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Abstract The formalism of generalized contexts for quantum histories is used to investigate the possibility to consider the survival probability as the probability of no decay property at a given time conditional to no decay property at an earlier time. A negative result is found for an isolated system. The inclusion of two quantum measurement instruments at two different times makes possible to interpret the survival probability as a conditional probability of the whole system.

Keywords Quantum histories · Decay process · Quantum logic

1 Introduction

In many physical systems it is necessary to consider expressions that involve properties at different times. For example, a property of a microscopic system at a given time before a measurement has to be related with a property of the pointer of the instrument after the measurement. A formalism of quantum histories is useful to describe this type of situations. In a series of papers that were written from 1984, Griffiths [1], Omnès [2], Gell-Mann and Hartle [3] developed an interpretation of quantum mechanics known as *consistent histories*. The notion of *history* is defined in these papers as a sequence of properties at different times. The histories that can participate in the description of a physical system must satisfy a consistency condition that guarantees well defined probabilities.

One of us and Vanni [4] developed a formalism of *generalized contexts*. It is a history formalism based on the notion of time translation of properties. The compatibility condition imposed for properties at different times is the commutation of the corresponding projectors

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when they are translated to a common time. This condition is a natural extension of the usual notion of context of properties at a fixed time, and it replaces the consistency conditions of references [1–3].

Situations related with quantum measurements can be faced without the need of the collapse hypothesis, and therefore quantum histories can be a valuable contribution to non-collapse interpretations of quantum mechanics. Our formalism of generalized contexts not only avoid collapse, but also some difficulties of the formalism of consistent histories [4].

The aim of this paper is to apply the formalism of generalized contexts to describe decay processes. In the usual approach [5], a quasi-stationary state of a quantum system with continuous spectrum is defined when the mean value of energy has a small dispersion. In the orthodox interpretations, which postulate the collapse of the state vector due to the process of measurement, the survival probability of the initially prepared quasi-stationary state is defined as the probability for the system of still be found in the same quasi-stationary state in a future time. The time for which the survival probability is e^{-1} is called lifetime.

For a description of a decay process which emphasizes the relevant properties of the system and its probabilities, it is possible to define the property of *decay* (or *no decay*), mathematically represented by suitable projectors. The survival probability may be identified with the probability of the property of no decay at a given time t_2, conditional to the same property of no decay at a previous time t_1, for an arbitrary state prepared at a different time t_0. As the conjunction of properties at different times is involved, it seems natural to appeal to a formalism of *quantum histories*. R. Omnès proposed to describe the decay process with the theory of consistent histories. He found the problem that the consistency condition is not satisfied [6].

In this paper we propose to analyze the decay process with our formalism of generalized contexts. In Sect. 2 we review the usual definitions of decay process, survival probability and lifetime in the orthodox interpretations. Modal interpretations are considered in Sect. 3. In Sect. 4.1 we show that our formalism of generalized contexts cannot be applied to the decay process without quantum measurement instruments. The unstable system together with two consecutive ideal quantum measurements of the non-decay or decay properties is analyzed with the formalism of generalized contexts in Sect. 4.2. The main conclusions are presented in Sect. 5, and in the Appendix we provide a brief summary of our formalism of generalized contexts.

2 Decay Process and Orthodox Interpretations

The decay processes has a characteristic time, named lifetime. To understand the relation of this characteristic time with the survival probability it is convenient to introduce the notion of a quasi-stationary state [5].

Let us consider a quantum system S with a Hamiltonian $\widehat{H}_S : \mathcal{H}_S \longrightarrow \mathcal{H}_S$ having continuous spectrum. We indicate by $|E\rangle$ the generalized eigenvector of the Hamiltonian with eigenvalue E ($\widehat{H}_S|E\rangle = E|E\rangle$, $0 < E < \infty$, $\int_0^\infty dE|E\rangle\langle E| = \widehat{I}_S$).

A *quasi-stationary state* $|\chi\rangle \in \mathcal{H}_S$ is defined as a linear combination of energy eigenvectors with a small dispersion ε from the mean value E_0 ($\varepsilon \ll E_0$). We can write

$$|\chi\rangle = \int_0^\infty dE|E\rangle\langle E|\chi\rangle,$$

where, for example, $\langle E|\chi\rangle$ may be

$$\langle E|\chi\rangle = \frac{1}{2\pi}\frac{\varepsilon}{(E-E_0)^2+\frac{1}{4}\varepsilon^2}.$$

In the Schrödinger representation, the state vector $|\chi\rangle$ at the time t_1 transforms into the state vector $|\varphi(t_2)\rangle = e^{-\frac{i}{\hbar}\widehat{H}_S(t_2-t_1)}|\chi\rangle$ at time t_2.

The projection (or collapse) postulate of the orthodox interpretations prescribes a non unitary reduction of the total state vector in the measurement process. The probability to measure the state $|\chi\rangle$ at the time $t_2 > t_1$ is

$$W(t_2-t_1) \equiv \left|\langle\chi|e^{-\frac{i}{\hbar}\widehat{H}_S(t_2-t_1)}|\chi\rangle\right|^2 = \int_0^\infty dE\left|\langle E|\chi\rangle\right|^2 e^{-\frac{i}{\hbar}E(t_2-t_1)}, \tag{1}$$

called the *survival probability* at the time t_2 of the quasi-stationary state $|\chi\rangle$ prepared at the time t_1. The *lifetime* T is defined in such a way that the survival probability (1) is decreased by a factor e ($W(T) = \frac{1}{e}W(0)$).

If $\langle E|\chi\rangle$ is a distribution with small dispersion ε with respect to the mean energy E_0 ($\varepsilon \ll E_0$), and we consider a time interval $(t_2 - t_1)$ which is not too large nor too small so that neither Zeno or Kalflin effects are involved, the survival probability has the approximated expression

$$W(t_2-t_1) \cong e^{-\frac{\varepsilon}{\hbar}(t_2-t_1)}. \tag{2}$$

The probability to find the initial state $|\chi\rangle$ is decreased by a factor e after a time interval equal to $\hbar/\varepsilon$, and therefore the approximated expression $T = \hbar/\varepsilon$ is obtained for the lifetime. The lifetime T and the energy indeterminacy $\Delta E = \varepsilon$ are related by the uncertainty relation $T\Delta E \simeq \hbar$.

For a system prepared in state $|\chi\rangle$ *at time* t_1, *the orthodox interpretations give the value* $W(t_2 - t_1)$ *to the probability for the system to collapse to state* $|\chi\rangle$ *when it is measured at a later time* t_2.

The orthodox interpretations emphasizes the role of the collapse of the state vector in the measurement, a process which has a strong "black box" character [7].

In the following sections we are going to abandon the postulate of collapse and consider the decay process from the point of view of interpretations which emphasize on the actualization of properties.

3 Decay Process and Modal Interpretations

The modal interpretations abandon the rule of standard quantum mechanics stating that a system must be in an eigenstate of an observable in order for that observable to have a definite value. There is no projection postulate, and the time evolution of the state vector is *always* generated by the Schrödinger equation. New rules are introduced specifying the possible properties that can be ascribed to a system for a given state.

In the case of a decay process, the quantum *no decay property* p_χ, represented by the projector $\widehat{\Pi}_\chi \equiv |\chi\rangle\langle\chi|$, can be associated with the quasi-stationary state vector $|\chi\rangle$ of the previous section. The *decay property* $\overline{p}_\chi$ is represented by $\widehat{\Pi}_{\overline{\chi}} = \widehat{I}_S - |\chi\rangle\langle\chi|$. The no decay and decay properties are exhaustive ($p_\chi \vee \overline{p}_\chi = \Omega$) and mutually exclusive ($p_\chi \wedge \overline{p}_\chi = \phi$). The properties ϕ, p_χ, $\overline{p}_\chi$ and Ω, represented by the projectors $\widehat{\Pi}_\phi = |0\rangle\langle 0|$, $\widehat{\Pi}_\chi = |\chi\rangle\langle\chi|$,

$\widehat{\Pi}_{\overline{\chi}} = \widehat{I}_S - |\chi\rangle\langle\chi|$ and $\widehat{\Pi}_\Omega = \widehat{I}_S$, form a distributive lattice, i.e. a context of properties in the usual sense.

If the state of the quantum system S at the time t_1 is represented by the vector $|\chi\rangle$, the probability of the property p_χ is equal to one at this time, as can be easily obtained using the Born rule

$$\Pr(p_\chi, t_1) = Tr\big(\widehat{\rho}(t_1)\widehat{\Pi}_\chi\big) = Tr\big(|\chi\rangle\langle\chi|\chi\rangle\langle\chi|\big) = \big|\langle\chi|\chi\rangle\big|^2 = 1,$$

and therefore we can safely say that the system S *has* the property p_χ at the time t_1.

At time $t_2 > t_1$, the state vector is $|\varphi(t_2)\rangle = e^{-\frac{i}{\hbar}\widehat{H}_S(t_2-t_1)}|\chi\rangle$, and there is no certainty about the property p_χ, because its probability is smaller than one

$$\begin{aligned}\Pr(p_\chi, t_2) &= \mathrm{Tr}\big(\widehat{\rho}(t_2)\widehat{\Pi}_\chi\big) = \mathrm{Tr}\big(\big|\varphi(t_2)\big\rangle\big\langle\varphi(t_2)\big|\chi\big\rangle\langle\chi|\big)\\ &= \big|\big\langle\chi\big|\varphi(t_2)\big\rangle\big|^2 = \big|\langle\chi|e^{-\frac{i}{\hbar}\widehat{H}_S(t_2-t_1)}|\chi\rangle\big|^2 = W(t_2-t_1) < 1.\end{aligned} \qquad (3)$$

The expression for $W(t_2 - t_1)$ was already obtained in the previous section, but modal interpretations give to it a different interpretation: *for the isolated system S prepared in the state $|\chi\rangle$ at time t_1, it is the probability to have the property p_χ at a later time t_2.*

It should be noted the different roles played by the vector $|\chi\rangle$ at different times: it is a quasi-stationary state vector at time t_1, and it is a property of no decay at a later time t_2. Moreover, it should be noted that in this approach the prepared state is not arbitrary.

In modal interpretations, it seems reasonable to search for the possibility of considering the survival probability of the property p_χ at both times t_1 and t_2, for an arbitrary state prepared at a previous time t_0 ($t_0 < t_1 < t_2$). In the following section we discuss this possibility using the formalism of generalized contexts.

4 Decay Process and Quantum Histories

The central idea of quantum histories is to abandon the fundamental role of measurements and external observers of the orthodox interpretations, and to study temporal sequences of quantum properties, represented by time dependent projectors. Families of compatible histories have to be selected in such a way that each history of the family has a well defined probability of realization or actualization.

In the following subsections we are going to consider the decay process in the formalism of *generalized contexts*, developed by Vanni and one of us [4] (a brief summary of this formalism is included in the Appendix). In this approach, the compatibility condition imposed to properties at different times is that the corresponding projectors commute when translated to a common time. This condition is an extension of the usual notion of context of properties at a fixed time.

4.1 Decay Process Without Quantum Instruments

The formalism of generalized contexts, which allows to consider the properties p_χ and $\overline{p}_\chi$ for two different times t_1 and t_2, seems a good candidate to search for the possibility of considering the survival probability as a conditional probability of properties of an isolated microscopic system S, with no intervention of measurement instruments or external observers.

In such a formalism, the probability for the system S *to have* the property p_χ at the time t_2, conditional *to have had* the property p_χ at the earlier time t_1, would be given by

$$\Pr(p_\chi, t_2 | p_\chi, t_1) = \frac{\Pr\{(p_\chi, t_2) \wedge (p_\chi, t_1)\}}{\Pr(p_\chi, t_1)}, \tag{4}$$

which could be identified with the survival probability $W(t_2 - t_1)$.

With this identification and the linear approximation of (2), the lifetime can be related to conditional probability

$$\frac{1}{T} \cong \frac{1 - \Pr(p_\chi, t_2 | p_\chi, t_1)}{(t_2 - t_1)}.$$

But expression (4) for the conditional probability is meaningful only if we can include the property $(p_\chi, t_2) \wedge (p_\chi, t_1)$ in a set of properties at two times (histories) for which probabilities are well defined.

In the formalism of generalized contexts the properties (p_χ, t_1) and (p_χ, t_2) are admitted if they are represented by commuting projectors when translated to a common time [4] (see also the Appendix).

Choosing t_2 as the common time, we should verify that $\widehat{U}(t_2, t_1)\widehat{\Pi}_\chi \widehat{U}^{-1}(t_2, t_1)$ and $\widehat{U}(t_2, t_1)\widehat{\Pi}_{\overline{\chi}}\widehat{U}^{-1}(t_2, t_1)$ commute with the projectors $\widehat{\Pi}_\chi$ and $\widehat{\Pi}_{\overline{\chi}}$ ($\widehat{U}(t_2, t_1) = e^{-\frac{i}{\hbar}\widehat{H}_S(t_2 - t_1)}$). Let us try to verify if

$$\big[\widehat{U}(t_2, t_1)\widehat{\Pi}_\chi \widehat{U}^{-1}(t_2, t_1); \widehat{\Pi}_\chi\big] = 0. \tag{5}$$

If the time interval $\Delta t = t_2 - t_1$ is very small, we have $\widehat{U}(t_2, t_1) \cong \widehat{I} - \frac{i}{\hbar}\widehat{H}_S \Delta t$. Up to first order in Δt, (5) gives

$$\big[[\widehat{H}_S; \widehat{\Pi}_\chi]; \widehat{\Pi}_\chi\big] = 0.$$

This condition can also be written as

$$\widehat{H}_S|\chi\rangle\langle\chi| - 2|\chi\rangle\langle\chi|\widehat{H}_S|\chi\rangle\langle\chi| + |\chi\rangle\langle\chi|\widehat{H}_S = 0.$$

This last equation is only verified if $|\chi\rangle$ is an eigenvector of $\widehat{H}_S$, but in this case $|\chi\rangle$ would be a stationary state with no decay.

The composed property $(p_\chi, t_2) \wedge (p_\chi, t_1)$ cannot be part of the universe of discourse of a set of histories organized in a generalized context. Therefore, *the identification of the survival probability* $W(t_2 - t_1)$ *with the conditional probability* $\Pr(p_\chi, t_2 | p_\chi, t_1)$ *is meaningless*.

The same difficulties where found by Omnès (see [6], pp. 176 to 180) in his attempt to include the composed property $(p_\chi, t_2) \wedge (p_\chi, t_1)$ in a set of consistent histories. He suggested to overcome the problem by considering special states and times for which the consistency condition is approximately valid, but in his approach it is necessary to deal with the concept of approximated logics.

In the following subsection we are going to consider consecutive measurements (at times t_1 and t_2) of the decaying system with quantum instruments. We will show that it is possible to organize the properties corresponding to the pointer positions of both measurement instruments in the universe of discourse of a generalized context.

4.2 Decay Process with Measurement Instruments

Let us consider an ideal measurement of the properties p_χ and $\overline{p}_\chi$ of the system S, performed by the interaction with a measurement instrument A during a small time interval $[t_1, t_1+\Delta t]$. We will consider the properties p_{A_0}, p_{A_χ} and $p_{A_{\overline{\chi}}}$ of the measurement instrument A. They are represented by the projectors $\widehat{\Pi}_{A_0} = |A_0\rangle\langle A_0|$, $\widehat{\Pi}_{A_\chi} = |A_\chi\rangle\langle A_\chi|$ and $\widehat{\Pi}_{A_{\overline{\chi}}} = |A_{\overline{\chi}}\rangle\langle A_{\overline{\chi}}|$ of the Hilbert space $\mathcal{H}_A$. The property p_{A_0} is the initial position of the pointer, while p_{A_χ} and $p_{A_{\overline{\chi}}}$ are the positions of the pointer correlated with the properties p_χ (no decay) and $\overline{p}_\chi$ (decay) of the system S.

We assume that the Hamiltonian generating the time evolution of the composed system $S+A$ is

$$\widehat{H} = \widehat{H}_S \otimes \widehat{I}_A + \widehat{I}_S \otimes \widehat{H}_A + \widehat{H}_{SA}.$$

To make the calculations simpler we also assume that $\widehat{H} \cong \widehat{H}_{SA}$ during the time interval $[t_1, t_1+\Delta t]$ and $\widehat{H} \cong \widehat{H}_S \otimes \widehat{I}_A$ for all values of time not included in $[t_1, t_1+\Delta t]$.

In the short time interval of the measurement process, the evolution of the composed system is given by the unitary transformation $\widehat{U}(\Delta t) \cong e^{-\frac{i}{\hbar}\widehat{H}_{SA}\Delta t}$, producing the correlation

$$\big|\Psi(t_1)\big\rangle \equiv \big|\varphi(t_1)\big\rangle|A_0\rangle \xrightarrow{\widehat{U}(\Delta t)} \big|\Psi(t_1+\Delta t)\big\rangle \equiv \big(\widehat{\Pi}_\chi\big|\varphi(t_1)\big\rangle\big)|A_\chi\rangle + \big(\widehat{\Pi}_{\overline{\chi}}\big|\varphi(t_1)\big\rangle\big)|A_{\overline{\chi}}\rangle,$$

where $|\Psi(t_1)\rangle$ ($|\Psi(t_1+\Delta t)\rangle$) is the state vector of the system $S+A$ immediately before (after) the measurement process.

Using the last equation we obtain

$$\Pr(p_\chi, t_1) = \mathrm{Tr}\big\{\big|\Psi(t_1)\big\rangle\big\langle\Psi(t_1)\big|(\widehat{\Pi}_\chi \otimes \widehat{I}_A)\big\} = \big|\big\langle\chi\big|\varphi(t_1)\big\rangle\big|^2,$$

$$\Pr(p_{A_\chi}, t_1+\Delta t) = \mathrm{Tr}\big\{\big|\Psi(t_1+\Delta t)\big\rangle\big\langle\Psi(t_1+\Delta t)\big|\big(\widehat{I}_S \otimes |A_\chi\rangle\langle A_\chi|\big)\big\} = \big|\big\langle\chi\big|\varphi(t_1)\big\rangle\big|^2,$$

where we notice that the probability of the property p_χ of the system S at the time t_1 has the same value as the probability of the pointer indicating A_χ at the time $t_1+\Delta t$.

We are going to consider two consecutive measurements of the system S with two identical instruments A and B. In a previous paper we discussed the logic of consecutive measurements applied to the deduction of the projection postulate [8]. The interaction between S and A (S and B) occurs in the time interval $[t_1, t_1+\Delta t]$ ($[t_2, t_2+\Delta t]$), where $t_1+\Delta t < t_2$. The system S is in a state represented by the vector $|\varphi(t_1)\rangle \in \mathcal{H}_S$ at the time t_1. At the same time, the instruments A and B are in there reference states represented by the vectors $|A_0\rangle \in \mathcal{H}_A$ and $|B_0\rangle \in \mathcal{H}_B$. Therefore, the state of the composed system $S+A+B$ at the time t_1 is represented by the vector

$$\big|\Phi(t_1)\big\rangle \equiv \big|\varphi(t_1)\big\rangle|A_0\rangle|B_0\rangle \in \mathcal{H}_S \otimes \mathcal{H}_A \otimes \mathcal{H}_B. \tag{6}$$

The Hamiltonian of the isolated system $S+A+B$ is

$$\widehat{H} = \widehat{H}_S \otimes \widehat{I}_A \otimes \widehat{I}_B + \widehat{I}_S \otimes \widehat{H}_A \otimes \widehat{I}_B + \widehat{I}_S \otimes \widehat{I}_A \otimes \widehat{H}_B + \widehat{H}_{SA} \otimes \widehat{I}_B + \widehat{H}_{SB} \otimes \widehat{I}_A \tag{7}$$

For simplicity we assume that the dominant term is $\widehat{H}_{SA} \otimes \widehat{I}_B$ ($\widehat{H}_{SB} \otimes \widehat{I}_A$) for the time interval $[t_1, t_1+\Delta t]$ ($[t_2, t_2+\Delta t]$), while the free terms dominate outside these two time intervals.

The properties of the instruments A and B are represented by projectors having the following form

$$\widehat{\Pi}^{(A)} = \widehat{I}_S \otimes \widehat{\pi}^{(A)} \otimes \widehat{I}_B, \quad \widehat{\pi}^{(A)} : \mathcal{H}_A \longrightarrow \mathcal{H}_A$$
$$\widehat{\Pi}^{(B)} = \widehat{I}_S \otimes \widehat{I}_A \otimes \widehat{\pi}^{(B)}, \quad \widehat{\pi}^{(B)} : \mathcal{H}_B \longrightarrow \mathcal{H}_B$$

In our formalism of generalized contexts [4], the properties of the instrument A at the time $t_1 + \Delta t$ are compatible with the properties of the instrument B at the time $t_2 + \Delta t$ if the time translation of both properties to a common time are represented by commuting projectors. If the common time is chosen to be $t_2 + \Delta t$, we have to prove the commutation of the projectors $\widehat{U}(t_2 + \Delta t, t_1 + \Delta t)\widehat{\Pi}^{(A)}\widehat{U}^{-1}(t_2 + \Delta t, t_1 + \Delta t)$ and $\widehat{\Pi}^{(B)}$.

Taking into account the dominant contributions in each time interval we obtain

$$\begin{aligned} \widehat{U}(t_1 + \Delta t, t_1) &= \widehat{U}_{SA} \otimes \widehat{I}_B \\ \widehat{U}(t_2, t_1 + \Delta t) &= \widehat{U}_S \otimes \widehat{U}_A \otimes \widehat{U}_B \\ \widehat{U}(t_2 + \Delta t, t_2) &= \widehat{U}_{SB} \otimes \widehat{I}_A \end{aligned} \tag{8}$$

where

$$\widehat{U}_{SA} \equiv e^{-\frac{i}{\hbar}\widehat{H}_{SA}\Delta t}, \qquad \widehat{U}_{SB} \equiv e^{-\frac{i}{\hbar}\widehat{H}_{SB}\Delta t},$$
$$\widehat{U}_S \equiv e^{-\frac{i}{\hbar}\widehat{H}_S(t_2 - t_1 - \Delta t)}, \qquad \widehat{U}_A \equiv e^{-\frac{i}{\hbar}\widehat{H}_A(t_2 - t_1 - \Delta t)}, \qquad \widehat{U}_B \equiv e^{-\frac{i}{\hbar}\widehat{H}_B(t_2 - t_1 - \Delta t)}.$$

Therefore

$$\begin{aligned} &\widehat{\Pi}^{(B)}\widehat{U}(t_2 + \Delta t, t_1 + \Delta t)\widehat{\Pi}^{(A)}\widehat{U}^{-1}(t_2 + \Delta t, t_1 + \Delta t) \\ &\quad = \big(\widehat{I}_S \otimes \widehat{I}_A \otimes \widehat{\pi}^{(B)}\big)(\widehat{U}_{SB} \otimes \widehat{I}_A)(\widehat{U}_S \otimes \widehat{U}_A \otimes \widehat{U}_B)\big(\widehat{I}_S \otimes \widehat{\pi}^{(A)} \otimes \widehat{I}_B\big) \\ &\qquad \times \big(\widehat{U}_S^{-1} \otimes \widehat{U}_A^{-1} \otimes \widehat{U}_B^{-1}\big)\big(\widehat{U}_{SB}^{-1} \otimes \widehat{I}_A\big) \\ &\quad = \big(\widehat{I}_S \otimes \widehat{I}_A \otimes \widehat{\pi}^{(B)}\big)\big(\widehat{I}_S \otimes \widehat{U}_A\widehat{\pi}^{(A)}\widehat{U}_A^{-1} \otimes \widehat{I}_B\big) = \big(\widehat{I}_S \otimes \widehat{U}_A\widehat{\pi}^{(A)}\widehat{U}_A^{-1} \otimes \widehat{I}_B\big)\big(\widehat{I}_S \otimes \widehat{I}_A \otimes \widehat{\pi}^{(B)}\big) \\ &\quad = (\widehat{U}_{SB} \otimes \widehat{I}_A)(\widehat{U}_S \otimes \widehat{U}_A \otimes \widehat{U}_B)\big(\widehat{I}_S \otimes \widehat{\pi}^{(A)} \otimes \widehat{I}_B\big)\big(\widehat{U}_S^{-1} \otimes \widehat{U}_A^{-1} \otimes \widehat{U}_B^{-1}\big)\big(\widehat{U}_{SB}^{-1} \otimes \widehat{I}_A\big) \\ &\qquad \times \big(\widehat{I}_S \otimes \widehat{I}_A \otimes \widehat{\pi}^{(B)}\big) \\ &\quad = \widehat{U}(t_2 + \Delta t, t_1 + \Delta t)\widehat{\Pi}^{(A)}\widehat{U}^{-1}(t_2 + \Delta t, t_1 + \Delta t)\widehat{\Pi}^{(B)}. \end{aligned}$$

This is the proof of the compatibility of properties of the measuring instrument A at the time $t_1 + \Delta t$ and properties of the measuring instrument B at the time $t_2 + \Delta t$. Therefore both classes of properties can be included in the same universe of discourse.

For the instrument A we are specially interested in p_{A_χ} and $p_{A_{\overline{\chi}}}$, correlated with properties p_χ and $p_{\overline{\chi}}$ of the system S at the time t_1. For the instrument B, at the time $t_2 + \Delta t$, the relevant properties are p_{B_χ} and $p_{B_{\overline{\chi}}}$, correlated with properties p_χ and $p_{\overline{\chi}}$ of the system S at time t_2.

The formalism of generalized contexts can be applied to compute the conditional probability

$$\Pr\big\{(p_{B_\chi}, t_2 + \Delta t)|(p_{A_\chi}, t_1 + \Delta t)\big\} = \frac{\Pr\{(p_{B_\chi}, t_2 + \Delta t) \wedge (p_{A_\chi}, t_1 + \Delta t)\}}{\Pr(p_{A_\chi}, t_1 + \Delta t)}.$$

If we choose the time translation of the properties to time t_1 we obtain

$$\begin{aligned}
&\Pr\left\{(p_{B_\chi}, t_2+\Delta t)\wedge(p_{A_\chi}, t_1+\Delta t)\right\}\\
&\quad=\operatorname{Tr}\left\{\widehat{\rho}(t_1)\widehat{U}(t_1, t_2+\Delta t)\widehat{\Pi}_\chi^{(B)}\widehat{U}^{-1}(t_1, t_2+\Delta t)\widehat{U}(t_1, t_1+\Delta t)\widehat{\Pi}_\chi^{(A)}\widehat{U}^{-1}(t_1, t_1+\Delta t)\right\}\\
&\quad=\operatorname{Tr}\left\{\widehat{\rho}(t_1)\widehat{U}(t_1, t_2+\Delta t)\widehat{\Pi}_\chi^{(B)}\widehat{U}(t_2+\Delta t, t_1+\Delta t)\widehat{\Pi}_\chi^{(A)}\widehat{U}(t_1+\Delta t, t_1)\right\},
\end{aligned}$$

where

$$\begin{aligned}
\widehat{\rho}(t_1)&=\left|\varphi(t_1)\right\rangle\!\left\langle\varphi(t_1)\right|\otimes|A_0\rangle\langle A_0|\otimes|B_0\rangle\langle B_0|,\\
\widehat{\Pi}_\chi^{(A)}&=\widehat{I}_S\otimes|A_\chi\rangle\langle A_\chi|\otimes\widehat{I}_B,\\
\widehat{\Pi}_\chi^{(B)}&=\widehat{I}_S\otimes\widehat{I}_A\otimes|B_\chi\rangle\langle B_\chi|.
\end{aligned}$$

Taking into account (8) and assuming for simplicity $\widehat{H}_A=\widehat{H}_B=0$ in (7), we obtain

$$\Pr\left\{(p_{B_\chi}, t_2+\Delta t)|(p_{A_\chi}, t_1+\Delta t)\right\}=\left|\langle\chi|e^{-\frac{i}{\hbar}\widehat{H}_S(t_2-t_1)}|\chi\rangle\right|^2=W(t_2-t_1) \tag{9}$$

The expression given in this last equation for the conditional probability is formally identical to the survival probability obtained in (1), from the point of view of orthodox interpretations. It also coincides with the result obtained in (3), from the point of view of modal interpretations. However, the physical interpretation of (9) is very different: *it is the probability of the instrument B to have the pointer property* p_{B_χ} *at the time* $t_2+\Delta t$, *conditional to the instrument A having the pointer property* p_{A_χ} *at the previous time* $t_1+\Delta t$.

5 Conclusions

For a system prepared in a quasi-stationary state the orthodox interpretations name survival probability to the probability of finding the system collapsed in the same quasi-stationary state when measured at a later time.

The modal interpretations of quantum mechanics do not postulate the collapse of the state vector and do not attribute a privileged role to the quantum measurements, but rather consider the probability of actualization of the possible properties. For a system initially *prepared* in a quasi-stationary state, isolated after its preparation, the survival probability can be viewed as the probability for the system to have at a later time the property represented by the projector corresponding to the initial quasi-stationary state. This approach needs to consider a prepared quasi-stationary state at a given time and the properties of decay or no decay at a later time.

We investigated the possibility of considering the survival probability only in terms of decay or no decay properties for an arbitrary prepared state. For this purpose we used our formalism of generalized contexts, developed to deal with quantum histories (i.e. time sequences of properties).

Our first attempt was to interpret the survival probability as the probability of a system prepared in an arbitrary state to have the no decay property at a given time conditional to have the same property at a previous time. The system was considered to be isolated after its preparation, i.e. with no measurement instruments. We obtained a negative result: there is no generalized context including the decay and no decay properties at two different times.

Finally, we considered the decay process of a system interacting at two different times with other two quantum systems. These two systems are considered ideal measurements instruments for the decay and no decay properties. In this case it is possible to construct a generalized context of properties including the pointer indications of the first instrument at a given time and the pointer indications of the second one at a later time. Therefore, it is possible to interpret the survival probability as the probability to measure the no decay property at a given time conditional to have measured the same no decay property at an earlier time.

Although quantum histories interpretations do not assume the existence of external observers, we have shown trough our example of a decay process that in general it is not possible, using the formalism of generalized contexts, to speak about properties of a system at two different times. However, if the measurements are represented by quantum interactions, it is always possible to speak in a consistent way (i.e. with well defined probabilities) about the pointer indications at the two different times.

Appendix: Generalized Contexts

For the sake of completeness we present in this section a brief summary of our formalism of generalized contexts [4, 9].

Let us represent a quantum property p at time t by the pair $(p;t)$, or equivalently by $(\widehat{\Pi}_p;t)$, where $\widehat{\Pi}_p$ is the projector representing the property p in the Hilbert space $\mathcal{H}$ of the system. The time translation of the property p at time t to time t' is defined by the pair $(p';t')$, or by $(\widehat{\Pi}_{p'};t')$, where p' is the quantum property represented by $\widehat{\Pi}_{p'} \equiv \widehat{U}(t',t)\,\widehat{\Pi}_p\,\widehat{U}^{-1}(t',t)$. The unitary operator $\widehat{U}(t',t) = \exp(-i\widehat{H}(t'-t)/\hbar)$ is the time evolution operator generated by the Hamiltonian operator $\widehat{H}$ of the system. The relation between time translated pairs is transitive, reflexive and symmetric and, therefore, it is an equivalence relation. We use $[(p;t)]$ (or $[(\widehat{\Pi}_p;t)]$) to name the class of pairs equivalent to $(p;t)$ (or to $(\widehat{\Pi}_p;t)$). It is interesting to note that the Born rule assigns the same probability to all the pairs of the same equivalence class in a given state, i.e.

$$\Pr(p;t) = \mathrm{Tr}(\widehat{\rho}_t\widehat{\Pi}_p) = \mathrm{Tr}(\widehat{\rho}_{t'}\widehat{\Pi}_{p'}) = \Pr\big(p';t'\big) = \Pr\big[(p;t)\big].$$

By definition, the equivalence class $[(\widehat{\Pi}^{(1)};t_1)]$ *implies* the equivalence class $[(\widehat{\Pi}^{(2)};t_2)]$ if the representative elements of the classes at a common time t_0 verify the implication of the usual formalism of quantum mechanics, i.e.

$$\begin{aligned} &\widehat{\Pi}^{(1,0)}\mathcal{H} \subset \widehat{\Pi}^{(2,0)}\mathcal{H}, \\ &\widehat{\Pi}^{(1,0)} \equiv \widehat{U}(t_0,t_1)\widehat{\Pi}^{(1)}\widehat{U}^{-1}(t_0,t_1), \qquad \widehat{\Pi}^{(2,0)} \equiv \widehat{U}(t_0,t_2)\widehat{\Pi}^{(2)}\widehat{U}^{-1}(t_0,t_2). \end{aligned} \tag{10}$$

This implication is a transitive, reflexive and antisymmetric relation, being therefore an order relation.

The *conjunction* (*disjunction*) of two classes $[(\widehat{\Pi};t)]$ and $[(\widehat{\Pi}';t')]$ can be obtained as the greatest lower (least upper) bound, i.e.

$$\begin{aligned} \big[(\widehat{\Pi};t)\big] \wedge \big[\big(\widehat{\Pi}';t'\big)\big] &= \mathrm{Inf}\big\{\big[(\widehat{\Pi};t)\big],\big[\big(\widehat{\Pi}';t'\big)\big]\big\}, \\ \big[(\widehat{\Pi};t)\big] \vee \big[\big(\widehat{\Pi}';t'\big)\big] &= \mathrm{Sup}\big\{\big[(\widehat{\Pi};t)\big],\big[\big(\widehat{\Pi}';t'\big)\big]\big\}. \end{aligned}$$

The *negation* of an equivalence class $[(\widehat{\Pi};t)]$ is defined by

$$\overline{[(\widehat{\Pi};t)]} = [(\widehat{\overline{\Pi}};t)] = [((\widehat{I}-\widehat{\Pi});t)].$$

With the implication, disjunction, conjunction and negation previously obtained, the set of equivalence classes has the structure of an orthocomplemented nondistributive lattice.

The usual concept of context is a subset of all possible simultaneous properties which can be organized as a meaningful description of a quantum system at a given time, and can be endowed with a boolean logic with well-defined probabilities. Our formalism supplies a prescription to obtain, from the nondistributive lattice of equivalence classes of pairs, the valid descriptions involving properties at different times, which we called *generalized contexts*.

Let us consider a context of properties at time t_1, generated by atomic properties $p_j^{(1)}$ represented by projectors $\widehat{\Pi}_j^{(1)}$ verifying

$$\widehat{\Pi}_i^{(1)}\widehat{\Pi}_j^{(1)} = \delta_{ij}\,\widehat{\Pi}_i^{(1)}, \qquad \sum_{j\in\sigma^{(1)}}\widehat{\Pi}_j^{(1)} = \widehat{I},\ i,j\in\sigma^{(1)}.$$

Let us also consider a context of properties at time t_2, generated by atomic properties $p_\mu^{(2)}$ represented by projectors $\widehat{\Pi}_\mu^{(2)}$ verifying

$$\widehat{\Pi}_\mu^{(2)}\widehat{\Pi}_\nu^{(2)} = \delta_{\mu\nu}\widehat{\Pi}_\mu^{(2)}, \qquad \sum_{\mu\in\sigma^{(2)}}\widehat{\Pi}_\mu^{(2)} = \widehat{I},\ \mu,\nu\in\sigma^{(2)}.$$

We wish to represent with our formalism a universe of discourse able to incorporate expressions like "the property $p_j^{(1)}$ at time t_1 *and* the property $p_\mu^{(2)}$ at time t_2". The conjunction of the classes with representative elements $\widehat{\Pi}_i^{(1)}$ at t_1 and $\widehat{\Pi}_\mu^{(2)}$ at t_2 is also the conjunction of the classes with representative elements $\widehat{\Pi}_i^{(1,0)} \equiv \widehat{U}(t_0,t_1)\widehat{\Pi}_i^{(1)}\widehat{U}^{-1}(t_0,t_1)$ and $\widehat{\Pi}_\mu^{(2,0)} \equiv \widehat{U}(t_0,t_2)\widehat{\Pi}_\mu^{(2)}\widehat{U}^{-1}(t_0,t_2)$ at the common time t_0.

In usual quantum theory the conjunction of simultaneous properties represented by non-commuting operators has no meaning. So, it seems natural to consider quantum descriptions of a system, involving the properties generated by the projectors $\widehat{\Pi}_i^{(1)}$ at time t_1 and $\widehat{\Pi}_\mu^{(2)}$ at time t_2, *only for the cases* in which the projectors $\widehat{\Pi}_i^{(1)}$ and $\widehat{\Pi}_\mu^{(2)}$ commute when translated to a common time t_0, i.e.

$$\widehat{\Pi}_i^{(1,0)}\widehat{\Pi}_\mu^{(2,0)} - \widehat{\Pi}_\mu^{(2,0)}\widehat{\Pi}_i^{(1,0)} = 0.$$

If this is the case, for the equivalence class of composite properties representing "the property $p_j^{(1)}$ at time t_1 *and* the property $p_\mu^{(2)}$ at time t_2" we obtain

$$h_{i\mu} = [(\widehat{\Pi}_i^{(1)};t_1)] \wedge [(\widehat{\Pi}_\mu^{(2)};t_2)] = \Big[\Big(\lim_{n\to\infty}(\widehat{\Pi}_i^{(1,0)}\widehat{\Pi}_\mu^{(2,0)})^n;t_0\Big)\Big] = [(\widehat{\Pi}_i^{(1,0)}\widehat{\Pi}_\mu^{(2,0)};t_0)].$$

As we can see, the conjunction of properties at different times t_1 and t_2 is equivalent to a single property, represented by the projector $\widehat{\Pi}_{i\mu}^{(0)} \equiv \widehat{\Pi}_i^{(1,0)}\widehat{\Pi}_\mu^{(2,0)}$ at a single time t_0.

If the different contexts at times t_1 and t_2 produce commuting projectors $\widehat{\Pi}_i^{(1,0)}$ and $\widehat{\Pi}_\mu^{(2,0)}$ at the common time t_0, it is easy to prove that

$$\widehat{\Pi}_{i\mu}^{(0)}\widehat{\Pi}_{j\nu}^{(0)} = \delta_{ij}\delta_{\mu\nu}\widehat{\Pi}_{i\mu}^{(0)}, \qquad \sum_{i\mu}\widehat{\Pi}_{i\mu}^{(0)} = \widehat{I}.$$

Therefore, we realize that the composite properties $h_{i\mu}$, represented at time t_0 by the complete and exclusive set of projectors $\widehat{\Pi}_{i\mu}^{(0)}$, can be interpreted as the atomic properties generating a usual context in the sense described above. More general properties are obtained from the atomic ones by means of the disjunction operation. For instance, we can represent the property "$p_j^{(1)}$ at time t_1 *and* $p_\mu^{(2)}$ at time t_2, with j and μ having any value in the subsets $\Delta^{(1)} \subset \sigma^{(1)}$ and $\Delta^{(2)} \subset \sigma^{(2)}$" as

$$h_{\Delta^{(1)},\Delta^{(2)}} = \left[\left(\sum_{i\in\Delta^{(1)}} \sum_{\mu\in\Delta^{(2)}} \widehat{\Pi}_{i\mu}^{(0)}; t_0\right)\right].$$

The set of properties obtained in this way is an orthocomplemented and distributive lattice.

If the state of the system at time t_0 is represented by $\widehat{\rho}_{t_0}$, the Born rule gives the following expression for the probability of the class of properties $h_{\Delta^{(1)},\Delta^{(2)}}$,

$$\Pr(h_{\Delta^{(1)},\Delta^{(2)}}) = \sum_{i\in\Delta^{(1)}} \sum_{\mu\in\Delta^{(2)}} \operatorname{Tr}\big(\widehat{\rho}_{t_0}\widehat{\Pi}_{i\mu}^{(0)}\big).$$

As a natural extension of the notion of context, we *postulate* that a description of a physical system involving properties at two different times t_1 and t_2 is valid if these properties are represented by commuting projectors when they are translated to a single time t_0. We will call each one of those valid descriptions *generalized context*. On each generalized context, the probabilities given by the Born rule are well-defined (i.e. they are positive, normalized and additive) and, therefore, they may be meaningful in terms of frequencies.

In summary, our formalism is based on the notion of time-translation, allowing to transform the properties at a sequence of different times into properties at a single common time. A usual context of properties is first considered for each time of the sequence. If the projectors representing the atomic properties of each context commute when they are translated to a common time, the contexts at different times can be organized to lead to a generalized context of properties. A generalized context of properties is a distributive and orthocomplemented lattice, a boolean logic with well-defined implication, negation, conjunction and disjunction. This logic can be used to speak and make inferences about the selected properties of the system at different times. Well-defined probabilities on the elements of the lattice of properties are obtained by means of the well-known Born rule.

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