (\mathcal{P}_{<\infty}([s] \times \mathbb{N}^n))^+ \to \mathcal{M}_{<\infty}([s] \times \mathbb{N}^n)$ such that if $\mathbf{X}^\frown X$ is a bad sequence in $(\mathcal{P}_{<\infty}([s] \times \mathbb{N}^n), \leq_{\mathrm{maj}})$, $\mathbf{X}$ is nonempty and X is a nonempty set, then $f_{s,n}(\mathbf{X}^\frown X) <_{\mathrm{ms}} f_{s,n}(\mathbf{X})$. Furthermore if $\mathbf{X}$ is g, t-controlled then $|f_{s,n}(\mathbf{X})| < \bar{g}(|\mathbf{X}| - 1 + t)$, for $\bar{g}(x) = (s \cdot \hat{g}(2x))^{n(x+1)} + 1$, where $\hat{g}$ is as in Theorem 5.*

Proof. The first part follows from Theorem 22 and Theorem 5 by taking $(A, \leq_A)$ to be $([s] \times \mathbb{N}^n, \leq^{\mathrm{d}}_{\mathrm{pr}})$ and $(B, \leq_B)$ to be $([s] \times \mathbb{N}^n, \leq_{\mathrm{lex}})$ (as usual $|\cdot|_{[s]\times\mathbb{N}^n}$ is taken to be $|\cdot|_\infty$).

For the second part, we verify that $\bar{g}(x)$ satisfies the conditions of Theorem 22, when the function $\tilde{g}$ of Theorem 22 is set to $\hat{g}$. On the one hand, since $[g(x)]_{[s]\times\mathbb{N}^n} \leq s \cdot g(x)^n$ and $\tilde{g} \geq g$ is increasing, it is clear that $\bar{g}(x) > (s \cdot g(x)^n)^{x+1} \geq ([g(x)]_{[s]\times\mathbb{N}^n})^{x+1}$. On the other, it is straightforward that $\bar{g}(x) > \hat{g}(2x)$. □

Corollary 28 *(Upper bound for $L^{\mathrm{maj}}_{s,n,g}$). For any primitive recursive g there is a primitive recursive $\tilde{g}$ such that $L^{\mathrm{maj}}_{s,n,g} \leq L^{\mathrm{ms}}_{s,n,\tilde{g}}$. Hence if g has a primitive recursive upper bound then $L^{\mathrm{maj}}_{s,n,g}$ has an upper bound in $\mathfrak{F}_{\omega^n \cdot s}$.*

Proof. It follows from Corollary 27 and Theorem 21. □

Corollary 29 *(Upper bound for $L^{\mathrm{maj}}_{n,g}$). If g has a primitive recursive upper bound then $L^{\mathrm{maj}}_{n,g}$ has an upper bound in $\mathfrak{F}_{\omega^n}$.*

Proof. Immediate from Corollary 28 taking $s = 1$ and observing that $(0, \bar{x}) \leq^{\mathrm{d}}_{\mathrm{pr}} (0, \bar{y})$ iff $\bar{x} \leq_{\mathrm{pr}} \bar{y}$. □

7. Applications

Jurdziínki and Lazić [21] showed that for the class of incrementing tree counter automata (ITCA) as well as the class of alternating top-down tree one register automata (ATRA), the emptiness problem—i.e. deciding whether the language accepted by an automaton of such classes is empty—over finite data trees is decidable. Figueira [30] later showed that for some extensions of ATRA decidability still holds. All these proofs go along the lines of interpreting the automaton execution as a downward well-structured transition system, then showing that it is reflexive-downward-compatible with respect to a wqo between sets of configurations, and finally applying Finkel and Schnoebelen results [32] (mainly Prop. 5.4). That wqo is precisely the majoring order over the disjoint product ordering.

From [21], we know that the computational complexity of such decision procedures is lower-bounded by a non-primitive recursive function. For the upper-bound for ITCA's, an algorithm can be given in a manner analogous to [20, §VII.B.] for finding the *levels* (a finite set of configurations) reachable from the initial level—the emptiness problem is then reduced to testing whether the empty level is amongst them. The complexity of such an algorithm is mainly determined by the length of a bad sequence of levels $\mathbf{V} = V_0, V_1, \ldots, V_m$. In more detail, suppose an ITCA $\mathcal{C}$ has k counters and a finite set of states Q. Then a level of $\mathcal{C}$ is a finite set of tuples of the form $\langle q, v\rangle$, where $q \in Q$ and $v = \langle a_1, \ldots, a_k\rangle \in \mathbb{N}^k$ is the current value of the k counters. The levels are ordered by the majoring ordering with the underlying disjoint product ordering over $[|Q|] \times \mathbb{N}^k$ (mapping the set of states Q to $\{0, \ldots, |Q| - 1\}$). The complexity of the emptiness problem can be bounded by the length of the longest bad sequence in $(\mathcal{P}_{<\infty}([|Q|] \times \mathbb{N}^k), \leq^{\mathrm{d}}_{\mathrm{maj}})$. Regarding how $\mathbf{V}$ is controlled, the analysis is almost the same as in [20, §VII .B.]. Let $V_0 = \{\langle 0, \bar{0}\rangle\}$ and $V_i = \{c_1, \ldots, c_{p_i}\}$. From Definition 3 we have that $|V_i| = \max_j\{|c_j|_\infty\}$. The change from V_i to V_{i+1} may involve a change of state or increment of c_j's counters' values by one. The 'state part' of c_j is controlled by the constant $|Q|$ and the 'counters part' is controlled by the successor function. Hence, the bad sequence of sets is $g, 0$-controlled by $g(x) = x + 1 + |Q|$. Now, applying Corollary 28 we conclude

Proposition 30. *The time complexity of the emptiness problem for an ITCA with k counters and a finite set of states Q is upper bounded by a function in $\mathfrak{F}_{\omega^k \cdot |Q|}$.*

This immediately gives us an upper bound for the emptiness problem for ATRA. From [21, Thm. 3.1] we have that emptiness for ATRA follows from a PSPACE-reduction to emptiness for ITCA. For a fixed alphabet, if the ATRA $\mathcal{A}$ has s states then the ITCA $\mathcal{C}$ constructed in the reduction has $k(s) \stackrel{\mathrm{def}}{=} 2^s - 1 + 2^{4s}$ many counters[1] and $f \in O(2^s)$ many states. Hence we conclude

Proposition 31. *The time complexity of the emptiness problem for an ATRA with s states is upper bounded by a function in $\mathfrak{F}_{\omega^{k(s)} \cdot f(s)}$, for some $f \in O(2^s)$.*

8. Conclusions

Upper bounds for controlled *decreasing* sequences in a well-order are easier to obtain than for controlled *bad* sequences in a wqo. We studied the length of controlled decreasing sequences of two well-orders: lexicographic and multiset. For these,

[1] In [21] there is typo in the number of counters in the auxiliary array c'. Where it says $2^{|Q|^4}$, it should read $2^{4|Q|}$.

control function	well quasi-order	length function	upper bound	tight?
$g \in \mathfrak{F}_\gamma$	$([s] \times \mathbb{N}^n, \leq_{\mathrm{lex}})$	$L^{\mathrm{lex}}_{s,n,g}$	$\mathfrak{F}_{\gamma+n-1}$	✓
$g \in \mathfrak{F}_\gamma$	$([s] \times \mathbb{N}^n, \leq^{\mathrm{d}}_{\mathrm{pr}})$	$L^{\mathrm{pr}}_{s,n,g}$	$\mathfrak{F}_{\gamma+n-1}$	✓
g prim. rec.	$(\mathcal{M}_{<\infty}([s] \times \mathbb{N}^n), \leq_{\mathrm{ms}})$	$L^{\mathrm{ms}}_{s,n,g}$	$\mathfrak{F}_{\omega^n \cdot s}$	✓
g prim. rec.	$(\mathcal{P}_{<\infty}([s] \times \mathbb{N}^n), \leq^{\mathrm{d}}_{\mathrm{maj}})$	$L^{\mathrm{maj}}_{s,n,g}$	$\mathfrak{F}_{\omega^n \cdot s}$	?

Fig. 4. Summary of main results.[2]

the upper bounds in the Fast Growing Hierarchy had to be crafted from scratch. We also studied three well quasi-orders: product, disjoint product and majoring. The length of controlled bad sequences over these wqo's had been determined by linearizing them into the previous well-orders, and hence avoiding a direct classification of the length function in the Fast Growing Hierarchy. The case of the disjoint product had been previously analyzed in [20]. However we gave a straightforward and elementary proof which keeps away the "sum of powers of $\mathbb{N}$" approach. This last approach—being noticeably more understandable than previous proofs, and also leading to a sharper result—still needs some rather technical lemmas.

Motivated by the study of the *disjoint* product wqo, we analyzed, for all the previous wqo's, not only the space $\mathbb{N}^n$ but also $[s] \times \mathbb{N}^n$. In general we first addressed the former to then adapt it to the latter. By the characteristics of the wqo's studied here, in all the cases, the most general results are those $[s] \times \mathbb{N}^n$.

For the lexicographic, (disjoint) product and multiset case, our upper bounds are tight. For the majoring ordering the question of tightness remains open. In Fig. 4 we summarize our main results.

As applications we stated complexity upper bounds for the emptiness problem for ITCA and ATRA automata.

For future research we plan to investigate the length of controlled bad sequences of the *minoring ordering*, which is defined over a quasi-order $(X, \leq)$ as follows: $A \leq^{(\leq)}_{\min} B \overset{\mathrm{def}}{\Leftrightarrow} (\forall y \in B)(\exists x \in A)\ x \leq y$, and which turns out to be a wqo under some assumptions on $(X, \leq)$. For some easy observations on this ordering, see [33].

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