Probabilities for time-dependent properties in classical and quantum mechanics

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We present a formalism which allows one to define probabilities for expressions that involve properties at different times for classical and quantum systems and we study its lattice structure. The formalism is based on the notion of time translation of properties. In the quantum case, the properties involved should satisfy compatibility conditions in order to obtain well-defined probabilities. The formalism is applied to describe the double-slit experiment.

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I. INTRODUCTION

For each state of a quantum system at a given time, the Born rule can be used to compute well-defined probabilities on a special set of properties called a context. Properties belonging to different contexts are not considered in the same description of a quantum system.

In the orthodox interpretation of quantum mechanics, the choice of the context of properties is dictated by the classical measurement instruments acting on the quantum system. A realistic interpretation of quantum mechanics should consider the measurement instruments also as quantum systems interacting with the quantum system to be measured. In this type of interpretation an additional rule has to be given to select the relevant context of properties. For the pointer variables of the measurement instrument, decoherence is commonly used as a selection rule of privileged classical variables.

In this paper we are not going to consider the criterion for the selection of a context at a given time, but rather we are going to discuss the compatibility of different contexts at different times. Quite often it is necessary to deal with properties at different times: It is necessary to relate some properties of a microscopic system before the measurement process with the pointer position after the measurement process or, in the double-slit experiment, it is necessary to argue about the impossibility to assert through which slit has passed the electron that produces a dot on the photographic plate [1].

For the conjunction of properties at different times, Ballentine suggested using the Heisenberg representation, but he did not discuss the conditions for well-defined probabilities (see Ref. [2], Sec. 9.6). Consistent or decoherent histories, involving sequences of properties at different times, have been considered by Griffiths [3], Omnès [4], and Gell-Mann and Hartle [5]. Properties at different times are imposed with a consistency condition which is state dependent.

In a previous paper [6], two of us presented an approach based on the concept of time translation of properties. In this approach, well-defined probabilities are obtained if the properties translated to a common time are represented by commuting projectors. This compatibility condition is a natural extension of the usual notion of a context of properties at a fixed time, and it replaces the consistency conditions of Refs. [3–5]. This condition is independent of the state of the system, and it provides a simple procedure to search for Boolean lattices of properties as possible universes of discourse about a quantum system. Time translations of properties have been applied to quantum measurements [7] and to quantum decay processes [8].

In this paper we present in full detail the lattice structure and probability definitions for expressions involving properties at different times for classical and quantum systems. We also apply our formalism to analyze the double-slit experiment.

In Sec. II we present the lattice of time-dependent classical properties, showing that the probabilities involving properties at different times are obtained from the properties translated to a common time. The concept of time translation is used in Sec III to obtain the lattice of time-dependent quantum properties and the corresponding probabilities. In Secs. IV and V we apply our formalism to discuss the famous double-slit experiment. We show that, with no measurement instruments at the slits, the impossibility to assert that the particle has passed through one of the slits can be deduced from our formalism. When measurement instruments are included, the elimination of interferences can be obtained with no reference to the projection postulate. The conclusions of the paper are in Sec. VI. To prevent the reader from being distracted from the main line of the arguments of our work, definitions and theorems about the lattice structure have been included in Appendices A and B.

II. CLASSICAL MECHANICS

A. Probabilities for properties at a fixed time

In classical mechanics the states of a physical system are represented by points in the phase space Γ (i.e., the space of generalized coordinates and momentums). The dynamics of the state is determined by the Hamilton equations. A state represented by the point $x \in \Gamma$ at the time t evolves into a state represented by the point $x' = S_{t't}\, x$ at the time t', where the map $S_{t't} : \Gamma \longrightarrow \Gamma$ is invertible ($S_{t't}^{-1} = S_{tt'}$) and volume preserving.

Each property of the physical system is identified with a subset of Γ. The system *has* (*does not have*) the property $C \subset \Gamma$ if it is in a state represented by a point $x \in C$ ($x \notin C$).

The set of all properties of the system is the power set $\mathcal{P}(\Gamma)$ of the phase space Γ. $\mathcal{P}(\Gamma)$ with the *order relation* given by the inclusion ($\subset$) is a Boolean lattice (i.e., it is orthocomplemented and distributive). The *infimum* and the *supremum* are given respectively by the intersection and the union [$\mathrm{Inf}(C_1,C_2) = C_1 \cap C_2$ and $\mathrm{Sup}(C_1,C_2) = C_1 \cup C_2$]. The *null element* is the empty set $\phi \subset \Gamma$ and the *universal element* is the phase space Γ. The *complement* of a set $C \subset \Gamma$ is the set $\Gamma - C$.

In some cases there is no precision about the point representing the state at a given time. Therefore, there is no certainty about the system having or not having a property, but only about the probability to have it.

In these cases it is necessary to appeal to what is known as a probability distribution. It is represented by a function $\rho_t : \Gamma \longrightarrow \mathbb{R}$, non-negative and normalized ($\int_\Gamma \rho_t(x)dx = 1$). If $\rho_t(x)$ is the state probability density at time t, the probability density at time t' is given by

$$\rho_{t'}(x) = \rho_t\left(S_{t't}^{-1}x\right). \tag{1}$$

Using the density function ρ_t it is possible to define a probability on the set of all properties. Classical statistical mechanics gives for the probability of a property C at time t the following expression:

$$\mathrm{Pr}_t(C) = \int_C \rho_t(x)dx,$$

which satisfies the Kolmogorov axioms (i) $\mathrm{Pr}_t(C) = \int_C \rho_t(x)dx \geqslant 0$ for all properties C, (ii) $\mathrm{Pr}_t(\Gamma) = \int_\Gamma \rho_t(x)dx = 1$, and (iii) if $C \cap C' = \phi$, then $\mathrm{Pr}_t(C \cup C') = \mathrm{Pr}_t(C) + \mathrm{Pr}_t(C')$.

In classical statistical mechanics $\mathrm{Pr}_t : \mathcal{P}(\Gamma) \longrightarrow \mathbb{R}$ gives the probabilities for properties at a fixed time t. In what follows we are going to present a formalism suitable for including different properties at different times in a probabilistic classical description of a physical system.

B. Probabilities for properties at different times

A property C at time t will be called an *event* and it will be identified with the pair (C,t). The space of all events of a given system will be denoted by E. In this section we are going to define the probability associated with these events.

According to classical statistical mechanics the *probability* for a property C at time t is given by

$$\mathrm{Pr}_t(C) = \int_C \rho_t(x)dx.$$

Using Eq. (1) we obtain

$$\mathrm{Pr}_t(C) = \int_C \rho_{t'}(S_{t't}\, x)dx = \int_{S_{t't}\, C} \rho_{t'}(x)dx = \mathrm{Pr}_{t'}(C'), \tag{2}$$

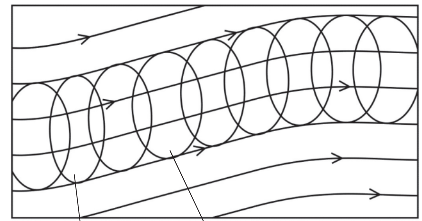

FIG. 1. The equivalence class $[C,t]$.

where $C' = S_{t't}C$. We give to the property $C' = S_{t't}C$ the interpretation of the *time translation*of property C from time t to time t'.

Expression (2) shows that in classical statistical mechanics the probability Pr_t for the property C at time t has the same value as the probability $\mathrm{Pr}_{t'}$ for the time translated property $C' = S_{t't}C$ at time t'. However, it should be noted that $\mathrm{Pr}_t(C)$ and $\mathrm{Pr}_{t'}(C')$ are obtained from different probability distributions ρ_t and $\rho_{t'}$.

The time translation defines a relation $\backsim$ on the space of events E [$(C',t') \backsim (C,t)$ if and only if $C' = S_{t't}\, C$]. This relation is transitive, reflexive, and symmetric, and is an equivalence relation on E.

The space E can be partitioned in equivalence classes. We denote by $[C,t]$ the *class of events* which are equivalent to the single event (C,t). The set of all equivalence classes $E/\backsim$ will be denoted by $[E]$. An equivalence class of events is represented in Fig. 1 for a two-dimensional phase space.

Events belonging to the same equivalence class are not essentially different. They are obtained one from the other by time evolution and they have the same value of probability in classical statistical mechanics.

In order to compute well-defined probabilities of properties at different times we are going to endow the set of equivalence classes $[E]$ with a Boolean lattice structure.

A partial order relation $\leqslant$ between two classes of events in $[E]$ is defined in the following way:

$$[C_1,t_1] \leqslant [C_2,t] \quad \text{if and only if } S_{t_0t_1}C_1 \subseteq S_{t_0t_2}C_2,$$

where t_0 is an arbitrary time. This definition is independent of the arbitrary time t_0, because if $S_{t_0t_1}C_1 \subseteq S_{t_0t_2}C_2$, then $S_{t_0't_0}S_{t_0t_1}C_1 \subseteq S_{t_0't_0}S_{t_0t_2}C_2$, and therefore $S_{t_0't_1}C_1 \subseteq S_{t_0't_2}C_2$.

From this order relation we obtain

$$[C,t] \wedge [C',t'] \equiv \mathrm{Inf}([C,t],[C',t']) = [(S_{t_0t}C \cap S_{t_0t'}C'),t_0],$$
$$[C,t] \vee [C',t'] \equiv \mathrm{Sup}([C,t],[C',t']) = [(S_{t_0t}C \cup S_{t_0t'}C'),t_0].$$

The null element of $[E]$ is $[\phi,t]$, where $\phi \subset \Gamma$ is the empty set and t is an arbitrary time. The universal element is $[\Gamma,t]$, where t is an arbitrary time. The complement of $[C,t]$ is $\overline{[C,t]} \equiv [\Gamma - C,t]$. The set $[E]$ of equivalence classes of events, with the just defined order relation, is a *Boolean lattice*.

Generalizing the probability already given in classical statistical mechanics, we can now define a probability on the lattice $[E]$ of equivalence classes of events

$$\mathrm{Pr}[C,t] \equiv \int_C \rho_t(x)dx.$$

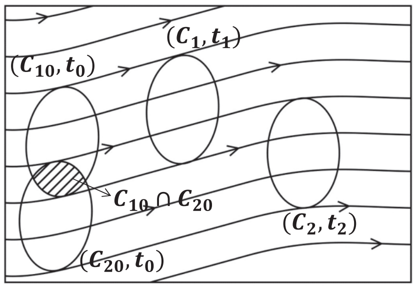

FIG. 2. The properties for $[C_1,t_1] \wedge [C_2,t_2]$.

It is easy to see that $\Pr : [E] \longrightarrow \mathbb{R}$ is well defined, i.e., it does not depend on the representative element of the class. Given $(C_1,t_1) \sim (C_2,t_2)$, then $S_{t_2t_1} C_1 = C_2$, and

$$\Pr[C_1,t_1] = \int_{C_1} \rho_{t_1}(x)dx = \int_{S_{t_2t_1} C_1} \rho_{t_1}\left(S_{t_2t_1}^{-1} x'\right) \left|\frac{\partial x}{\partial x'}\right| dx'$$
$$= \int_{C_2} \rho_{t_2}(x')dx' = \Pr[C_2,t_2].$$

Moreover it is easy to prove that $\Pr : [E] \longrightarrow \mathbb{R}$ satisfies the Kolmogorov axioms.

We have obtained a lattice structure, suitable for dealing with probabilities involving different properties at different times. The probability for the conjunction is given by

$$\begin{aligned} \Pr([C_1,t_1] \wedge [C_2,t_2]) &= \Pr([C_{10},t_0] \wedge [C_{20},t_0]) \\ &= \Pr([C_{10} \cap C_{20},t_0]) \\ &= \int_{C_{10}\cap C_{20}} \rho_{t_0}(x)dx, \end{aligned} \quad (3)$$

where $C_{10} = S_{t_0t_1} C_1$ and $C_{20} = S_{t_0t_2} C_2$ are the time translation of properties C_1 and C_2 to the common time t_0. The properties involved in Eq. (3) are represented in Fig. 2 for the particular case of a two-dimensional phase space.

For the disjunction we obtain

$$\begin{aligned} \Pr([C_1,t_1] \vee [C_2,t_2]) &= \Pr([C_{10},t_0] \vee [C_{20},t_0]) \\ &= \Pr([C_{10} \cup C_{20},t_0]) \\ &= \int_{C_{10}\cup C_{20}} \rho_{t_0}(x)\, dx. \end{aligned}$$

In this section we presented a formalism to assign probabilities to expressions involving different properties at different times for a classical system. In the next section we will develop a formalism for quantum systems.

III. QUANTUM MECHANICS

A. Probabilities for properties at a fixed time

A Hilbert space $\mathcal{H}$ and a Hamiltonian operator $\widehat{H} : \mathcal{H} \longrightarrow \mathcal{H}$ are associated with each isolated physical system. A state of the system at time t is represented by a non-negative, self-adjoint operator $\widehat{\rho}_t : \mathcal{H} \longrightarrow \mathcal{H}$, with $\mathrm{Tr}(\widehat{\rho}_t) = 1$.

The time evolution of the state is generated by the Liouville–von Neumann equation. If $\widehat{\rho}_t$ is the density operator representing the state at the time t, the state at a different time t' is given by

$$\widehat{\rho}_{t'} = \widehat{U}(t',t)\, \widehat{\rho}_t\, \widehat{U}(t',t)^{-1}, \quad \widehat{U}(t',t) = e^{-(i/\hbar)\widehat{H}(t'-t)}. \quad (4)$$

Each property of the quantum system is identified with a closed vector subspace V of the Hilbert space $\mathcal{H}$. For each closed subspace V there exists only one orthogonal projection operator $\widehat{\Pi}_V : \mathcal{H} \longrightarrow \mathcal{H}$ such that $V = \widehat{\Pi}_V \mathcal{H}$, and therefore the property can also be identified with the projector $\widehat{\Pi}_V$.

The set of all closed vector subspaces of a Hilbert space $\mathcal{H}$, with the *partial order relation* given by the set inclusion ($\subset$), is an orthocomplemented nondistributive lattice. The infimum and the supremum of V and V' are given by

$$\mathrm{Inf}(V,V') = V \cap V', \quad \mathrm{Sup}(V,V') = V + V'.$$

The *zero property* is identified with the subspace $\{0_{\mathcal{H}}\}$, where $0_{\mathcal{H}}$ is the zero element of $\mathcal{H}$, and the *universal property*is identified with the whole space $\mathcal{H}$. The complement of a property V is the orthogonal complement $V^{\perp}$ of the subspace V in $\mathcal{H}$.

A very special feature of quantum mechanics is that not all the possible properties can be simultaneously considered in a description of the system. There is no accepted probability formula for the conjunction of properties whose corresponding projectors do not commute [9,10]. When the projectors do commute, the corresponding properties are called *compatible*. Only compatible properties can be included in a description of a quantum system.

The standard rule to obtain a Boolean lattice of properties in quantum mechanics is to start with a set B of mutually orthogonal closed subspaces of $\mathcal{H}$ which expand the whole Hilbert space, i.e.,

$$B = \Bigg\{ V_i \mid i \in \sigma,\ V_i \text{ is a closed subspace of } \mathcal{H},$$
$$V_i \perp V_j \text{ if } i \neq j,\ \sum_{i\in\sigma} V_i = \mathcal{H} \Bigg\}, \quad (5)$$

where σ is a set of indices.

From this set of *atomic properties* B, a *context* C_B of properties can be obtained as the set of all subspaces which are the sums and intersections of elements of B:

$$C_B = \{0, V_i, V_i + V_j, V_i + V_j + V_k, \ldots, \mathcal{H};\ i,j,k,\ldots \in \sigma\}. \quad (6)$$

The context of properties C_B, *generated* by the set of atomic properties B, with the partial order relation defined by the inclusion ($\subset$), is a Boolean lattice.

The projectors $\widehat{\Pi}_i$ corresponding to the subspaces $V_i \in B$ satisfy the relations

$$\sum_{i\in\sigma} \widehat{\Pi}_i = \widehat{I}, \quad \widehat{\Pi}_i \widehat{\Pi}_j = \delta_{ij} \widehat{\Pi}_j, \quad i,j \in \sigma, \quad (7)$$

where $\widehat{I}$ is the identity operator in $\mathcal{H}$. Moreover, any pair of properties of the context C_B are compatible, i.e., they are represented by commutative projectors.

If $\widehat{\rho}_t$ is the state operator for the system at time t, the Born rule can be used to compute

$$\Pr_t(V) = \mathrm{Tr}(\widehat{\rho}_t \widehat{\Pi}_V), \quad (8)$$

for each property V. If we restrict the properties V to be elements of a context C_B, the function $\Pr_t$ satisfies the Kolmogorov axioms (i) $\Pr_t(V) \geqslant 0$, (ii) $\Pr_t(\mathcal{H}) = 1$, and

(iii) if $V_1 \cap V_2 = 0$, then $\Pr_t(V_1 + V_2) = \Pr_t(V_1) + \Pr_t(V_2)$. Therefore, the function $\Pr_t : C_B \longrightarrow \mathbb{R}$ is a well-defined probability on the context of properties C_B.

In this section we have described how to construct a context C_B of properties from the whole set of properties and we have defined a probability on the context for a fixed value of time. In the next section we will develop a formalism to deal with different properties at different times.

B. Probabilities for properties at different times

Following the steps of the classical case we define an *event* as a property V at a given time t, identified with the pair (V,t). We also define the set E of all possible events for a system, i.e.,

$$E = \{(V,t) \mid t \in \mathbb{R} \text{ and } V \text{ is a closed vector subspace of } \mathcal{H}\}.$$

We will say that (V',t') is the *time translation* of the event (V,t) if $V' = \widehat{U}(t',t)V$. It is interesting to notice that the Born rule gives the same probability to the property V at time t and to the property V' at time t'. Let us consider two events (V,t) and (V',t') such that $\widehat{U}(t',t)V = V'$. The associated projectors satisfy $\widehat{\Pi}_{V'} = \widehat{U}(t',t)\widehat{\Pi}_V\, \widehat{U}^{-1}(t',t)$. If $\widehat{\rho}_t$ and $\widehat{\rho}_{t'}$ are the state operators for the times t and t', the Born rule (8) and Eq. (4) give

$$\begin{aligned}\Pr_{t'}(V') &= \mathrm{Tr}(\widehat{\rho}_{t'}\widehat{\Pi}_{V'}) = \mathrm{Tr}[\widehat{\rho}_{t'}\widehat{U}(t',t)\widehat{\Pi}_V\widehat{U}^{-1}(t',t)]\\ &= \mathrm{Tr}[\widehat{U}(t,t')\widehat{\rho}_{t'}\widehat{U}^{-1}(t,t')\,\widehat{\Pi}_V] = \mathrm{Tr}(\widehat{\rho}_t\widehat{\Pi}_V) = \Pr_t(V).\end{aligned}$$

This result strongly suggests that the events connected by a time translation should not be considered as essentially different. Moreover, the relation $(V',t') \backsim (V,t)$ defined by $V' = \widehat{U}(t',t)V$ is an equivalence relation (see Proposition 1 in Appendix B). Therefore, each element of E belongs to only one set of equivalent events. We will *denote* by $[V,t]$ the class of events which are equivalent to the event (V,t), i.e.,

$$[V,t] \equiv \{(V',t') \mid (V',t') \sim (V,t)\}.$$

We also call $[E] \equiv E/\sim$ the set formed by all equivalence classes of events,

$$[E] = E/\sim = \{[V,t] \mid (V,t) \in E\}.$$

To endorse a lattice structure to the set $[E]$, we need to define a partial order relation ($\leqslant$). We propose the following *definition*:

$$[V_1,t_1] \leqslant [V_2,t_2] \quad \text{if and only if } \widehat{U}(t_2,t_1)V_1 \subset V_2.$$

In Appendix B (Propositions 2 and 3) we prove that $\leqslant$ is a well-defined relation on $[E]$, i.e., it does not depend on the representative elements of the equivalence classes, and also that it is a partial order relation on $[E]$.

It is also easy to prove that for each pair of elements $[V,t],[V',t'] \in [E]$, $\mathrm{Sup}\{[V,t],[V',t']\}$ and $\mathrm{Inf}\{[V,t],[V',t']\}$ exist and they are given by

$$\begin{aligned}[V,t] \vee [V',t'] &\equiv \mathrm{Sup}\{[V,t],[V',t']\}\\ &= [\widehat{U}(t_0,t)V + \widehat{U}(t_0,t')V',t_0],\\ [V,t] \wedge [V',t'] &\equiv \mathrm{Inf}\{[V,t],[V',t']\}\\ &= [\widehat{U}(t_0,t)V \cap \widehat{U}(t_0,t')V',t_0],\end{aligned} \tag{9}$$

where t_0 is an arbitrary fixed time. The proofs are given in Propositions 4 and 5 of Appendix B. As for each pair of elements of $[E]$ there are a supremum and an infimum, then $([E], \leqslant)$ is a lattice.

The lattice $([E], \leqslant)$ has zero and universal elements given by $[\{0_{\mathcal{H}}\},t]$ and $[\mathcal{H},t]$, respectively (see Proposition 6 in Appendix B). It is also a complemented lattice, with $[V^{\perp},t]$ the complement of $[V,t]$ (see Proposition 7 of Appendix B).

As mentioned above, we need a complemented and distributive lattice in order to define a probability function. Even though $([E], \leqslant)$ is a complemented lattice, it is not distributive if $\dim \mathcal{H} \geqslant 2$ (see Proposition 8 in Appendix B).

It is possible to obtain a Boolean sublattice of $([E], \leqslant)$ starting from an ordinary context of properties C_B having the form given by Eqs. (5) and (6). For a given fixed value t_0 of time, we prove in Proposition 9 of Appendix B that the set of equivalence classes $[E]_B \subset [E]$ given by

$$[E]_B \equiv \{[V,t_0] \in [E] \mid V \in C_B\}$$

is a Boolean sublattice of $([E], \leqslant)$. As C_B is generated by B, we will say that the lattice $[E]_B$ is generated by B.

Once we have a Boolean sublattice $[E]_B \subset [E]$, a well-defined probability $\Pr : [E]_B \longrightarrow \mathbb{R}$ can be defined as a generalization of the Born rule

$$\Pr[V,t_0] \equiv \mathrm{Tr}\big(\widehat{\rho}_{t_0}\widehat{\Pi}_V\big),$$

where $\widehat{\rho}_{t_0}$ is the state of the system at time t_0 and $\widehat{\Pi}_V$ is the projector corresponding to $V \in C_B$ (see Proposition 10 of Appendix B).

The sublattice $[E]_B$ is only a trivial generalization of an ordinary context of properties at a fixed time t. However, our main interest is to include different properties at different times in the description of the quantum system. Therefore, we are going to consider contexts at two different times.

Let us consider a context of properties $C_{B^{(1)}}$ at the time t_1, generated by the atomic properties

$$B^{(1)} = \Big\{V_i^{(1)}\Big|\, i \in \sigma^{(1)}, V_i^{(1)} \perp V_j^{(1)} \text{ if } i \neq j, \sum_{i\in\sigma^{(1)}} V_i^{(1)} = \mathcal{H}\Big\},$$

where the projectors $\widehat{\Pi}_i^{(1)}$ corresponding to the atomic properties $V_i^{(1)}$ satisfy the equations

$$\widehat{\Pi}_i^{(1)}\widehat{\Pi}_j^{(1)} = \delta_{ij}\widehat{\Pi}_j^{(1)}, \quad \sum_{i\in\sigma^{(1)}} \widehat{\Pi}_i^{(1)} = \widehat{I}.$$

We also consider another context of properties $C_{B^{(2)}}$ at the time t_2, generated by the atomic properties $B^{(2)} = \{V_\mu^{(2)} \mid \mu \in \sigma^{(2)}, V_\mu^{(2)} \perp V_\nu^{(2)} \text{ if } \mu \neq \nu, \sum_{\mu\in\sigma^{(2)}} V_\mu^{(2)} = \mathcal{H}\}$, where the corresponding projectors $\widehat{\Pi}_\mu^{(2)}$ satisfy

$$\widehat{\Pi}_\mu^{(2)}\widehat{\Pi}_\nu^{(2)} = \delta_{\mu\nu}\widehat{\Pi}_\nu^{(2)}, \quad \sum_{\mu\in\sigma^{(2)}} \widehat{\Pi}_\mu^{(2)} = \widehat{I}.$$

Properties $V^{(1)} \in C_{B^{(1)}}$ and $V^{(2)} \in C_{B^{(2)}}$ can be written in the form

$$V^{(1)} = \sum_{j\in\sigma_{V^{(1)}}} V_j^{(1)}, \quad \sigma_{V^{(1)}} \subset \sigma^{(1)},$$

$$V^{(2)} = \sum_{\mu\in\sigma_{V^{(2)}}} V_\mu^{(2)}, \quad \sigma_{V^{(2)}} \subset \sigma^{(2)},$$

with the corresponding projectors $\widehat{\Pi}^{(1)} = \sum_{j\in\sigma_{V^{(1)}}} \widehat{\Pi}_j^{(1)}$ and $\widehat{\Pi}^{(2)} = \sum_{\mu\in\sigma_{V^{(2)}}} \widehat{\Pi}_\mu^{(2)}$.

Taking into account Eqs. (9) the expression "property $V^{(1)}$ at time t_1 *and* property $V^{(2)}$ at time t_2" can be identified with the equivalence class

$$\begin{aligned}[V^{(1)},t_1] \wedge [V^{(2)},t_2] &= \mathrm{Inf}\{[V^{(1)},t_1],[V^{(2)},t_2]\}\\ &= [V^{(1,0)} \cap V^{(2,0)},t_0],\end{aligned}$$

and the expression "property $V^{(1)}$ at time t_1 or property $V^{(2)}$ at time t_2" can be identified with

$$\begin{aligned}[V^{(1)},t_1] \vee [V^{(2)},t_2] &= \mathrm{Sup}\{[V^{(1)},t_1],[V^{(2)},t_2]\}\\ &= [V^{(1,0)} + V^{(2,0)},t_0].\end{aligned}$$

In the previous equations $V^{(1,0)}$ and $V^{(2,0)}$ are the time translation of properties $V^{(1)}$ and $V^{(2)}$ to a common time t_0 [i.e., $V^{(1,0)} = \widehat{U}(t_0,t_1)V^{(1)}$ and $V^{(2,0)} = \widehat{U}(t_0,t_2)V^{(2)}$].

It seems natural at this point to consider descriptions involving properties $V^{(1)}$ at t_1 generated by the atomic properties $V_i^{(1)}$ $(i \in \sigma^{(1)})$ and properties $V^{(2)}$ at t_2 generated by the atomic properties $V_\mu^{(2)}$ $(\mu \in \sigma^{(2)})$, only if $V^{(1)}$ and $V^{(2)}$ are *compatible* when they are time translated to a common time t_0. This will be the case if the corresponding time translated projectors $\widehat{\Pi}_i^{(1,0)} = \widehat{U}(t_0,t_1)\widehat{\Pi}_i^{(1)}\widehat{U}^{-1}(t_0,t_1)$ and $\widehat{\Pi}_\mu^{(2,0)} = \widehat{U}(t_0,t_2)\widehat{\Pi}_\mu^{(2)}\widehat{U}^{-1}(t_0,t_2)$ commute (i.e., $[\widehat{\Pi}_i^{(1,0)},\widehat{\Pi}_\mu^{(2,0)}] = 0$).

Therefore the projectors $\widehat{\Pi}_{i\mu}^{(0)} \equiv \widehat{\Pi}_i^{(1,0)}\widehat{\Pi}_\mu^{(2,0)}$ satisfy

$$\begin{aligned}\widehat{\Pi}_{i\mu}^{(0)}\widehat{\Pi}_{j\nu}^{(0)} &= \widehat{\Pi}_i^{(1,0)}\widehat{\Pi}_\mu^{(2,0)}\widehat{\Pi}_j^{(1,0)}\widehat{\Pi}_\nu^{(2,0)}\\ &= \widehat{\Pi}_i^{(1,0)}\widehat{\Pi}_j^{(1,0)}\widehat{\Pi}_\mu^{(2,0)}\widehat{\Pi}_\nu^{(2,0)}\\ &= \delta_{ij}\delta_{\mu\nu}\widehat{\Pi}_{j\nu}^{(0)}.\end{aligned}$$

These projectors $\widehat{\Pi}_{i\mu}^{(0)}$, with $i \in \sigma^{(1)}$ and $\mu \in \sigma^{(2)}$, correspond to a set $B^{(0)}$ of atomic properties $V_{i\mu}^{(0)} \equiv V_i^{(1,0)} \cap V_\mu^{(2,0)}$, the generators of a context $C_{B^{(0)}}$ at the time t_0. All the expressions involving properties of the contexts $C_{B^{(1)}}$ at t_1 and $C_{B^{(2)}}$ at t_2 can be written in terms of properties generated by $B^{(0)}$ at the single time t_0. The set of equivalence classes of events obtained in this way is a Boolean lattice and the Born rule can be generalized to define

$$\begin{aligned}\Pr([V^{(1)},t_1] \wedge [V^{(2)},t_2]) &\equiv \Pr([V^{(1,0)} \cap V^{(2,0)},t_0])\\ &= \mathrm{Tr}\big(\widehat{\rho}_{t_0}\widehat{\Pi}^{(1,0)}\widehat{\Pi}^{(2,0)}\big),\\ \Pr([V^{(1)},t_1] \vee [V^{(2)},t_2]) &\equiv \Pr([V^{(1,0)} + V^{(2,0)},t_0])\\ &= \mathrm{Tr}\big(\widehat{\rho}_{t_0}\big\{\widehat{\Pi}^{(1,0)}+\widehat{\Pi}^{(2,0)}-\widehat{\Pi}^{(1,0)}\widehat{\Pi}^{(2,0)}\big\}\big)\end{aligned}$$

This is a well-defined probability, satisfying the Kolmogorov conditions. In this way we have obtained a quantum formalism for computing the probabilities of expressions involving properties at different times.

This formalism can be summarized in the following two postulates, which we propose to incorporate to generalize the usual formalism of quantum mechanics:

Postulate 1. Descriptions of quantum systems involving different properties at different times can be considered only if the set of properties at each time belong to a single context, and if the projectors corresponding to the generators of the contexts at each time commute when they are time translated to a single common time.

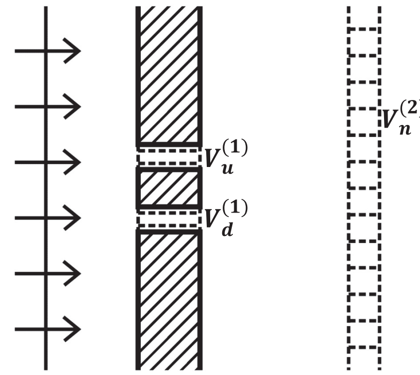

FIG. 3. The double-slit experiment.

Postulate 2. The probability for the conjunction or disjunction of properties at different times is obtained by first time translating the properties to a common time and then computing the probability for the ordinary conjunction or disjunction using the Born rule.

Our formalism will be applied in the next section to describe the double-slit experiment with and without a measurement instrument detecting which slit the particle has passed through.

IV. THE DOUBLE-SLIT EXPERIMENT WITHOUT MEASUREMENT INSTRUMENTS

In this section we analyze the experiment in which a quantum particle is passing through a double slit. We assume there are no measurement instruments detecting the particle.

Let us consider the regions $V_u^{(1)}$, $V_d^{(1)}$, and $V_n^{(2)}$ of Fig. 3. We want to discuss the possibility of giving a description of the system in which it would be meaningful to state that "the particle is in the region $V_u^{(1)}$ (the upper slit) at time t_1, and it is in the region $V_n^{(2)}$ of the vertical zone at time t_2."We are going to prove that the quoted expression involves properties at two different times which are not compatible, i.e., they do not satisfy Postulate 1 formulated at the end of the previous section.

We first define the relevant properties at times t_1 and t_2. For the time t_1 we consider the properties represented by the projectors

$$\widehat{\Pi}_u^{(1)} = \int_{V_u^{(1)}} d^3r\ |\overline{r}\rangle\langle\overline{r}|, \quad \widehat{\Pi}_d^{(1)} = \int_{V_d^{(1)}} d^3r|\overline{r}\rangle\langle\overline{r}|,$$

corresponding to the particle located in the upper or the lower slit. These two properties are represented by orthogonal projectors and may be part of an ordinary context for the time t_1.

For the time t_2 we need properties represented by the projectors

$$\widehat{\Pi}_n^{(2)} = \int_{V_n^{(2)}} d^3r|\overline{r}\rangle\langle\overline{r}|,$$

corresponding to the particle located in the region $V_n^{(2)}$.

We are going to prove that the projectors corresponding to both slits at t_1 and the projectors corresponding to the different regions $V_n^{(2)}$ at t_2 do not commute when they are translated to a common time.

If the arbitrary common time is chosen to be $t_0 = t_1$, the projectors $\widehat{\Pi}_n^{(2)}$ translated from t_2 to t_1 are given by

$$\widehat{\Pi}_n^{(1)} = \widehat{U}(t_1,t_2)\widehat{\Pi}_n^{(2)}\widehat{U}^{-1}(t_1,t_2).$$

To prove that $\widehat{\Pi}_u^{(1)}$ and $\widehat{\Pi}_n^{(1)}$ do not commute, it is enough to verify the noncommutation when the projectors act on a particular vector $|\phi_1\rangle$, which we choose located in the double slit. For this vector we obtain

$$\begin{aligned}\widehat{\Pi}_u^{(1)}\widehat{\Pi}_n^{(1)}|\phi_1\rangle &= \widehat{\Pi}_u^{(1)}\widehat{U}^{-1}(t_2,t_1)\widehat{\Pi}_n^{(2)}\widehat{U}(t_2,t_1)|\phi_1\rangle \\ &= \widehat{\Pi}_u^{(1)}\widehat{U}^{-1}(t_2,t_1)\widehat{\Pi}_n^{(2)}|\phi_2\rangle.\end{aligned}$$

The vector $|\phi_2\rangle$, obtained from $|\phi_1\rangle$ using the Schrödinger equation, has zones of destructive interference. The wave function $\langle \overline{r}|\phi_2\rangle$ vanishes for some region $V_{n^*}^{(2)}$and therefore

$$\widehat{\Pi}_u^{(1)}\widehat{\Pi}_{n^*}^{(1)}|\phi_1\rangle = 0.$$

We now consider the product of the projectors in different order,

$$\widehat{\Pi}_{n^*}^{(1)}\widehat{\Pi}_u^{(1)}|\phi_1\rangle = \widehat{U}^{-1}(t_2,t_1)\widehat{\Pi}_{n^*}^{(2)}\widehat{U}(t_2,t_1)\widehat{\Pi}_u^{(1)}|\phi_1\rangle.$$

The vector $\widehat{\Pi}_u^{(1)}|\phi_1\rangle$ represents the state we would have at t_1 when only the upper slit is open. Its time evolution $\widehat{U}(t_2,t_1)\widehat{\Pi}_u^{(1)}|\phi_1\rangle$ does not have destructive interferences, as it will not vanish when acted upon by $\widehat{U}^{-1}(t_2,t_1)\widehat{\Pi}_{n^*}^{(2)}$. Then, $\widehat{\Pi}_{n^*}^{(1)}\widehat{\Pi}_u^{(1)}|\phi_1\rangle$ do not vanish. As we have found a vector $|\phi_1\rangle$ for which $\widehat{\Pi}_u^{(1)}\widehat{\Pi}_{n^*}^{(1)}|\phi_1\rangle \neq \widehat{\Pi}_{n^*}^{(1)}\widehat{\Pi}_u^{(1)}|\phi_1\rangle$, we conclude that $\widehat{\Pi}_{n^*}^{(1)}$ and $\widehat{\Pi}_u^{(1)}$ do not commute.

According to our Postulate 1, we conclude that it is not possible to give a description of the quantum system suitable to talk about which slit passed the particle before reaching one of the regions of the vertical zone. This impossibility is *deduced* from Postulate 1.

V. THE DOUBLE-SLIT EXPERIMENT WITH MEASUREMENT INSTRUMENTS

We now consider a modified double-slit experiment with a detector A located in the slit zone, with its pointer variable indicating a_u (a_d) if the particle is detected passing through the upper (lower) slit. The detector A interacts with the particle in the small time interval $(t_1,t_1+\Delta_1)$. A second detector B is located in the vertical zone to the right of the double slit with a pointer variable indicating the value b_n if the particle is detected in the zone $V_n^{(2)}$. The detector B interacts with the particle in the small time interval $(t_2,t_2+\Delta_2)$. The system $S+A+B$, composed of the particle and the detectors, is initially in a state represented by the vector $|\varphi_{t_1}\rangle|a_0\rangle|b_0\rangle \in \mathcal{H}_S\otimes\mathcal{H}_A\otimes\mathcal{H}_B$, where $|a_0\rangle$ and $|b_0\rangle$ are the initial states of the instruments.

The following equations give the state vector of the composed system for different times:

$$\begin{aligned}\Psi(t_1) &= |\varphi_{t_1}\rangle|a_0\rangle|b_0\rangle,\\ \Psi(t_1+\Delta_1) &= \big(\widehat{\Pi}_u^{(1)}|\varphi_{t_1}\rangle\big)|a_u\rangle|b_0\rangle + \big(\widehat{\Pi}_d^{(1)}|\varphi_{t_1}\rangle\big)|a_d\rangle|b_0\rangle,\\ \Psi(t_2) &= \big[\widehat{U}(t_2,t_1+\Delta_1)\widehat{\Pi}_u^{(1)}|\varphi_{t_1}\rangle\big]|a_u\rangle|b_0\rangle\\ &\quad + \big[\widehat{U}(t_2,t_1+\Delta_1)\widehat{\Pi}_d^{(1)}|\varphi_{t_1}\rangle\big]|a_d\rangle|b_0\rangle,\\ \Psi(t_2+\Delta_2) &= \sum_n\big[\widehat{\Pi}_n^{(2)}\widehat{U}(t_2,t_1+\Delta_1)\widehat{\Pi}_u^{(1)}|\varphi_{t_1}\rangle\big]|a_u\rangle|b_n\rangle\\ &\quad + \sum_n\big[\widehat{\Pi}_n^{(2)}\widehat{U}(t_2,t_1+\Delta_1)\widehat{\Pi}_d^{(1)}|\varphi_{t_1}\rangle\big]|a_d\rangle|b_n\rangle,\end{aligned}$$

where $\widehat{U}(t_2,t_1+\Delta_1)$ is the free time evolution operator for the particle from $t_1+\Delta_1$ to t_2.

We are now going to consider the possibility of giving a description involving the pointer indication of detector A at time $t_1+\Delta_1$ and the pointer indication of detector B at time $t_2+\Delta_2$. The relevant properties for the time $t_1+\Delta_1$ are represented by the projectors

$$\widehat{\Pi}_{a_u} = \widehat{I}_S\otimes|a_u\rangle\langle a_u|\otimes\widehat{I}_B,\quad \widehat{\Pi}_{a_d} = \widehat{I}_S\otimes|a_d\rangle\langle a_d|\otimes\widehat{I}_B.$$

For the time $t_2+\Delta_2$ the relevant properties correspond to the projectors

$$\widehat{\Pi}_{b_n} = \widehat{I}_S\otimes\widehat{I}_A\otimes|b_n\rangle\langle b_n|.$$

To know if these properties can be included in a description of the system according to our Postulate 1, we should verify if the corresponding projectors commute when translated to a common time, which we choose to be t_1. We obtain

$$\begin{aligned}\widehat{\Pi}_{a_u}^{(1)}\widehat{\Pi}_{b_n}^{(1)} &= \widehat{U}^{-1}(t_1+\Delta_1,t_1)\widehat{\Pi}_{a_u}\widehat{U}(t_1+\Delta_1,t_1)\\ &\quad\times\widehat{U}^{-1}(t_2+\Delta_2,t_1)\widehat{\Pi}_{b_n}\widehat{U}(t_2+\Delta_2,t_1)\\ &= \widehat{U}^{-1}(t_1+\Delta_1,t_1)\widehat{\Pi}_{a_u}\widehat{U}(t_1+\Delta_1,t_2+\Delta_2)\\ &\quad\times\widehat{\Pi}_{b_n}\widehat{U}(t_2+\Delta_2,t_1)\\ &= \widehat{U}^{-1}(t_1+\Delta_1,t_1)\widehat{U}(t_1+\Delta_1,t_2+\Delta_2)\\ &\quad\times\widehat{\Pi}_{a_u}\widehat{\Pi}_{b_n}\widehat{U}(t_2+\Delta_2,t_1)\\ &= \widehat{U}^{-1}(t_2+\Delta_2,t_1)\widehat{\Pi}_{a_u}\widehat{\Pi}_{b_n}\widehat{U}(t_2+\Delta_2,t_1),\end{aligned}$$

$$\begin{aligned}\widehat{\Pi}_{b_n}^{(1)}\widehat{\Pi}_{a_u}^{(1)} &= \widehat{U}^{-1}(t_2+\Delta_2,t_1)\widehat{\Pi}_{b_n}\widehat{U}(t_2+\Delta_2,t_1)\\ &\quad\times\widehat{U}^{-1}(t_1+\Delta_1,t_1)\widehat{\Pi}_{a_u}\widehat{U}(t_1+\Delta_1,t_1)\\ &= \widehat{U}^{-1}(t_2+\Delta_2,t_1)\widehat{\Pi}_{b_n}\widehat{U}(t_2+\Delta_2,t_1+\Delta_1)\\ &\quad\times\widehat{\Pi}_{a_u}\widehat{U}(t_1+\Delta_1,t_1)\\ &= \widehat{U}^{-1}(t_2+\Delta_2,t_1)\widehat{\Pi}_{b_n}\widehat{\Pi}_{a_u}\widehat{U}(t_2+\Delta_2,t_1)\\ &= \widehat{U}^{-1}(t_2+\Delta_2,t_1)\widehat{\Pi}_{a_u}\widehat{\Pi}_{b_n}\widehat{U}(t_2+\Delta_2,t_1),\end{aligned}$$

and therefore we find that $\widehat{\Pi}_{b_n}^{(1)}$ and $\widehat{\Pi}_{a_u}^{(1)}$ commute. The same result is obtained for $\widehat{\Pi}_{b_n}^{(1)}$ and $\widehat{\Pi}_{a_d}^{(1)}$.

Therefore, according to Postulate 1, it is possible to give a description of the composed system $S+A+B$ involving the pointer indication of instrument A at the time $t_1+\Delta_1$ *and* the pointer indication of instrument B at the time $t_2+\Delta_2$.

According to our Postulate 2, the probability for the particle to be detected in the upper slit by instrument A is

$$\begin{aligned}\Pr\big(\big[\widehat{\Pi}_{a_u}^{(1)}\mathcal{H},t_1\big]\big) &= \mathrm{Tr}\big(\widehat{\rho}_{t_1}\widehat{\Pi}_{a_u}^{(1)}\big) = \langle\varphi_{t_1}|\langle a_0|\langle b_0|\widehat{U}^{-1}(t_1+\Delta_1,t_1)\\ &\quad\times\widehat{\Pi}_{a_u}\widehat{U}(t_1+\Delta_1,t_1)|\varphi_{t_1}\rangle|a_0\rangle|b_0\rangle\\ &= \langle\varphi_{t_1}|\widehat{\Pi}_u^{(1)}|\varphi_{t_1}\rangle. \qquad (10)\end{aligned}$$

Moreover, the probability for the particle to be detected in the upper slit by A at time $t_1+\Delta_1$ *and* to be detected in the volume $V_n^{(2)}$ by B at the time $t_2 + \Delta_2$ is

$$\begin{aligned}\Pr\left([\widehat{\Pi}_{b_n}^{(1)}\widehat{\Pi}_{a_u}^{(1)}\mathcal{H},t_1]\right) &= \mathrm{Tr}\left(\widehat{\rho}_{t_1}\widehat{\Pi}_{b_n}^{(1)}\widehat{\Pi}_{a_u}^{(1)}\right)\\ &= \langle\varphi_{t_1}|\langle a_0|\langle b_0|\widehat{U}^{-1}(t_2+\Delta_2,t_1)\\ &\quad\times\widehat{\Pi}_{a_u}\widehat{\Pi}_{b_n}\widehat{U}(t_2+\Delta_2,t_1)|\varphi_{t_1}\rangle|a_0\rangle|b_0\rangle\\ &= \langle\varphi_{u,t_2}|\widehat{\Pi}_n^{(2)}|\varphi_{u,t_2}\rangle, \qquad (11)\end{aligned}$$

where $|\varphi_{u,t_2}\rangle \equiv \widehat{U}(t_2,t_1+\Delta_1)\widehat{\Pi}_u^{(1)}|\varphi_{t_1}\rangle$.

Taking into account Eqs. (10) and (11), the probability for the particle to be detected in the volume $V_n^{(2)}$ by instrument B at the time $t_2 + \Delta_2$, *conditional* on having been detected in the upper slit by instrument A at the time $t_1 + \Delta_1$, is

$$\begin{aligned}\Pr\left([\widehat{\Pi}_{b_n}^{(1)}\widehat{\Pi}_{a_u}^{(1)}\mathcal{H},t_1]\,|\,[\widehat{\Pi}_{a_u}^{(1)}\mathcal{H},t_1]\right) &\equiv \frac{\langle\varphi_{u,t_2}|\widehat{\Pi}_n^{(2)}|\varphi_{u,t_2}\rangle}{\langle\varphi_{t_1}|\widehat{\Pi}_u^{(1)}|\varphi_{t_1}\rangle}\\ &= \langle\widetilde{\varphi_{u,t_2}}|\widehat{\Pi}_n^{(2)}|\widetilde{\varphi_{u,t_2}}\rangle,\end{aligned}$$

where $|\widetilde{\varphi_{u,t_2}}\rangle \equiv \widehat{U}(t_2,t_1+\Delta_1)|\widetilde{\varphi_{u,t_1}}\rangle$ and $|\widetilde{\varphi_{u,t_1}}\rangle = \frac{\widehat{\Pi}_u^{(1)}|\varphi_{t_1}\rangle}{||\widehat{\Pi}_u^{(1)}|\varphi_{t_1}\rangle||}$.

The state vector $|\widetilde{\varphi_{u,t_2}}\rangle$ is the *free* time evolution from time $t_1 + \Delta_1$ to time t_2 of the normalized state vector $|\widetilde{\varphi_{u,t_1}}\rangle$, emerging at time $t_1 + \Delta_1$ from the upper slit. This is a well-known result, showing no interference pattern. Usually it is obtained from the collapse postulate. With our formalism we obtained the same result, but without invoking the collapse postulate.

VI. CONCLUSIONS

In this paper we have developed a formalism for a description of classical and quantum systems involving different properties at different times. We first presented the classical case, where each property is identified with a subset of the phase space, and we defined an event as a property at a given time. The time translation of an event is obtained from the time evolution generated the by Hamilton equations.

Events connected by a time translation are not considered as essentially different and they belong to the same equivalence class. A Boolean lattice structure is defined on the equivalence classes of events. With this structure, the probability for expressions involving different properties at different times is well defined as a generalization of the usual probability of classical statistical mechanics.

We followed a similar approach for the quantum case, where the properties are identified with closed subspaces of the Hilbert space and events are properties at a given time. The time translation of these events is generated by the Liouville–von Neumann equation. Events which are connected by a time translation have the same probability given by the Born rule and satisfy an equivalence relation. A very special feature of ordinary quantum mechanics is that it is necessary to restrict to a context the set of properties that can simultaneously be considered. As this context is Boolean the Born rule gives probabilities satisfying the Kolmogorov axioms.

When properties at different times are involved, it is not only necessary to consider properties that at each time belong to a context, but also to impose a compatibility condition between properties at different times, namely, when they are translated to a common time, their corresponding projectors should commute (Postulate 1). With this postulate a Boolean lattice of equivalence classes of events can be obtained and a probability can be defined using Postulate 2.

Finally, we applied our formalism to describe the well-known double-slit experiment. Without measurement instruments, we deduced from Postulate 1 of our formalism the impossibility of a description of the trajectory of the particle.

We also considered a different physical process in which there is a measurement instrument at the double slit and another measurement instrument at a vertical zone in front of the slits. In this case we described the particle and both measurement instruments as parts of a compound quantum system. We found that the pointer indications of the measurement instruments are compatible properties at different times (i.e., they satisfy Postulate 1). The conditional probability we obtained is the same as the one obtained using the collapse postulate, but we obtained it without this postulate.

Using Postulate 2 we compute the probability to detect the particle in front of the double slit, conditional on having detected it going through one of the slits. We reestablish noninterference results without invoking the collapse of the wave function.

Our description of the double-slit experiment shows that this formalism is suitable for dealing with physical situations involving properties at different times. Preliminary results for the delayed choice experiment are encouraging us to extend our work along this line.

APPENDIX A: DEFINITIONS

Equivalence relation. Given a set A and a binary relation R on A, we say that $R \subset A \times A$ is an equivalence relation if it satisfies the following properties:

Reflexivity: $\forall a \in A, aRa$.

Symmetry: $\forall a,b \in A, aRb \Rightarrow bRa$.

Transitivity: $\forall a,b,c \in A, aRb, bRc \Rightarrow aRc$.

Order relation. Given a set A and a binary relation R on A, we say that $R \subset A \times A$ is an order relation if it satisfies the following properties:

Reflexivity: $\forall a \in A, aRa$.

Antisymmetry: $\forall a,b \in A, aRb, bRa \Rightarrow a = b$.

Transitivity: $\forall a,b,c \in A, aRb, bRc \Rightarrow aRc$.

Lattice. Given a set A and an order relation $\leqslant$ on A, the pair $(A, \leqslant)$ is a lattice if it satisfies the following properties: (i) $\forall a,b \in A, \exists s \in A s = \mathrm{Sup}\{a,b\} \equiv a \vee b$ and (ii) $\forall a,b \in A, \exists t \in A / t = \mathrm{Inf}\{a,b\} \equiv a \wedge b$, where $\mathrm{Sup}\{a,b\}$ and $\mathrm{Inf}\{a,b\}$ are the supremum and the infimum of $\{a,b\}$, respectively.

A lattice satisfies a number of properties. The more important are the following:

Idempotency: $a \wedge a = a$, $a \vee a = a$.

Commutativity: $a \wedge b = b \wedge a$, $a \vee b = b \vee a$.

Associativity: $a \wedge (b \wedge c) = (a \wedge b) \wedge c$, $a \vee (b \vee c) = (a \vee b) \vee c$.

Absorption: $a \wedge (a \vee b) = a$, $a \vee (a \wedge b) = a$.

Distributive inequality: $a \wedge (b \vee c) \geqslant (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) \leqslant (a \vee b) \wedge (a \vee c)$.

Distributive lattice. A lattice $(A, \leqslant)$ is distributive if the distributive inequalities are equalities, i.e., $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)\ \forall a,b,c \in A$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)\ \forall a,b,c \in A$.

Null and universal element of a lattice. Let $(A, \leqslant)$ be a lattice.

An element $a \in A$ is a null element if $\forall b \in A$ it is $a \leqslant b$.

If a null element exists, then it is unique and it is denoted as 0.

An element $a \in A$ is a universal element if $\forall b \in A$ it is $b \leqslant a$.

If a universal element exists, then it is unique and it is denoted as u.

Complemented lattice. $(A, \leqslant)$ is a complemented lattice if it is a lattice with null element 0, universal element u, and with the following property: $\forall a \in A$, $\exists a' \in A/\mathrm{Inf}\{a,a'\} = 0$ and $\mathrm{Sup}\{a,a'\} = u$. The element a' is called a complement of a.

In a complemented lattice an element can have more than one complement. However, if the complemented lattice is also distributive, each element has a unique complement, called $a^{\perp}$, and it satisfies $(a^{\perp})^{\perp} = a$. The last condition implies that the lattice is orthocomplemented. The lattices that are complemented and distributive are called Boolean lattices.

Probability. Given a Boolean lattice $(A, \leqslant)$, then $P : A \to \mathbb{R}$ is a probability if

(i) $P(a) \geqslant 0$, $\forall a \in A$.

(ii) $P(u) = 1$, where $u \in A$ is the universal element of A.

(iii) If $a \wedge b = 0$, then $P(a \vee b) = P(a) + P(b)$.

APPENDIX B: PROPOSITIONS

In this section we prove some of the results that are stated in the paper. The basic definitions needed to understand the propositions are summarized in the Appendix A.

Proposition 1. $\sim \subset E \times E$ is an equivalence relation.

Reflexivity. $\widehat{U}(t,t)V = \widehat{I}V = V$, therefore $(V,t) \sim (V,t)$.

Symmetry. If $(V',t') \sim (V,t)$, then $V' = \widehat{U}(t',t)V$. Therefore $V = \widehat{U}^{-1}(t',t)V' = \widehat{U}(t,t')V'$ and then $(V,t) \sim (V',t')$.

Transitivity. If $(V,t) \sim (V',t')$ and $(V',t') \sim (V'',t'')$, then $\widehat{U}(t',t)V = V'$ and $\widehat{U}(t'',t')V' = V''$. Therefore $\widehat{U}(t'',t')\widehat{U}(t',t)V = \widehat{U}(t'',t)V = V''$, then $(V,t) \sim (V'',t'')$.

Therefore, $\sim$ is an equivalence relation. ■

Proposition 2. $\leqslant$ is a well-defined relation in $[E]$.

If $[V_1,t_1] \leqslant [V_2,t_2]$, then $\widehat{U}(t_2,t_1)V_1 \subset V_2$. Given $(V_1',t_1') \sim (V_1,t_1)$ and $(V_2',t_2') \sim (V_2,t_2)$ we have $\widehat{U}(t_1,t_1')V_1' = V_1$ and $\widehat{U}(t_2,t_2')V_2' = V_2$. Therefore, $\widehat{U}(t_2,t_1)V_1 = \widehat{U}(t_2,t_1)\widehat{U}(t_1,t_1')V_1' = \widehat{U}(t_2,t_1')V_1' \subset V_2 = \widehat{U}(t_2,t_2')V_2'$. Then $\widehat{U}(t_2',t_2)\widehat{U}(t_2,t_1')V_1' \subset V_2'$, hence $\widehat{U}(t_2',t_1')V_1' \subset V_2'$. Therefore $[V_1',t_1'] \leqslant [V_2',t_2']$. ■

Proposition 3. $\leqslant \subset [E] \times [E]$ is an order relation.

Reflexivity. $\widehat{U}(t,t)V = V$, then $[V,t] \leqslant [V,t]$.

Transitivity. If $[V,t] \leqslant [V',t']$ and $[V',t'] \leqslant [V'',t'']$, then $\widehat{U}(t',t)V \subset V'$ and $\widehat{U}(t'',t')V' \subset V''$. Therefore $\widehat{U}(t'',t')\widehat{U}(t',t)V \subset \widehat{U}(t'',t')V' \subset V''$. Then $\widehat{U}(t'',t)V \subset V''$, and therefore $[V,t] \leqslant [V'',t'']$.

Antisymmetry. If $[V,t] \leqslant [V',t']$ and $[V',t'] \leqslant [V,t]$, then $\widehat{U}(t',t)V \subset V'$ and $\widehat{U}(t,t')V' \subset V$. Therefore $\widehat{U}(t',t)V \subset V'$ and $V' \subset \widehat{U}(t',t)V$. Then $\widehat{U}(t',t)V = V'$, hence $[V,t] = [V',t']$. As $\leqslant$ is reflexive, antisymmetric, and transitive, it is an order relation. ■

Proposition 4. $\mathrm{Sup}\{[V,t],[V',t']\} = [\widehat{U}(t_0,t)V + \widehat{U}(t_0,t')V',t_0]$.

(i) $\widehat{U}(t_0,t)V \subset \widehat{U}(t_0,t)V + \widehat{U}(t_0,t')V'$. Then $[V,t] \leqslant [\widehat{U}(t_0,t)V + \widehat{U}(t_0,t')V',t_0]$.

(ii) $\widehat{U}(t_0,t')V' \subset \widehat{U}(t_0,t)V + \widehat{U}(t_0,t')V'$. Then $[V',t'] \leqslant [\widehat{U}(t_0,t)V + \widehat{U}(t_0,t')V',t_0]$.

(iii) Consider $[V'',t'']$ satisfying $[V,t] \leqslant [V'',t'']$ and $[V',t'] \leqslant [V'',t'']$. Then $\widehat{U}(t'',t)V \subset V''$ and $\widehat{U}(t'',t')V' \subset V''$. Therefore $\widehat{U}(t'',t)V + \widehat{U}(t'',t')V' \subset V''$, and then $\widehat{U}(t'',t_0)(\widehat{U}(t_0,t)V + \widehat{U}(t_0,t')V') \subset V''$. Hence, $[\widehat{U}(t_0,t)V + \widehat{U}(t_0,t')V',t_0] \leqslant [V'',t'']$. Therefore, $\mathrm{Sup}\{[V,t],[V',t']\} = [\widehat{U}(t_0,t)V + \widehat{U}(t_0,t')V',t_0]$. ■

Proposition 5. $\mathrm{Inf}\{[V,t],[V',t']\} = [\widehat{U}(t_0,t)V \cap \widehat{U}(t_0,t')V',t_0]$.

(i) $\widehat{U}(t_0,t)V \cap \widehat{U}(t_0,t')V' \subset \widehat{U}(t_0,t)V$. Then $[\widehat{U}(t_0,t)V \cap \widehat{U}(t_0,t')V',t_0] \leqslant [\widehat{U}(t_0,t)V,t_0] = [V,t]$.

(ii) $\widehat{U}(t_0,t)V \cap \widehat{U}(t_0,t')V' \subset \widehat{U}(t_0,t')V$. Then $[\widehat{U}(t_0,t)V \cap \widehat{U}(t_0,t')V',t_0] \leqslant [\widehat{U}(t_0,t')V,t_0] = [V,t']$.

(iii) Consider $[V'',t'']$ satisfying $[V'',t''] \leqslant [V,t]$ and $[V'',t''] \leqslant [V',t']$. Then $V'' \subset \widehat{U}(t'',t)V$ and $V'' \subset \widehat{U}(t'',t')V'$. Therefore $V'' \subset [\widehat{U}(t'',t)V \cap \widehat{U}(t'',t')V'] = \widehat{U}(t'',t_0)[\widehat{U}(t_0,t)V \cap \widehat{U}(t_0,t')V']$. Then $[V'',t''] \leqslant [\widehat{U}(t_0,t) V \cap \widehat{U}(t_0,t')V',t_0]$. Therefore, $\mathrm{Inf}\{[V,t],[V',t']\} = [\widehat{U}(t_0,t) V \cap \widehat{U}(t_0,t')V',t_0]$. ■

Proposition 6. $[\{0_{\mathcal{H}}\},t]$ and $[\mathcal{H},t]$ are the zero and the universal elements of $([E], \leqslant)$.

We have $\widehat{U}(t',t)\{0_{\mathcal{H}}\} = \{0_{\mathcal{H}}\} \subset V'$ for every subspace V' of $\mathcal{H}$. Then $[\{0_{\mathcal{H}}\},t] \leqslant [V',t']$ for all $[V',t'] \in [E]$. Therefore $[\{0_{\mathcal{H}}\},t]$ is the zero element of the lattice $[E]$.

For each subspace V' we have $\widehat{U}(t,t')V' \subset \mathcal{H}$, and $[V',t'] \leqslant [\mathcal{H},t]$. Hence $[\mathcal{H},t]$ is the universal element of $[E]$. ■

Proposition 7. $([E], \leqslant)$ is a complemented lattice.

Given any $[V,t] \in [E]$ we obtain

$\mathrm{Sup}([V,t],[V^{\perp},t]) = [\widehat{U}(t,t)V + V^{\perp},t] = [V + V^{\perp},t] = [\mathcal{H},t]$.

$\mathrm{Inf}([V,t],[V^{\perp},t]) = [\widehat{U}(t,t)V \cap V^{\perp},t] = [V \cap V^{\perp},t] = [\{0_{\mathcal{H}}\},t]$.

Then, $[V^{\perp},t]$ is a complement of $[V,t]$.

Therefore, $([E], \leqslant)$ is a complemented lattice. ■

The previous results could have been obtained taking into account that $([E], \leqslant)$ and the lattice formed by the closed vector subspaces of $\mathcal{H}$ (with the inclusion as order relation) are isomorphic. However, the previous proofs are useful to understand the structure of $([E], \leqslant)$.

Proposition 8. If $\dim H \geqslant 2$,$([E], \leqslant)$ is not a distributive lattice.

If $\dim \mathcal{H} \geqslant 2$, we can consider two nonvanishing linearly independent vectors u and v and the nonvanishing vector $w = u + v$. We will call by U, V, and W the one-dimensional subspaces of $\mathcal{H}$ spanned by u, v, and w, respectively. U, V, and W are closed subspaces of $\mathcal{H}$, then $[W,t]$, $[U,t]$, and $[V,t]$ are elements of $[E]$, where t is an arbitrary time.

We obtain $[W,t] \wedge ([U,t] \vee [V,t]) = [W,t] \wedge [U + V,t] = [W,t]$ and also $([W,t] \wedge [U,t]) \vee ([W,t] \wedge [V,t]) = [\{0_{\mathcal{H}}\},t] \vee [\{0_{\mathcal{H}}\},t] = [\{0_{\mathcal{H}}\},t]$. Therefore $[W,t] \wedge ([U,t] \vee [V,t]) \neq ([W,t] \wedge [U,t]) \vee ([W,t] \wedge [V,t])$, hence $([E], \leqslant)$ is not a distributive lattice. ■

Proposition 9. Let us consider a set B of mutually orthogonal closed subspaces of H which expand the whole Hilbert space, i.e. $B \equiv \{V_i \mid i \in \sigma, V_i$ is a closed subspace of $H, V_i \perp V_j$ if $i \neq j, \sum_{i\in\sigma} V_i = H\}$. The set B generates the context C_B. Then $[E]_B \equiv \{[V,t_0] \in [E] \ / \ V \in C_B\}$ with the order relation $\leqslant$ is a Boolean sublattice of $([E], \leqslant)$.

Let us consider $[V,t_0]$ and $[V',t_0] \in [E]_B$. Then, V and V' are elements of the Boolean sublattice C_B. Therefore $V + V' \in C_B$ and $V \cap V' \in C_B$. Moreover, $[V + V',t_0] = [V,t_0] \vee [V',t_0] \in [E]_B$ and $[V \cap V',t_0] = [V,t_0] \wedge [V',t_0] \in [E]_B$. Therefore $([E]_B, \leqslant)$ is a lattice.

$\{0_{\mathcal{H}}\}$ and $\mathcal{H}$ are in C_B, then $[\{0_{\mathcal{H}}\},t_0] \in [E]_B$ and $[\mathcal{H},t_0] \in [E]_B$.

If $[V,t_0] \in [E]_B$, then $V \in C_B$ and $V^{\perp} \in C_B$. Therefore $[V^{\perp},t_0] \in [E]_B$.

Hence, $([E]_B, \leqslant)$ is a complemented lattice.

Now consider $[V_1,t_0]$, $[V_2,t_0]$, and $[V_3,t_0]$, three arbitrary elements of $[E]_B$. Then V_1, V_2, and V_3 are in C_B. As C_B is a distributive lattice of subspaces of the Hilbert space $\mathcal{H}$, we have $V_1 \cap (V_2 + V_3) = (V_1 \cap V_2) + (V_1 \cap V_3)$ and $V_1 + (V_2 \cap V_3) = (V_1 + V_2) \cap (V_1 + V_3)$.

Then $[V_1,t_0] \wedge ([V_2,t_0] \vee [V_3,t_0]) = [V_1,t_0] \wedge [(V_2 + V_3),t_0] = [V_1 \cap (V_2 \cup V_3),t_0] = [(V_1 \cap V_2) \cup (V_1 \cap V_3),t_0] = [(V_1 \cap V_2),t_0] \vee [(V_1 \cap V_3),t_0] = ([V_1,0] \wedge [V_2,t_0]) \vee ([V_1,t_0] \wedge [V_3,t_0])$.

We also obtain $[V_1,t_0] \vee ([V_2,t_0] \wedge [V_3,t_0]) = ([V_1,t_0] \vee [V_2,t_0]) \wedge ([V_1,t_0] \vee [V_3,t_0])$. Therefore, $([E]_B, \leqslant)$ is a Boolean sublattice of $([E], \leqslant)$. ■

Proposition 10. Given $B \equiv \{V_i \mid i \in \sigma, V_i$ is a closed subspace of $\mathcal{H}, V_i \perp V_j$ if $i \neq j, \sum_{i\in\sigma} V_i = \mathcal{H}\}$ and $[E]_B = \{[V,t_0] \in B \ / \ V \in C_B\}$, for each state $\widehat{\rho}_{t_0}$,$\Pr : [E]_B \to R$ given by $\Pr([V,t_0]) = \mathrm{Tr}(\widehat{\rho}_{t_0}\widehat{\Pi}_V)$ is a probability.

To prove that Pr is a probability we have to prove the following conditions:

(i) Given $[V,t_0] \in [E]_B$ we consider the projector $\widehat{\Pi}_V$ associated with the subspace V.

Consider $B_1 = \{|\varphi_i\rangle, \ i \in I\}$ and $B_2 = \{|\varphi_j\rangle, \ j \in J\}$ orthonormal bases of the subspaces V and $V^{\perp}$, respectively. Then $B_3 = B_1 \cup B_2$ is an orthonormal base of $\mathcal{H}$.

As $\widehat{\Pi}_V$ is the orthogonal projector associated with V, we have the following relations:

$\widehat{\Pi}_V|\varphi_i\rangle = |\varphi_i\rangle$, if $|\varphi_i\rangle \in B_1$.

$\widehat{\Pi}_V|\varphi_j\rangle = 0$, if $|\varphi_j\rangle \in B_2$.

Then $\Pr([V,t_0]) = \mathrm{Tr}(\widehat{\rho}_{t_0}\widehat{\Pi}_V) = \sum_{|\varphi\rangle\in B_3} \langle\varphi|\widehat{\rho}_{t_0}\widehat{\Pi}_V|\varphi\rangle = \sum_{|\varphi\rangle\in B_1} \langle\varphi|\widehat{\rho}_{t_0}|\varphi\rangle \geqslant 0$, because $\widehat{\rho}_{t_0}$ is positive.

(ii) Consider the projector $\widehat{\Pi}_{\mathcal{H}} = \widehat{I}$ associated with the subspace $\mathcal{H}$ and consider $B = \{|\varphi_k\rangle, \ k \in K\}$ the orthonormal base of $\mathcal{H}$. Then, $\Pr([\mathcal{H},t_0]) = \mathrm{Tr}(\widehat{\rho}_{t_0}\widehat{\Pi}_{\mathcal{H}}) = \mathrm{Tr}(\widehat{\rho}_{t_0}\widehat{I}) = Tr(\widehat{\rho}_{t_0}) = 1$.

(iii) Consider $[V_1,t_1]$, $[V_2,t_2] \in [E]_B$ such that $[V_1,t_1] \wedge [V_2,t_2] = [\{0_{\mathcal{H}}\},t_0]$.

$[V_1,t_1] = [\widetilde{V}_1,t_0]$, with $\widetilde{V}_1 = \widehat{U}(t_0,t_1)V_1$.

$[V_2,t_2] = [\widetilde{V}_2,t_0]$, with $\widetilde{V}_2 = \widehat{U}(t_0,t_2)V_2$.

Consider the projectors Π_1, Π_2, $\widetilde{\Pi}_1$, and $\widetilde{\Pi}_2$ associated with the subspaces V_1, V_2, $\widetilde{V}_1$, and $\widetilde{V}_2$, respectively. These projectors are related in the following way:

$$\Pi_1 = \widehat{U}(t_1,t_0)\widetilde{\Pi}_1\widehat{U}(t_0,t_1), \quad \Pi_2 = \widehat{U}(t_2,t_0)\widetilde{\Pi}_2\widehat{U}(t_0,t_2).$$

As $\widetilde{V}_1, \widetilde{V}_2 \in C_B$, then the projectors of $\widetilde{V}_1 + \widetilde{V}_2$ and $\widetilde{V}_1 \cap \widetilde{V}_2$ are given by

$$\widetilde{\Pi}_{\widetilde{V}_1+\widetilde{V}_2} = \widetilde{\Pi}_1 + \widetilde{\Pi}_2 - \widetilde{\Pi}_1\widetilde{\Pi}_2, \quad \widetilde{\Pi}_{\widetilde{V}_1\cap\widetilde{V}_2} = \widetilde{\Pi}_1\widetilde{\Pi}_2.$$

Moreover, $[\widetilde{V}_1,t_0] \wedge [\widetilde{V}_2,t_0] = [\widetilde{V}_1 \cap \widetilde{V}_2,t_0] = [\{0_{\mathcal{H}}\},t_0]$, then $\widetilde{V}_1 \cap \widetilde{V}_2 = \{0_{\mathcal{H}}\}$. Hence, $\widetilde{\Pi}_{\widetilde{V}_1\cap\widetilde{V}_2} = \widetilde{\Pi}_1\widetilde{\Pi}_2 = 0$. Then $\widetilde{\Pi}_{\widetilde{V}_1+\widetilde{V}_2} = \widetilde{\Pi}_1 + \widetilde{\Pi}_2$.

Therefore,

$$\begin{aligned}
&\Pr([V_1,t_1] \vee [V_2,t_2]) \\
&= \Pr([\widetilde{V}_1,t_0] \vee [\widetilde{V}_2,t_0]) = \Pr([\widetilde{V}_1 + \widetilde{V}_2,t_0]) \\
&= \mathrm{Tr}[\widehat{\rho}_{t_0}(\widetilde{\Pi}_1 + \widetilde{\Pi}_2)] = \mathrm{Tr}(\widehat{\rho}_{t_0}\widetilde{\Pi}_1) + \mathrm{Tr}(\widehat{\rho}_{t_0}\widetilde{\Pi}_2) \\
&= \mathrm{Tr}[\widehat{\rho}_{t_0}\widehat{U}(t_0,t_1)\Pi_1\widehat{U}(t_1,t_0)] + \mathrm{Tr}[\widehat{\rho}_{t_0}\widehat{U}(t_0,t_2)\Pi_2\widehat{U}(t_2,t_0)] \\
&= \mathrm{Tr}(\widehat{\rho}_{t_1}\Pi_1) + \mathrm{Tr}(\widehat{\rho}_{t_2}\Pi_2) = \Pr([V_1,t_1]) + \Pr([V_2,t_2]).
\end{aligned}$$

Then, $\Pr([V,t_0]) = \mathrm{Tr}(\widehat{\rho}_{t_0}\widehat{\Pi}_V)$ is a well-defined probability. ■

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