MATRIX GEGENBAUER POLYNOMIALS: THE $2 \times 2$ FUNDAMENTAL CASES

INÉS PACHARONI AND IGNACIO ZURRIÁN

Abstract. In this paper, we exhibit explicitly a sequence of $2 \times 2$ matrix valued orthogonal polynomials with respect to a weight $W_{p,n}$, for any pair of real numbers $p$ and $n$ such that $0 < p < n$. The entries of these polynomials are expressed in terms of the Gegenbauer polynomials $C_\lambda^k$. Also the corresponding three-term recursion relations are given and we make some studies of the algebra of differential operators associated with the weight $W_{p,n}$.

1. Introduction

The theory of matrix valued orthogonal polynomials, without any consideration of differential equations, goes back to [18] and [19]. In [3], the study of the matrix valued orthogonal polynomials that are eigenfunctions of certain second order symmetric differential operators was started. The first explicit examples of such polynomials were given in [8], [9], [7], [10] and [4]. See also [5], [6], [1], [2], and the references given there.

On the two dimensional sphere $S^2 = \text{SO}(3)/\text{SO}(2)$, the harmonic analysis with respect to the action of the orthogonal group is contained in the classical theory of the spherical harmonics. In spherical coordinates, the zonal spherical functions on $S^2$ are the Legendre polynomials. More generally, in the case of the $n$-dimensional sphere $S^n$ the zonal spherical functions are given in terms of Gegenbauer (or ultraspherical) polynomials of parameter $(n-1)/2$.

This fruitful connection between orthogonal polynomials and representation theory of compact Lie groups is also established in the matrix case: the matrix valued spherical functions of any $K$-type are closely related to matrix valued orthogonal polynomials. In this way, several examples of matrix orthogonal polynomials which are eigenfunctions of a symmetric differential operator have been obtained by focusing on a group representation approach. See for example [9], [11], [22], [23], [21] and more recently [16] and [24].

The examples of matrix orthogonal polynomials introduced in this paper are motivated by the spherical functions of fundamental $K$-types associated with the $n$-dimensional spheres $S^n \simeq G/K$, where $(G, K) = (\text{SO}(n+1), \text{SO}(n))$. These matrix valued spherical functions were studied in detail in [27] and [29]. The “group parameters” of the fundamental $K$-types are $p, n \in \mathbb{N}$ such that $0 < p < [n/2]$ and they give rise to $2 \times 2$ matrix valued orthogonal polynomials.

In this paper we go beyond these group parameters and we extend these parameters continuously. We would like to remark that the group representation theory is a natural source of examples of matrix valued orthogonal polynomials. We keep this in mind in spite of the fact that the results obtained in this paper are self-contained, the proofs are of analytic nature and they do not depend on any previous results on spherical functions.

Given a weight matrix $W$, it is very natural to study the algebra $\mathcal{D}(W)$, of all differential operators that have a sequence of matrix valued orthogonal polynomials with respect to $W$ as eigenfunctions, see [3]. In the classical cases of Hermite, Laguerre and Jacobi weights, the structure of this algebra is well understood: it is a polynomial algebra in a second order differential operator, see [20]. In particular, it is a commutative...
algebra. In the matrix case, the first attempt to go beyond the issue of the existence of one nontrivial element in $\mathcal{D}(W)$ and to study the full algebra is undertaken in [2], with the assistance of symbolic computation, for a few weights $W$. The first deep study of the algebra $\mathcal{D}(W)$ can be founded in [26], where the author worked out one of the examples introduced in [2]. We refer the reader to [13] for basic definitions and main results concerning the algebra $\mathcal{D}(W)$. The present paper leads to understand completely a second and more promising example of $\mathcal{D}(W)$ in a forthcoming paper, [23]. There are very few examples of non-commutative algebras that arise in a natural setup at the intersection of harmonic analysis and algebras. The study of such examples for the algebra $\mathcal{D}(W)$ considered here is one step in that direction. ++As a consequence of this work, together with F.A. Grünbaum, in [12] we extend to a matrix setup a result that traces its origin and its importance to the work of Claude Shannon in laying the mathematical foundations of information theory, and to a remarkable series of papers by D. Slepian, H. Landau and H. Pollak.

To the best of our knowledge, this is the first example showing in a non-commutative setup that a bispectral property implies that the corresponding global operator of “time and band limiting” admits a commuting local operator. This is a noncommutative analog of the famous prolate spheroidal wave operator.

Now we discuss briefly the content of the paper. In Section 2 we recall the general notions of matrix valued orthogonal polynomials and some results from [13] about the algebra $\mathcal{D}(W)$.

In Section 3 we introduce our sequence $\{P_w\}_{w \in \mathbb{N}_0}$ of $2 \times 2$ matrix valued polynomials on $[-1, 1]$ whose entries are given in terms of the classical Gegenbauer polynomials, for real parameters $p$ and $n$ such that $0 < p < n$, see [4]. We prove that these polynomials satisfy $P_w D = \Lambda_w P_w$, where $D$ is a (right-hand side) hypergeometric differential operator and the eigenvalue is a diagonal matrix. This differential operator $D$ is symmetric with respect to the matrix weight $W$ introduced in [12]. We use these facts to prove that the polynomials $\{P_w\}_{w \in \mathbb{N}_0}$ are orthogonal with respect to the weight matrix $W = W_{p,n}$ (Theorem 3.6).

We also connect our weight matrix $W_{p,n}$ with the weight considered in [15], where the authors give examples of matrix valued Gegenbauer polynomials, extending for an arbitrary parameter $\nu$ the results in [16] for $\nu = 1$. See Remark 3.7.

In Section 4 we prove a three-term recursion relation satisfied by $\{P_w\}_{w \in \mathbb{N}_0}$. Section 5 is focused on the study of the algebra $\mathcal{D}(W)$. In our case $\mathcal{D}(W)$ is a noncommutative algebra. We provide a basis $\{D_1, D_2, D_3, D_4, I\}$ of the subspace of the differential operators in $\mathcal{D}(W)$ of order at most two. The differential operators $D_1$ and $D_2$ are symmetric operators, while $D_3$ and $D_4$ are not. We conjecture that $D_1, D_2, D_3, D_4$ generates the algebra $\mathcal{D}(W)$.

2. Background on matrix valued orthogonal polynomials

Let $W = W(x)$ be a weight matrix of size $N$ on the real line, that is a complex $N \times N$ matrix valued integrable function on the interval $(a, b)$ such that $W(x)$ is positive definite almost everywhere and with finite moments of all orders. Let $\text{Mat}_N(\mathbb{C})$ be the algebra of all $N \times N$ complex matrices and let $\text{Mat}_N(\mathbb{C})[x]$ be the algebra over $\mathbb{C}$ of all polynomials in the indeterminate $x$ with coefficients in $\text{Mat}_N(\mathbb{C})$. We consider the following Hermitian sesquilinear form in the linear space $\text{Mat}_N(\mathbb{C})[x]$

$$\langle P, Q \rangle = \langle P, Q \rangle_W = \int_a^b P(x)W(x)Q(x)^* \, dx.$$ 

The following properties are satisfied, for all $P, Q, R \in \text{Mat}_N(\mathbb{C})[x], a, b \in \mathbb{C}, T \in \text{Mat}_N(\mathbb{C})$

1. $\langle aP + bQ, R \rangle = a\langle P, R \rangle + b\langle Q, R \rangle$,
2. $\langle TP, R \rangle = T\langle P, R \rangle$,
3. $\langle P, Q \rangle^* = \langle Q, P \rangle$,
4. $\langle P, P \rangle \geq 0$. Moreover, if $\langle P, P \rangle = 0$, then $P = 0$.

Let us denote $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Given a weight matrix $W$ one can construct sequences of matrix valued orthogonal polynomials, that is sequences $\{P_n\}_{n \in \mathbb{N}_0}$, where $P_n$ is a polynomial of degree $n$ with nonsingular
leading coefficient and \((P_n, P_m) = 0\) for \(n \neq m\). We observe that there exists a unique sequence of monic orthogonal polynomials \(\{Q_n\}_{n \in \mathbb{N}_0}\) in \(\text{Mat}_N(\mathbb{C})[x]\). By following a standard argument, given for instance in [18] or [19], one shows that the monic orthogonal polynomials \(\{Q_n\}_{n \in \mathbb{N}_0}\) satisfy a three-term recursion relation

\[
xQ_n(x) = A_n Q_{n-1}(x) + B_n Q_n(x) + Q_{n+1}(x), \quad n \in \mathbb{N}_0,
\]

where \(Q_{-1} = 0\) and \(A_n, B_n\) are matrices depending on \(n\) and not on \(x\).

Two weights \(W\) and \(\widetilde{W}\) are said to be similar if there exists a nonsingular matrix \(M\), which does not depend on \(x\), such that

\[
\widetilde{W}(x) = MW(x)M^*, \quad \text{for all } x \in (a, b).
\]

Notice that if \(\{P_n\}_{n \geq 0}\) is a sequence of orthogonal polynomials with respect to \(W\), and \(M \in \text{GL}_N(\mathbb{C})\), then \(\{P_nM^{-1}\}_{n \geq 0}\) is orthogonal with respect to \(W = MWM^*\). A weight matrix \(W\) reduces to a smaller size if there exists a nonsingular matrix \(M\) such that

\[
MW(x)M^* = \begin{pmatrix} W_1(x) & 0 \\ 0 & W_2(x) \end{pmatrix}, \quad \text{for all } x \in (a, b),
\]

where \(W_1\) and \(W_2\) are weights of smaller size.

For a given weight matrix and a sequence of orthogonal polynomials, it may be of interest the study of the differential operators having these polynomials as eigenfunctions. Let \(D\) be a right-hand side ordinary differential operator with matrix polynomial coefficients \(F_i(x)\) of degree less than or equal to \(i\) of the form

\[
D = \sum_{i=0}^{s} \partial^i F_i(x), \quad \partial = \frac{d}{dx},
\]

with the action of \(D\) on a polynomial function \(P(x)\) given by

\[
(PD)(x) = \sum_{i=0}^{s} \partial^i (P)(x)F_i(x).
\]

We say that the differential operator \(D\) is symmetric if \((PD, Q) = (Q, PD)\), for all \(P, Q \in \text{Mat}_N(\mathbb{C})[x]\). It is a matter of careful integration by parts to see that the condition of symmetry for a differential operator of order two is equivalent to a set of three differential equations involving the weight \(W\) and the coefficients of the differential operator \(D\).

**Proposition 2.1** ([18] or [19]). Let \(W(x)\) be a smooth weight matrix supported on \((a, b)\). Let \(D = \partial^2 F_2(x) + \partial F_1(x) + F_0\). Then \(D\) is symmetric with respect to \(W\) if and only if

\[
\left\{
\begin{array}{l}
F_2 W = WF_2^* \\
2(F_2 W)' - F_1 W = WF_1^* \\
(F_2 W)'' - (F_1 W)' + F_0 W = WF_0^*
\end{array}
\right.
\]

with the boundary conditions

\[
\lim_{x \to a, b} F_2(x)W(x) = 0, \quad \lim_{x \to a, b} (F_1(x)W(x) - WF_1^*(x)) = 0.
\]

We consider the following subalgebra of the algebra of all right-hand side differential operators with coefficients in \(\text{Mat}_N(\mathbb{C})[x]\),

\[
\mathcal{D} = \{D = \sum_{i=0}^{s} \partial^i F_i : s \in \mathbb{N}_0, F_i \in \text{Mat}_N(\mathbb{C})[x], \deg F_i \leq i\}.
\]
Proposition 2.2 ([13], Propositions 2.6 and 2.7). Let $W = W(x)$ be a weight matrix of size $N \times N$ and let \( \{Q_n\}_{n \geq 0} \) be the sequence of monic orthogonal polynomials in $\operatorname{Mat}_N(\mathbb{C})[x]$. If $D$ is a right-hand side ordinary differential operator of order $s$, as in [1], such that

\[
Q_n D = \Lambda_n Q_n, \quad \text{for all } n \in \mathbb{N}_0,
\]

with $\Lambda_n \in \operatorname{Mat}_N(\mathbb{C})$, then $F_i = F_i(x) = \sum_{j=0}^{i} x^j F_j$, $F_j \in \operatorname{Mat}_N(\mathbb{C})$, is a polynomial and $\deg(F_i) \leq i$. Moreover $D$ is determined by the sequence $\{\Lambda_n\}_{n \geq 0}$ and

\[
\Lambda_n = \sum_{i=0}^{s} [n]_i F_i, \quad \text{for all } n \geq 0,
\]

where $[n]_i = n(n-1) \cdots (n-i+1)$, $[n]_0 = 1$.

Given a matrix weight $W$, the algebra

\[
\mathcal{D}(W) = \{ D \in \mathcal{D} : P_n D = \Lambda_n(D) P_n, \quad \Lambda_n(D) \in \operatorname{Mat}_N(\mathbb{C}), \quad \text{for all } n \in \mathbb{N}_0 \}
\]

is introduced in [13], where $\{P_n\}_{n \in \mathbb{N}_0}$ is any sequence of matrix valued orthogonal polynomials with respect to $W$.

We observe that the definition of $\mathcal{D}(W)$ depends only on the weight matrix $W$ and not on the particular sequence of orthogonal polynomials, since two sequences $\{P_w\}_{w \in \mathbb{N}_0}$ and $\{Q_w\}_{w \in \mathbb{N}_0}$ of matrix orthogonal polynomials with respect to the weight $W$ are related by $P_w = M_w Q_w$, for $w \in \mathbb{N}_0$, with $\{M_w\}_{w \in \mathbb{N}_0}$ invertible matrices (see [13 Corollary 2.5]).

Proposition 2.3 ([13], Proposition 2.8). For each $n \in \mathbb{N}_0$, the mapping $D \mapsto \Lambda_n(D)$ is a representation of $\mathcal{D}(W)$ in $\operatorname{Mat}_N(\mathbb{C})$. Moreover, the sequence of representations $\{\Lambda_n\}_{n \in \mathbb{N}_0}$ separates the elements of $\mathcal{D}(W)$.

We remark that the result in Proposition 2.3 says that the map

\[
D \mapsto (\Lambda_0(D), \Lambda_1(D), \Lambda_2(D), \ldots)
\]

is an injective morphism of $\mathcal{D}(W)$ into $\operatorname{Mat}_N(\mathbb{C})^{\mathbb{N}_0}$, the direct product of infinite copies, indexed by $\mathbb{N}_0$, of the algebra $\operatorname{Mat}_N(\mathbb{C})$. In particular, if $D_1, D_2 \in \mathcal{D}(W)$ then

\[
D_1 = D_2 \quad \text{if and only if} \quad \Lambda_n(D_1) = \Lambda_n(D_2) \quad \text{for all } n \in \mathbb{N}_0.
\]

For any $D \in \mathcal{D}(W)$ there exists a unique differential operator $D^* \in \mathcal{D}(W)$, the adjoint of $D$ in $\mathcal{D}(W)$, such that

\[
\langle PD, Q \rangle = \langle P, QD^* \rangle,
\]

for all $P, Q \in \operatorname{Mat}_N(\mathbb{C})[x]$. See Theorem 4.3 and Corollary 4.5 in [13]. The map $D \mapsto D^*$ is a *-operation in the algebra $\mathcal{D}(W)$. Moreover, it is shown that $\mathcal{S}(W)$, the set of all symmetric operators in $\mathcal{D}(W)$, is a real form of the space $\mathcal{D}(W)$, i.e.

\[
\mathcal{D}(W) = \mathcal{S}(W) \oplus i\mathcal{S}(W),
\]

as real vector spaces. In particular, the algebra $\mathcal{D}(W)$, together with the involution, is completely determined by $\mathcal{S}(W)$.

Corollary 2.4. A differential operator $D \in \mathcal{D}(W)$ is a symmetric operator if and only if

\[
\Lambda_n(D)\langle Q_n, Q_n \rangle = \langle Q_n, Q_n \rangle \Lambda_n(D)^*
\]

for all $n \in \mathbb{N}_0$.

Also it is worth to recall the following important result from [13].

Proposition 2.5 (Proposition 2.10). If $D \in \mathcal{D}$ is symmetric then $D \in \mathcal{D}(W)$. 
3. Matrix valued orthogonal polynomials associated with the \( n \)-dimensional spheres

Motivated by the results obtained in [27] we introduce the following family of polynomials, for \( w \in \mathbb{N}_0 \),

\[
P_w(x) = P_w^{n,p}(x) = \left( \frac{1}{n+1} C_w^{n+3}(x) + \frac{1}{n+2} C_w^{n+1}(x) \right) \left( \frac{1}{n+1} C_w^{n+1}(x) + \frac{1}{n+2} C_w^{n+3}(x) \right),
\]

with parameters \( p, n \in \mathbb{R} \) such that \( 0 < p < n \). Here \( C_n^{\lambda}(x) \) denotes the \( n \)-th Gegenbauer polynomial

\[
C_n^{\lambda}(x) = \frac{(2\lambda)_w}{w!} {}_2F_1 \left( -w, w + 2\lambda; \frac{1-x}{2} \right), \quad x \in [-1, 1],
\]

where \((a)_w = a(a+1) \ldots (a+w-1)\) denotes the Pochhammer symbol. As usual, we assume \( C_n^{\lambda}(x) = 0 \) if \( w < 0 \). We recall that \( C_n^{\lambda} \) is a polynomial of degree \( w \), with leading coefficient \( \frac{2^{w}(\lambda+w)}{w!} \).

Let us observe that \( \text{deg}(P_w) = w \) and the leading coefficient of \( P_w \) is a nonsingular scalar matrix

\[
\frac{2^{w}(n+1)}{(n+1)w!} \text{Id} = \frac{1}{w!} 2^{w-1}(\frac{n+3}{2})_{w-1} \text{Id}.
\]

We start by proving that the polynomials \( P_w \) given in (4) are eigenfunctions of the following differential operator \( D \).

**Theorem 3.1.** For each \( w \in \mathbb{N}_0 \), the matrix polynomial \( P_w \) is an eigenfunction of the differential operator

\[
D = \partial^2 (1 - x^2) - \partial \left( (n+2)x + 2 \left( \begin{array}{c} n+1 \\ 1 \end{array} \right) \right) - \left( \begin{array}{c} p \\ 0 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right),
\]

with eigenvalue

\[
\Lambda_w(D) = \left( \begin{array}{c} -w(w+n+1) - p \\ 0 \end{array} \right) \left( \begin{array}{c} 0 \\ -w(w+n+1) - n + p \end{array} \right).
\]

**Proof.** We need to verify that

\[
P_w D = \Lambda_w P_w.
\]

We will need to use the following properties of the Gegenbauer polynomials (for the first three see [14] page 40, and for the last one see [25], page 83, equation (4.7.27))

\[
(1 - x^2) \frac{d^2}{dx^2} C_m^\lambda(x) - (2\lambda + 1)x \frac{d}{dx} C_m^\lambda(x) + m(m + 2\lambda) C_m^\lambda(x) = 0,
\]

\[
\frac{d}{dx} C_m^\lambda(x) = 2\lambda C_{m-1}^{\lambda+1}(x),
\]

\[
2(m + \lambda)x C_m^\lambda(x) = (m + 1) C_{m+1}^\lambda(x) + (m + 2\lambda - 1) C_{m-1}^\lambda(x),
\]

\[
\frac{m + 2\lambda - 1}{2(\lambda - 1)} C_{m+1}^\lambda(x) = C_{m+1}^\lambda(x) - x C_m^\lambda(x).
\]

Also, combining (8) and (9), we have

\[
(m + \lambda) C_{m+1}^{\lambda-1}(x) = (\lambda - 1) \left( C_{m+1}^\lambda(x) - C_{m-1}^\lambda(x) \right).
\]
The entry \((1,1)\) of the matrix \(P_w D - \Lambda_w P_w\) is
\[
(1 - x^2)(P_w)''_{11} - (n + 2)x(P_w)'_{11} - 2(P_w)'_{12} + w(w + n + 1)(P_w)_{11}
\]
\[
= (1 - x^2) \left( \frac{n+1}{n+1} C_w^2 + \frac{1}{p+w}C_w^2 \right)'' - (n + 2)x \left( \frac{n+1}{n+1} C_w^2 + \frac{1}{p+w}C_w^2 \right)'
\]
\[
- \frac{2}{p+w} \left( \frac{n+3}{w-1} \right)' + w(w + n + 1) \left( \frac{n+1}{n+1} C_w^2 + \frac{1}{p+w}C_w^2 \right) .
\]

Applying (6) for \(\lambda = \frac{1}{2}(n + 1), \lambda = \frac{1}{2}(n + 3)\) and (7) for \(\lambda = \frac{1}{2}(n + 3)\), with \(m = w\), we have that the entry \((1,1)\) of \(P_w D - \Lambda_w P_w\), multiplied by \((p + w)/2\) is
\[
-(n + 3) \frac{n+5}{w-2} + (n + 3)x \frac{n+5}{w-3} + (w + n + 1) \frac{n+3}{w-2} = 0,
\]
this last identity follows from equation (9) with \(\lambda = \frac{n+5}{2}\) and \(m = w - 3\). Repeating the previous verification, by changing \(p\) by \(n - p\), it follows that the entry \((2,2)\) of \(P_w D - \Lambda_w P_w\) is also zero.

The entry \((1,2)\) of \(P_w D - \Lambda_w P_w\) is
\[
(1 - x^2)(P_w)'_{12} - (n + 2)x(P_w)'_{12} - 2(P_w)'_{11} + (w(w + n + 1) - n + 2p)(P_w)_{12},
\]
if we multiply it by \((p + w)\) we get
\[
(1 - x^2) \left( \frac{n+3}{w-1} \right)'' - (n + 2)x \left( \frac{n+3}{w-1} \right)' + (w + n + 1) - n + 2p \frac{n+3}{w-1} - 2 \frac{p+w}{n+1} \left( \frac{n+1}{w} \right)' - 2 \left( \frac{n+3}{w-2} \right)'.
\]
Applying (6) for \(\lambda = (n + 3)/2, m = w - 1, \lambda = (n + 1)/2, m = w\) and \(\lambda = (n + 3)/2, m = w - 1\), one obtain that (11) is
\[
2x \left( \frac{n+3}{w-2} \right)' - 2(w - 1) \frac{n+3}{w-2} + 2(n + 3) \frac{n+5}{w-2} .
\]
Now, applying (7) and (9), this expression becomes
\[
2(n + 3) \left( \frac{n+5}{w-1} - \frac{n+5}{w-3} \right) - 2(2w + n + 1) \frac{n+3}{w-1} ,
\]
which is equal to zero by (10) with \(\lambda = \frac{n+5}{2}\) and \(m = w - 2\). This concludes that the entry \((1,2)\) of \(P_w D - \Lambda_w P_w\) is zero. To complete the proof of the theorem we need to verify that the entry \((2,1)\) is also zero. This is obtained making exactly the same computations, by changing \(p\) by \(n - p\).

We introduce the weight matrix
\[
W(x) = W_{p,n} = (1 - x^2)^{p-1} \begin{pmatrix} px^2 + n - p & -nx \\ -nx & (n - p)x^2 + p \end{pmatrix}, \quad x \in [-1,1].
\]

**Proposition 3.2.** For \(n \neq 2p\), the weight \(W(x)\) does not reduce to a smaller size.

**Proof.** Assume that there exists a nonsingular matrix \(M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}\) such that
\[
MW(x)M^* = \begin{pmatrix} w_1(x) & 0 \\ 0 & w_2(x) \end{pmatrix}.
\]
The entry \((1,2)\) of \(MW(x)M^*\) is
\[
x^2(p m_{11} m_{21} + (n - p) m_{12} m_{22}) - (m_{11} m_{22} + m_{12} m_{21}) n x + (n - p) m_{11} m_{21} + p m_{12} m_{22},
\]
from here we see that
\[
m_{11}m_{22} + m_{12}m_{21} = 0,
\]
(13)
\[
p m_{11}m_{21} + (n - p)m_{12}m_{22} = 0,
\]
(14)
By combining equations (13) and (14) we have that \((n - 2p)m_{11}m_{21} = 0\). The assumption \(n \neq 2p\), together with (9), implies \(\det(M) = 0\), which is a contradiction. \(\square\)

\textbf{Remark 3.3.} For \(n = 2p\), the weight matrix \(W\) reduces to two scalar weights. The corresponding scalar polynomials are Jacobi polynomials \(P_{\alpha,\beta}^m\) with \((\alpha, \beta) = (n/2 + 1, n/2 - 1)\) and \((\alpha, \beta) = (n/2 - 1, n/2 + 1)\), respectively. In fact, by taking \(M = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\) we have that
\[
MW(x)M^* = 2p(1 - x^2)^{p-1/2} \begin{pmatrix} (1 - x)^2 & 0 \\ 0 & (1 + x)^2 \end{pmatrix}.
\]

\textbf{Remark 3.4.} We have that the weight matrices \(W_{p,n}\) and \(W_{n-p,n}\) are similar. In fact, by taking \(M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) we get
\[
MW_{p,n}M^* = W_{n-p,n}.
\]

From Proposition 2.1 and following straightforward computations, one can prove the following result.

\textbf{Proposition 3.5.} The differential operator
\[
D = \partial^2 (1 - x^2) - \partial \left( (n + 2)x + 2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) - \begin{pmatrix} p & 0 \\ 0 & n - p \end{pmatrix}
\]
is symmetric with respect to the weight function \(W(x)\).

In the scalar case, if \(D\) is a symmetric differential operator with respect to \(W\) and \(\{P_w\} \in \mathbb{N}_0\) is a family of eigenfunctions of \(D\) with different eigenvalues, then the sequence \(\{P_w\}_{w \in \mathbb{N}_0}\) is automatically orthogonal with respect to \(W\). In the matrix case this is not always true since
\[
\lambda_w(P_w, P_{w'}) = (P_wD, P_{w'}) = (P_w, P_{w'}D) = (P_w, P_{w'})\Lambda_{w'}
\]
does not imply that \((P_w, P_{w'}) = 0\), for \(w \neq w'\). Therefore, we prove the orthogonality in the next theorem.

\textbf{Theorem 3.6.} When \(n \neq 2p\) the matrix polynomials \(\{P_w\}_{w \in \mathbb{N}_0}\) are orthogonal polynomials with respect to the matrix valued inner product
\[
\langle P, Q \rangle = \int_{-1}^1 P(x)W(x)Q(x)^\ast dx.
\]

\textbf{Proof.} We know that \(P_w\) is a polynomial of degree \(w\) and its leading coefficient is a nonsingular diagonal matrix (see (3)). We only have to verify that for \(w \neq w'\), \((P_w, P_{w'})_W = 0\). Since \(P_w\) is an eigenfunction of the differential operator \(D\), which is symmetric with respect to \(W\), we have that (13) holds with
\[
\Lambda_w = \begin{pmatrix} \lambda_{w,1} & 0 \\ 0 & \lambda_{w,2} \end{pmatrix} = \begin{pmatrix} -w(w+n+1) - p & 0 \\ 0 & -w(w+n+1) - n + p \end{pmatrix},
\]
see Theorem 3.1. Therefore, for \(i, j = 1, 2\) we have \(\lambda_{w,i}(P_{w,i}, P_{w',j}) = \lambda_{w',j}(P_{w,i}, P_{w',j})\), where \(P_{w,i}\) is the \(i\)-th row of the polynomial \(P_w\), and
\[
\langle P_{w,i}, P_{w',j} \rangle = \int_{-1}^1 P_{w,i}(x)W(x)P_{w',j}(x)^\ast dx \in \mathbb{C}.
\]

It is not difficult to verify that \(\lambda_{w,i} = \lambda_{w',j}\), for \(w \neq w'\) or \(i \neq j\). Then we have
\[
\langle P_{w,i}, P_{w',j} \rangle = 0, \quad \text{for } w \neq w' \text{ or } i \neq j.
\]
Therefore \((P_w, P_{w'}) = 0\), for \(w \neq w'\), which concludes the proof of the theorem. \(\square\)
Remark 3.7. Recently, in [15] the authors study some families on matrix valued polynomials, depending on one real parameter \( \nu > 0 \), of arbitrary size \((2\ell + 1) \times (2\ell + 1)\) with \( \ell \in \frac{1}{2}\mathbb{N} \). These weights are not irreducible. For \( \ell = 1, \frac{3}{2}, 2 \) appears some irreducible \(2 \times 2\) blocks \( W^{(\nu)}_+\) and \( W^{(\nu)}_-\). See Remark 2.8 (ii) there.

The case \( \ell = 3/2 \) does not match with the examples considered in this paper. The cases \( \ell = 1 \) and \( \ell = 2 \) are particular cases of our weight matrices \( W_{p,n} \) by choosing our parameters \((p,n) = (\nu,2\nu+1)\) and \((p,n) = (\nu,2\nu+3)\), for \( \ell = 1 \) and \( \ell = 2 \) respectively. In fact, with \( L = \begin{pmatrix} 0 & \sqrt{\tau} \\ -1 & 0 \end{pmatrix} \) and \( D = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \) we get

\[
W^{(\nu)}_+ = \frac{(\nu + 2)}{(2\nu + 1)} L W_{\nu,2\nu+1} L^* \quad \text{for } \ell = 1,
\]

\[
W^{(\nu)}_- = \frac{(\nu + 4)(\nu + 2)}{(2\nu + 1)(2\nu + 3)} D W_{\nu,2\nu+3} D^* \quad \text{for } \ell = 2.
\]

The case \( \nu = 1 \) was previously studied in [16] and [17].

4. Three-term recursion relation

The main result of this section is a three-term recursion relation satisfied by the sequence of orthogonal polynomials studied in this paper. We give a proof by using some properties of the Gegenbauer polynomials.

**Theorem 4.1.** The orthogonal polynomials \( \{P_w\}_{w \in \mathbb{N}_0} \) satisfy the three-term recursion relation

\[
x P_w(x) = A_w P_{w-1}(x) + B_w P_w(x) + C_w P_{w+1}(x),
\]

where

\[
A_w = \begin{pmatrix}
\frac{(n+w)(p+w-1)(n-p+w+1)}{(p+w)(n-p+w)(2w+n+1)} & 0 \\
0 & \frac{(n+w)(p+w-1)(n-p+w-1)}{(p+w)(n-p+w)(2w+n+1)}
\end{pmatrix},
\]

\[
B_w = \begin{pmatrix}
0 & \frac{-p}{(p+w)(p+w+1)} \\
\frac{-p}{(n-p+w)(n-p+w+1)} & 0
\end{pmatrix}, \quad C_w = \frac{w+1}{2w+n+1} I.
\]

**Proof.** To verify the \((1,1)\)-entry of the equation in the statement of the theorem we need to prove that

\[
x \left( \frac{1}{n+1} C_{w-1}^{\frac{n+w}{n+1}}(x) + \frac{1}{p+w} C_{w-2}^{\frac{n+w}{p+w}}(x) \right) = \frac{(n+w)(p+w-1)(n-p+w+1)}{(2w+n+1)(p+w)(n-p+w)} \left( \frac{1}{n+1} C_{w-1}^{\frac{n+w}{n+1}}(x) + \frac{1}{p+w} C_{w-2}^{\frac{n+w}{p+w}}(x) \right)
\]

\[
- \frac{p}{(p+w)(p+w+1)(n-p+w)} C_{w-1}^{\frac{n+w}{p+w}}(x) + \frac{w+1}{2w+n+1} \left( \frac{1}{n+1} C_{w+1}^{\frac{n+1}{n+1}}(x) + \frac{1}{p+w-1} C_{w-1}^{\frac{n+1}{p+w-1}}(x) \right).
\]

By replacing the identities given by (8) for \( \lambda = \frac{n+1}{2}, m = w \) and \( \lambda = \frac{n+3}{2}, m = w - 2 \), one obtain that (17) is equivalent to

\[
\frac{(w+n)}{(n+1)(2w+n+1)} \left( -1 + \frac{(p+w-1)(n-p+w-1)}{(p+w)(n-p+w)} \right) C_{w-1}^{\frac{n+w}{n+1}}(x)
\]

\[
+ \left( \frac{p}{(p+w)(p+w+1)(n-p+w)} + \frac{w+1}{(2w+n+1)(p+w+1)} - \frac{w-1}{(p+w)(2w+n+1)} \right) C_{w-1}^{\frac{n+3}{w-3}}(x) = 0.
\]

Thus, by using the relation (10) for \( \lambda = \frac{n+3}{2} \) and \( m = w - 2 \), the identity in (18) follows after some straightforward computations.
Now we verify that the equation for the \((1, 2)\)-entry in the statement of the theorem holds. We need to verify that the following identity holds

\[
\frac{1}{p+w} x C_{w-1}^{n+3}(x) = \frac{(n+w)(n-p+w+1)}{(p+w)(2w+n+1)(n-p+w)} C_{w-2}^{n+3}(x)
\]

\[- \frac{p}{(p+w)(p+w+1)} \left( \frac{1}{n+1} C_{w}^{n+3}(x) + \frac{1}{n-p+w} C_{w-2}^{n+3}(x) \right) + \frac{w}{(2w+n+1)(p+w+1)} C_{w}^{n+3}(x).
\]

From (10) for \(\lambda = \frac{n+3}{2}\) and \(m = w - 1\) we have that the right-hand side of (19) is

\[
\frac{n+3}{2} \frac{w}{(p+w)(2w+n+1)} C_{w-2}^{n+3}(x) + \frac{w}{(p+w)(2w+n+1)} C_{w}^{n+3}(x).
\]

Therefore, (19) is proved, since it is equivalent to (8) with \(\lambda = \frac{n+3}{2}\) and \(m = w - 1\).

For the entries \((2, 2)\) and \((2, 1)\) we proceed in a similar way, by observing that we need to do the same computations as in the cases \((1, 1)\) and \((1, 2)\) respectively, by changing \(p\) by \(n - p\). This concludes the proof of the theorem.

\(\square\)

The sequence of monic orthogonal polynomials is given by

\[
Q_w = \frac{w!(n+1)}{2w \left( \frac{n+1}{2} \right)_w} P_w, \quad w \in \mathbb{N}_0.
\]

The first polynomials of the sequence \(\{Q_w\}_{w=0}^\infty\) are

\[
Q_0 = 1, \quad Q_1 = \left( \frac{x}{n+1} \right), \quad Q_2 = \left( \frac{x^2 - \frac{p}{(n+3)(p+2)}}{\frac{n-p}{n+2}} \right),
\]

\[
Q_3 = \left( \frac{x^3 - \frac{3p+1}{(n+3)(p+2)} x}{\frac{n-p}{n+3} \frac{n-p+1}{n+3}} \right).
\]

Remark 4.2. Observe that from (16) and (20) we have that \(\langle Q_w, Q_w \rangle\) is always a diagonal matrix. Moreover one can verify that

\[
\langle Q_w, Q_w \rangle = \|Q_w\|^2 = \frac{\pi^{w/2} \Gamma(n/2 + 1 + [w/2])}{w!(n + 2w + 1) \Gamma((n + 3)/2)} \prod_{k=1}^{[w/2]} \left( \frac{(n+2k+1)}{(n+2k+2)} \right) \left( \frac{p(n-p+w+1)}{p+1} \right) \left( \frac{0}{n-p+w} \right).
\]

5. The algebra \(D(W)\)

In this section we discuss some properties of the structure of the algebra \(D(W)\), defined in (3), for our weight matrix \(W(x)\) introduced in (12). We are not interested in the cases when \(p = n - p\), since the weight reduces to classical scalar weights, see Remark 3.3. We observe that in our example, the polynomials \(\{P_w\}_{w=0}^\infty\), given in (4), and the monic orthogonal polynomials \(\{Q_w\}_{w=0}^\infty\) have the same sequence of eigenvalues, they are related by a scalar multiple, see (20).

First of all we observe that the space of differential operators of order zero in \(D(W)\) consists of scalar multiplies of the identity operator. In fact, a differential operator of order zero, having the sequence of monic orthogonal polynomials \(\{Q_w\}_{w=0}^\infty\) as eigenfunctions, is a constant matrix \(L\) such that

\[
Q_w L = \Lambda_w Q_w, \quad \text{for all } w \in \mathbb{N}_0.
\]

From (2) we have that \(\Lambda_w = L\) for every \(w\). When \(w = 1\), we obtain that the entries of \(L\) satisfy \(L_{11} = L_{22}\) and \((p + 1)L_{12} = (n - p + 1)L_{21}\). Thus, looking at the case \(w = 2\) we get \((n - 2p)L_{12} = 0\). Therefore we obtain that any operator of order zero \(L\) in \(D(W)\) is a multiple of the identity matrix.
Now we study differential operators of order at most two in the algebra $D(W)$. Let $\{Q_w\}_{w \in \mathbb{N}_0}$ the sequence of monic orthogonal polynomials with respect to $W$ and $D$ a differential operator of order at most two in $D(W)$. From Proposition 2.2 we have

$$D = \partial^2(A_2x^2 + A_1x + A_0) + \partial(B_1x + B_0) + C \in D(W)$$

if and only if

$$Q_wD = (w(w-1)A_2 + wB_1 + C)Q_w, \quad \text{for all } w \in \mathbb{N}_0.$$ 

Here $A_2, A_1, A_0, B_1, B_0, C$ are $2 \times 2$ complex matrices. Let us denote $Q_{w,j}$ the coefficients of the polynomial $Q_w$, i.e., $Q_w = \sum_{j=0}^n Q_{w,j}x^j$, with $Q_{w,w} = I$. Therefore $D \in D(W)$ if and only if

$$j(j-1)Q_{w,j}A_2 + j(j+1)Q_{w,j+1}A_1 + (j+1)(j+2)Q_{w,j+2}A_0 + jQ_{w,j}B_1$$

$$+ (j+1)Q_{w,j+1}B_0 + Q_{w,j}C - (w(w-1)A_2 + wB_1 + C)Q_{w,j} = 0$$

for all $w \in \mathbb{N}_0$ and $j = 0, \ldots, w$. For $j = w-1$ and $j = 0$ we respectively obtain

$$(w-1)(w-2)Q_{w-1,w}A_2 + w(w-1)A_1 + (w-1)Q_{w-1,w-1}B_1 + wB_0 + Q_{w-1,w}C$$

$$= 0$$

and

$$2Q_{w,2}A_0 + Q_{w,1}B_0 + Q_{w,0}C - (w(w-1)A_2 + wB_1 + C)Q_{w,0} = 0.$$ 

Now from (21) considering $w = 1$ and $w = 2$, and from (22) considering $w = 2$, we respectively obtain

$$B_0 = (B_1 + C)Q_{1,0} - Q_{1,1}C, \quad 2A_1 = (2A_2 + 2B_1 + C)Q_{2,1} - Q_{2,2}B_1 - 2B_0 - Q_{2,1}C,$$

$$2A_0 = (2A_2 + 2B_1 + C)Q_{2,0} - Q_{2,1}B_0 - Q_{2,0}C.$$ 

From the expression of $Q_1$ and $Q_2$, given at the end of Section 4 we know that

$$Q_{1,0} = \begin{pmatrix} 0 & \frac{1}{p+1} \\ \frac{2}{n-p+2} & 0 \end{pmatrix}, \quad Q_{2,1} = \begin{pmatrix} 0 & \frac{2}{p+2} \\ \frac{1}{p+1} & 0 \end{pmatrix}, \quad Q_{2,0} = -p \begin{pmatrix} \frac{1}{(p+2)} & 0 \\ 0 & \frac{1}{(n-p+2)} \end{pmatrix}.$$ 

By using (20) and (4) it is easy to see that

$$Q_{w,w-1} = \begin{pmatrix} 0 & \frac{w}{n-p+w} \\ \frac{w}{p+w} & 0 \end{pmatrix}, \quad \text{for all } w \in \mathbb{N}.$$ 

To determine the matrices $A_2 = (a_{ij}), B_1 = (b_{ij})$ and $C = (c_{ij})$, we first combine the entries in the diagonal of the matrix (21) to obtain

$$2(n+2)a_{21} = \frac{(n+p+2)b_{21} - 2c_{21}}{p+1} + \frac{(p+2)(p+w)(2c_{12} - (n-p)b_{12})}{(n-p+1)(n-p+2)(n-p+w)},$$

$$2(n+2)a_{12} = \frac{(2n+p+2)b_{12} - 2c_{12}}{n-p+1} + \frac{(n-p+2)(n-p+w)(2c_{21} - p b_{21})}{(p+1)(p+2)(p+w)}.$$ 

Since these identities are valid for any integer $w \geq 3$ we conclude that, if $n \neq 2p$ then $2c_{12} = (n-p)b_{12}$ and $2c_{21} = p b_{21}$. Therefore $b_{21} = 2(p+1)a_{21}$ and $b_{12} = 2(n-p+1)a_{12}$. By combining the nondiagonal entries of (21) we have

$$(n-2p+1)((n+2)a_{11} - b_{11}) = (n-2p-1)((n+2)a_{22} - b_{22})$$

and

$$c_{11} - c_{22} = (p+1)(p+2)a_{22} - p(p+1)a_{11} + pb_{11} - (p+1)b_{22}.$$ 

Equation (22) with $w = 3$ says that

$$2Q_{3,2}A_0 + Q_{3,1}B_0 + Q_{3,0}C - (6A_2 + 3B_1 + C)Q_{3,0} = 0.$$
Now, by using the expression of $Q_3 = x^3 + Q_{3,2}x^2 + Q_{3,1}x + Q_{3,0}$ given at the end of Section II it is not difficult to see that $b_{11} = (n+2)a_{11}$. Thus $b_{22} = (n+2)a_{22}$, and $c_{1} - c_{2} = p(n-p+1)a_{11} - (p+1)(n-p)a_{22}$.

Therefore, the matrices $A_2, A_1, A_0, B_1, B_0, C$ are given in terms of the entries of $A_2$ and $c_{1}$, as we state in the following theorem.

**Theorem 5.1.** The differential operators of order at most two in $\mathcal{D}(W)$ are of the form

$$D = \partial^2 F_2(x) + \partial F_1(x) + F_0,$$

where

$$F_2(x) = x^2 \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} + x \begin{pmatrix} a_{12} - a_{21} & a_{11} - a_{22} \\ a_{22} - a_{11} & a_{21} - a_{12} \end{pmatrix} + \begin{pmatrix} a_{22} & a_{21} \\ a_{12} & a_{11} \end{pmatrix},$$

$$F_1(x) = x \begin{pmatrix} (n+2)a_{11} & 2(n-p+1)a_{12} \\ 2(p+1)a_{21} & (n+2)a_{22} \end{pmatrix} + \begin{pmatrix} -pa_{21} + (n-p+2)a_{12} & (n-p+2)a_{11} - (n-p)a_{22} \\ -pa_{11} + (p+2)a_{22} & (p+2)a_{21} - (n-p)a_{12} \end{pmatrix},$$

$$F_0 = \begin{pmatrix} p(n-p+1)a_{11} + c & (n-p)(n-p+1)a_{12} \\ p(p+1)a_{21} & (p+1)(n-p)a_{22} + c \end{pmatrix},$$

with $a_{11}, a_{12}, a_{21}, a_{22}, c$ arbitrary complex numbers.

**Proof.** We have already proved that any differential operator of order at most two in $\mathcal{D}(W)$ is of this form for some constant $a_{11}, a_{12}, a_{21}, a_{22}, c \in \mathbb{C}$. Let $\mathcal{D}_2$ be the complex vector space of the differential operators in $\mathcal{D}(W)$ of order at most two. Then we have that $\dim \mathcal{D}_2 \leq 5$.

From Proposition 2.1 it is not difficult to see that a differential operator $D$ of order two, with coefficients given by (23), is a symmetric operator if and only if

$$a_{11}, a_{22}, c \in \mathbb{R} \quad \text{and} \quad pa_{21} = (n-p)\overline{a_{12}}.$$

From Proposition 2.5, any symmetric operator $D \in \mathcal{D}$ belongs to the algebra $\mathcal{D}(W)$. Thus there exists (at least) five $\mathbb{R}$-linearly independent symmetric operators in $\mathcal{D}_2$. Therefore $\dim \mathcal{D}_2 = 5$ and this concludes the proof of the theorem.

**Corollary 5.2.** There are no operators of order one in the algebra $\mathcal{D}(W)$.

The elements of the sequence $\{Q_w\}_w$ are eigenfunctions of the operators $D \in \mathcal{D}(W)$ and they satisfy $Q_wD = \Lambda_w(D)Q_w$ for $w \in \mathbb{N}_0$. We explicitly state the eigenvalues $\Lambda_w$ using formula (2): for a differential operator $D = \partial^2 F_2 + \partial F_1 + F_0$ we have

$$\Lambda_w(D) = w(w-1)F_2^2 + wF_1 + F_0,$$

with $F_i$ ($i=1,2,3$) the leading coefficient of the polynomial coefficient $F_i$ of the differential operator $D$. Therefore we get

**Corollary 5.3.** Let $D \in \mathcal{D}(W)$, defined as in Theorem 5.1. The monic orthogonal polynomials $\{Q_w\}_w$ satisfy

$$Q_wD = \Lambda_w(D)Q_w,$$

where the eigenvalue $\Lambda_w(D)$ is given by

$$\Lambda_w(D) = \begin{pmatrix} (w+p)(w+n-p+1)a_{11} + c & (w+n-p)(w+n-p+1)a_{12} \\ (w+p)(w+p+1)a_{21} & (w+n-p)(w+p+1)a_{22} + c \end{pmatrix}.$$
\[ D_2 = \partial^2 \left( \frac{-1}{x} - \frac{x}{x^2} \right) + \partial \left( \frac{0}{p + 2} - \frac{p - n}{(n + 2)x} + \frac{0}{0} \frac{0}{(p + 1)(n - p)} \right), \]
\[ D_3 = \partial^2 \left( \frac{-x}{x^2} - \frac{1}{x} \right) + \partial \left( \frac{-p}{2(p + 1)x} + \frac{0}{p + 2} + \frac{0}{p(p + 1)} \right), \]
\[ D_4 = \partial^2 \left( \frac{x}{x^2} - \frac{x}{-1} - \frac{x}{-x} \right) + \partial \left( \frac{n - p + 2}{0} - \frac{2(n - p + 1)x}{p - n} + \frac{0}{0} \frac{0}{(n - p)(n - p + 1)} \right). \]

The corresponding eigenvalues are
\[ \Lambda_w(D_1) = \left( \begin{array}{cc} (w + p)(w + n - p + 1) & 0 \\ 0 & 0 \end{array} \right), \quad \Lambda_w(D_2) = \left( \begin{array}{cc} 0 & 0 \\ 0 & (w + p + 1)(w + n - p) \end{array} \right), \]
\[ \Lambda_w(D_3) = \left( \begin{array}{cc} 0 & 0 \\ 0 & (w + n - p)(w + n - p + 1) \end{array} \right), \quad \Lambda_w(D_4) = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right). \]

Remark 5.4. The differential operator \( D \) appearing in Theorem 3.1 is \( D = D_1 - D_2 + p(n - p)I \).

We observe here that, for example,
\[ \Lambda_w(D_1)\Lambda_w(D_3) \neq \Lambda_w(D_3)\Lambda_w(D_1), \quad \text{for all } w \in \mathbb{N}_0. \]

By using Proposition 2.3 we obtain that \( D_1 D_3 \neq D_3 D_1 \), which in turn implies the following result.

Corollary 5.5. The algebra \( D(W) \) is not commutative.

By following the same argument, through the sequence of eigenvalues, we obtain the following relations among the differential operators \( D_1, D_2, D_3, D_4 \).
\[ D_1D_2 = 0, \quad D_2D_1 = 0, \quad D_1D_3 = 0, \quad D_3D_1 = 0, \quad D_2D_4 = 0, \quad D_4D_2 = 0, \quad D_3D_2 = 0, \quad D_4D_3 = 0, \quad D_1D_4 = 0, \]
\[ D_3D_1 = D_2D_3 - (n - 2p)D_3, \quad D_1D_4 = D_4D_2 - (n - 2p)D_4, \quad D_3D_4 = D_2^2 - (n - 2p)D_2, \quad D_4D_3 = D_2^2 + (n - 2p)D_1. \]

Conjecture 5.6.

(1) There are no operators of odd order in \( D(W) \).

(2) The second order differential operators in \( D(W) \) generate the algebra \( D(W) \).

For a differential operator of order two \( D = \partial^2 F_2 + \partial F_1 + F_0 \in D(W) \), the explicit expression of the adjoint operator \( D^* \) is
\[ D^* = \partial^2 G_2 + \partial G_1 + G_0, \]
where the polynomials \( G_i, i = 0, 1, 2 \), are defined by
\[ G_0 = \langle Q_0, Q_0 \rangle \Lambda_0(D)^* Q_0, \quad G_1 = \langle Q_1, Q_1 \rangle \Lambda_1(D)^* \langle Q_1, Q_1 \rangle^{-1} Q_1(x) - Q_1(x)G_0, \]
\[ G_2 = \langle Q_2, Q_2 \rangle \Lambda_2(D)^* \langle Q_2, Q_2 \rangle^{-1} Q_2(x) - \partial \langle Q_2 \rangle G_1(x) - Q_2(x)G_0, \]
see Theorem 4.3 in [13].

Also from Corollary 4.5 in [13], we obtain the expression for the corresponding eigenvalues for the adjoint operator \( D^* \), in terms of the eigenvalues of the differential operator \( D \) and the norm of the polynomials \( Q_w \),
\[ \Lambda_w(D^*) = \langle Q_w, Q_w \rangle \Lambda_w(D)^* \langle Q_w, Q_w \rangle^{-1}, \quad \text{for all } w. \]

By using the expressions of \( \langle Q_i, Q_i \rangle \), given at the end of Section 4, and making straightforward computations, we can verify that
\[ D_1^* = D_1, \quad D_2^* = D_2, \quad D_3^* = \frac{1}{n-p}D_4. \]

Therefore
\[ E_3 = (n-p)D_3 + pD_4 \quad \text{and} \quad E_4 = i((n-p)D_3 - pD_4) \]
are also symmetric operators, because for any $D \in D(W)$ the operators $D + D^*$ and $i(D - D^*)$ are symmetric operators. Explicitly,

$$E_3 = (n - p)D_3 + pD_4 = \partial^2 \left( \frac{-nx(n - 2p)}{x^2(n - p) + p} \frac{p^2 - n + p}{x(n - 2p)} \right) + \partial \left( \frac{2p}{(p + 1)(n - p)x} \frac{2p(n - p + 1)x}{2(n - p)} \right)$$

$$+ \left( \begin{array}{cc}
0 & p(n - p)(n - p + 1) \\
(p(p + 1)(n - p)) & 0
\end{array} \right),$$

$$-iE_4 = (n - p)D_3 - pD_4 = \partial^2 \left( \frac{-nx}{x^2(n - p) + p} \frac{-x^2p - n + p}{nx} \right) + \partial \left( \frac{-2p(n - p + 1)}{2(p + 1)(n - p)x} \frac{-2p(n - p + 1)x}{2(n - p)(p + 1)} \right)$$

$$+ \left( \begin{array}{cc}
0 & -p(n - p)(n - p + 1) \\
(p(p + 1)(n - p)) & 0
\end{array} \right).$$

The corresponding eigenvalues are

$$\Lambda_w(E_3) = \left( \begin{array}{cc}
0 & p(n - p + w)(n - p + w + 1) \\
(n - p)(p + w)(p + w + 1) & 0
\end{array} \right),$$

$$\Lambda_w(-iE_4) = \left( \begin{array}{cc}
0 & -p(n - p + w)(n - p + w + 1) \\
(n - p)(p + w)(p + w + 1) & 0
\end{array} \right).$$

**Remark 5.7.** In [10] the authors study matrix valued orthogonal polynomials related to spherical functions on the group $(SU(2) \times SU(2), SU(2))$. The weight matrix is $W_{\nu}^+(\nu)$, with $\nu = 1$ in the notation of Remark 5.7.

Let us denote $\bar{D}_1$, $\bar{D}_2$ and $\bar{D}_3$ the differential operators $D_1$, $D_2$ and $D_3$ appearing in Theorem 8.1 in [10]. Then we have the following relations with our operators $D_1$, $D_2$, $D_3$ and $D_4$ for the case $n = 3$ and $p = 1$:

$$\bar{D}_1 = L(D_1 + D_2 - 3)L^{-1}, \quad \bar{D}_2 = LD_2L^{-1}, \quad \bar{D}_3 = -\sqrt{2}L(2D_3 + D_4)L^{-1}.$$

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**References**


CIEM-FAMAF, UNIVERSIDAD NACIONAL DE CÓRDoba, 5000 Córdoba, ARGENTINA.

E-mail address: pacharon@famaf.unc.edu.ar, zurrian@famaf.unc.edu.ar