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Entanglement in a spin ring with anisotropic interactions

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The relationship between entanglement and anisotropy is studied in small spin chains with periodic boundary conditions. The Hamiltonian of the spin chains is given by a slight modification of the dipolar Hamiltonian. The effect of the anisotropy is analyzed using the concurrence shared by spin pairs, but the study is not restricted to nearest-neighbor (NN) entanglement. It is shown that, under rather general conditions, the inclusion of anisotropic terms diminishes the entanglement shared between the spins of the chain irrespective of its range or its magnetic character.

Keywords: Entanglement; spin chain; long-range interaction.

1. Introduction

The study of rings, chains, or small clusters of quantum spins has been fueled by its applications in different areas such as molecular magnetism and, more recently, quantum computing.[1,2] In the case of molecular magnets, the system contains several magnetic ions whose coupled spins generate a collective spin. Recently this collective spin has been pointed out as a feasible physical system to implement qubits. This is possible since the two lowest energy states are well separated from the other states,

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enabling the manipulation of the qubit basis states. In the work of Bertaina *et al.*[2] the Rabi oscillations associated to the $S = 1/2$ and $S = 3/2$ collective states have been observed. This is quite remarkable, since the system considered was a cluster of 15 vanadium ions, known as V_{15}, each of which has $S = 1/2$. Another molecular magnet that enables the observation of coherent dynamics is the antiferromagnetic ring Cr_7Ni molecule[3] and more recently, long-lived Rabi oscillations has been observed in antiferromagnetic quantum spin chains $(TMTTF)_2X$, with $X = AsF_6$, PF_6, SbF_6. These last systems can be well modeled with an isotropic Heisenberg Hamiltonian, with a dimerization parameter in the case of the composes $(TMTTF)_2PF_6 6$ and $(TMTTF)_2AsF_6$.

The calculation of the energy spectrum of the systems mentioned above is a rather difficult task, the dimension of the Hilbert space associated to the V_{15} cluster is 32,768 (and far larger in the case of the Cr_7Ni cluster). There is lot of work related to the calculation of the spectrum of molecular magnets, mainly those using group-theoretical approaches (see Tsukerblat[4] and references therein). The group theoretical approach enables to reduce the dimension of the problem to be solved numerically, anyway this method does not provide the eigen-states of the problem, so if one is interested in the entanglement shared between the spins of the molecule or cluster it is necessary to resort to other methods. Another problem to study molecular magnets or spins clusters lies in the fact that these systems present long-range interactions, i.e. the spins do not interact only with its closest neighbors but with a set of spins that can be scattered over the cluster.

There is only a handful of results about the behavior of entanglement in systems with long-range interactions (see Gaudiano *et al.*[5] and references therein) and those refer mainly to mean field models as the Lipkin–Meshkov–Glick model,[6] which is of the XY type in an external field, or an equivalent of the BCS model,[7] which is of the XX type in an external field. Reference 8 presents a study of entanglement for nearest and next-nearest neighbors (NNs) for the ground state of a Heisenberg chain with nearest and competing ferromagnetic next-NN interactions. For a review on the subject see Ref. 9. A previous work by us[5] attempts to clarify the behavior of the entanglement on a Heisenberg-like spin model with a exchange interaction which effectively depends on the distance between the spins. Reference 5 deals with the bipartite subsystem entanglement over all the possible distances between the constituents spins on a ring and for the whole spectrum. Interestingly, as can be learned from Ref. 5, at some extent the detailed dependence of the interactions with the distance is unimportant: except for a small number of singular points the properties of the spectrum and the associated orto-projectors of isotropic long-range Heisenberg-like Hamiltonians, whose exchange interactions decay with the distance as a power law, are very much the same. The mean field and the Haldane–Shastry model are particular examples of "non-typical behavior",[5] in particular the Haldane–Shastry model is integrable.[11,12]

In the case of cluster or molecular magnets both elements, long-range interactions and anisotropy are crucial to understand its magnetic properties.[10] In the case of

molecular magnets, the adiabatic change of the magnetization at low temperature is governed by the discrete energy-level structure, in this sense the adiabatic change of the magnetization requires some interactions that produce gaps between the lowest lying energy levels, i.e. interactions that do not commute with the magnetization.[10] It is in this regime, when the physical behavior is governed for what happens near a level crossing, that the dipolar interaction becomes relevant, otherwise the spin Hamiltonian is dominated by the isotropic exchange between NNs.

In this paper, we address the problem of how the entanglement is shared between spins arranged in a polygon, with both long-range interactions and anisotropy. A well known model Hamiltonian that has both features is the dipolar one, where the antiferromagnetic isotropic exchange term decays with the distance and the anisotropic term is ferromagnetic. To study the effect of the anisotropy, a parameter that regulates the strength of this interaction is introduced. Changing this parameter enables us to go from a completely isotropic model to the dipolar one, and results in a richer phase space. The arrangement of the spins makes natural to impose periodic boundary conditions, this not only precludes the apparition of border effects but stabilizes the results when rings of different number of spins are analyzed. The model can be studied with, or without, an external transverse field.

The anisotropic interaction mostly considered, in studies focusing in the relationship between entanglement and anisotropy, is the Dzyaloshinskii–Moriya (DM) interaction. Antisymmetric superexchange interactions in spin Hamiltonians which describe quantum antiferromagnetic systems were introduced phenomenologically by Dzyaloshinskii.[13] Moriya showed that such interactions arise naturally in perturbation theory due to the spin-orbit coupling in magnetic systems with low symmetry.[14] Quantum information techniques have been used to characterize the quantum critical properties of different models with DM interactions since the work by Jafari *et al.,*[15] where the scaling properties of the entanglement on an antiferromagnetic Ising model with DM interaction were obtained. The concurrence between next NNs and between end spins in a one-dimensional (1D) frustrated ferromagnetic model with DM interactions,[16] the entanglement entropy of an antiferromagnetic XY model with DM interactions,[17] the ground state fidelity in bond-alternative Ising chains with DM interactions,[18] and the concurrence of finite Ising chains with antiferromagnetic NNs interactions[19] are good examples of the efforts made to analyze the effect of anisotropic interactions over the entanglement content of the ground state of quantum spin chains models. *Despite the differences between the DM interaction and the dipolar anisotropy term, we will compare our results with those available in the literature whenever that is possible.*

Another issue that leads us to study a model with long-range interaction and anisotropy is the possibility of finding a system which presents (varying an adequate parameter) entanglement with different numbers of neighbors. Most commonly, in magnetic spin models the number of spins that share entanglement with a given spin is a constant for the ground state over the parameter space. For instance, the Ising

model with transverse field presents entanglement between nearest and next-NNs in the whole parameter space, which is given by the exchange interaction and the external field.[20] Some particular models, finite spin rings with different non-collinear anisotropic Ising interaction, show that changing the strength of an external magnetic field applied to it enables to select the number of neighbors that share entanglement with a given spin.[21]

The model is presented in the next section, together with some information on its spectral properties. In Sec. 3, we deal with the pair, or two-site, entanglement. We pay particular attention to the relationship between the concurrence of two-spin subsystems and the two-site spin-correlation functions, trying to emphasize the qualitative aspects that seem to be present in all the models with anisotropic interactions. Both Secs. 2 and 3, deal with the model without external field, while the behavior of the pair entanglement when an external constant field is applied is the subject of Sec. 4. In the concluding section, Sec. 5, we highlight those features observed which we consider interesting and provide an argument explaining the short range of the pair entanglement in the ground state in models with DM interactions. Besides, we briefly discuss in which kind of systems and parameter range, it can be plausible to observe two-spin entanglement with a behavior similar to the one described in Sec. 4.

2. The model

We consider the following slightly modified version of the dipolar Hamiltonian

$$H = \sum_{i<j} \left[\frac{\sigma_i \cdot \sigma_j}{r_{ij}^3} - 3f \frac{(\sigma_i \cdot r_{ij})(\sigma_j \cdot r_{ij})}{r_{ij}^5} \right] - h \sum_i \sigma_i^z, \tag{1}$$

where σ_i^α, $\alpha = x, y, z$ are the Pauli matrices, r_{ij} is the distance between spins i and j, the spins are arranged in a regular polygon which lies in the $x - y$ plane, $\mathbf{r}_{ij}$ is the vector that goes from spin i to spin j, h is an external magnetic field that points in the $\hat{\mathbf{z}}$ direction and f is the anisotropy parameter that enables to change from an anisotropic model ($f = 1$ is the dipolar model) to an isotropic one ($f = 0$ is a generalized Heisenberg model, which has been studied in Gaudiano *et al.*[5]). Through the paper, the distance between NNs is equal to one. Obviously the number of vertices of the polygon is equal to the number of spins, N. Despite the rather particular dependence on the distance, the Hamiltonian Eq. (1) has many of the elements present in Hamiltonians describing molecular magnets: anisotropy, long-range interactions and, at least for a portion of the phase space, antiferromagnetism.

The Hamiltonian in Eq. (1) is dimensionless so the external field h is measured in terms of the gyromagnetic ratio of each spin and the cubic power of the NN distance. As we want to focus in general physical features that can be derived for systems with such Hamiltonian we do not consider a specific set of interaction parameters. Anyway, later on, we will discuss possible realistic experimental setups to implement the Hamiltonian in Eq. (1).

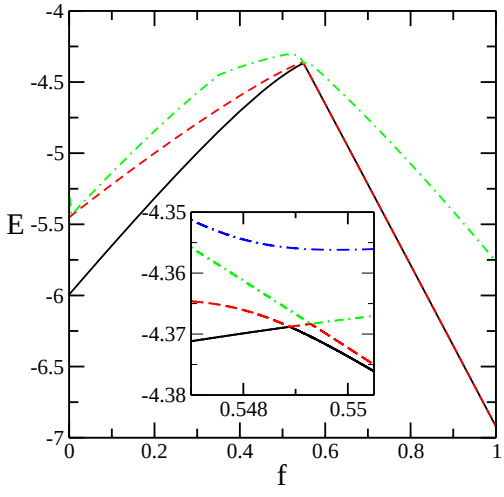

Fig. 1. The four lowest lying eigenvalues versus the anisotropy parameter f for a chain with $N = 8$ spins. The ground state energy (solid black line) shows a sharp cusp near $f \sim 0.549$. The cusp sharpens when the chain size is increased. The inset shows a detailed view of the zone near the cusp of the ground state energy, where a number of avoided-crossings between the different eigenvalues can be appreciated.

The energy spectrum and the eigenfunctions can be obtained numerically for small values of N up to 11, without any special assumptions. Anyway, as we are interested in finite rings we will present results mostly for $N = 10$.

Figure 1 shows the first four energy levels as a function of f for $N = 8$. It is very noticeable the abrupt change of behavior around $f \simeq 0.548$. For $f > 0.548$ the ground state is not degenerate, but it is very close to the first excited state. The same can be said of the second and the third excited states, they form an approximate multiplet. The inset of Fig. 1 shows in detail the structure of the lowest-lying energy levels near this critical point. A remarkable fact about the spectrum of the model is that its eigenvalues are only single or doubly degenerated, for even N both degeneracies are present, but for odd N *all* the eigenvalues are doubly degenerated. For odd N the double degeneracy of each level is a consequence of the Kramers degeneracy theorem,[22] nevertheless it is interesting that a rather simple anisotropic interaction is able to broke all the symmetries that produce a highly degenerate spectrum for the isotropic ($f = 0$ case) (see, for example, the work of Gaudiano *et al.*[5]).

The double degeneracy of the eigenvalues does not introduce further difficulties in order to analyze even or odd polygons, but it must be taken into account if the entanglement of the state were to be calculated (for a discussion about the calculation of the entanglement of a degenerate eigen-energy see Gaudiano *et al.*[5]).

Despite that in this work, we do not attempt to completely characterize the phase transition associated to this critical point studying the behavior of the model for $N \to \infty$, it is possible to correlate the change of behavior of the ground state energy with the physics of the model studying the spin correlation functions.

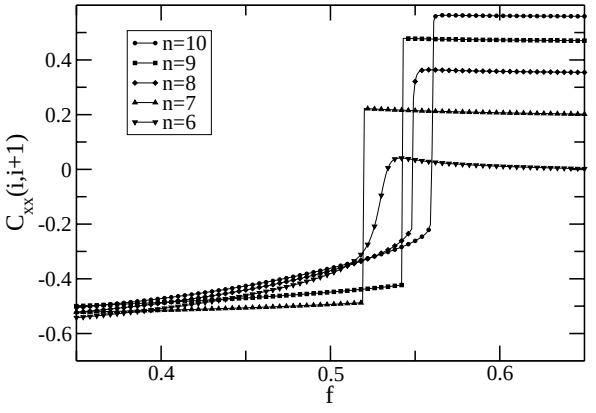

Fig. 2. The spin correlation function versus the anisotropy parameter f calculated for several chain lengths. The transitions from negative to positive values of the spin correlation function signals the presence of a critical point f_{c1}^N.

Figure 2 shows the behavior of the NN spin correlation function $C_{xx}(i, i+1) = \langle \sigma_i^x \sigma_{i+1}^x \rangle$ as a function of f and different polygon size, as we have assumed periodical boundary conditions the spin correlation function does not depend on i. For $N \geq 5$ there is a transition from negative values of the spin correlation function $C_{xx}(i, i+1) = \langle \sigma_i^x \sigma_{i+1}^x \rangle$ to positive ones. Since the transition is rather sharp it is possible to find a well defined critical point, f_{c1}^N, which depends on the chain size N and is given by the solution of $C_{xx}(i, i+1)|_{f_{c1}} = 0$. For small values of N there are odd–even effects that can be appreciated from Fig. 2, for instance, $f_{c1}^{10} > f_{c1}^{8} > f_{c1}^{9} > f_{c1}^{6} > f_{c1}^{7}$. These features become less and less noticeable when the chain size increases.

The abrupt change observed in the spin correlation function can be understood as follows: for *any* value of N, and for f small enough the model must behave as an antiferromagnetic system, i.e. the isotropic antiferromagnetic term in the Hamiltonian dictates the behavior of the system. For large enough f **and** N the anisotropic ferromagnetic term of the Hamiltonian competes efficiently with the antiferromagnetic one producing the change of behavior observed around f_{c1}^N. It is clear that at $c1$ the ground state presents clear features of a finite-size precursor of a phase transition.

Interestingly, there is another value of f that needs a closer look because the degeneracy of the ground state energy at this point is different than for any other value of f. This point corresponds to $f = 2/3 = f_{2/3}$, which is a point of high symmetry of the Hamiltonian, and at this point the ground state becomes degenerate for any value of N. As the ground state and the first excited state are nearly degenerate around $f_{2/3}$ it is rather difficult to discern if the critical point is associated to a crossings of levels, or if the two levels are degenerate just at $f_{2/3}$ without crossing each other. The presence of $f = 2/3$ is not revealed by the spin correlation function $\langle \sigma_i^x \sigma_{i+1}^x \rangle$, but for $\langle \sigma_i^z \sigma_{i+1}^z \rangle$ which changes its sign at $f_{2/3}$. Figure 3 shows the behavior of this spin correlation function near $f = 2/3$. Since the eigenvalues and the

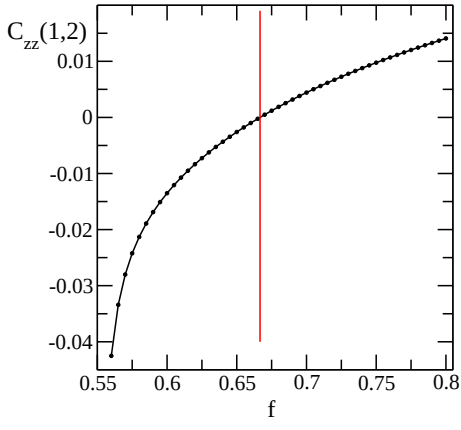

Fig. 3. The spin correlation function $C_{zz}(i, i+1)$ versus the anisotropy parameter f. The vertical red line corresponds to $f = 2/3$.

eigenvectors of the Hamiltonian depend analytically on f, it is possible to discern between the possibility of a crossing or another situation just calculating the inner product between eigen-states at both sides of the point $f = 2/3$. Calling ψ_0^- and ψ_1^- to the ground and first excited state, respectively, to the left of $f = 2/3$, and ψ_0^+, ψ_1^+ to the ground and first excited state to the right, in can be obtained numerically that $(\psi_0^-, \psi_0^+) \simeq 1$ and $(\psi_1^-, \psi_1^+) \simeq 1$ too, which lead us to conclude that the point $f = 2/3$ does not correspond to a crossing of eigenvalues.

The results presented above can be summarized as follows. There are two points f_{c1}^N and $f_{2/3}$, that separate the segment $f \in (0, 1]$ in three parts

$$\begin{aligned}
&\text{for } f \in (0, f_{c1}^N) \rightarrow \langle \sigma_i^x \sigma_{i+1}^x \rangle < 0 \quad \text{and} \quad \langle \sigma_i^z \sigma_{i+1}^z \rangle < 0 \\
&\text{for } f \in (f_{c1}^N, f_{2/3}) \rightarrow \langle \sigma_i^x \sigma_{i+1}^x \rangle > 0 \quad \text{and} \quad \langle \sigma_i^z \sigma_{i+1}^z \rangle < 0 \\
&\text{for } f \in \left(f_{2/3}, 1\right] \rightarrow \langle \sigma_i^x \sigma_{i+1}^x \rangle > 0 \quad \text{and} \quad \langle \sigma_i^z \sigma_{i+1}^z \rangle > 0. \qquad (2)
\end{aligned}$$

The transitions between the regions can be further characterized studying the entanglement of the ground state via the concurrence[23] and the fidelity as has been proposed by Zanardi.[24] We want to point out here that $f = 0$ is a singular point, since the degeneracies of the spectrum for $f = 0$ are quite different from the degeneracies of the model with $f \neq 0$. As we have already mentioned above, for $f \neq 0$ the degeneracies present in the spectrum are one and two, conversely the spectrum of the model with $f = 0$ is highly degenerate, see Gaudiano *et al.*[5]

3. Entanglement for Zero External Field

The first work to point out that quantum information quantities such as the concurrence are well suited to study quantum phase transitions was done by Osterloh *et al.* in 2002.[20] They dealt with the bipartite entanglement shared between the spins

of an Ising model with transversal field. Using finite size scaling methods they calculated the critical parameter and critical exponent from the derivative of the concurrence. Anyway, in their work there was no indication of why the quantity to be analyzed was the derivative of the concurrence (except that the concurrence was a finite quantity for all values of the external field).

The relationship between bipartite entanglement and quantum phase transitions was clarified, at least for quantum spin models, by the work of Wu *et al.*[25] More recently, Zanardi *et al.* have proposed the ground state overlap as a good quantity to reveal the presence of critical behavior in quantum spin models.[26] The idea behind both approaches is simple: some quantity shows non-analytic behavior when a quantum phase transition happens. Moreover, the work of Wu *et al.* gives precisely the recipe of which quantity shows the non-analytic behavior: if the quantum phase transition is of first-order the entanglement measure used to obtain the bipartite entanglement is a discontinuous function, in the case of a second-order transition the derivative of the entanglement measure becomes the non-analytical function to be considered.

3.1. *Concurrence*

Figure 4 shows the behavior of the concurrence in the ground state for NNs spins as a function of f for several values of N. The change of behavior around $f \simeq 0.548$ is quite noticeable. For $f \to 0$ the concurrence goes to ≈ 0.41 which is consistent with the concurrence calculated in Ref. 5 for an isotropic model (the isotropic model corresponds exactly to the model analyzed in this paper with $f = 0$). The NNs concurrence is rather steep near $f \simeq f_{c1}^{N}$, and, probably, discontinuous in the limit of $N \to \infty$. Anyway, here we do not attempt to characterize the critical behavior for larger systems. A more striking result appears studying the next nearest (and beyond)

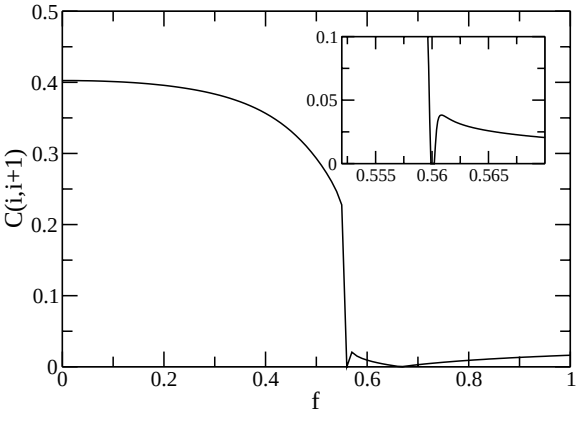

Fig. 4. The NN concurrence versus f for $N = 10$. The curve shows a very steep behavior near $f \simeq 0.558$. The inset shows a zoom of the region near the critical point $f \simeq 0.558$.

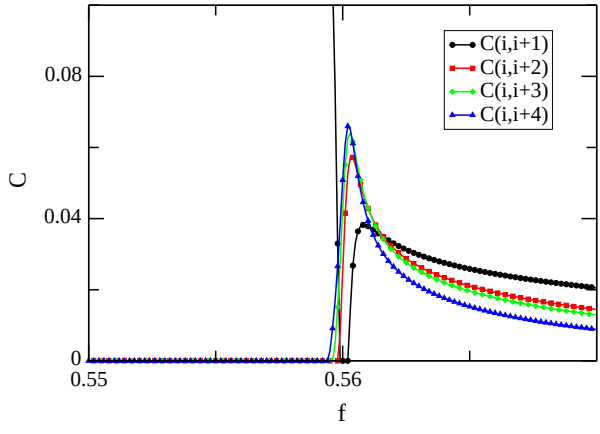

Fig. 5. The concurrence $C(i, i+j)$ versus the parameter f, for $j = 1, 2, 3$ and 4. The maximum value of the concurrence for neighbors beyond the nearest one is rather small when compared with the NN concurrence in the antiferromagnetic region, $f < f_{c1}^{10}$. For $f > f_{c1}^{10}$ the entanglement shared with the different neighbors is basically the same, while the concurrence $C(i, i+j) \sim 0$ when $f < f_{c1}^{10}$ for the second, third and fourth neighbors.

concurrence. For $f < f_{c1}^{N}$ (in the antiferromagnetic phase) the concurrence is different from zero only for NNs bipartite subsystems. For $f > f_{c1}^{N}$ the concurrence is different from zero for all distances, as shows Fig. 5.

3.2. *Ground state overlap*

As has been shown by Zanardi and Paunović,[26] and Cozzini *et al.*[27] the overlap function $|\langle\psi(f+\delta f)|\psi(f-\delta f)\rangle|$ is a good quantity to detect the presence of quantum critical points and transitions, where f is the parameter of the Hamiltonian of the system that drives the transition, $\psi(f)$ is the ground state of the system, and δf is small enough. For a good quantity, it is understood that $\langle\psi(f+\delta f)|\psi(f-\delta f)\rangle$ is a smooth function of f and δf. The method detects different kinds of models and quantum phase transitions, for example for the Dicke model and the XY spin chain (see Ref. 26) and for second-order matrix product state quantum phase transitions (see Ref. 27).

Figure 6 shows the ground state overlap function near the critical point $f \simeq f_{c1}^{N}$, for $N = 8$. The overlap is shown in the small region around $f \simeq f_{c1}^{N}$ where it differs appreciably from one, this region decreases when $\delta \to 0$ as can be seen from the several curves shown in Fig. 6, which correspond to different values of δf. The overlap changes from values around one to values close to zero rather suddenly, and remains very small ($\sim 10^{-13}$) for an interval. The behavior of the overlap observed in Fig. 6 is slightly different from the one observed in, for example, the works by Zanardi[26] and Cozzini.[27] They analyzed systems with larger number of components than the ones considered in this work, resulting in a rather smooth behavior of the overlap in

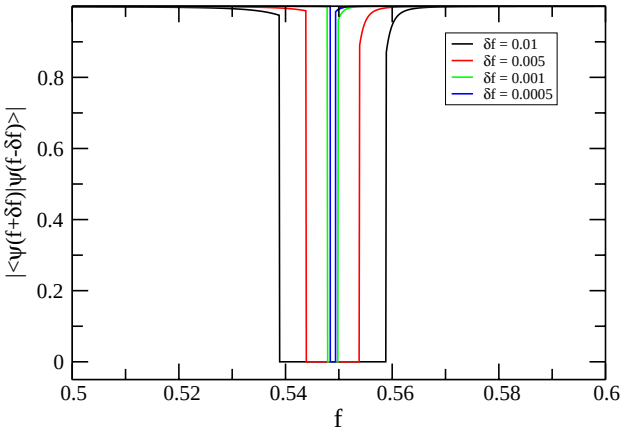

Fig. 6. The ground state overlap versus f, for $N = 8$ and different increments δf. The overlap is approximately equal to one except near f_{c1}^{N}.

contradistinction with the "box-shaped" curves shown in Fig. 6. It is clear that for $\delta f \to 0$ the overlap goes to zero in a smaller and smaller region around the critical point, pointing to it more and more precisely.

It is worth to point that the critical points obtained using the ground state overlap, the spin correlation function and the concurrence are the same, for fixed N, up to a relative error of order 10^{-4}.

4. Entanglement with an External Applied Field

As it has been shown in Sec. 3, the model shows a transition from short ranged entanglement, with only NNs sharing entanglement, to long ranged entanglement, where a given spin shares almost the same amount of entanglement with all its neighbors, see Fig. 5. We have found this feature for all the spin chain sizes (up to $N = 12$) analyzed. On the other hand, this long-range entanglement becomes weaker and weaker as the chain length increases.

In this Section, we analyze the effect of an external magnetic field h over the entanglement, we look for regions of the (f, h) space in which a given spin shares entanglement with its neighbors. Our motivation comes from certain tasks in quantum information in which it is necessary to control the entanglement between different pairs of spins. Of course, for an eigenstate of a translationally invariant spin chain, if a spin at site i shares entanglement with a spin at site j then all pairs separated by distance $|i - j|$ do share entanglement too. Other point of interest is how the entanglement decays with the distance.

We consider an external magnetic field applied in the z direction, i.e. perpendicular to the plane in which lies the ring of spins. In this case, it is clear that for large enough external magnetic field the quantum state should become disentangled since the spins should tend to be aligned with the field if this is large enough. For fixed f in

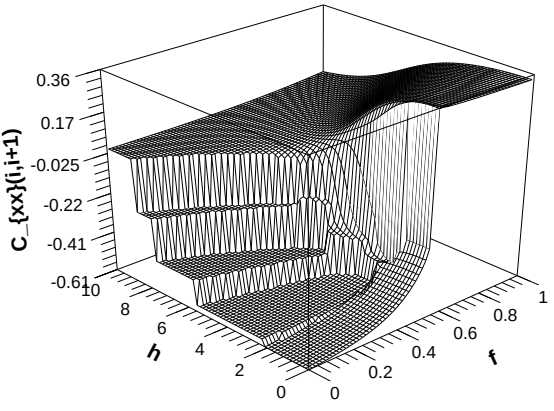

Fig. 7. The spin correlation function $\langle \sigma_i^x \sigma_{i+1}^x \rangle$ versus the anisotropy parameter and the strength of the external magnetic field for a spin chain with $N = 8$. The plane $h = 0$ shows the curve already shown in Fig. 2. For weak enough magnetic fields the ferromagnetic and antiferromagnetic zones can be easily appreciated, the spin correlation function is positive or negative, respectively.

the ferromagnetic region, the NN concurrence should behave very much alike as the NN concurrence of the Ising model with transverse field.[20] As in Sec. 3, we proceed first to analyze the behavior of the NN spin correlation function $\langle \sigma_i^x \sigma_{i+1}^x \rangle$ and then the concurrence as a function of both f and h.

Figure 7 shows the NN spin correlation function as a function of f and h. The figure shows clearly that for $f > f_{c1}^N$, which corresponds to the ferromagnetic region for the model without magnetic field, the spin correlation function smoothly decays to zero when h grows and for f fixed, while for $f < f_{c1}^N$, which correspond to the antiferromagnetic region, the spin correlation function proceeds through a number of steps until its value reaches zero. This steps are very similar to those observed in the ferromagnetic XX model with NN exchange interaction and transverse external field[28] or the Heisenberg model in an external field.[29] The steps are flat near $f = 0$ and slightly curved near the critical point f_{c1}^N. We call $f_{c1}^N(h)$ the curve that separates the region in which $\langle \sigma_i^x \sigma_{i+1}^x \rangle < 0$ from the region in which $\langle \sigma_i^x \sigma_{i+1}^x \rangle > 0$. Figure 8 shows several cuts of the surface depicted in Fig. 7. The cuts correspond to different values of the external field, and the steps in the spin correlation function are clearly visible for $h = 5, 6, 7, 8$.

Figure 9 shows the NN concurrence as a function of f and h. As the concurrence is a positive definite quantity, the transition between the ferro and antiferro region, given by the curve $f_{c1}^N(h)$ is mostly signaled by the steep change in the value of the concurrence. In the antiferromagnetic region, the steps associated to the steps of the spin correlation function are quite noticeable. On the other hand fo fixed f in the ferro region the concurrence is a smooth function with a single maximum.

The main difference between the ferromagnetic and antiferromagnetic regions are best put out by looking at the behavior of $C(i, i+1)$, $C(i, i+2)$, $C(i, i+3)$ and so on

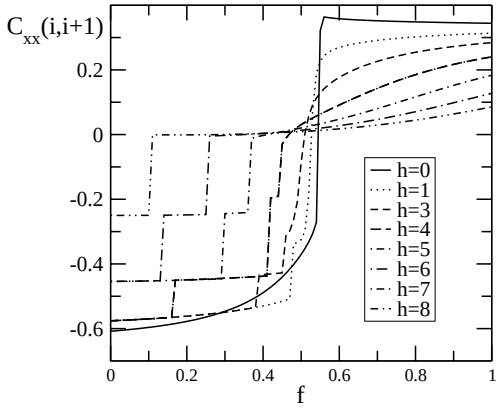

Fig. 8. The NN spin correlation function versus f for different values of the external field. Near $f = 1$ the curves are, from top to bottom, $h = 0, 1, 3, 4, 5, 6, 7, 8$. For large enough values of the external fields the spin correlation shows the steps characteristics of antiferromagnetic models as the XX model. Inside each step the spin correlation changes its value very slowly, but it is not a constant function.

($C(i, i + j)$ stands for the concurrence between spins separated by distance j). These concurrences are shown in Fig. 10. As can be appreciated there is a small region in the (f, h) space where the concurrences $C(i, i + j)$ are all different from zero. In the antiferromagnetic region (which can be easily identified by the presence of the well defined steps) the number of neighbors that share entanglement with a given spin can be tuned by changing f or h. This number can be equal to 1, 2 or $\lfloor N/2 \rfloor$, where $\lfloor x \rfloor$ stands for the the largest integer not greater than x, i.e. is the floor function.

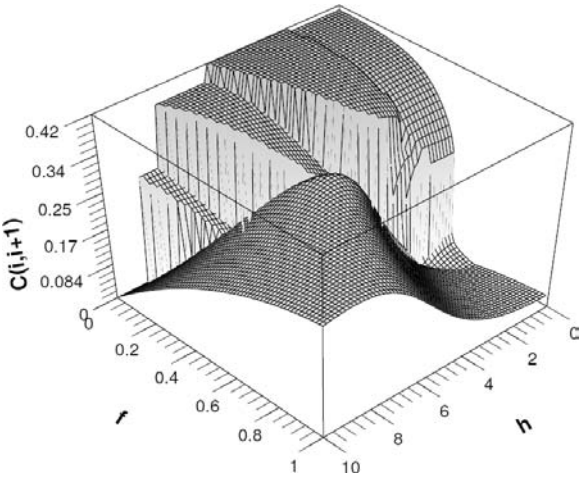

Fig. 9. The NN concurrence versus f and h. The ferromagnetic region can be clearly appreciated as the hill-like surface that appears for f large enough. For f close to zero there is a number of steps that signals the antiferromagnetic behavior and whose height decreases when the magnetic field strength increases.

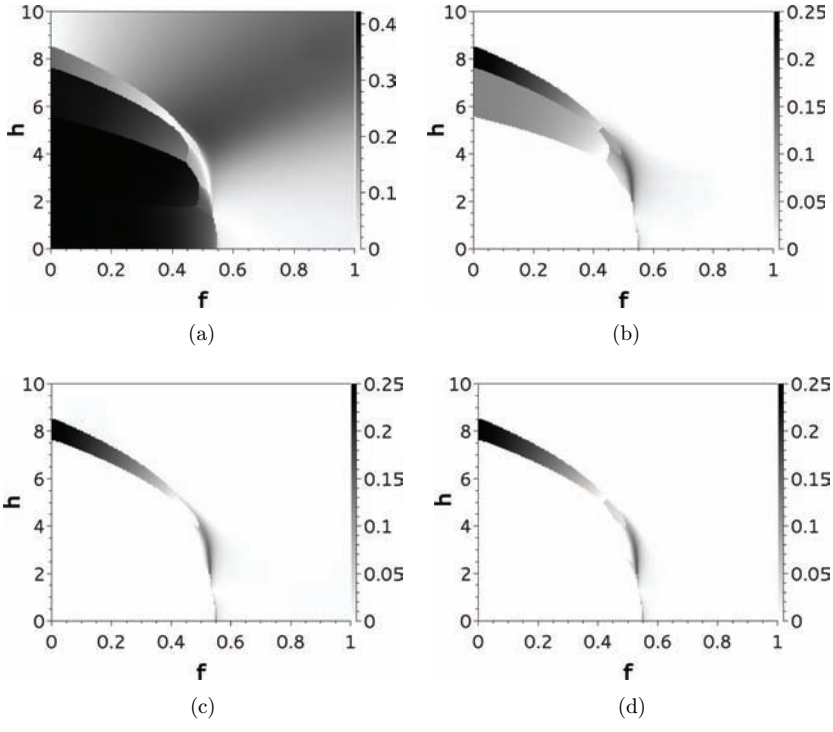

Fig. 10. (Color online) Gray scale map of the concurrences $C(i, i+j)$ for $j = 1, 2, 3$ and 4 (panel (a), (b), (c) and (d), respectively) in the (f, h) space. The darker zones correspond to regions with larger entanglement. It is rather easy to identify the region where the entanglement is not zero for different neighbors, in particular it is appreciable the region where a given spin shares bipartite entanglement with all its neighbors. This region corresponds to the zone where $C(i, i+4) \neq 0$.

On the other hand, the long ranged entanglement found in the ferromagnetic region for $h = 0$, see Fig. 5, also can be found for $h \neq 0$ in a very small region of the (f, h) space.

5. Discussion and Conclusions

The spin correlation function steps found in the antiferromagnetic region are quite similar to the ones founded in frustrated two-leg $s = \frac{1}{2}$ ladder models.[30,31] Besides, there are other models which have eigen-states where a given spin is equally entangled with a number of neighbors, for example the frustrated magnetic model in a kagomé lattice, see Ref. 31.

Given the similarities stated in the paragraph above, it is natural to ask what other similar traits can be found between the model analyzed in this work and the ones found in the literature. There is a problem to start with the comparisons: most of the models previously analyzed focus on, as has been already mentioned in the introduction, anisotropic terms DM-like. It is well known that the DM anisotropy

induces a quantum phase transition when it is added to NN models and depending the strength of the DM term the ground state corresponds to a chiral phase or an antiferromagnetic one.[15] So, we proceeded to explore two other models, (i) a model with Hamiltonian given by Eq. (1) but with $f < 0$, and (ii) a model with Hamiltonian

$$H = \sum_{i<j} \frac{\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j}{r_{ij}^3} - D \sum_i \left(\sigma_i^x \sigma_{i+1}^y - \sigma_i^y \sigma_{i+1}^y \right), \tag{3}$$

i.e. an isotropic Heisenberg-like term with a DM anisotropy. To avoid reiterative information we do not include further figures for these models, and proceed to comment the results.

The model with "antiferromagnetic dipolar interaction", i.e. the case (a) of the above paragraph, does not show any critical points for $f < 0$ and the NN spin correlation function and concurrence are smooth non-increasing functions of $|f|$. This is also the case in the model with a DM term, except for a few critical points the NN spin correlation function and concurrence are smooth non-increasing functions of D. One of this critical points is related to the transition from a chiral phase to an antiferromagnetic one.[15] Besides, if an external magnetic field h is applied, the NN functions again show steps whose height decreases when the strength of the field is increased, quite similarly to the features observed in the model of Eq. (1). A major difference between the dipolar model and the model with DM interactions appears in the behavior of the concurrences beyond the NN. The model with DM interactions and external field does not show bipartite entanglement beyond the NN, i.e. $C(i, i+j) = 0$, for $j > 1$ and for all the values of h and D explored. This is compatible with the findings shown in Ref. 30.

It is interesting to ponder in what kind of experimental setups and parameter regions it is plausible to study some of the entanglement scenarios depicted in this work. In a typical Nuclear Magnetic Resonance setup the magnetic dipole–dipole interaction is on the order of the tens of KHz, while the Zeeman term owed to the external magnetic field is on the order of tens of MHz for a magnetic field strength of several Teslas. So, at least in principle, the scenario depicted in Fig. 7 could be observed for fields of 10^{-2} or 10^{-3} Teslas. Ultracold atomic gases in optical lattices have been used to simulate condensed matter systems and, in particular, spin Hamiltonians. In these gases, the long-range interaction strength is low enough to enable the manipulation of the quantum states with field strengths of 10^{-2} or 10^{-3} Teslas. We estimate that the aforementioned scenario can be found in systems where the Zeeman energy of one magnetic dipole is similar to the dipolar interaction between NNs.

All in all, it can be said that the most immediate effect of the anisotropy smoothly reduce the NN concurrence value (with respect to the one observed in the isotropic case), regardless of the type of anisotropy involved or its range.

Anyway, a word of caution must be mentioned. Since we studied the model described by Eq. (1) in *rings*, when the number of sites in the ring increases and the ring

becomes more like a chain the distance between pairs of spins beyond the NN changes, as well as the angles between successive segments of the chain. It is reasonable to expect that the different quantities studied are smooth functions of the distances between neighbors and the angles between successive segments of the chain. The properties of the bipartite entanglement beyond NNs in two-dimensional lattices with different coordination numbers is currently under study.

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