THE OBSTACLE PROBLEM FOR THE INFINITY FRACTIONAL LAPLACIAN

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Abstract. Given g an α-Hölder continuous function defined on the boundary of a bounded domain Ω and given ψ a continuous obstacle defined in $\overline{\Omega}$, in this article, we find u an α-Hölder extension of g in Ω with $u \geq \psi$. This function u minimizes the α-Hölder semi-norm of all possible extensions with these properties and it is a viscosity solution of the associated obstacle problem for the infinity fractional Laplace operator.

1. Introduction

Let Ω be an open, bounded domain of $\mathbb{R}^N$ and $\alpha \in (0,1)$. In this paper we will consider the infinite fractional Laplace operator given by

$$Lu(x) = \sup_{y\in\overline{\Omega},\, y\neq x} \frac{u(y)-u(x)}{|y-x|^\alpha} + \inf_{y\in\overline{\Omega},\, y\neq x} \frac{u(y)-u(x)}{|y-x|^\alpha}\,, \text{ for } x \in \Omega\,.$$

Motivated by the results of Chambolle, Lindgren and Monneau (see [6]), we will be interested in solutions of the associated Dirichlet obstacle problem. Concretely, given an α-Hölder function g defined on $\partial\Omega$ and a continuous obstacle ψ defined on $\overline{\Omega}$, we aim to prove the existence and uniqueness of at least a super infinity fractional harmonic function constrained to lie above the obstacle and to take the datum on $\partial\Omega$. More precisely, we consider the following obstacle problem

$$(1)\qquad \begin{cases} -Lu(x) = 0, & \text{in } \{x\in\Omega : u(x) > \psi(x)\}, \\ -Lu(x) \geq 0, & \text{in } \{x\in\Omega : u(x) = \psi(x)\}, \\ u(x) \geq \psi(x), & \text{if } x\in\Omega, \\ u(x) = g(x), & \text{if } x\in\partial\Omega, \end{cases}$$

and we will study the existence and uniqueness of a viscosity solution that seems to be natural in this framework.

By a viscosity subsolution (resp. supersolution) of (1) we mean an upper semicontinuous (resp. lower semicontinuous) function u from $\overline{\Omega}$ to $\mathbb{R}$ satisfying that $u \leq g$ (resp. $u \geq g$) on $\partial\Omega$ and the following property: $\forall \varphi \in C^1(\Omega)\cap C(\overline{\Omega})$ such that

$$u \leq \varphi\,,\ \text{in } \overline{\Omega}\,(\text{resp. } u \geq \varphi)\,,$$

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$$u(x_0) = \varphi(x_0)\,, \text{ for some } x_0 \in \Omega\,,$$

then

$$\min\{-L\varphi(x_0), \varphi(x_0) - \psi(x_0)\} \leq 0\,(\text{ resp. } \geq 0)\,.$$

A viscosity solution is a function which is both a subsolution and a supersolution.

We observe that, if u is a continuos function defined on $\overline{\Omega}$ satisfying $u = g$ on $\partial\Omega$, then we easily deduce, from the above definition, the following characterization of viscosity sub and super solution. Concretely, u is a viscosity subsolution of (1) if for any $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$ such that

$$u \leq \varphi\,, \text{ in } \overline{\Omega}\,,$$

$$\psi(x_0) < u(x_0) = \varphi(x_0)\,, \text{ for some } x_0 \in \Omega\,,$$

then

$$-L\varphi(x_0) \leq 0\,.$$

On the other hand, u is a viscosity supersolution of (1) if for any $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$ such that

$$u \geq \varphi\,, \text{ in } \overline{\Omega}\,,$$

$$u(x_0) = \varphi(x_0)\,, \text{ for some } x_0 \in \Omega\,,$$

then

$$-L\varphi(x_0) \geq 0 \qquad \text{and} \qquad u(x_0) \geq \psi(x_0).$$

It is interesting to note that the obstacle function ψ is always a viscosity subsolution of (1). See [11] for more information about viscosity solutions.

We also emphasize that in order to have a solution of our obstacle problem (1), it is necessary (due to the boundary conditions) that

$$\psi(x) \leq g(x)\,,\ \forall x \in \partial\Omega\,, \tag{2}$$

holds true.

Problem (1) involves a boundary problem in the fractional setting. This kink of problems have been studied extensively, see for instance [7] and [8]. Specifically, Problem (1) involves the infinity fractional Laplace operator The infinity Laplace operator was considered widely in the literature in the local case, see [1], [2], [5], [12] [14], and [15]; as well as in the nonlocal case (especially fractional), see for instance [4], [6] and [10]. Moreover, in [4], [12] and [15] it is studied the existence of a solution for some obstacle problems. On one hand, in [12], the authors consider the (local) infinity Laplace operator. In particular, they propose a game which involves an obstacle function and they prove that certain limit of some specific values functions is a viscosity solution of the obstacle problem for the infinity Laplacian. On the other side, in [4], the authors consider a nonlocal tug of war game. Motivated by [14] the authors consider here a nonlocal version of the game.

Recently, in [6], given an α-Hölder continuous function g defined on $\partial\Omega$, it is obtained a viscosity solution for the Dirichlet problem

$$\begin{cases} -Lu(x) = 0, & \text{if } \quad x \in \Omega, \\ u(x) = g(x), & \text{if } \quad x \in \partial\Omega, \end{cases}$$

where L is the infinity fractional Laplace operator defined before. Our aim is to extend these results considering the study of the obstacle problem (1).

Specifically, if we will denote the α-Hölder semi-norm of a function u defined on Ω by [1]

$$[u]_\alpha = \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(y)-u(x)|}{|y-x|^\alpha}\,,$$

our main results are the following theorems.

Theorem 1. *Let Ω be an open, bounded domain of $I\!R^N$, $g \in C(\partial\Omega)$ and $\psi \in C(\overline{\Omega})$ such that (2) holds true. Then there exists at most a viscosity solution u of the obstacle problem (1).*

Theorem 2. *Let Ω be an open, bounded and Lipschitz domain of $I\!R^N$, $\alpha \in (0,1)$. If $g \in C^{0,\alpha}(\partial\Omega)$ and $\psi \in C(\overline{\Omega})$ satisfy (2), then there exists a unique viscosity solution u of the obstacle problem (1) which belongs to $C^{0,\alpha}(\overline{\Omega})$. Moreover, the solution u is the best α-Hölder continuous extension of the datum g which lies above the obstacle ψ, in the sense that*

$$[u]_\alpha \leq [z]_\alpha\,,$$

for any arbitrary α-Hölder extension z of the datum g which satisfies $z \geq \psi$.

Following the arguments of [6], a possible approach to study our problem (1) is to approximate our infinity Laplace operator L with a sequence of approximate operators (see Section 2 below). In this sense, in what follows, given $p > N$, $\frac{N}{p} < \alpha < 1$ and $s := \alpha - \frac{N}{p}$, we consider the fractional Sobolev space $W^{s,p}(\Omega)$ defined by

$$W^{s,p}(\Omega) = \left\{ u \in L^p(\Omega) : \int_\Omega\int_\Omega \left(\frac{|u(y)-u(x)|}{|y-x|^\alpha}\right)^p dydx < +\infty \right\},$$

and we recall that $(W^{s,p}(\Omega), ||.||_{s,p})$ is a Banach space, where

$$||u||_{s,p} = \left(\int_\Omega |u|^p + \int_\Omega\int_\Omega \left(\frac{|u(y)-u(x)|}{|y-x|^\alpha}\right)^p dydx\right)^{1/p}, \quad u \in W^{s,p}(\Omega)\,.$$

We define the functional $E_p : W^{s,p}(\Omega) \longrightarrow I\!R$ by

$$E_p(u) = \int_\Omega\int_\Omega \left(\frac{|u(y)-u(x)|}{|y-x|^\alpha}\right)^p dydx\,, \tag{3}$$

and we study the minimization problem in a specific set. Observe that the operator of the Euler Lagrange equation associated to this functional is

$$L_p u(x) = \int_\Omega \left(\frac{|u(y)-u(x)|}{|y-x|^\alpha}\right)^{p-1} \frac{\mathrm{sig}(u(y)-u(x))}{|y-x|^\alpha}\, dy\,, \tag{4}$$

where $\mathrm{sig}(x) = \frac{x}{|x|}$ for $x \neq 0$. At least formally, we emphasize that the operator $(L_p(u))^{1/(p-1)}$ should tend to our infinity fractional Laplace operator L when p goes to ∞. We remark that this formal limit procedure only works when the right hand side is zero (when it is not zero one may expect a different limit equation). This will be the key point in our approach. We want to prove that the unique minimum (belonging to a suitable set) u_p of E_p is a viscosity solution of the obstacle problem associated to the operator L_p. Afterwards, we want to pass to the limit when p tends to infinity. We

[1] Recall that $(C^{0,\alpha}(\Omega), ||.||_\infty + [.]_\alpha)$ is a Banach Space

will prove that the limit of the sequence u_p of approximate solutions is a viscosity solution of (1).

The article is organized as follows: in Section 2, we study the properties of the approximate obstacle problems (associated to the approximate operators L_p) and in Section 3, we prove our main results.

2. Approximate problems

We consider the approximate operators L_p given by (4) and we study in this section the approximate obstacle problems

$$\begin{cases} -L_p u(x) = 0, & \text{in } \{x \in \Omega : u(x) > \psi(x)\}, \\ -L_p u(x) \geq 0, & \text{in } \{x \in \Omega : u(x) = \psi(x)\}, \\ u(x) \geq \psi(x), & \text{if } x \in \Omega, \\ u(x) = g(x), & \text{if } x \in \partial\Omega. \end{cases} \tag{5}$$

In the next lemmas, we prove that the functional E_p given by (3) has a unique minimum (in a specific set) which is a viscosity solution of the approximate obstacle problem (5).

Lemma 1. *Let Ω be an open, bounded and Lipschitz domain of $I\!R^N$, $\alpha \in (0,1)$, $p > \frac{2N}{\alpha}$. If $g \in C^{0,\alpha}(\partial\Omega)$, $\psi \in C(\overline{\Omega})$ and (2) holds, then the functional E_p given by (3) takes a unique minimum u_p in the set*

$$X_{g,\psi} = \{v \in W^{s,p}(\Omega) \,:\, v \geq \psi \text{ en } \overline{\Omega}\,,\, v = g \text{ en } \partial\Omega\}\,.$$

Moreover, u_p belongs to $C(\overline{\Omega})$.

Proof. Firstly we observe that, any α-Hölder extension of g which lies above the obstacle ψ belongs to the set $X_{g,\psi}$. Thus, this set is not empty (see [15, Proposition 3.3] for the existence of this extension). In addition, if we take $u \in X_{g,\psi}$ and we fix $k \geq ||\psi||_\infty + ||g||_\infty$, then $T_k(u) \in X_{g,\psi}$ and

$$E_p(u) \geq E_p(T_k(u))\,,$$

where the function $T_k : I\!R \longrightarrow I\!R$ is defined by

$$T_k(s) = \begin{cases} -k\,, & \text{si } s < -k\,, \\ s\,, & \text{si } |s| \leq k\,, \\ k\,, & \text{si } s > k\,. \end{cases}$$

As a consequence, we have that

$$\min_{X_{g,\psi}} E_p(u) = \min_{X^k_{g,\psi}} E_p(u)\,,$$

where

$$X^k_{g,\psi} = \{u \in X_{g,\psi} \,:\, ||u||_\infty \leq k\}.$$

Since the set $X^k_{g,\psi}$ is weakly closed (with the weak topology of $W^{s,p}(\Omega)$), to prove the existence of a minimum in this set, we will study the coercivity and the weak lower semicontinuity of the functional E_p.

On the one hand, we take a sequence $\{u_n\} \subset X^k_{g,\psi}$ such that $||u_n||_{s,p} \to +\infty$. Since $||u_n||_\infty \leq k$, this necessarily means that $E_p(u_n) \to +\infty$. That is, our functional is coercive.

On the other hand, we take a sequence $\{u_n\} \subset X^k_{g,\psi}$ such that u_n weakly converges to a function u in $W^{s,p}(\Omega)$. Since $W^{s,p}(\Omega)$ is compactly embedded in $L^p(\Omega)$ (see [9, Corollary 1.2]) and the norm $||.||_{s,p}$ is a w.l.s.c. function, then

$$\liminf E_p(u_n) \geq E_p(u)$$

as we desired.

Consequently, the functional E_p has a minimum u_p in the set $X_{g,\psi}$ and again by [9, Theorem 8.2], $u_p \in C(\overline{\Omega})$. Moreover, since the functional is convex, this minimum is unique. □

Lemma 2. *Let Ω be an open, bounded and Lipschitz domain of $\mathbb{R}^N$, $g \in C^{0,\alpha}(\partial\Omega)$ and $\psi \in C(\overline{\Omega})$ satisfying* (2)*. If $p > 2N/\alpha$, then the minimum u_p (given by Lemma 1) is a viscosity solution of* (5)*.*

Proof. Firstly, recall that the minimum u_p (given by Lemma 1) belongs to $C(\overline{\Omega})$ and satisfies $u_p = g$ on $\partial\Omega$ and $u_p \geq \psi$ in $\overline{\Omega}$. We will prove that u_p is a viscosity sub and super solution. On the one hand, we claim that u_p is a viscosity subsolution of (5). Indeed, we take $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$ such that

$$u_p \leq \varphi\,, \ \text{ in } \overline{\Omega}\,,$$

$$\psi(x_0) < u_p(x_0) = \varphi(x_0)\,, \ \text{for some } x_0 \in \Omega\,,$$

and we prove $-L\varphi(x_0) \leq 0$. Without loss of generality, we suppose that φ touches u_p only at the point x_0; otherwise it is sufficient to replace φ by $\varphi(x) + \delta|x - x_0|^2$ with δ small enough. We define the functions

$$\varphi^\epsilon = \max(u_p, \varphi - \epsilon)\,,$$

and

$$\varphi_\epsilon = \min(u_p, \varphi - \epsilon).$$

Since, we suppose that $\varphi(x_0) = u_p(x_0) > \psi(x_0)$, for ϵ small enough, $\varphi_\epsilon \geq \psi$ in Ω and moreover, $\varphi_\epsilon = u_p$ on $\partial\Omega$. Hence, φ_ϵ belongs to $X_{g,\psi}$ and using that u_p is a minimum of E_p in this set, we have

$$E_p(\varphi_\epsilon) \geq E_p(u_p)\,.$$

From this inequality and using the following convexity inequality (see [6, Lemma 6.2])

$$|\max(a, c) - \max(b, d)|^p + |\min(a, c) - \min(b, d)|^p \leq |a - b|^p + |c - d|^p\,,$$

for all $p \geq 1$, we deduce that

$$E_p(\varphi_\epsilon) + E_p(\varphi^\epsilon) \leq E_p(u_p) + E_p(\varphi) \leq E_p(\varphi_\epsilon) + E_p(\varphi)\,,$$

that is

$$E_p(\varphi^\epsilon) \leq E_p(\varphi)\,.$$

The convexity of E_p implies

$$E_p((1-t)\varphi + t\varphi^\epsilon) \leq (1-t)E_p(\varphi) + tE_p(\varphi^\epsilon) \leq E_p(\varphi)\,,$$

and then we have

$$\frac{E_p((1-t)\varphi + t\varphi^\epsilon) - E_p(\varphi)}{t} \leq 0\,.$$

Let call

$$f(t) = E_p((1-t)\varphi + t\varphi^\epsilon).$$

From the above inequality and using the convexity of the function f, we have

$$f'(0) \leq \frac{f(t)-f(0)}{t} \leq 0\,,$$

and then

$$p\int_\Omega\int_\Omega H(x,y)\,dydx \leq 0\,,$$

where

$$H(x,y)=\left|\frac{\varphi(y)-\varphi(x)}{|y-x|^\alpha}\right|^{p-1}\frac{\operatorname{sgn}(\varphi(y)-\varphi(x))}{|y-x|^\alpha}(\varphi^\epsilon(y)-\varphi(y)+\epsilon-\varphi^\epsilon(x)+\varphi(x)-\epsilon)\,.$$

Therefore, a change of variable implies

$$\int_\Omega (\varphi^\epsilon-\varphi+\epsilon)(x)(-L_p\varphi(x))\,dx \leq 0\,.$$

Now we argue by contradiction. Suppose that $-L_p\varphi(x_0)>0$. By continuity, which holds under our assumptions, there is a small ball $B_r(x_0)$ such that $-L_p\varphi>0$ in $B_r(x_0)$. Since $\varphi^\epsilon=\max(u_p,\varphi-\epsilon)$, for ϵ small enough, we have $\operatorname{supp}(\varphi^\epsilon-\varphi+\epsilon)\subset B_r(x_0)$. We also observe that $\varphi^\epsilon-\varphi+\epsilon\geq 0$. Consequently, we deduce

$$0<\int_{B_r(x_0)}(\varphi^\epsilon-\varphi+\epsilon)(x)(-L_p\varphi(x))\,dx \leq 0\,,$$

which is a contradiction.

In the same way, one can prove that u is a viscosity supersolution. $\square$

3. Proofs of the main results

Proof of Theorem 1. Suppose that u and v are two viscosity solutions of the obstacle problem (1) and define the set

$$W=\{x\in\Omega\,:\,u(x)>v(x)\}\,.$$

We claim that W is an empty set. Indeed, arguing by contradiction, we suppose that W is not empty. Since $v\geq\psi$ in Ω, we have $v\geq\psi$ and $u>\psi$ in W. Consequently, the functions u and v satisfy

$$\begin{cases} -Lu=0 & \text{in} \quad W\,,\\ u=u & \text{on} \quad \partial W\,,\end{cases} \qquad\qquad \begin{cases} -Lv\geq 0 & \text{in} \quad W\,,\\ v=u & \text{on} \quad \partial W\,,\end{cases}$$

which implies, using the comparison principle [6, Proposition 11.2], that $v\geq u$ in W. This is a contradiction and the claim is proved. Reversing the role of u and v gives that $u=v$ which conclude the proof. $\square$

To prove Theorem 2 we need the following technical result.

Lemma 3. [6, Lemma 6.5] *For $\varphi\in C^1(\Omega)$, $p\geq 1$ and $\alpha\in(0,1)$, we define*

$$f_p(y)=\frac{\varphi(y)-\varphi(x_p)}{|y-x_p|^\alpha} \qquad \textit{and} \qquad f(y)=\frac{\varphi(y)-\varphi(x_0)}{|y-x_0|^\alpha},$$

where $x_p\longrightarrow x_0\in\Omega$ as $p\longrightarrow\infty$. Then,

$$\lim_{p\longrightarrow\infty}\left|\left|\frac{f_p^+(y)}{|y-x_p|^{\alpha/p}}\right|\right|_{L^p(\Omega)}=||f^+||_{L^\infty(\Omega)}\,,$$

with $f^{\pm}(x) = \max(\pm f(x), 0)$. *The same also holds for* f_p^-.

Proof of Theorem 2. Let $\{u_p\}$ be a sequence of viscosity solutions u_p of (5) given by Lemma 1. Our aim is to pass to the limit when p goes to infinite. Firstly, given $\alpha \in (0,1)$, we prove that the sequence $\{u_p\}$ is bounded in $W^{s,q}(\Omega)$ for any $q > 2N/\alpha$. Indeed, by construction, there is a positive constant k such that

$$||u_p||_\infty \leq k\,,\ \forall p\,.$$

Now, we take any $p > 2N/\alpha$ and we fix a number q such that $2N/\alpha < q < p$. Let z be a Hölder extension of g such that $z \geq \psi$ (see [15, Proposition 3.3]). Since the functional E_p takes a unique minimum u_p in the set $X_{g,\psi}$, then

$$E_p(u_p) \leq E_p(z) \leq |\Omega|^2 [z]_\alpha^p,$$

and by Hölder inequality

$$E_q(u_p) \leq E_p(u_p)^{\frac{q}{p}} |\Omega|^{\frac{2(p-q)}{p}} \leq |\Omega|^2 [z]_\alpha^q\,, \tag{6}$$

which implies that the sequence $\{u_p\}$ is bounded in $W^{s,q}(\Omega)$. By the Sobolev embedding (see [9, Theorem 8.2]) we deduce that, up to a subsequence, u_p strongly converges to a function u in $C(\overline{\Omega})$. Moreover, since $u_p = g$ on $\partial\Omega$, and $u_p \geq \psi$ in Ω, then we also have that the function u satisfies

$$u = g\,,\ \text{on } \partial\Omega\,,\ \ u \geq \psi\,,\ \text{in } \Omega\,.$$

Now, we will prove that u is a viscosity sub and super solution of problem (1). On the one hand, we claim that u is a viscosity subsolution. Indeed, we take $\varphi \in C^1(\Omega) \cap C(\overline{\Omega})$ such that

$$u \leq \varphi\,,\ \text{in } \overline{\Omega}\,,$$

$$\psi(x_0) < u(x_0) = \varphi(x_0)\,,\ \text{for some } x_0 \in \Omega\,,$$

and, without loss of generality, we suppose that φ touches u only at the point x_0 (x_0 is a strict maximum of $u - \varphi$). Hence,

$$M_p := \sup_{\overline{\Omega}} (u_p - \varphi) = (u_p - \varphi)(x_p),$$

where

$$x_p \mapsto x_0\,,\quad M_p \mapsto 0.$$

Moreover, since $\varphi(x_0) > \psi(x_0)$, we can suppose that $\varphi(x_p) > \psi(x_p)$ for p large enough. This shows that

$$\begin{cases} u_p \leq \varphi_p := \varphi + M_p\,, \\ \psi(x_p) < u_p(x_p) = \varphi(x_p). \end{cases}$$

The fact that u_p is a viscosity solution implies

$$0 \geq -L_p \varphi_p(x_p) = -L_p \varphi(x_p),$$

that is,

$$0 \geq -\int_\Omega \left| \frac{\varphi(y) - \varphi(x_p)}{|y - x_p|^\alpha} \right|^{p-1} \frac{\operatorname{sgn}(\varphi(y) - \varphi(x_p))}{|y - x_p|^\alpha}\, dy\,,$$

or equivalently,

$$\left\| \left(\frac{\varphi(y) - \varphi(x_p)}{|y - x_p|^{\alpha + \frac{\alpha}{p-1}}} \right)^+ \right\|_{L^{p-1}(\Omega)} \geq \left\| \left(\frac{\varphi(y) - \varphi(x_p)}{|y - x_p|^{\alpha + \frac{\alpha}{p-1}}} \right)^- \right\|_{L^{p-1}(\Omega)}.$$

Thanks to Lemma 3, we can pass to the limit in this inequality to obtain

$$\sup_{y\in\Omega}\left(\max\left(\frac{\varphi(y)-\varphi(x_0)}{|y-x_0|^\alpha},0\right)\right)+\inf_{y\in\Omega}\left(\min\left(\frac{\varphi(y)-\varphi(x_0)}{|y-x_0|\alpha},0\right)\right)\geq 0.$$

Since φ is C^1 at x_0, it is clear that $L^+\varphi(x_0)\geq 0$ and $L^-\varphi(x_0)\leq 0$, where

$$L^+\varphi(x_0)=\sup_{y\in\Omega,\ y\neq x_0}\frac{\varphi(y)-\varphi(x_0)}{|y-x_0|^\alpha}\,,\quad L^-\varphi(x_0)=\inf_{y\in\Omega,\ y\neq x_0}\frac{\varphi(y)-\varphi(x_0)}{|y-x_0|^\alpha}.$$

Summing up, we deduce

$$-L\varphi(x_0)\leq 0\,,$$

as we desired.

Finally, in the same way, one can proved that u is a viscosity super solution. Therefore, u is a viscosity solution of the obstacle problem (1).

To conclude, we characterize the function limit u. In order to do it, let z be any Hölder extension of g such that $z\geq\psi$. By (6), we have

$$E_q(u_p)\leq|\Omega|^2[z]_\alpha^q,$$

which implies, passing to the limit as p goes to ∞,

$$(E_q(u))^{1/q}\leq|\Omega|^{2/q}[z]_\alpha.$$

As a consequence, when q tends to ∞, we obtain

$$\left|\left|\frac{u(y)-u(x)}{|y-x|^\alpha}\right|\right|_{L^\infty(\Omega\times\Omega)}\leq[z]_\alpha,$$

i.e., we have proved that $u\in C^{0,\alpha}(\overline{\Omega})$ and moreover

$$[u]_\alpha\leq[z]_\alpha\,,$$

for any arbitrary Hölder extension z of the datum g which satisfies $z\geq\psi$, as we desired.

□

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