DISTINGUISHED PRE-NICHOLS ALGEBRAS

IVÁN ANGIONO

Abstract. We formally define and study the distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}(V)$ of a braided vector space of diagonal type V with finite-dimensional Nichols algebra $\mathcal{B}(V)$. The algebra $\widetilde{\mathcal{B}}(V)$ is presented by fewer relations than $\mathcal{B}(V)$, so it is intermediate between the tensor algebra $T(V)$ and $\mathcal{B}(V)$. Prominent examples of distinguished pre-Nichols algebras are the positive parts of quantized enveloping (super)algebras and their multiparametric versions. We prove that these algebras give rise to new examples of Noetherian pointed Hopf algebras of finite Gelfand-Kirillov dimension. We investigate the kernel (in the sense of Hopf algebras) of the projection from $\widetilde{\mathcal{B}}(V)$ to $\mathcal{B}(V)$, generalizing results of De Concini and Procesi on quantum groups at roots of unity.

1. INTRODUCTION

1.1. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$. The quantized enveloping algebra $U_q(\mathfrak{g})$ was introduced in [Dr, J] by deforming the relations of the enveloping algebra $U(\mathfrak{g})$. If the parameter q is not a root of unity, then $U_q(\mathfrak{g})$ (defined as in [DP]) has similar properties to $U(\mathfrak{g})$. But if q is a root of unity, then $U_q(\mathfrak{g})$ differs significantly from $U(\mathfrak{g})$, For instance there is a Hopf pairing between $U_q^+(\mathfrak{g})$ (the subalgebra generated by the E_i's) and $U_q^-(\mathfrak{g})$ (the subalgebra generated by the F_i's) but is degenerate and its radical is the ideal generated by powers of roots vectors E_α^N, respectively F_α^N, $\alpha \in \Delta_+$. Here E_α, F_α, $\alpha \in \Delta_+$, are obtained from E_i, F_i by applying Lusztig isomorphisms $T_i \in \operatorname{Aut} U_q(\mathfrak{g})$ [DP, Section 9]. The quotient of $U_q(\mathfrak{g})$ by the ideal generated by E_α^N, K_α^N, F_α^N is a finite-dimensional Hopf algebra known as the *small quantum group* or *Frobenius-Lusztig kernel* $\mathfrak{u}_q(\mathfrak{g})$. Now the induced Hopf pairing between $\mathfrak{u}_q^+(\mathfrak{g})$ and $\mathfrak{u}_q^-(\mathfrak{g})$ is non-degenerate, so that $\mathfrak{u}_q^+(\mathfrak{g})$ is a Nichols algebra [AS1]. The kernel of the natural projection $U_q(\mathfrak{g}) \to \mathfrak{u}_q(\mathfrak{g})$ (in the Hopf algebra sense) is the central Hopf subalgebra Z_0 generated by E_α^N, K_α^N, F_α^N; Z_0 is the algebra of functions of a Poisson group [DP, Chapter 19] that plays roles in the representation theory of $U_q(\mathfrak{g})$ [DP, DPRR] and in the classification of finite-dimensional pointed Hopf algebras [AS2].

1.2. A *braided vector space* is a pair (V, c), where V is a vector space and $c \in \operatorname{Aut}(V \otimes V)$ satisfies $(c \otimes \operatorname{id})(\operatorname{id} \otimes c)(c \otimes \operatorname{id}) = (\operatorname{id} \otimes c)(c \otimes \operatorname{id})(\operatorname{id} \otimes c)$.

The Nichols algebra of a braided vector space (V, c) is a graded braided Hopf algebra $\mathcal{B}(V) = T(V)/\mathcal{J}(V) = \oplus_{n \geq 0} \mathcal{B}^n(V)$ with the property that all primitive elements are in degree 1. The study of Nichols algebras is crucial in the classification program of Hopf algebras [AS1]. If there are a basis $(v_i)_{1 \leq i \leq \theta}$ of V and a matrix $(q_{ij}) \in (\mathbb{C}^\times)^{\theta \times \theta}$ such that $c(v_i \otimes v_j) = q_{ij}\, v_j \otimes v_i$ for all $1 \leq i, j \leq \theta$, then (V, c) is of *diagonal type*. Finite-dimensional Nichols algebras of (braided vector spaces of) diagonal type are classified in [H2]; the defining relations of them are described in [A2, A3].

A pre-Nichols algebra[1] of a braided vector space (V, c) is any graded braided Hopf algebra intermediate between $T(V)$ and $\mathcal{B}(V)$, that is any braided Hopf algebra of the form $T(V)/\mathcal{I}$ where $\mathcal{I} \subseteq \mathcal{J}(V)$ is a homogeneous Hopf ideal. Pre-Nichols algebras form a partially ordered set with order given by projection, with $T(V)$ the minimal and $\mathcal{B}(V)$ the maximal points. Pre-Nichols algebras appear naturally in the computation of the deformations or liftings of [Ma, AAGMV]. Let V be a braided vector space of diagonal type with finite-dimensional $\mathcal{B}(V)$. In this paper we define and investigate the *distinguished* pre-Nichols algebra $\widetilde{\mathcal{B}}(V)$. Actually they are already present without name in [A3, Proposition 3.3]. The distinguished pre-Nichols algebra $\widetilde{\mathcal{B}}(V)$ can be realized in the category of Yetter-Drinfeld modules over $\mathbf{k}\mathbb{Z}^\theta$, where $\theta = \dim V$; let $U(V)$ be the quantum double of the bosonization $\widetilde{\mathcal{B}}(V)\#\mathbf{k}\mathbb{Z}^\theta$. Similarly, let $\mathfrak{u}(V)$ be the quantum double of the bosonization $\mathcal{B}(V)\#\mathbf{k}\mathbb{Z}^\theta$. Here are some properties of $\widetilde{\mathcal{B}}(V)$, justifying the adjective distinguished (to our understanding):

1.2.1.   Let $i \in \mathbb{I}_\theta$ and $\rho_i : V \mapsto V'$ the corresponding reflection in the Weyl groupoid of (V, c). The Lusztig isomorphism $T_i : \mathfrak{u}(V) \to \mathfrak{u}(V')$ can be lifted to a Lusztig isomorphism $T_i : U(V) \to U(V')$ [A3, Proposition 3.26]. We conjecture that $\widetilde{\mathcal{B}}(V)$ is minimal among the pre-Nichols algebras admitting Lusztig isomorphisms.

1.2.2.   If $\mathcal{B}(V) = \mathfrak{u}_q^+(\mathfrak{g})$, then $\widetilde{\mathcal{B}}(V) = U_q^+(\mathfrak{g})$.

1.2.3.   By a general result of [Kh], every pre-Nichols algebra has a restricted (that is, with heights) PBW basis. We prove the existence of a PBW basis with the same generators (root vectors), but different heights, as $\mathcal{B}(V)$, obtained by applying several T_i's, see Theorem 3.6.

1.2.4.   The pointed Hopf algebra $U(V)$ is Noetherian of finite Gelfand-Kirillov dimension, see Theorems 3.19 and 3.20. We conjecture[2] that $\widetilde{\mathcal{B}}(V)$ is minimal among the pre-Nichols algebras with these properties. Observe that $\widetilde{\mathcal{B}}(V)$ is not a domain in general.

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[1]This terminology is due to Akira Masuoka.
[2]jointly with N. A.

1.2.5. The powers of root vectors that are non-zero in the $\widetilde{\mathcal{B}}(V)$ but zero in $\mathcal{B}(V)$ generate a subalgebra $Z^+(V)$ on $\widetilde{\mathcal{B}}(V)$, that coincides with the intersection of the kernels of the skew-derivations associated to the coproduct of $U^+(V)$, see Theorem 4.13. Correspondingly there is a normal Hopf subalgebra $Z(V)$ of $U(V)$, see Theorem 4.15; $U(V)$ is a finite free $Z(V)$-module.

1.3. The organization of the article is the following. In Section 2 we recall notions and basic properties of generalized root systems and Lusztig isomorphisms of quantum doubles of Nichols algebras.

In Section 3 we study the distinguished pre-Nichols algebras $\widetilde{\mathcal{B}}(V)$. First we prove the existence of Lusztig isomorphisms for the quantum double $U(V)$ and PBW bases whose PBW generators are obtained by applying Lusztig isomorphisms. Then we obtain an algebra filtration on $\widetilde{\mathcal{B}}(V)$ and $U(V)$ such that the associated graded algebra is a quantum polynomial algebra, so $\widetilde{\mathcal{B}}(V)$ and $U(V)$ are Noetherian of finite Gelfand-Kirillov dimension.

In Section 4 we study the subalgebra $Z(V)$ of $U(V)$ generated by powers of root vectors. First we show that each element of $Z(V)$ commutes up to scalars with each homogeneous element of $U(V)$, so $U(V)$ is a free $Z(V)$-module. Next we obtain a formula relating the coproduct on $\widetilde{\mathcal{B}}(V)$ with Lusztig isomorphisms, close to [HS1, Theorem 4.2]. We present a recursive formula for the coproduct of powers of root vectors in $\widetilde{\mathcal{B}}(V)$, with a view towards the computation of the liftings of $\mathcal{B}(V)$, as proposed in [AAGMV].

Finally, we compute the coproduct of $Z(V)$ for braidings of super type A and of type $\mathfrak{br}(2;5)$ in Section 5.

1.4. Here are some questions on, and potential applications of, distinguished pre-Nichols algebras that support our interest on them.

1.4.1. If the diagonal braiding (V,c) is given by a symmetric matrix, then the algebra $Z(V)$ is commutative. Let $G(V)$ be the algebraic group $\mathrm{Spec}\, Z(V)$. There exists a correspondence between (almost all) braidings of diagonal type whose Nichols algebras are finite-dimensional, and finite-dimensional contragredient Lie superalgebras in positive characteristic [AA]. The group $G(V)$ should be related with the Lie superalgebra corresponding to (V,c); it should allow a geometric approach to the representation theory of $U(V)$ as in [DP].

1.4.2. The Hopf algebra $U(V)$ is the 'quantum' analogue of the enveloping Lie superalgebra in the previous point while $\mathfrak{u}(V)$ is the one of its restricted enveloping superalgebra. Therefore, as for Lie algebras in positive characteristic, we can start by studying the representation theory of the algebra $U(V)$ and pass to $\mathfrak{u}(V)$.

2. Preliminaries

2.1. **Conventions.** We work over an algebraically closed field $\mathbf{k}$ of characteristic zero. Tensor products, algebras, coalgebras and Hopf algebras are taken over $\mathbf{k}$. For each $\theta \in \mathbb{N}$, let $\mathbb{I}_\theta = \{1, 2, \ldots, \theta\}$; when θ is clear from the context, we simply set $\mathbb{I} = \mathbb{I}_\theta$.

Given a matrix $(q_{ij})_{1 \leq i,j \leq \theta} \in \mathbf{k}^{\theta \times \theta}$, let $\widetilde{q_{ij}} = q_{ij} q_{ji}$, $i \neq j$.

We consider the q-polynomial numbers in $\mathbb{Z}[\mathbf{q}]$, $n \in \mathbb{N}$, $0 \leq i \leq n$,

$$(n)_{\mathbf{q}} = \sum_{j=0}^{n-1} \mathbf{q}^j, \qquad (n)_{\mathbf{q}}! = \prod_{j=1}^{n} (j)_{\mathbf{q}}, \qquad \binom{n}{i}_{\mathbf{q}} = \frac{(n)_{\mathbf{q}}!}{(n-i)_{\mathbf{q}}!(i)_{\mathbf{q}}!}.$$

$(n)_q$, $(n)_q!$, $\binom{n}{i}_q$ are the evaluations of the polynomials in $\mathbf{q} = q \in \mathbf{k}$.

Given $\mathbf{a} = (a_1, \ldots, a_\theta) \in \mathbb{N}_0^\theta$, we use the notation: $\mathbf{t}^{\mathbf{a}} = t_1^{a_1} \cdots t_\theta^{a_\theta}$. The *height* of $\mathbf{a}$ is $\operatorname{ht}(\mathbf{a}) = \sum_{j=1}^{\theta} a_j$. If $W = \oplus_{\mathbf{a} \in \mathbb{N}_0^\theta} W_{\mathbf{a}}$ is an $\mathbb{N}_0^\theta$-graded vector space, then its *Hilbert series* is $\mathfrak{H}_W = \sum_{\mathbf{a} \in \mathbb{N}_0^\theta} \dim W_{\mathbf{a}}\, \mathbf{t}^{\mathbf{a}} \in \mathbb{N}_0[[t_1, \ldots, t_\theta]]$.

We denote by $\{\alpha_i\}_{1 \leq i \leq \theta}$ the canonical basis on $\mathbb{Z}^\theta$. Given a bicharacter $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbf{k}^\times$ and $w \in \operatorname{Aut}(\mathbb{Z}^\theta)$, $w^* \chi$ denotes the bicharacter

$$w^* \chi(\beta, \gamma) = \chi(w^{-1}(\beta), w^{-1}(\gamma)), \qquad\qquad \beta, \gamma \in \mathbb{Z}^\theta.$$

Let H be a Hopf algebra with bijective antipode $\mathcal{S}$. We use the Sweedler notation for the comultiplication, $\Delta(h) = h_{(1)} \otimes h_{(2)}$, $h \in H$, and for left H-comodules X, $\lambda(x) = x_{(-1)} \otimes x_{(0)} \in X \otimes H$, $x \in X$.

We denote by $^H_H \mathcal{YD}$ the category of left Yetter-Drinfeld modules over H. Recall that this is a braided tensor category, where the braiding for $M, N \in\, ^H_H \mathcal{YD}$ is $c = c_{M,N} : M \otimes N \to N \otimes M$,

$$c(m \otimes n) = m_{(-1)} \cdot n \otimes m_{(0)}, \qquad\qquad m \in M, n \in N.$$

The inverse braiding is $c^{-1} = c_{M,N}^{-1} : N \otimes M \to M \otimes N$,

$$c^{-1}(n \otimes m) = m_{(0)} \otimes \mathcal{S}^{-1}(m_{(-1)}) \cdot n, \qquad\qquad m \in M, n \in N.$$

In particular, if $V \in\, ^H_H \mathcal{YD}$, then $(V, c_{V \otimes V})$ is a braided vector space. If (V, c) is of diagonal type with basis $E_1, \ldots, E_\theta$ and matrix (q_{ij}), Γ is a group and $g_i \in \Gamma$, $\chi_j \in \widehat{\Gamma}$ are such that $\chi_j(g_i) = q_{ij}$, then (V, c) can be realized as a Yetter-Drinfeld module over $H = \mathbf{k}\Gamma$ by defining the action and coaction:

(1) $\lambda(E_i) = g_i \otimes E_i, \qquad g \cdot E_i = \chi_i(g) E_i, \qquad 1 \leq i \leq \theta, g \in \Gamma.$

For example take $\Gamma = \mathbb{Z}^\theta$, $g_i = \alpha_i$ and $\chi_j \in \widehat{\mathbb{Z}^\theta}$ such that $\chi_j(\alpha_i) = q_{ij}$ [AS1].

Let R be a Hopf algebra in $^H_H \mathcal{YD}$ with product m, coproduct $\underline{\Delta}$ and antipode $\mathcal{S}_R$. We use the following variation of Sweedler notation: $\underline{\Delta}(r) = r^{(1)} \otimes r^{(2)}$, $r \in R$. $R\#H$ denotes the *bosonization* of R by H; we refer to [AS1, Section 1.5] for the definition.

The *braided commutator* and the *braided adjoint action* of x on y, $x, y \in R$, are defined as follows

$$[x, y]_c = m(\mathrm{id}_{R \otimes R} - c)(x \otimes y),$$
$$\mathrm{ad}_c x(y) = m(m \otimes \mathcal{S}_R)(\mathrm{id} \otimes c)(\underline{\Delta} \otimes \mathrm{id})(x \otimes y).$$

The set of primitive elements is $\mathcal{P}(R) = \{r \in R : \underline{\Delta}(r) = r \otimes 1 + 1 \otimes r\}$. Notice that $\mathrm{ad}_c x(y) = [x, y]_c$ if $x \in \mathcal{P}(R)$.

2.2. Normal coideal subalgebras and Hopf ideals.

A *right coideal subalgebra* B of a Hopf algebra H is a subalgebra B such that $\Delta(B) \subset B \otimes H$. A right coideal subalgebra B is *normal* if it is stable under the right adjoint action $\mathrm{ad}_r(h)(b) = \mathcal{S}(h_{(1)})bh_{(2)}$, $b \in B$, $h \in H$. If $B \subset H$ is a right coideal subalgebra, then the *normal right coideal subalgebra* $N(B)$ is the subalgebra generated by $\{\mathcal{S}(h_{(1)})bh_{(2)} : h \in H, b \in B\}$.

There is a correspondence between quotient Hopf algebras and normal right coideal subalgebras under some mild conditions.

Theorem 2.1. [T, Theorem 3.2] *The maps*

$$B \text{ a normal right coideal subalgebra} \rightsquigarrow \mathcal{I}(B) = HB^+,$$
$$I \text{ a Hopf ideal} \rightsquigarrow \mathcal{X}(I) = H^{\mathrm{co}\, H/I},$$

restrict to mutually inverse bijective correspondences between the set of normal right coideal subalgebras B such that H is right B-faithfully flat and the set of Hopf ideals I such that H is H/I-coflat. $\square$

There is an analogous bijection for connected Hopf algebras in ${}^H_H \mathcal{YD}$.

Proposition 2.2. [AAGMV, Proposition 3.6] *Let R be a connected Hopf algebra in ${}^H_H \mathcal{YD}$.*

(i) *R is a free left and right module over every right coideal subalgebra, and is a cofree left and right comodule over every quotient left module coalgebra.*

(ii) *The maps $B \mapsto R/RB^+$, $T \mapsto {}^{\mathrm{co}\, T} R$ give a bijection between the set of right coideal subalgebras B of R and the set of quotient left R-module coalgebras T of R.*

(iii) *If B and T correspond to each other via the bijection in* (ii) *, then there exists a left T-colinear and right B-linear isomorphism $T \otimes B \simeq R$.* $\square$

2.3. Weyl groupoids and generalized root systems.

We recall the notion of generalized root systems using the notation in [AA], see also [CH1].

2.3.1. *Basic data.* Given $\mathbb{I} = \mathbb{I}_\theta$, $\mathcal{X} \neq \emptyset$ and $\rho : \mathbb{I} \to \mathbb{S}_{\mathcal{X}}$, the pair $(\mathcal{X}, \rho)$ is called a *basic datum* of rank $|\mathcal{X}|$ and type θ if $\rho_i^2 = \mathrm{id}$ for all $i \in \mathbb{I}$.

Let $\mathcal{Q}_\rho$ be the quiver $\{\sigma_i^x := (x, i, \rho_i(x)) : i \in \mathbb{I}, x \in \mathcal{X}\}$ over $\mathcal{X}$, with target and source $t(\sigma_i^x) = x$, $s(\sigma_i^x) = \rho_i(x)$. In any quotient of the free groupoid $F(\mathcal{Q}_\rho)$, we adopt the convention

$$(2) \qquad \sigma_{i_1}^x \sigma_{i_2} \cdots \sigma_{i_t} = \sigma_{i_1}^x \sigma_{i_2}^{\rho_{i_1}(x)} \cdots \sigma_{i_t}^{\rho_{i_{t-1}} \cdots \rho_{i_1}(x)}.$$

That is, the implicit superscripts are the only possible to have compositions.

2.3.2. *Coxeter groupoids.* A *Coxeter datum* is a triple $(\mathcal{X}, \rho, \mathbf{M})$, where $(\mathcal{X}, \rho)$ is a basic datum of type $\mathbb{I}$ and $\mathbf{M} = (\mathbf{m}^x)_{x \in \mathcal{X}}$, $\mathbf{m}^x = (m_{ij}^x)_{i,j \in \mathbb{I}}$, is a family of Coxeter matrices such that

$$(3) \qquad s((\sigma_i^x \sigma_j)^{m_{ij}^x}) = x, \qquad\qquad i, j \in \mathbb{I}, \qquad\qquad x \in \mathcal{X}.$$

The *Coxeter groupoid* $\mathcal{W}(\mathcal{X}, \rho, \mathbf{M})$ [HY1, Definition 1] is the groupoid generated by $\mathcal{Q}_\rho$ with relations

$$(4) \qquad\qquad (\sigma_i^x \sigma_j)^{m_{ij}^x} = \mathrm{id}_x, \qquad\qquad i, j \in \mathbb{I}, x \in \mathcal{X}.$$

In particular, if $i = j$, then (4) means that either σ_i^x is an involution when $\rho_i(x) = x$, or else that σ_i^x is the inverse arrow of $\sigma_i^{\rho_i}(x)$ when $\rho_i(x) \neq x$.

2.3.3. *Generalized root systems.* Let $\mathcal{C} = (C^x)_{x \in \mathcal{X}}$ be a family of generalized Cartan matrices $C^x = (c_{ij}^x)_{i,j \in \mathbb{I}}$ with row invariance

$$(5) \qquad\qquad c_{ij}^x = c_{ij}^{\rho_i(x)} \qquad\qquad \text{for all } x \in \mathcal{X},\ i, j \in \mathbb{I}.$$

We define $s_i^x \in GL_\theta(\mathbb{Z})$ by

$$(6) \qquad s_i^x(\alpha_j) = \alpha_j - c_{ij}^x \alpha_i, \qquad\qquad j \in \mathbb{I}, \qquad\qquad i \in \mathbb{I}, x \in \mathcal{X}.$$

Then (5) means that s_i^x is the inverse of $s_i^{\rho_i(x)}$. A *generalized root system* (GRS for short) [HY1, Definition 1] is a collection $\mathcal{R} := \mathcal{R}(\mathcal{X}, \rho, \mathcal{C}, \Delta)$, where $(\mathcal{X}, \rho)$ is a basic datum of type $\mathbb{I}$, $\mathcal{C}$ is as above, and $\Delta = (\Delta^x)_{x \in \mathcal{X}}$ is a family of subsets $\Delta^x \subset \mathbb{Z}^\mathbb{I}$ such that for all $x \in \mathcal{X}$, $i \neq j \in \mathbb{I}$,

$$(7) \qquad\qquad \Delta^x = \Delta_+^x \cup \Delta_-^x, \qquad \Delta_+^x := \Delta^x \cap \mathbb{N}_0^I,\ \Delta_-^x := -\Delta_+^x;$$

$$(8) \qquad\qquad \Delta^x \cap \mathbb{Z}\alpha_i = \{\pm \alpha_i\};$$

$$(9) \qquad\qquad s_i^x(\Delta^x) = \Delta^{\rho_i(x)};$$

$$(10) \qquad (\rho_i \rho_j)^{m_{ij}^x}(x) = (x), \qquad\qquad m_{ij}^x := |\Delta^x \cap (\mathbb{N}_0 \alpha_i + \mathbb{N}_0 \alpha_j)|.$$

We call Δ_+^x, Δ_-^x the set of *positive*, respectively *negative*, roots.

Let $\mathbf{M} = (M^x)_{x \in \mathcal{X}}$, $M^x = (m_{ij}^x)_{i,j \in \mathbb{I}}$. The *Weyl groupoid* of $\mathcal{R}$ is $\mathcal{W} = \mathcal{W}(\mathcal{X}, \rho, \mathbf{M})$. We can describe this groupoid by using [HY1, Theorem 1]. Let $\mathcal{G} = \mathcal{X} \times GL_\theta(\mathbb{Z}) \times \mathcal{X}$, $\varsigma_i^x = (x, s_i^x, \rho_i(x))$, $i \in \mathbb{I}$, $x \in \mathcal{X}$, and $\mathcal{W}' = \mathcal{W}(\mathcal{X}, \rho, \mathcal{C})$ the subgroupoid of $\mathcal{G}$ generated by all the ς_i^x. There exists a morphism of quivers $\mathcal{Q}_\rho \to \mathcal{G}$, $\sigma_i^x \mapsto \varsigma_i^x$ with image $\mathcal{W}'$. This induces an isomorphism of groupoids $\mathcal{W} \to \mathcal{W}(\mathcal{X}, \rho, \mathcal{C})$.

If $w = \sigma_{i_1}^x \cdots \sigma_{i_m}$ and $\alpha \in \mathbb{Z}^\theta$, then we define $w(\alpha) = s_{i_1}^x \cdots s_{i_m}(\alpha)$, so $w(\Delta^x) = \Delta^y$, by (9). The set of *real roots* at x is

$$(11) \qquad\qquad (\Delta^{\mathrm{re}})^x = \bigcup_{y \in \mathcal{X}} \{w(\alpha_i) :\ i \in \mathbb{I},\ w \in \mathcal{W}(y, x)\}.$$

The *length* of $w \in \mathcal{W}(x, \mathcal{X})$ is

$$\ell(w) = \min\{m \in \mathbb{N}_0 :\ \exists i_1, \ldots, i_n \in \mathbb{I} \text{ such that } w = \sigma_{i_1}^x \cdots \sigma_{i_m}\}.$$

An expression $w = \sigma_{i_1}^x \cdots \sigma_{i_m}$ is *reduced* if $m = \ell(w)$.

Lemma 2.3. [HY1, Corollary 3] *Let $m \in \mathbb{N}$, $x, y \in \mathcal{X}$, $i_1, \ldots, i_m, j \in I$, $w = \sigma_{i_1}^x \cdots \sigma_{i_m} \in \mathrm{Hom}(Y, X)$, where $\ell(w) = m$. Then*

- *$\ell(w\sigma_j) = m + 1$ if and only if $w(\alpha_j) \in \Delta_+^x$,*
- *$\ell(w\sigma_j) = m - 1$ if and only if $w(\alpha_j) \in \Delta_-^x$.* □

Proposition 2.4. [CH1, Prop. 2.12] *If $w = \sigma_{i_1}^x \cdots \sigma_{i_N} \in \mathcal{W}$ is such that $\ell(w) = N$, then all the roots $\beta_j = s_{i_1}^x \cdots s_{i_{j-1}}(\alpha_{i_j}) \in \Delta^x$ are positive and pairwise different. In particular, if $\mathcal{R}$ is finite and w is an element of maximal length, then $\Delta_+^x = \{\beta_j | 1 \le j \le N\}$. Hence $\Delta^x = (\Delta^{\mathrm{re}})^x$.* □

2.3.4. *GRS of Nichols algebras with finite sets of roots.* Let V be a braided vector space of diagonal type with matrix (q_{jk}). Let $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbf{k}^\times$ be the bicharacter such that $\chi(\alpha_j, \alpha_k) = q_{jk}$ for all $j, k \in \mathbb{I}$. The set Δ_+^V of degrees of a PBW basis of $\mathcal{B}(V)$, counted with their multiplicities as in [H1], does not depend on the PBW basis. For each $i \in \mathbb{I}$ let $c_{ii}^V = 2$,

$$-c_{ij}^V := \min \left\{ n \in \mathbb{N}_0 : (n+1)_{q_{ii}}(1 - q_{ii}^n q_{ij} q_{ji}) = 0 \right\}$$

$$(12) \qquad = \max\{ n \in \mathbb{N}_0 : n\alpha_i + \alpha_j \in \Delta_+^V \}, \qquad\qquad j \neq i$$

$s_i^V \in \mathrm{Aut}(\mathbb{Z}^\theta)$ such that $s_i^V(\alpha_j) = \alpha_j - c_{ij}^V \alpha_i$. Let $\mathcal{X}$ be the set of braided vector spaces (V, c) of diagonal type such that Δ_+^V is finite. Set $\Delta^V = \Delta_+^V \cup (-\Delta_+^V)$. Define $\rho_i(V)$ as the braided vector space with associated bicharacter $\rho_i(\chi)(\alpha, \beta) = \chi(s_i^V(\alpha), s_i^V(\beta))$, $\alpha, \beta \in \mathbb{Z}^\theta$. $\mathcal{R} := \mathcal{R}(\mathcal{X}, \rho, \mathcal{C}, (\Delta^x)_{x \in \mathcal{X}})$ is the GRS attached to $\mathcal{X}$. Indeed $\Delta_+^{\rho_i(V)} = s_i\left(\Delta_+^V \setminus \{\alpha_i\}\right) \cup \{\alpha_i\}$ by [H1], so (9) holds, and [HS1, Theorems 6.2, 6.9] completes the proof.

2.4. **Lusztig Isomorphisms of Nichols algebras of diagonal type.** We recall now Lusztig type isomorphisms [H3] of Hopf algebras related with Nichols algebras of diagonal type. They can be thought of as generalizations of the isomorphisms of quantized enveloping algebras in [L2].

Fix a bicharacter $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \to \mathbf{k}^\times$, $q_{ij} = \chi(\alpha_i, \alpha_j)$ and (V, c) the braided vector space of diagonal type with braiding matrix (q_{ij}) for a basis $(E_i)_{1 \le i \le \theta}$. *From now on we assume that $V \in \mathcal{X}$; in particular all the c_{ij}^V do exist.* Set (V, c) as an element of $_{\mathbf{k}\mathbb{Z}^\theta}^{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$ as in (1), so $T(V)$ is a Hopf algebra in this category. We recall definitions and results from [H3, Section 4.1].

Let $\mathcal{U}(V)$ be the algebra generated by elements E_i, F_i, K_i, K_i^{-1}, L_i, L_i^{-1}, $1 \le i \le \theta$, and relations

$$\begin{aligned} XY &= YX, & X, Y &\in \{K_i^\pm, L_i^\pm : 1 \le i \le \theta\}, \\ K_i K_i^{-1} &= L_i L_i^{-1} = 1, & E_i F_j - F_j E_i &= \delta_{i,j}(K_i - L_i). \\ K_i E_j &= q_{ij} E_j K_i, & L_i E_j &= q_{ji}^{-1} E_j L_i, \\ K_i F_j &= q_{ij}^{-1} F_j K_i, & L_i F_j &= q_{ji} F_j L_i. \end{aligned}$$

If (W, c) denotes the braided vector space of diagonal type corresponding to the transpose of the matrix $(q_{ij})_{i,j \in \mathbb{I}}$, then $\mathcal{U}(V)$ is the quantum double of

$T(V)\#\mathbf{k}\mathbb{Z}^{\theta}$ and $T(W)\#\mathbf{k}\mathbb{Z}^{\theta}$, see [H3, Proposition 4.6]. $\mathcal{U}(V)$ admits a Hopf algebra structure, where the comultiplication satisfies

$$\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \qquad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i,$$
$$\Delta(L_i^{\pm 1}) = L_i^{\pm 1} \otimes L_i^{\pm 1}, \qquad \Delta(F_i) = F_i \otimes L_i + 1 \otimes F_i.$$

Then $\mathcal{U}(V)$ is a $\mathbb{Z}^{\theta}$-graded Hopf algebra such that

$$\deg(K_i) = \deg(L_i) = 0, \qquad \deg(E_i) = \alpha_i, \qquad \deg(F_i) = -\alpha_i.$$

We fix the following notation:
- $\mathcal{U}^+(V)$ (respectively, $\mathcal{U}^-(V)$) is the subalgebra generated by E_i (respectively, F_i), $1 \le i \le \theta$;
- $\mathcal{U}^{\ge 0}(V)$ (respectively, $\mathcal{U}^{\le 0}(V)$) is the subalgebra generated by E_i, K_i, K_i^{-1} (respectively, F_i, L_i, L_i^{-1}), $1 \le i \le \theta$,
- $\mathcal{U}^{+0}(V)$ (respectively, $\mathcal{U}^{-0}(V)$) is the subalgebra generated by K_i, K_i^{-1} (respectively, L_i, L_i^{-1}), $1 \le i \le \theta$, which are isomorphic to $\mathbf{k}\mathbb{Z}^{\theta}$ as Hopf algebras;
- $\mathcal{U}^0(V)$ is the subalgebra generated by K_i, K_i^{-1}, L_i and L_i^{-1}, which is isomorphic to $\mathbf{k}\mathbb{Z}^{2\theta}$ as Hopf algebras.

Then $\mathcal{U}^+(V)$ is isomorphic to $T(V)$ and $\mathcal{U}^{\ge 0}(V)$ is isomorphic to $T(V)\#\mathbf{k}\mathbb{Z}^{\theta}$ as Hopf algebras. For each homogeneous element $E \in \mathcal{U}^+(V)_n$ and $k \in \{0, 1, \ldots, n\}$, $\underline{\Delta}_{n-k,k}(E)$ is the component of $\underline{\Delta}(E)$ in $\mathcal{U}^+(V)_{n-k} \otimes \mathcal{U}^+(V)_k$.

We consider some skew-derivations as in [H3, 4.2].

Proposition 2.5. *There exist linear endomorphisms ∂_i^K, ∂_i^L of $\mathcal{U}^+(V)$, $i \in \mathbb{I}$, such that for each $E \in \mathcal{U}^+(V)_n$, $n \in \mathbb{N}$,*

$$\underline{\Delta}_{n-1,1}(E) = \sum_{i=1}^{\theta} \partial_i^K(E) \otimes E_i, \qquad \underline{\Delta}_{1,n-1}(E) = \sum_{i=1}^{\theta} E_i \otimes \partial_i^L(E),$$

Moreover, $EF_i - F_iE = \partial_i^K(E)K_i - L_i\partial_i^L(E)$ and

$$\partial_i^K(1) = \partial_i^L(1) = 0, \qquad \partial_i^K(EE') = \partial_i^K(E)(K_i \cdot E') + E\partial_i^K(E'),$$
$$\partial_i^K(E_j) = \partial_i^L(E_j) = \delta_{i,j}, \qquad \partial_i^L(EE') = \partial_i^L(E)E' + (L_i^{-1} \cdot E)\partial_i^L(E').$$

for all $E, E' \in \mathcal{U}^+(V)$, $j \in \mathbb{I}$. $\square$

Corollary 2.6. *Let $m, n \in \mathbb{N}$, $E \in \mathcal{U}^+(V)_m$, $F \in \mathcal{U}^-(V)_n$.*

(i) *If $m \ge n$, then $FE \in EF + \sum_{j=0}^{n-1} \mathcal{U}^+(V)_{m-n+j}\mathcal{U}^0(V)\mathcal{U}^-(V)_j$.*

(ii) *If $m < n$, then $FE \in EF + \sum_{j=0}^{m-1} \mathcal{U}^+(V)_j\mathcal{U}^0(V)\mathcal{U}^-(V)_{n-m+j}$.*

Proof. (i) We argue by induction on n. If $n = 1$, then F is a linear combination of F_i's and Proposition 2.5 applies. Let $n > 1$. We may assume that $F = F_iF'$, $F' \in \mathcal{U}^+(V)_{n-1}$. By inductive hypothesis,

$$FE = F_iF'E \in F_iEF' + \sum_{j=0}^{n-2} F_i\mathcal{U}^+(V)_{m-n+1+j}\mathcal{U}^0(V)\mathcal{U}^-(V)_j,$$

and the proof follows by Proposition 2.5.

(ii) We may assume that $F = F_{i_1} \ldots F_{i_{n-m}} F'$, $F' \in \mathcal{U}^+(V)_m$. By (i) ,

$$FE = F_{i_1} \ldots F_{i_{n-m}} F' E \in F_{i_1} \ldots F_{i_{n-m}} E F'$$

$$+ \sum_{j=0}^{m-1} F_{i_1} \ldots F_{i_{n-m}} \mathcal{U}^+(V)_j \mathcal{U}^0(V) \mathcal{U}^-(V)_j,$$

and the proof follows again by Proposition 2.5. $\qquad \square$

Fix $i \in \mathbb{I}$. For each $j \neq i$ we set as in [H3]:

$$E^+_{j,0(i)} = E^-_{j,0(i)} = E_j, \qquad\qquad F^+_{j,0(i)} = F^-_{j,0(i)} = F_j,$$

and recursively,

(13)
$$\begin{aligned}
E^+_{j,m+1(i)} &:= E_i E^+_{j,m(i)} - (K_i \cdot E^+_{j,m(i)}) E_i = (\mathrm{ad}_c E_i)^{m+1} E_j, \\
F^+_{j,m+1(i)} &:= F_i F^+_{j,m(i)} - (L_i \cdot F^+_{j,m(i)}) F_i = (\mathrm{ad}_c F_i)^{m+1} F_j, \\
E^-_{j,m+1(i)} &:= E_i E^-_{j,m(i)} - (L_i \cdot E^-_{j,m(i)}) E_i, \\
F^-_{j,m+1(i)} &:= F_i F^-_{j,m(i)} - (K_i \cdot F^-_{j,m(i)}) F_i.
\end{aligned}$$

When i is explicit, we simply denote $E^{\pm}_{j,m(i)}$ by $E^{\pm}_{j,m}$.

Remark 2.7. [H3, Lemma 4.23, Corollary 4.25] For all $N \in \mathbb{N}$,

(14)
$$E^+_{j,N} = \sum_{s=0}^{N} (-1)^s q^s_{ij} q^{s(s-1)/2}_{ii} \binom{N}{s}_{q_{ii}} E^{N-s}_i E_j E^s_i,$$

(15)
$$E^N_i F_i - F_i E^N_i = (N)_{q_{ii}} (q^{1-N}_{ii} K_i - L_i) E^{N-1}_i,$$

(16)
$$E^+_{j,N} F_i - F_i E^+_{j,N} = (N)_{q_{ii}} (q^{N-1}_{ii} \widetilde{q_{ij}} - 1) L_i E^+_{j,N-1}.$$

The coproduct satisfies the following identities:

(17)
$$\underline{\Delta}\left(E^N_i\right) = \sum_{s=0}^{N} \binom{N}{s}_{q_{ii}} E^s_i \otimes E^{N-s}_i,$$

(18)
$$\underline{\Delta}\left(E^+_{j,N}\right) = E^+_{j,N} \otimes 1 + \sum_{s=0}^{N-1} \left(\prod_{r=N-s}^{N-1} (r)_{q_{ii}} (1 - q^r_{ii} \widetilde{q_{ij}}) \right) E^s_i \otimes E^+_{j,N-s},$$

$$\underline{\Delta}\left(E^-_{j,N}\right) = 1 \otimes E^-_{j,N}$$

(19)
$$+ \sum_{s=0}^{N-1} q^s_{ij} \left(\prod_{r=N-s}^{N-1} (r)_{q_{ii}} \left(1 - q^{-r}_{ii} \widetilde{q_{ij}}^{-1}\right) \right) E^-_{j,N-s} \otimes E^s_i.$$

Let $\mathcal{J}^{\pm}(V)$ be the ideal of $\mathcal{U}^{\pm}(V)$ such that $\mathfrak{u}^+(V) = \mathcal{U}^+(V)/\mathcal{J}^+(V)$, $\mathfrak{u}^-(V) = \mathcal{U}^-(V)/\mathcal{J}^-(V)$ are isomorphic to $\mathcal{B}(V)$, $\mathcal{B}(W)$, respectively, and set $\mathfrak{u}(V) = \mathcal{U}(V)/(\mathcal{J}^-(V) + \mathcal{J}^+(V))$. Then $\mathfrak{u}(V)$ is the quantum double of the algebras $\mathcal{B}(V)\#\mathbf{k}\mathbb{Z}^{\theta}$ and $\mathcal{B}(W)\#\mathbf{k}\mathbb{Z}^{\theta}$ by [H3, Theorem 5.8].

We need another quotients of $\mathcal{U}(V)$ in order to introduce Lusztig isomorphisms. First we recall [A3, Definition 2.6]. An element $i \in \mathbb{I}$ is a *Cartan vertex* of V if $\widetilde{q_{ij}} = q_{ii}^{c_{ij}^V}$ for all $j \neq i$. The set of *Cartan roots* is

(20) $\mathcal{O}(V) := \{s_{i_1}^V \dots s_{i_k}(\alpha_i) : i \text{ is a Cartan vertex of } \rho_{i_k} \dots \rho_{i_1}(V)\}.$

Set $N_i = \operatorname{ord} q_{ii}$. $\mathcal{J}_i^+(V)$, $\mathcal{J}_i^-(V)$ are the ideals of $\mathcal{U}^+(V)$, respectively $\mathcal{U}^-(V)$, generated by

(a) $E_i^{N_i}$, respectively $F_i^{N_i}$, if i is not a Cartan vertex,
(b) $E_{j,-c_{ij}^V+1}^+$, respectively $F_{j,-c_{ij}^V+1}^+$, for each i such that $N_i \geq -c_{ij}^V + 1$.

Set also

$$\mathcal{U}_i(V) := \mathcal{U}(V)/\left(\mathcal{J}_i^+(V) + \mathcal{J}_i^-(V)\right), \qquad \mathcal{U}_i^\pm(V) := \mathcal{U}^\pm(V)/\mathcal{J}_i^\pm(V).$$

Let $\underline{E}_j$, $\underline{F}_j$, $\underline{K}_j$, $\underline{L}_j$ be the generators of $\mathcal{U}(\rho_i(V))$. Set

(21) $\displaystyle \lambda_j(V) := (-c_{ij}^V)_{q_{ii}} \prod_{s=0}^{-c_{ij}^V-1} (q_{ii}^s \widetilde{q_{ij}} - 1), \qquad j \neq i.$

Theorem 2.8. [H3, Lemma 6.5, Theorem 6.12] *There exist algebra maps*

(22) $T_i, T_i^- : \mathcal{U}(V) \to \mathcal{U}_i(\rho_i(V))$

univocally determined by the following conditions:

$$T_i(K_i) = T_i^-(K_i) = \underline{K}_i^{-1}, \qquad\qquad T_i(K_j) = T_i^-(K_j) = \underline{K}_i^{-c_{ij}^V}\underline{K}_j,$$

$$T_i(L_i) = T_i^-(L_i) = \underline{L}_i^{-1}, \qquad\qquad T_i(L_j) = T_i^-(L_j) = \underline{L}_i^{-c_{ij}^V}\underline{L}_j,$$

$$T_i(E_i) = \underline{F}_i\underline{L}_i^{-1}, \qquad\qquad T_i(E_j) = \underline{E}_{j,-c_{ij}^V}^+,$$

$$T_i(F_i) = \underline{K}_i^{-1}\underline{E}_i, \qquad\qquad T_i(F_j) = \lambda_i(\rho_i(V))^{-1}\underline{F}_{j,-c_{ij}^V}^+,$$

$$T_i^-(E_i) = \underline{K}_i^{-1}\underline{F}_i, \qquad\qquad T_i^-(E_j) = \lambda_i(\rho_i(V)^{-1})^{-1}\underline{E}_{j,-c_{ij}^V}^-,$$

$$T_i^-(F_i) = \underline{E}_i\underline{L}_i^{-1}, \qquad\qquad T_i^-(F_j) = \underline{F}_{j,-c_{ij}^V}^-.$$

Such morphisms induce algebra isomorphisms (denoted by the same name):

(23) $T_i, T_i^- : \mathfrak{u}(V) \to \mathfrak{u}(\rho_i(V))$ *such that* $T_i T_i^- = T_i^- T_i = \operatorname{id}$. $\square$

2.5. **Lusztig isomorphisms and PBW bases.** Fix a reduced expression $w = \sigma_{i_1}^V \sigma_{i_2} \cdots \sigma_{i_M}$ of the element of maximal length of $\mathcal{W}(V)$. If $1 \leq k \leq M$ then set $\beta_k := s_{i_1}^V \cdots s_{i_{k-1}}(\alpha_{i_k})$, $q_k := \chi(\beta_k, \beta_k)$, $N_k = \operatorname{ord} q_k \in \mathbb{N} \cup \{\infty\}$. By Proposition 2.4 $\beta_k \neq \beta_l$ if $k \neq l$, and $\Delta_+^V = \{\beta_k | 1 \leq k \leq M\}$. Let

(24) $E_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}) \in \mathfrak{u}(V)_{\beta_k}^+, \quad F_{\beta_k} = T_{i_1} \cdots T_{i_{k-1}}(F_{i_k}) \in \mathfrak{u}(V)_{\beta_k}^-.$

and for each $\mathbf{a} = (a_1, \dots, a_M) \in \mathbb{N}_0^M$,

(25) $\mathbf{E^a} = E_{\beta_M}^{a_M} E_{\beta_{M-1}}^{a_{M-1}} \cdots E_{\beta_1}^{a_1}, \qquad\qquad \mathbf{F^a} = F_{\beta_M}^{a_M} F_{\beta_{M-1}}^{a_{M-1}} \cdots F_{\beta_1}^{a_1}.$

Theorem 2.9. [HY2, Theorems 4.5, 4.8, 4.9] *The sets*

$$\{\mathbf{E^a} \mid \mathbf{a} \in \mathbb{N}_0^M, \, 0 \le a_k < N_k, \, 1 \le k \le M\},$$

$$\{\mathbf{F^a} \mid \mathbf{a} \in \mathbb{N}_0^M, \, 0 \le a_k < N_k, \, 1 \le k \le M\},$$

are bases of the vector spaces $\mathfrak{u}^+(V)$, $\mathfrak{u}^-(V)$, respectively. □

3. Distinguished pre-Nichols algebras

From now on we assume that $\mathcal{B}(V)$ *is finite-dimensional.*

3.1. We now consider some intermediate quotients between $\mathcal{U}(V)$ and $\mathfrak{u}(V)$.

Definition 3.1. Let $\mathcal{I}(V)$ be the ideal of $T(V)$ generated by all the relations in [A3, Theorem 3.1], except the power root vectors $E_\alpha^{N_\alpha}$, $\alpha \in \mathcal{O}(V)$, plus the quantum Serre relations $(\mathrm{ad}_c E_i)^{1-c_{ij}^V} E_j$ for those $i \ne j$ such that $q_{ii}^{c_{ij}^V} = q_{ij}q_{ji} = q_{ii}$. Then $\widetilde{\mathcal{B}}(V) = T(V)/\mathcal{I}(V)$ is the *distinguished pre-Nichols algebra* of (V, c).

We identify $\mathcal{I}(V)$ as an ideal $\mathcal{I}^+(V)$ of $\mathcal{U}^+(V)$. Let $\mathcal{I}^-(V)$ be the corresponding ideal of $\mathcal{U}^-(V)$. We denote by $U(V)$ the quotient of $\mathcal{U}(V)$ by the ideal generated by $\mathcal{I}^+(V)$ and $\mathcal{I}^-(V)$. By abuse of notation we denote by E_i, F_i, $K_i^\pm$, $L_i^\pm$ the generators of $U(V)$. $U^+(V) = \widetilde{\mathcal{B}}(V)$, $U^-(V)$ are, respectively, the subalgebras of $U(V)$ generated by E_i, F_i. Let $U^0(V)$ be the subalgebra generated by $K_i^\pm$, $L_i^\pm$.

Proposition 3.2. [A3, Proposition 3.3] *$U(V)$ is a Hopf algebra and there exist a canonical Hopf algebra morphism $\pi_V : U(V) \twoheadrightarrow \mathfrak{u}(V)$ such that $\pi_V(U^\pm(V)) = \mathfrak{u}^\pm(V)$. The multiplication $m : U^+(V) \otimes U^0(V) \otimes U^-(V) \to U(V)$ gives an isomorphism of graded vector spaces.* □

$U(V)$ is the quantum double of $\widetilde{\mathcal{B}}(V)\#\mathbf{k}\mathbb{Z}^\theta$ and $\widetilde{\mathcal{B}}(W)\#\mathbf{k}\mathbb{Z}^\theta$ since there exists a Hopf pairing induced by the one between $T(V)\#\mathbf{k}\mathbb{Z}^\theta$ and $T(W)\#\mathbf{k}\mathbb{Z}^\theta$.

Remark 3.3. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$, $C = (c_{ij}) \in \mathbb{Z}^{\mathbb{I} \times \mathbb{I}}$ its finite Cartan matrix and $D = \mathrm{diag}(d_1, \ldots, d_\theta)$ such that DA is symmetric. If q is a root of unity of odd order, then the symmetric matrix $(q_{ij})_{i,j \in \mathbb{I}}$, $q_{ij} = q^{d_i c_{ij}}$, defines a braiding of Cartan type. The small quantum group $\mathfrak{u}_q(\mathfrak{g})$ is the quotient of $\mathfrak{u}(V)$ by the central elements $K_i - L_i^{-1}$, $1 \le i \le \theta$, while the quantized enveloping algebra $U_q(\mathfrak{g})$ is obtained from $U(V)$ analogously.

The Lusztig isomorphisms descend to the family of algebras $U(V)$.

Proposition 3.4. [A3, Proposition 3.26] *The maps* (22) *induce algebra isomorphisms $T_i, T_i^- : U(V) \to U(\rho_i(V))$ such that $T_i T_i^- = T_i^- T_i = \mathrm{id}_{U(V)}$.* □

Remark 3.5. Using the Lusztig isomorphisms we deduce that

$$E_\alpha^{N_\alpha}, F_\alpha^{N_\alpha} \ne 0 \text{ for all } \alpha \in \mathcal{O}(V), \quad E_\alpha^{N_\alpha} = F_\alpha^{N_\alpha} = 0 \text{ for all } \alpha \notin \mathcal{O}(V),$$

on $U(V)$. The Hilbert series of $U^{\pm}(V)$ is:

$$\mathcal{H}_{U^{\pm}(V)} = \left(\prod_{\alpha \in \Delta_+^V \setminus \mathcal{O}(V)} (t^{\alpha})_{N_{\alpha}} \right) \cup \left(\prod_{\alpha \in \mathcal{O}(V)} \frac{1}{1 - t^{\alpha}} \right).$$

Indeed $U^+(V)$ has a PBW basis of Lyndon hyperwords as in [Kh] with the same PBW generators of $\mathfrak{u}^+(V)$, see the proof of [A3, Theorem 3.1].

Set $E_{\alpha}, \mathbf{E^a} \in U^+(V)$, $F_{\alpha}, \mathbf{F^a} \in U^-(V)$, $\alpha \in \Delta_+^V$, $\mathbf{a} \in \mathbb{N}_0^M$, as in (24), (25).

Theorem 3.6. *The sets*

$$\{ \mathbf{E^a} \,|\, \mathbf{a} \in \mathbb{N}_0^M, \, 0 \le a_k < N_k \text{ if } \beta_k \notin \mathcal{O}(V) \},$$
$$\{ \mathbf{F^a} \,|\, \mathbf{a} \in \mathbb{N}_0^M, \, 0 \le a_k < N_k \text{ if } \beta_k \notin \mathcal{O}(V) \},$$

determine bases of the vector space $U^+(V)$, $U^-(V)$, respectively.

Proof. The expression of the Hilbert series reduces the problem to the linearly independence of this set, which is proved following the same recursion as [HY2, Theorem 4.5]. $\qquad\qquad\square$

3.2. Let $\mathcal{B}_i$ be the algebra generated by E_i on $U(V)$. This is a braided graded Hopf algebra. Its graded dual $\mathcal{B}_i^*$ is also a graded braided Hopf algebra. There exists a projection $\pi_{i,V} : U^+(V) \twoheadrightarrow \mathcal{B}_i$ of braided graded Hopf algebras annihilating all the E_j for $j \ne i$; the inclusion $\iota_{i,V} : \mathcal{B}_i \hookrightarrow U^+(V)$ is a section for this projection.

Remark 3.7. Let i be a Cartan vertex. $\mathcal{B}_i^*$ has a basis $\{ \mathfrak{E}_i^{(n)} : n \in \mathbb{N}_0 \}$, where $\mathfrak{E}_i^{(n)}(E_i^m) = \delta_{n,m}$. The algebra and the coalgebra structures satisfy:

$$(26) \qquad \mathfrak{E}_i^{(j)} \cdot \mathfrak{E}_i^{(k)} = \binom{j+k}{j}_{q_{ii}} \mathfrak{E}_i^{(k+j)}, \qquad \underline{\Delta}(\mathfrak{E}_i^{(n)}) = \sum_{j=0}^{n} \mathfrak{E}_i^{(j)} \otimes \mathfrak{E}_i^{(n-j)},$$

with unit $\mathfrak{E}_i^{(0)} = 1$, and counit $\varepsilon(\mathfrak{E}_i^{(n)}) = \delta_{n,0}$. In particular $\mathfrak{E}_i^{(j)} \cdot \mathfrak{E}_i^{(N_i-j)} = 0$ if $1 \le j \le N_i - 1$. As an algebra $\mathcal{B}_i^*$ is generated by $\mathfrak{E}_i^{(1)}$, $\mathfrak{E}_i^{(N_i)}$.

Remark 3.8. There exist left and right actions of $\mathcal{B}_i^*$ on $U^+(V)$ given by

$$(27) \qquad \mathfrak{E} \triangleright Y = Y_{(1)} \mathfrak{E}(\pi_{i,V}(Y_{(2)})), \qquad Y \triangleleft \mathfrak{E} = \mathfrak{E}(\pi_{i,V}(Y_{(1)})) Y_{(2)},$$

for each $\mathfrak{E} \in \mathcal{B}_i^*$, $Y \in U^+(V)$. In particular,

$$(28) \qquad \mathfrak{E}_i^{(1)} \triangleright Y = \partial_i^K(Y), \qquad Y \triangleleft \mathfrak{E}_i^{(1)} = \partial_i^L(Y), \qquad \text{for all } Y \in U^+(V).$$

Lemma 3.9. *For all $X \in U^+(V)_{\beta}$, $Y \in U^+(V)$, $t \in \mathbb{N}$,*

$$(29) \qquad (XY) \triangleleft \mathfrak{E}_i^{(t)} = \sum_{r=0}^{t} \chi(\beta - r\alpha_i, \alpha_i)^{t-r} \left(X \triangleleft \mathfrak{E}_i^{(r)} \right) \left(Y \triangleleft \mathfrak{E}_i^{(t-r)} \right).$$

Proof. Notice that

$$\sum_{t\geq 0} E_i^t \otimes (XY) \triangleleft \mathfrak{E}_i^{(t)} = (\pi_{i,V} \otimes \mathrm{id})\underline{\Delta}(XY) = (\pi_{i,V} \otimes \mathrm{id})\left(\underline{\Delta}(X) \cdot \underline{\Delta}(Y)\right)$$

$$= \left(\sum_{r\geq 0} E_i^r \otimes X \triangleleft \mathfrak{E}_i^{(r)}\right) \cdot \left(\sum_{s\geq 0} E_i^s \otimes Y \triangleleft \mathfrak{E}_i^{(s)}\right)$$

$$= \sum_{r,s\geq 0} \chi(\beta - r\alpha_i, \alpha_i)^s \, E_i^{r+s} \otimes (X \triangleleft \mathfrak{E}_i^{(r)})(Y \triangleleft \mathfrak{E}_i^{(s)}).$$

Then compare the terms of the form $E_i^t \otimes -$. $\qquad\square$

3.3. Let $U_{\pm i}^+(V)$ be the subalgebra generated by $E_{j,N}^\pm$, $j \neq i$, $N \in \mathbb{N}_0$. Given $\alpha = \sum_{i=1}^\theta n_i \alpha_i \in \mathbb{Z}^\theta$, set $K_\alpha = \prod_{i=1}^\theta K_i^{n_i}$, $L_\alpha = \prod_{i=1}^\theta L_i^{n_i} \in U^0(V)$.

Lemma 3.10. (i) $U_{+i}^+(V) = U^+(V)^{\mathrm{co}\,\pi_{i,V}}$, $U_{-i}^+(V) =^{\mathrm{co}\,\pi_{i,V}} U^+(V)$. *Then there exist isomorphisms of graded vector spaces* $U^+(V) \cong U_{\pm i}^+(V) \otimes \mathcal{B}_i$.

(ii) *If i is not a Cartan vertex, then* $\ker(\partial_i^K) = U_{+i}^+(V)$, $\ker(\partial_i^L) = U_{-i}^+(V)$.

(iii) *If i is a Cartan vertex, then* $\ker(\partial_i^K) = U_{+i}^+(V)\mathbf{k}\left[E_i^{N_i}\right]$, $\ker(\partial_i^L) = U_{-i}^+(V)\mathbf{k}\left[E_i^{N_i}\right]$.

Proof. The claims about $U_{+i}^+(V)$ follow by [A1, Lemma 2.4], [H3, Lemma 4.31], so we prove those about $U_{-i}^+(V)$. By (19) $U_{-i}^+(V)$ is a right coideal subalgebra contained in $B =^{\mathrm{co}\,\pi_{i,V}} U^+(V)$. We claim that $U_{-i}^+(V)\mathcal{B}_i$ is a left ideal of $U^+(V)$. Indeed if $E \in U_{-i}^+(V)$, then $E_j E \in U_{-i}^+(V)$ for $j \neq i$ since $E_j \in U_{-i}^+(V)$, and $E_i X \in U_{-i}^+(V)\mathcal{B}_i$ since

$$E_i E_{j,n}^- = E_{j,n+1}^- + \chi(n\alpha_i + \alpha_j, -\alpha_i)E_{j,n}^- E_i, \qquad j \neq i, n \in \mathbb{N}_0.$$

But $1 \in U_{-i}^+(V)\mathcal{B}_i$, so $U_{-i}^+(V)\mathcal{B}_i = U^+(V)$. By Proposition 2.2 (iii) , there exists an isomorphism $B \otimes \mathcal{B}_i \simeq U^+(V)$, so $B = U_{-i}^+(V)$ and the multiplication gives this isomorphism $U_{-i}^+(V) \otimes \mathcal{B}_i \simeq U^+(V)$. Therefore for each $X \in U^+(V)$ there exist unique $X_n \in U_{-i}^+(V)$ such that $X = \sum_{n\geq 0} X_n E_i^n$. If X has degree β, then $\partial_i^L(X) = \sum_{n\geq 1}(n)_{q_{ii}}\chi(n\alpha_i - \beta, \alpha_i) X_n E_i^{n-1}$ since $U_{-i}^+(V) \subseteq \ker \partial_i^L$. If $X \in \ker \partial_i^L$, then $X_n = 0$ for all $n \notin \mathbb{N}N_i$. $\qquad\square$

Remark 3.11. $U_{+i}^+(V)$ is a braided Hopf algebra in $_{\mathcal{B}_i \#\mathbf{k}\mathbb{Z}^\theta}^{\mathcal{B}_i \#\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$. Indeed Lemma 3.10(iii) says that $U_{+i}^+(V) = U^+(V)^{\mathrm{co}\,\pi_{i,V}}$, and by [AHS, Lemma 3.1]

$$U^+(V)^{\mathrm{co}\,\pi_{i,V}} = (U^+(V)\#\mathbf{k}\mathbb{Z}^\theta)^{\mathrm{co}\,\pi_{i,V}\# \mathrm{id}}.$$

The action and coaction satisfy, for each $E \in U_{+i}^+(V)_\beta$,

$$(30) \qquad E_i \rightharpoonup E = (\mathrm{ad}_c E_i)E, \qquad \lambda(E) = \sum_{n\geq 0} E_i^n K_{\beta - n\alpha_i} \otimes E \triangleleft \mathfrak{E}_i^{(n)}.$$

Remark 3.12. If R is braided Hopf algebra R in ${}^{\mathbf{k}\mathbb{Z}^\theta}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$, then there exists an structure of braided Hopf algebra R^{bop} as in [AG, Proposition 2.2.4] with underlying Yetter-Drinfeld module R and

$$m^{\mathrm{bop}} = m \circ c_{R,R}, \qquad \underline{\Delta}^{\mathrm{bop}} = c^{-1}_{R,R} \circ \underline{\Delta}, \qquad \mathcal{S}^{\mathrm{bop}} = \mathcal{S}.$$

This applies for $U^+(V)$, $\mathcal{B}_i$, and $\pi_{i,V} : U^+(V)^{\mathrm{bop}} \twoheadrightarrow \mathcal{B}^{\mathrm{bop}}_i$ is a Hopf algebra map in ${}^{\mathbf{k}\mathbb{Z}^\theta}_{\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$. Consider the Hopf algebras $U^+(V)^{\mathrm{bop}}\#\mathbf{k}\mathbb{Z}^\theta$, $\mathcal{B}^{\mathrm{bop}}_i\#\mathbf{k}\mathbb{Z}^\theta$, and the Hopf algebra maps $\pi_{i,V}\#\,\mathrm{id}$, $\iota_{i,V}\#\,\mathrm{id}$. Then

$$(U^+(V)^{\mathrm{bop}}\#\mathbf{k}\mathbb{Z}^\theta)^{\mathrm{co}\,\pi_{i,V}\#\,\mathrm{id}} = (U^+(V)^{\mathrm{bop}})^{\mathrm{co}\,\pi_{i,V}} = {}^{\mathrm{co}\,\pi_{i,V}}\,U^+(V) = U^+_{-i}(V)$$

is a braided Hopf algebra in ${}^{\mathcal{B}^{\mathrm{bop}}_i\#\mathbf{k}\mathbb{Z}^\theta}_{\mathcal{B}^{\mathrm{bop}}_i\#\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$. The action and coaction satisfy

$$(31) \qquad E_i \rightharpoonup' E = [E, E_i]_c, \qquad \lambda'(E) = \sum_{n\geq 0} E^n_i K_{\beta - n\alpha_i} \otimes \mathfrak{E}^{(n)}_i \rhd E,$$

for each $E \in U^+_{-i}(V)_\beta$. But $U^+_{-i}(V)$ has the opposite product, so we take the braided Hopf algebra structure on $U^+_{-i}(V)$ obtained by applying the bop construction. The coproduct $\Delta_i : U^+_{-i}(V) \to U^+_{-i}(V)\underline{\otimes}U^+_{-i}(V)$ is given by

$$(32) \qquad \Delta_i(x) = x_{(1)} \otimes \iota_{i,V}\pi_{i,V}(\mathcal{S}^{-1}(x_{(2)}))x_{(3)}, \qquad x \in U^+_{-i}(V)$$

Here $U^+_{-i}(V)\underline{\otimes}U^+_{-i}(V)$ denotes the space $U^+_{-i}(V)\underline{\otimes}U^+_{-i}(V)$ with the algebra structure viewed as an element in ${}^{\mathcal{B}^{\mathrm{bop}}_i\#\mathbf{k}\mathbb{Z}^\theta}_{\mathcal{B}^{\mathrm{bop}}_i\#\mathbf{k}\mathbb{Z}^\theta}\mathcal{YD}$,

$$(x \otimes y) \cdot (w \otimes z) = \sum_{n=0}^{N_i-1} \chi(\beta, \gamma - n\alpha_i)\, x\left(\mathfrak{E}^{(n)}_i \rhd w\right)\underline{\otimes}\left(E^n_i \rightharpoonup' y\right)z,$$

for $x, y, w, z \in U^+_{-i}(V)$, where y, w are homogeneous of degrees $\beta, \gamma \in \mathbb{N}^\theta_0$.

Remark 3.13. By [H3, Lemma 6.7],

$$(33) \qquad q_{ii}\partial^L_i(T_i(X)) = -\chi(\beta, \alpha_i)^{-1}T_i(E_i \rightharpoonup' X),$$

$$(34) \quad T_i\left(K^{-1}_i \cdot \partial^K_i(X)\right) = -\left(\underline{E}\rightharpoonup T_i(X)\right),$$

$$(35) \qquad T_i(E^-_{j,n}) = \underline{q}^{\,n}_{ii}\left(\prod_{t=-c^V_{ij}-n}^{-c^V_{ij}-1}(t+1)_{\underline{q}_{ii}}(1 - \underline{q}^{\,t}_{ii}\underline{q}_{ij}\underline{q}_{ji})\right)\underline{E}^+_{j,-c^V_{ij}-n},$$

for each $X \in U^+_{-i}(V)_\beta$, $\beta \in \mathbb{N}^\theta_0$, $n \in \mathbb{N}_0$, where $(\underline{q}_{jk})_{j,k\in\mathbb{I}}$ is the matrix of $\rho_i(V)$. In particular $T_i(U^+_{-i}(V)) = U^+_{+i}(\rho_i(V))$.

3.4. Next results resemble [HY2, Lemma 4.6, Theorem 4.8].

Lemma 3.14. *The set*

$$(36) \qquad \{\mathbf{E^a} \,|\, \mathbf{a} \in \mathbb{N}_0^M, \, a_1 = 0, \, 0 \le a_k < N_k \text{ if } \beta_k \notin \mathcal{O}(V)\}$$

is a basis of the vector space $U_{+i}^+(V)$.

Proof. Set $i = i_1$. By Lemma 3.10 (i) , $E_{\beta_\ell} = \sum_{k \ge 0} X_k E_i^k$ for unique $X_k \in U_{+i}^+(V)$. By (35), $T_i^-(U_{+i}^+(V)) = U_{-i}^+(\rho_i(V))$, so $T_i^-(X_k) \in U^+(\rho_i(V))$. As

$$\sum_{k \ge 0} T_i^-(X_k)(K_i^{-1}F_i)^k = T_i^-(E_{\beta_\ell}) = T_{i_2}\dots T_{i_{\ell-1}}(E_{i_l}) \in U^+(V),$$

last statement of Proposition 3.2 implies that $X_k = 0$ for all $k > 0$, so $E_{\beta_\ell} = X_0 \in U_{+i}^+(V)$. Then the set (36) is contained in $U_{+i}^+(V)$. By Lemma 3.10 (i) and Theorem 3.6, this set should also generate $U_{+i}^+(V)$ because it generates a subspace with the same Hilbert series. $\square$

Proposition 3.15. *For each pair* $1 \le k < \ell \le M$,

$$(37) \quad E_{\beta_k}E_{\beta_\ell} - \chi(\beta_k, \beta_\ell)E_{\beta_\ell}E_{\beta_k} = \sum c_{a_{k+1},\dots,a_{\ell-1}} E_{\beta_{\ell-1}}^{a_{\ell-1}} \cdots E_{\beta_{k+1}}^{a_{k+1}} \in \mathfrak{u}^+(V),$$

$$(38) \quad F_{\beta_k}F_{\beta_\ell} - \chi(\beta_k, \beta_\ell)F_{\beta_\ell}F_{\beta_k} = \sum d_{a_{k+1},\dots,a_{\ell-1}} F_{\beta_{\ell-1}}^{a_{\ell-1}} \cdots F_{\beta_{k+1}}^{a_{k+1}} \in \mathfrak{u}^-(V),$$

for some scalars $c_{a_{k+1},\dots,a_{\ell-1}}, d_{a_{k+1},\dots,a_{\ell-1}} \in \mathbf{k}$.

Proof. Assume first $k = 1$. By Lemma 3.14 there exist $\mathsf{c_a} \in \mathbf{k}$ such that

$$E_{i_1}E_{\beta_\ell} - \chi(\alpha_{i_1}, \beta_\ell)E_{\beta_\ell}E_{i_1} = \sum_{\mathbf{a}:\, a_1 = 0} \mathsf{c_a}\,\mathbf{E^a}.$$

Let $V' = \rho_{i_\ell}\dots\rho_{i_1}V$. By applying $T_{i_\ell}^- \dots T_{i_1}^-$ to the left hand side we obtain an element of $U^{\le 0}(V')$. Then $\mathsf{c_a} = 0$ if some $a_j > 0$ for $j > \ell$ since

$$T_{i_\ell}^- \dots T_{i_1}^-(E_{\beta_j}) = T_{i_\ell}^- \dots T_{i_{j+1}}^-(K_{i_j}^{-1}F_{i_j}) \in U^{\le 0}(V') \qquad \text{if } j \le \ell,$$

$$T_{i_\ell}^- \dots T_{i_1}^-(E_{\beta_j}) = T_{i_{\ell+1}} \cdots T_{i_{j-1}}(E_{i_j}) \in U^+(V') \qquad \text{if } j > \ell.$$

Finally if $\mathsf{c}_{0,a_2,\dots,a_\ell,0,\dots,0} \ne 0$, then $\sum_{j=2}^\ell a_j\beta_j = \alpha_{i_1} + \beta_\ell$ since $U^+(V)$ is $\mathbb{Z}^\theta$-graded so $a_\ell = 0$. Therefore we get (37) for $c_{a_{k+1},\dots,a_{\ell-1}} = \mathsf{c}_{0,a_2,\dots,a_{\ell-1},0,\dots,0}$. The proof of (38) is analogous. $\square$

Now we introduce algebra filtrations of $U^+(V)$, $U^{\ge 0}(V)$ and $U(V)$ related with the PBW basis. We order $\mathbb{N}_0^M$, $\mathbb{N}_0^{2M+1}$ lexicographically. In particular $\delta_1 < \cdots < \delta_M$, where $\{\delta_j\}_{1 \le j \le M}$ denotes the canonical basis of $\mathbb{Z}^M$ to avoid confusion with the basis $\{\alpha_i\}_{1 \le i \le \theta}$ of $\mathbb{Z}^\theta$.

(i) In $U^+(V)$ set $U^+(V)(\mathbf{a})$ as the subspace spanned by $\mathbf{E^b}$, $\mathbf{b} \le \mathbf{a}$. This is vector space filtration of $U^+(V)$ so consider the $\mathbb{N}_0^M$-graded vector space

$$\operatorname{gr} U^+(V) = \oplus_{\mathbf{a} \in \mathbb{N}_0^M} U^+(V)_{\mathbf{a}}, \qquad U^+(V)_{\mathbf{a}} = U^+(V)(\mathbf{a}) / \sum_{\mathbf{b} < \mathbf{a}} U^+(V)(\mathbf{b}).$$

(ii) For $U^{\geq 0}(V)$, $U^{\geq 0}(V)(\mathbf{a})$ is the subspace spanned by $\mathbf{E}^{\mathbf{b}} K^{\alpha}$, $\mathbf{b} \leq \mathbf{a}$, $\alpha \in \mathbb{Z}^{\theta}$. In particular $U^{\geq 0}(V)(0) = U^{+0}(V)$.

(iii) For each $\mathbf{E}^{\mathbf{a}} K^{\alpha} L^{\beta} \mathbf{F}^{\mathbf{b}} \in U(V)$ set

$$d(\mathbf{E}^{\mathbf{a}} K^{\alpha} L^{\beta} \mathbf{F}^{\mathbf{b}}) = \left(\sum_{j=1}^{M} (a_j + b_j) \operatorname{ht}(\beta_j), a_1, \ldots, a_M, b_1, \ldots, b_M \right) \in \mathbb{N}_0^{2M+1}.$$

For each $\mathbf{u} \in \mathbb{N}_0^{2M+1}$ let $U(V)(\mathbf{u})$ be the subspace spanned by $\mathbf{E}^{\mathbf{a}} K^{\alpha} L^{\beta} \mathbf{F}^{\mathbf{b}}$, $d(\mathbf{E}^{\mathbf{a}} K^{\alpha} L^{\beta} \mathbf{F}^{\mathbf{b}}) \leq \mathbf{u}$. Then take the associated $\mathbb{N}_0^{2M+1}$-graded vector space

$$\operatorname{gr} U(V) = \oplus_{\mathbf{u} \in \mathbb{N}_0^{2M+1}} U(V)_{\mathbf{u}}, \qquad U(V)_{\mathbf{u}} = U(V)(\mathbf{u}) / \sum_{\mathbf{v} < \mathbf{u}} U(V)(\mathbf{v}).$$

Next result generalizes [DK, Proposition 1.7], [DP, Proposition 10.1]

Proposition 3.16. *The $\mathbb{N}_0^{M}$-filtrations on $U^{+}(V)$, $U^{\geq 0}(V)$ and the $\mathbb{N}_0^{2M+1}$-filtration on $U(V)$ are algebra filtrations.*

Proof. First we consider $U^{+}(V)$. We claim that $\mathbf{E}^{\mathbf{a}} \mathbf{E}^{\mathbf{b}} \in U^{+}(V)(\mathbf{a} + \mathbf{b})$. Let $\mathbf{a} = \delta_k$, $\mathbf{b} = \delta_\ell$, $1 \leq k, \ell \leq M$. Note that $U^{+}(V)(\delta_k) = \mathbf{k} E_{\beta_k}$, and $E_{\beta_k} E_{\beta_\ell} \in U^{+}(V)(\delta_k + \delta_\ell)$ if $k \geq \ell$ by definition. Also $E_{\beta_k} E_{\beta_\ell} \in U^{+}(V)(\delta_k + \delta_\ell)$ if $k < \ell$ by Proposition 3.15, since

$$\sum_{j=k+1}^{\ell+1} a_j \delta_j < \delta_k + \delta_\ell \qquad \text{so } E_{\beta_{\ell-1}}^{a_{\ell-1}} \cdots E_{\beta_{k+1}}^{a_{k+1}} \in \sum_{\mathbf{v} < \delta_k + \delta_\ell} U^{+}(V)(\mathbf{v}).$$

Using this case we can reorder the PBW generators of $\mathbf{E}^{\mathbf{a}} \mathbf{E}^{\mathbf{b}}$ for any $\mathbf{a}, \mathbf{b}$.

The proof for $U^{\geq 0}(V)$ follows from the previous case since K_i q-commutes with all the elements of $U^{+}(V)$. For $U(V)$ we argue as above and reduce the proof to the product between E_{β_k}, F_{β_ℓ}. By Corollary 2.6,

$$(39) \qquad F_{\beta_\ell} E_{\beta_k} \in E_{\beta_k} F_{\beta_\ell} + \sum_{\mathbf{v} : v_1 \leq \operatorname{ht}(\beta_\ell) + \operatorname{ht}(\beta_k) - 2} U(V)(\mathbf{v})$$

so $F_{\beta_\ell} E_{\beta_k} \in U(V)(\operatorname{ht}(\beta_\ell) + \operatorname{ht}(\beta_k), \delta_k, \delta_\ell)$. $\square$

We give a presentation of the corresponding graded algebras.

Corollary 3.17. (i) *The algebra* $\operatorname{gr} U^{+}(V)$ *is presented by generators* $\mathtt{E}_k$, $1 \leq k \leq M$ *and relations*

$$(40) \qquad \mathtt{E}_k \mathtt{E}_\ell = \chi(\beta_k, \beta_\ell) \mathtt{E}_\ell \mathtt{E}_k, \quad k < \ell, \qquad \mathtt{E}_k^{N_k} = 0, \quad \beta_k \notin \mathcal{O}(V).$$

(ii) *The algebra* $\operatorname{gr} U^{\geq 0}(V)$ *is presented by generators* $\mathtt{E}_k$, $1 \leq k \leq M$, $K_i^{\pm}$, $i \in \mathbb{I}$, *and relations* (40),

$$(41) \qquad K_i K_j = K_j K_i, \qquad\qquad\qquad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$(42) \qquad K_i \mathtt{E}_k = \chi(\alpha_i, \beta_k) \mathtt{E}_k K_i, \qquad\qquad 1 \leq k \leq M, \, i, j \in \mathbb{I}.$$

Proof. For (i) Let $\mathcal{F}$ be the free algebra generated by E_k, $1 \le k \le M$ and $\pi :$ $\mathcal{F} \to \operatorname{gr} U^+(V)$ the algebra map such that $\pi(\mathsf{E}_k) = E_{\beta_k}$. By (37), $E_{\beta_k} E_{\beta_\ell} = \chi(\beta_k, \beta_\ell) E_{\beta_\ell} E_{\beta_k}$ holds in $\operatorname{gr} U^+(V)$, and also $E_{\beta_k}^{N_k} = 0$ for $\beta_k \notin \mathcal{O}(V)$ by Remark 3.5. Then π factors through the algebra defined by the relations (40), which is the quotient of a q-polynomial algebra of M generators by $\mathsf{E}_k^{N_k}$ for those generators corresponding to $\beta_k \notin \mathcal{O}(V)$, so it has a basis

$$\{\mathsf{E}_M^{a_m} \dots \mathsf{E}_1^{a_1} | a_k \in \mathbb{N}_0, a_k < N_k \text{ if } \beta_k \notin \mathcal{O}(V)\},$$

As $\mathbf{E}^{\mathbf{a}} \in U^+(V)(\mathbf{a}) - \sum_{\mathbf{b} < \mathbf{a}} U^+(V)(\mathbf{b})$, $\operatorname{gr} U^+(V)$ has a basis corresponding to the images of (36). But $\pi(\mathsf{E}_M^{a_m} \dots \mathsf{E}_1^{a_1}) = \mathbf{E}^{\mathbf{a}}$ so π is an isomorphism. The proof of (ii) follows similarly. $\square$

Corollary 3.18. *The algebra* $\operatorname{gr} U(V)$ *is presented by generators* E_k, F_k, $1 \le k \le M$, K_i, K_i^{-1}, L_i, L_i^{-1}, $i \in \mathbb{I}$, *and relations*

$$
\begin{aligned}
XY &= YX, & X, Y &\in \{K_i^{\pm 1}, L_i^{\pm 1} : 1 \le i \le \theta\}, \\
K_i K_i^{-1} &= L_i L_i^{-1} = 1, & \mathsf{E}_k \mathsf{F}_\ell &= \mathsf{F}_\ell \mathsf{E}_k, \\
K_i \mathsf{E}_k &= \chi(\alpha_i, \beta_k) \mathsf{E}_k K_i, & L_i \mathsf{E}_k &= \chi(\beta_k, -\alpha_i) \mathsf{E}_k L_i, \\
K_i \mathsf{F}_k &= \chi(-\alpha_i, \beta_k) \mathsf{F}_k K_i, & L_i \mathsf{F}_k &= \chi(\beta_k, \alpha_i) \mathsf{F}_k L_i, \\
\mathsf{E}_k \mathsf{E}_\ell &= \chi(\beta_k, \beta_\ell) \mathsf{E}_\ell \mathsf{E}_k, \quad k < \ell, & \mathsf{E}_k^{N_k} &= 0, \quad \beta_k \notin \mathcal{O}(V), \\
\mathsf{F}_k \mathsf{F}_\ell &= \chi(\beta_\ell, \beta_k) \mathsf{F}_\ell \mathsf{F}_k, \quad k < \ell, & \mathsf{F}_k^{N_k} &= 0, \quad \beta_k \notin \mathcal{O}(V).
\end{aligned}
$$

Proof. The proof is analogous to the previous case if we check that $E_{\beta_k} F_{\beta_\ell} = F_{\beta_\ell} E_{\beta_k}$ in $\operatorname{gr} U(V)$; this relation follows by (39). $\square$

Theorem 3.19. *The algebras* $U^+(V)$, $U^{\ge 0}(V)$, $U(V)$ *are Noetherian.*

Proof. It suffices to prove that the graded algebras are Noetherian. As

- $\operatorname{gr} U^+(V)$ is the quotient of a quantum affine space on M generators by powers of some generators,
- $\operatorname{gr} U^{\ge 0}(V)$ is the localization (we add the inverses of the θ generators K_i) of the quotient of a quantum affine space on $M + \theta$ generators by powers of some generators, and
- $\operatorname{gr} U(V)$ is the localization of the quotient of a quantum affine space on $2M + 2\theta$ generators by powers of some generators,

the three algebras $\operatorname{gr} U^+(V)$, $\operatorname{gr} U^{\ge 0}(V)$, $\operatorname{gr} U(V)$ are Noetherian. $\square$

Now we compute the Gelfand-Kirillov dimension of these algebras. We refer to [KL] for the definition and properties.

Theorem 3.20. *The Gelfand-Kirillov dimension of* $U^+(V)$, $U^{\ge 0}(V)$, $U(V)$ *are, respectively,* $|\mathcal{O}(V)|$, $|\mathcal{O}(V)| + \theta$, $2|\mathcal{O}(V)| + 2\theta$.

Proof. By [KL, Proposition 6.6], $\operatorname{GKdim} U^+(V) = \operatorname{GKdim} \operatorname{gr} U^+(V)$. The subalgebra $S^+(V)$ of $\operatorname{gr} U^+(V)$ generated by E_k, with $\beta_k \in \mathcal{O}(V)$, is a quantum affine space in $|\mathcal{O}(V)|$ generators and $\operatorname{gr} U^+(V)$ is a free $S^+(V)$-module

of rank $\prod\limits_{k:\beta_k\notin\mathcal{O}(V)} N_k$, so $\mathrm{GKdim}\,\mathrm{gr}\,U^+(V) = \mathrm{GKdim}\,S^+(V) = |\mathcal{O}(V)|$. For $U^{\geq 0}(V)$, let $S^{\geq 0}(V)$ be the subalgebra of $\mathrm{gr}\,U^{\geq 0}(V)$ generated by K_i, $i \in \mathbb{I}$, E_k if $\beta_k \in \mathcal{O}(V)$; it is the localization of a quantum affine space with $|\mathcal{O}(V)| + \theta$ generators, and $\mathrm{gr}\,U^{\geq 0}(V)$ is a free $S^{\geq 0}(V)$-module of rank $\prod\limits_{k:\beta_k\notin\mathcal{O}(V)} N_k$, so $\mathrm{GKdim}\,U^{\geq 0}(V) = \mathrm{GKdim}\,\mathrm{gr}\,U^{\geq 0}(V) = \mathrm{GKdim}\,S^{\geq 0}(V) = |\mathcal{O}(V)| + \theta$. The proof for $U(V)$ follows analogously. $\square$

4. POWER ROOT VECTORS ON DISTINGUISHED PRE-NICHOLS ALGEBRAS

Now we study the subalgebra $Z(V)$ of $U(V)$ generated by $E_\alpha^{N_\alpha}$, $F_\alpha^{N_\alpha}$, $K_\alpha^{N_\alpha}$, $L_\alpha^{N_\alpha}$, $\alpha \in \mathcal{O}(V)$. First we describe the product of these elements. Then we give a general formula for the composition of the coproduct with T_i^V on the whole $U_{-i}^+(V)$, and finally apply this formula to show that $Z(V)$ is a Hopf subalgebra. In what follows

- $Z^+(V)$ and $Z^-(V)$ are the subalgebras generated by $E_\alpha^{N_\alpha}$, respectively $F_\alpha^{N_\alpha}$, $\alpha \in \mathcal{O}(V)$.
- $\Gamma(V)$ is the subgroup of $\mathbb{Z}^{2\theta}$ generated by $K_\alpha^{N_\alpha}$, $L_\alpha^{N_\alpha}$, $\alpha \in \mathcal{O}(V)$.
- $S(V)$ is a set of representatives of $\mathbb{Z}^{2\theta}/\Gamma(V)$.

The multiplication gives an isomorphism of $\mathbb{Z}^\theta$-graded vector spaces $Z^+(V) \otimes \mathbf{k}\Gamma(V) \otimes Z^-(V) \simeq Z(V)$.

4.1. **Algebra structure of $Z(V)$.** Power root vectors are central elements of quantized enveloping algebras $U_q(\mathfrak{g})$, q a root of unity [DP, Chapter 19]. In the general case they q-commute with the elements of $U(V)$.

Proposition 4.1. *Let $\beta \in \mathcal{O}(V)$.*
(i) *For each $X \in U^+(V)$ homogeneous of degree γ,*
$$E_\beta^{N_\beta} X = \chi(N_\beta\beta, \gamma)\, X E_\beta^{N_\beta}, \qquad\qquad F_\beta^{N_\beta} X = X F_\beta^{N_\beta}.$$
(ii) *For each $Y \in U^-(V)$ homogeneous of degree γ,*
$$E_\beta^{N_\beta} Y = Y E_\beta^{N_\beta}, \qquad\qquad F_\beta^{N_\beta} Y = \chi(\gamma, N_\beta\beta)\, Y F_\beta^{N_\beta}.$$

Proof. If $\beta = \alpha_i$, then $E_i^{N_i} F_j = F_j E_i^{N_i}$ for $j \neq i$, and $E_i^{N_i} F_i = F_i E_i^{N_i}$ by (15), since $(N_i)_{q_{ii}} = 0$. Therefore $E_i^{N_i} Y = Y E_i^{N_i}$ for all $Y \in U^-(V)$.

For each $j \neq i$, $0 = (\mathrm{ad}_c E_i)^{N_i} E_j = E_i^{N_i} E_j - q_{ij}^{N_i} E_j E_i^{N_i}$ by (14), since $N_i \geq 1 - a_{ij}$. Then $E_i^{N_i} X = \chi(N_i\alpha_i, \gamma) X E_i^{N_i}$ for all $X \in U^+(V)_\gamma$, $\gamma \in \mathbb{N}_0$.

If $\beta = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$, then $E_\beta = T_{i_1} \cdots T_{i_{k-1}}(\underline{E}_{i_k})$. Let $v = s_{i_1} \ldots s_{i_1}$. If j is such that $v(\alpha_j) \in \mathbb{N}_0^\theta$, then $T_{i_{k-1}}^- \ldots T_{i_1}^-(E_j) \in U^+(V')_{v(\alpha_j)}$ by [H3, Proposition 6.15], so we can use the previous case:

$$E_\beta^{N_\beta} E_j = T_{i_1} \cdots T_{i_{k-1}} \left(v^* \chi(N_{i_k}\alpha_{i_k}, v(\alpha_j)) \underline{E}_{i_k}^{N_{i_k}} T_{i_{k-1}}^- \ldots T_{i_1}^-(E_j) \right)$$
$$= \chi(N_\beta\beta, \alpha_j) E_j E_\beta^{N_\beta}.$$

As also $T^-_{i_{k-1}} \ldots T^-_{i_1}(F_j) \in U^-(V')_{-v(\alpha_j)}$, we have that

$$E^{N_\beta}_\beta F_j = T_{i_1} \cdots T_{i_{k-1}} \left(\underline{E}^{N_{i_k}}_{i_k} T^-_{i_{k-1}} \ldots T^-_{i_1}(F_j) \right)$$

$$= T_{i_1} \cdots T_{i_{k-1}} \left(T^-_{i_{k-1}} \ldots T^-_{i_1}(F_j) \underline{E}^{N_{i_k}}_{i_k} \right) = F_j E^{N_\beta}_\beta.$$

If j satisfies $v(\alpha_j) \in -\mathbb{N}^\theta_0$, then $s_{i_{t-1}} \cdots s_{i_1}(\alpha_j) \in \mathbb{N}^\theta_0$, $s_{i_t} \cdots s_{i_1}(\alpha_j) \in -\mathbb{N}^\theta_0$ for $t \geq 0$ minimal. Therefore $s_{i_{t-1}} \cdots s_{i_1}(\alpha_j) = \alpha_{i_t}$, so $T^-_{i_{t-1}} \ldots T^-_{i_1}(E_j) = c\, E_{i_t}$ for some $c \in \mathbf{k}^\times$, and

$$T^-_{i_{k-1}} \ldots T^-_{i_1}(E_j) = T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(c\, K^{-1}_t F_t) \in T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(K^{-1}_t) U^-(V'),$$

because $s_{i_{k-1}} \ldots s_{i_{t+1}}(-\alpha_t) = w(\alpha_j) \in -\mathbb{N}^\theta_0$, so

$$E^{N_\beta}_\beta E_j = T_{i_1} \cdots T_{i_{k-1}} \left(\underline{E}^{N_{i_k}}_{i_k} T^-_{i_{k-1}} \ldots T^-_{i_1}(E_j) \right)$$

$$= c\, T_{i_1} \cdots T_{i_{k-1}} \left(\underline{E}^{N_{i_k}}_{i_k} T^-_{i_{k-1}} \ldots T^-_{i_t}(cE_t) \right)$$

$$= c\, T_{i_1} \cdots T_{i_{k-1}} \left(\underline{E}^{N_{i_k}}_{i_k} T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(K^{-1}_t) T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(F_t) \right)$$

$$= c\, T_{i_1} \cdots T_{i_{k-1}} \big(w^*\chi(N_{i_k}\alpha_{i_k}, w(\alpha_j)) T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(K^{-1}_t)$$

$$\underline{E}^{N_{i_k}}_{i_k} T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(F_t) \big)$$

$$= c\, T_{i_1} \cdots T_{i_{k-1}} \big(w^*\chi(N_{i_k}\alpha_{i_k}, w(\alpha_j)) T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(K^{-1}_t)$$

$$T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(F_t) \underline{E}^{N_{i_k}}_{i_k} \big) = \chi(N_\beta\beta, \alpha_j) E_j E^{N_\beta}_\beta.$$

Analogously, $T^-_{i_{t-1}} \ldots T^-_{i_1}(E_j) = c'\, E_{i_t}$ for some $c' \in \mathbf{k}^\times$, so

$$T^-_{i_{k-1}} \ldots T^-_{i_1}(F_j) = T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(c'\, E_t L^{-1}_t) \in U^+(V') T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(L^{-1}_t),$$

because $s_{i_{k-1}} \ldots s_{i_{t+1}}(-\alpha_t) = w(\alpha_j) \in -\mathbb{N}^\theta_0$, and

$$E^{N_\beta}_\beta E_j = T_{i_1} \cdots T_{i_{k-1}} \left(\underline{E}^{N_{i_k}}_{i_k} T^-_{i_{k-1}} \ldots T^-_{i_1}(E_j) \right)$$

$$= T_{i_1} \cdots T_{i_{k-1}} \left(\underline{E}^{N_{i_k}}_{i_k} T^-_{i_{k-1}} \ldots T^-_{i_t}(cE_t) \right)$$

$$= c'\, T_{i_1} \cdots T_{i_{k-1}} \left(\underline{E}^{N_{i_k}}_{i_k} T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(E_t) T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(L^{-1}_t) \right)$$

$$= c'\, T_{i_1} \cdots T_{i_{k-1}} \big(w^*\chi(N_{i_k}\alpha_{i_k}, w(\alpha_j)) T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(E_t)$$

$$\underline{E}^{N_{i_k}}_{i_k} T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(L^{-1}_t) \big)$$

$$= c'\, T_{i_1} \cdots T_{i_{k-1}} \left(T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(E_t) T^-_{i_{k-1}} \ldots T^-_{i_{t+1}}(L^{-1}_t) \underline{E}^{N_{i_k}}_{i_k} \right)$$

$$= \chi(N_\beta\beta, \alpha_j) E_j E^{N_\beta}_\beta,$$

which completes the proof for $E^{N_\beta}_\beta$. The proof for $F^{N_\beta}_\beta$ is analogous. $\square$

Corollary 4.2. *For all $\alpha \in \mathcal{O}(V)$ and all $j \in \mathbb{I}$, $\partial^K_j(E^{N_\alpha}_\alpha) = \partial^L_j(E^{N_\alpha}_\alpha) = 0$.*

Proof. Let $i = i_1$. The proof for $\alpha = \alpha_i$ is direct. If $\alpha \neq \alpha_i$, then set $\beta = s_i(\alpha) \in \mathcal{O}(\rho_i(V))$, so $E_\alpha = T_i(E_\beta)$, $N_\alpha = N_\beta$. By Proposition 4.1,

$$\partial_i^K(E_\alpha^{N_\alpha})K_i - L_i^{-1}\partial_i^L(E_\alpha^{N_\alpha}) = E_\alpha^{N_\alpha}F_i - F_iE_\alpha^{N_\alpha}$$

$$= T_i\left(E_\beta^{N_\beta}E_iL_i^{-1} - E_iL_i^{-1}E_\beta^{N_\beta}\right)$$

$$= T_i\left(s_i^*\chi(N_\beta\beta, \alpha_i)E_iE_\beta^{N_\beta}L_i^{-1} - E_iL_i^{-1}E_\beta^{N_\beta}\right) = 0.$$

Then $\partial_i^K(E_\alpha^{N_\alpha}) = \partial_i^L(E_\alpha^{N_\alpha}) = 0$. For $j \neq i$,

$$\partial_j^K(E_\alpha^{N_\alpha})K_j - L_j^{-1}\partial_j^L(E_\alpha^{N_\alpha}) = E_\alpha^{N_\alpha}F_j - F_jE_\alpha^{N_\alpha}$$

$$= T_i\left(E_\beta^{N_\beta}F_i^{-a_{ij}^V}F_j - F_i^{-a_{ij}^V}F_jE_\beta^{N_\beta}\right) = 0,$$

so $\partial_j^K(E_\alpha^{N_\alpha}) = \partial_j^L(E_\alpha^{N_\alpha}) = 0$. $\qquad\square$

We deduce the freeness of $U(V)$ as $Z(V)$-module.

Theorem 4.3. *The set*

$$\left\{\left(\prod_{\beta \in \mathcal{O}(V)} E_\beta^{k_\beta N_\beta}\right)\gamma\left(\prod_{\beta \in \mathcal{O}(V)} F_\beta^{l_\beta N_\beta}\right) : k_\beta, l_\beta \in \mathbb{N}_0, \gamma \in \Gamma(V)\right\},$$

is a basis of $Z(V)$, where the order on the set $\mathcal{O}(V)$ is arbitrary.
   Moreover $U(V)$ is a free $Z(V)$-module with basis:

$$\left\{E_{\beta_M}^{a_M}E_{\beta_{M-1}}^{a_{M-1}}\cdots E_{\beta_1}^{a_1}sF_{\beta_M}^{b_M}F_{\beta_{M-1}}^{b_{M-1}}\cdots F_{\beta_1}^{b_1} | 0 \leq a_k, b_k < N_k, s \in S(V)\right\}.$$

Proof. By Proposition 4.1, the generators of the subalgebra $Z(V)$ q-commute with all the elements of $U(V)$. Then apply Theorem 3.6. $\qquad\square$

Remark 4.4. The algebra $Z(V)$ is not necessarily central since we can have $\chi(N_\beta\beta, \alpha_j) \neq 1$ for some $\beta \in \mathcal{O}(V)$, $1 \leq j \leq \theta$.

Lemma 4.5. *Let $\beta \in \mathcal{O}(V)$. For every $\alpha \in \mathbb{Z}^\theta$, $\chi(N_\beta\beta, \alpha)\chi(\alpha, N_\beta\beta) = 1$.*

Proof. If $\beta = \alpha_i$, then $\chi(N_i\alpha_i, \alpha_i)\chi(\alpha_i, N_i\alpha_i) = q_{ii}^{2N_i} = 1$ and for $j \neq i$,

$$\chi(N_i\alpha_i, \alpha_j)\chi(\alpha_j, N_i\alpha_i) = (q_{ij}q_{ji})^{N_i} = q_{ii}^{c_{ij}^V N_i} = 1,$$

so $\chi(N_i\alpha_i, \alpha)\chi(\alpha, N_i\alpha_i) = 1$ for every $\alpha \in \mathbb{Z}^\theta$. If $\beta \neq \alpha_i$, then $\beta = w(\alpha_i)$ for some $w \in \mathrm{Hom}(\mathcal{W}, V)$ and $\alpha_i \in \mathcal{O}(w^{-1}(V))$, so $N_\beta = N_i$. Therefore

$$\chi(N_\beta\beta, \alpha)\chi(\alpha, N_\beta\beta) = \chi(N_iw(\alpha_i), \alpha)\chi(\alpha, N_iw(\alpha_i))$$

$$= (w^{-1})^*\chi(N_i\alpha_i, w^{-1}(\alpha))(w^{-1})^*\chi(w^{-1}(\alpha), N_i\alpha_i) = 1,$$

for all $\alpha \in \mathbb{Z}^\theta$, by applying the previous step to $\alpha_i \in \mathcal{O}(w^{-1}(V))$. $\qquad\square$

4.2. **On the coproduct of** $U(V)$. We want to factorize the composition
$\underline{\Delta} \circ T_i : U_{-i}^+(V) \to U^+(V) \otimes U_{+i}^+(V)$ as in [HS1]. Similar formula was introduced in [L2, Proposition 5.3.4] for quantized enveloping algebras. Such factorization is interpreted as equivalences between the corresponding categories of Yetter-Drinfeld modules [BLS, HS2]. One of the factors is $T_i \otimes T_i$ but viewed as an algebra map between *braided* structures.

- $U_{-i}^+(V) \underline{\otimes} U_{-i}^+(V)$ denotes the tensor product $U_{-i}^+(V) \otimes U_{-i}^+(V)$ with the algebra structure in $_{\mathcal{B}_i^{\mathrm{bop}} \# \mathbf{k}\mathbb{Z}^\theta}^{\mathcal{B}_i^{\mathrm{bop}} \# \mathbf{k}\mathbb{Z}^\theta} \mathcal{YD}$, see Remark 3.12, and

- $U_{+i}^+(V) \underline{\otimes} U_{+i}^+(V)$ denotes the tensor product $U_{+i}^+(V) \otimes U_{+i}^+(V)$ with the algebra structure in $_{\mathcal{B}_i \# \mathbf{k}\mathbb{Z}^\theta}^{\mathcal{B}_i \# \mathbf{k}\mathbb{Z}^\theta} \mathcal{YD}$, see Remark 3.11.

Lemma 4.6. $T_i \otimes T_i : U_{-i}^+(V) \underline{\otimes} U_{-i}^+(V) \to U_{+i}^+(V) \underline{\otimes} U_{+i}^+(V)$ *is an algebra map.*

Proof. Let $x, y, w, z \in U_{-i}^+(V)$, where y, w are homogeneous of degrees $\beta, \gamma \in \mathbb{N}_0^\theta$, respectively. As $(\mathrm{ad}_c E_i)^{N_i} \equiv 0$,

$$T_i \otimes T_i(x \otimes y) \cdot T_i \otimes T_i(w \otimes z) = T_i(x) \left(T_i(y)_{(-1)} \rightharpoonup T_i(w) \right) \underline{\otimes} T_i(y)_{(0)} T_i(z)$$

$$\overset{(30)}{=} \sum_{n=0}^{N_i-1} s_i^* \chi(s_i(\beta) - n\alpha_i, s_i(\gamma)) \, T_i(x)(\mathrm{ad}_{\underline{c}} \underline{E}_i)^n T_i(w) \underline{\otimes} (T_i(y) \triangleleft \mathfrak{E}_i^{(n)}) T_i(z)$$

$$\overset{(26),(34)}{=} \sum_{n=0}^{N_i-1} \frac{(-1)^n q_{ii}^{\frac{n(n-1)}{2}} \chi(\beta + n\alpha_i, \gamma)}{(n)_{q_{ii}}! \chi(\alpha_i, \gamma)^n} \, T_i(x) T_i((\partial_i^K)^n w)$$
$$\underline{\otimes} \left((\partial_i^L)^n T_i(y) \right) T_i(z)$$

$$\overset{(26)}{=} \sum_{n=0}^{N_i-1} (-1)^n q_{ii}^{\frac{n(n-1)}{2}} \chi(\beta, \gamma) \, T_i(x) T_i(\mathfrak{E}_i^{(n)} \triangleright w) \underline{\otimes} \left((\partial_i^L)^n T_i(y) \right) T_i(z)$$

$$\overset{(33)}{=} \sum_{n=0}^{N_i-1} \frac{\chi(\beta, \gamma)}{\chi(\beta, \alpha_i)^n} \, T_i(x) T_i(\mathfrak{E}_i^{(n)} \triangleright w) \underline{\otimes} T_i \left(E_i^n \rightharpoonup' y \right) T_i(z)$$

$$= \sum_{n=0}^{N_i-1} \chi(\beta, \gamma - n\alpha_i) \, T_i \left(x \left(\mathfrak{E}_i^{(n)} \triangleright w \right) \right) \underline{\otimes} T_i \left((E_i^n \rightharpoonup' y) z \right)$$

$$= T_i \otimes T_i \left((x \otimes y) \cdot (w \otimes z) \right),$$

so the proof is complete. $\qquad \square$

Let $Y \in U_{+i}^+(V)_\beta$, $\beta \in \mathbb{N}_0^\theta$. As $Y \triangleleft \mathfrak{E}_i^{(n)} \in U(V)_{\beta - n\alpha_i}$, there exists $k \in \mathbb{N}$ such that $Y \triangleleft \mathfrak{E}_i^{(n)} = 0$ for all $n \geq k$. Then there exists a well-defined map $\mathfrak{R}_i : U_{+i}^+(V) \otimes U_{+i}^+(V) \to U^+(V) \otimes U_{+i}^+(V)$ such that

$$(43) \qquad \mathfrak{R}_i^V(x \otimes y) = \sum_{k \geq 0} x E_i^k \otimes y \triangleleft \mathfrak{E}_i^{(k)}, \qquad x, y \in U_{+i}^+(V).$$

Compare with [HS1, (3.4)].

Lemma 4.7. $\mathfrak{R}_i^V : U_{+i}^+(V) \underline{\otimes} U_{+i}^+(V) \to U^+(V) \otimes U_{+i}^+(V)$ *is an algebra map.*

Here $U^+(V) \otimes U_{+i}^+(V)$ is a subalgebra of $U^+(V) \otimes U^+(V)$, where $U^+(V) \otimes U^+(V)$ has the algebra structure such that $(x \otimes y)(x' \otimes y) = \chi(\beta, \gamma) xx' \otimes yy$ for $x, x', y, y' \in U^+(V)$, y, x' homogeneous of degree β, γ, respectively.

Proof. Set $x, y, w, z \in U_{+i}^+(V)$, y, w homogeneous of degrees $\beta, \gamma \in \mathbb{N}_0^\theta$. Then

$$(44) \qquad E_i^n w = \sum_{r=0}^n \binom{n}{r}_{q_{ii}} \chi(\alpha_i, \gamma)^{n-r} (\mathrm{ad}_c E_i)^r w E_i^{n-r}.$$

for all $n \in \mathbb{N}$, so

$$\mathfrak{R}_i^V(x \otimes y) \cdot \mathfrak{R}_i^V(w \otimes z) =$$

$$= \sum_{j,k \geq 0} \chi(\beta - j\alpha_i, \gamma + k\alpha_i)\, x E_i^j w E_i^k \otimes (y \triangleleft \mathfrak{E}_i^{(j)})(z \triangleleft \mathfrak{E}_i^{(k)})$$

$$\overset{(44)}{=} \sum_{j,k \geq 0} x \left(\sum_{r=0}^{\min\{N_i-1,j\}} \binom{j}{r}_{q_{ii}} \frac{\chi(\beta,\gamma)\chi(\beta,\alpha_i)^k}{\chi(\alpha_i,\gamma)^r q_{ii}^{jk}} (\mathrm{ad}_c E_i)^r w E_i^{k+j-r} \right)$$

$$\otimes (y \triangleleft \mathfrak{E}_i^{(j)})(z \triangleleft \mathfrak{E}_i^{(k)})$$

$$\overset{(26)}{=} \sum_{j,k \geq 0} \sum_{r=0}^{\min\{N_i-1,j\}} \frac{\chi(\beta, \gamma + k\alpha_i)}{\chi(\alpha_i,\gamma)^r q_{ii}^{jk}} x (\mathrm{ad}_c E_i)^r w E_i^{k+j-r}$$

$$\otimes ((y \triangleleft \mathfrak{E}_i^{(r)}) \triangleleft \mathfrak{E}_i^{(j-r)})(z \triangleleft \mathfrak{E}_i^{(k)}).$$

On the other hand,

$$\mathfrak{R}_i^V((x \otimes y) \cdot (w \otimes z)) =$$

$$\overset{(30)}{=} \mathfrak{R}_i^V \left(\sum_{r=0}^{N_i} \chi(\beta - r\alpha_i, \gamma)\, x (\mathrm{ad}_c E_i)^r w \otimes (y \triangleleft \mathfrak{E}_i^{(r)}) z \right)$$

$$\overset{(43)}{=} \sum_{r=0}^{N_i} \sum_{t \geq 0} \chi(\beta - r\alpha_i, \gamma)\, x (\mathrm{ad}_c E_i)^r w E_i^t \otimes \left((y \triangleleft \mathfrak{E}_i^{(r)}) z \right) \triangleleft \mathfrak{E}_i^{(t)}$$

$$\overset{(29)}{=} \sum_{r=0}^{N_i} \sum_{k,l \geq 0} \chi(\beta - r\alpha_i, \gamma)\chi(\beta - (r+l)\alpha_i, k\alpha_i)\, x (\mathrm{ad}_c E_i)^r w E_i^{k+l}$$

$$\otimes ((y \triangleleft \mathfrak{E}_i^{(r)}) \triangleleft \mathfrak{E}_i^{(l)}) z \triangleleft \mathfrak{E}_i^{(k)}$$

$$= \sum_{r=0}^{N_i} \sum_{k,l \geq 0} \frac{\chi(\beta, \gamma + k\alpha_i)}{\chi(\alpha_i,\gamma)^r q_{ii}^{(r+l)k}} x (\mathrm{ad}_c E_i)^r w E_i^{k+l} \otimes ((y \triangleleft \mathfrak{E}_i^{(r)}) \triangleleft \mathfrak{E}_i^{(l)}) z \triangleleft \mathfrak{E}_i^{(k)},$$

so $\mathfrak{R}_i^V(x \otimes y) \cdot \mathfrak{R}_i^V(w \otimes z) = \mathfrak{R}_i^V((x \otimes y) \cdot (w \otimes z))$. $\square$

Recall the map $\Delta_i : U^+_{-i}(V) \to U^+_{+i}(V) \underline{\otimes} U^+_{+i}(V)$ introduced in Remark 3.12. Now we prove the factorization of $\underline{\Delta} \circ T_i$.

Theorem 4.8. *For every $E \in U^+_{-i}(V)$,*

$$(45) \qquad \underline{\Delta} \circ T_i(E) = \mathfrak{R}^{\rho_i(V)}_i \circ (T_i \otimes T_i) \circ \Delta_i(E).$$

Proof. It is enough to prove the formula for $E = E^-_{j,n}$, $j \neq i$, $0 \leq n \leq -c^V_{ij}$, since they generate $U^+_{-i}(V)$ as an algebra and both $\underline{\Delta} \circ T_i$, $\mathfrak{R}^{\rho_i(V)}_i \circ (T_i \otimes T_i) \circ \Delta_i$ are algebra maps by Lemmas 4.6 and 4.7. Let

$$\kappa_{i,j;n} := \underline{q}^{\,n}_{ii} \left(\prod_{t=-c^V_{ij}-n}^{-c^V_{ij}-1} (t+1)_{\underline{q}_{ii}} (1 - \underline{q}^{\,t}_{ii} \underline{q}_{ij} \underline{q}_{ji}) \right).$$

By (35) and (18),

$$\kappa^{-1}_{i,j;n} \underline{\Delta} \circ T_i(E^-_{j,n}) = \underline{\Delta}(E^+_{j,-c^V_{ij}-n}) = E^+_{j,-c^V_{ij}-n} \otimes 1$$

$$+ \sum_{s=0}^{-c^V_{ij}-n-1} \binom{-c^V_{ij}-n}{s}_{\underline{q}_{ii}} \left(\prod_{r=1}^{s} (1 - \underline{q}^{\,-c^V_{ij}-n-r}_{ii} \underline{q}_{ij} \underline{q}_{ji}) \right) E^s_i \otimes E^+_{j,-c^V_{ij}-n-s}$$

On the other hand, by (32), (35), (43) and (26),

$$\kappa^{-1}_{i,j;n} \mathfrak{R}^{\rho_i(V)}_i \circ (T_i \otimes T_i) \circ \Delta_i(E^-_{j,n}) = \mathfrak{R}^{\rho_i(V)}_i \left(E^+_{j,-c^V_{ij}-n} \otimes 1 + 1 \otimes E^+_{j,-c^V_{ij}-n} \right)$$

$$= E^+_{j,-c^V_{ij}-n} \otimes 1 + \sum_{s=0}^{-c^V_{ij}-n-1} \frac{1}{s_{\underline{q}_{ii}}!} E^s_i \otimes (\partial^L_i)^s (E^+_{j,-c^V_{ij}-n}),$$

so $\underline{\Delta} \circ T_i(E^-_{j,n}) = \mathfrak{R}^{\rho_i(V)}_i \circ (T_i \otimes T_i) \circ \Delta_i(E^-_{j,n})$. $\qquad \square$

4.3. **Coproduct structure of $Z(V)$.** Next we prove that $Z(V)$ is a Hopf subalgebra of $U(V)$. We start by describing the left hand side of the coproduct $\underline{\Delta}$ of $Z^+(V)$. Let $v = s_{i_1} \cdots s_{i_k} \in \mathrm{Hom}(\mathcal{W}, V)$ be an element of length k, $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \in \Delta^V_+$, and as above $E_{\beta_j} = T_{i_1} \cdots T_{i_{j-1}}(E_{i_j})$.

Proposition 4.9. *If $\beta_k \in \mathcal{O}(V)$, then there exist $X(n_1, \ldots, n_{k-1}) \in U^+(V)$ such that*

$$(46) \qquad \underline{\Delta}(E^{N_{\beta_k}}_{\beta_k}) = E^{N_{\beta_k}}_{\beta_k} \otimes 1 + 1 \otimes E^{N_{\beta_k}}_{\beta_k}$$

$$+ \sum_{n_i \in \mathbb{N}_0} E^{n_{k-1}N_{\beta_{k-1}}}_{\beta_{k-1}} \ldots E^{n_1 N_{\beta_1}}_{\beta_1} \otimes X(n_1, \ldots, n_{k-1}).$$

Proof. The proof is by induction on k. If $k = 1$, then $E_{\beta_1} = E_{i_1}$, and $E^{N_{i_1}}_{i_1}$ is primitive. Assume that (46) holds for every v' of length less than k. Let

$v = s_{i_1}^V v'$, $i = i_1$, $\beta = \beta_k$, $\gamma_j := s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j}) \in \Delta_+^{\rho_i(V)}$, $j = 2, \cdots, k$, $\gamma = \gamma_k$, so $\beta_j = s_i(\gamma_j)$, $E_{\beta_j} = T_i(E_{\gamma_j})$, $N_\beta = N_\gamma$. By inductive hypothesis,

$$\underline{\Delta}(E_\gamma^{N_\gamma}) = E_\gamma^{N_\gamma} \otimes 1 + 1 \otimes E_\gamma^{N_\gamma}$$
$$+ \sum_{n_i \in \mathbb{N}} E_{\gamma_{k-1}}^{n_{k-1} N_{\gamma_{k-1}}} \dots E_{\gamma_2}^{n_2 N_{\gamma_2}} \otimes Y(n_1, \dots, n_{k-1}),$$

for some $Y(n_1, \dots, n_{k-1}) \in U^+(\rho_i(V))$. As $\partial_i^L(E_{\beta_j}^{N_{\beta_j}}) = 0$ by Corollary 4.2, we have that $E_i \rightharpoonup' E_{\gamma_j}^{N_{\gamma_j}} = 0$ by (33). By Lemma 4.5,

$$0 = \underline{\Delta}(E_i \rightharpoonup' E_\gamma^{N_\gamma}) = \underline{\Delta}(E_\gamma^{N_\gamma})(E_i \otimes 1 + 1 \otimes E_i)$$
$$- \chi(N_\gamma\gamma, \alpha_i)(E_i \otimes 1 + 1 \otimes E_i)\underline{\Delta}(E_\gamma^{N_\gamma})$$
$$= \sum_{n_i \in \mathbb{N}} E_i \rightharpoonup' \left(E_{\gamma_{k-1}}^{n_{k-1} N_{\gamma_{k-1}}} \dots E_{\gamma_2}^{n_2 N_{\gamma_2}} \right) \otimes Y(n_2, \dots, n_{k-1})$$
$$+ E_{\gamma_{k-1}}^{n_{k-1} N_{\gamma_{k-1}}} \dots E_{\gamma_2}^{n_2 N_{\gamma_2}} \otimes E_i \rightharpoonup' Y(n_2, \dots, n_{k-1})$$
$$+ \left(1 - \prod_{j=2}^{k-1} \chi(N_{\gamma_j}\gamma_j, \alpha_i)^{n_j} \chi(\alpha_i, N_{\gamma_j}\gamma_j)^{n_j} \right) \chi\left(N_\gamma\gamma - \sum_{j=2}^{k-1} N_{\gamma_j}\gamma_j, \alpha_i \right)$$
$$E_{\gamma_{k-1}}^{n_{k-1} N_{\gamma_{k-1}}} \dots E_{\gamma_2}^{n_2 N_{\gamma_2}} \otimes Y(n_2, \dots, n_{k-1}) E_i$$
$$= \sum_{n_i \in \mathbb{N}} E_{\gamma_{k-1}}^{n_{k-1} N_{\gamma_{k-1}}} \dots E_{\gamma_2}^{n_2 N_{\gamma_2}} \otimes E_i \rightharpoonup' Y(n_2, \dots, n_{k-1}),$$

so $E_i \rightharpoonup' Y(n_2, \dots, n_{k-1}) = 0$ for all $n_i \in \mathbb{N}_0$. Moreover $(\underline{\Delta} \otimes \mathrm{id})\underline{\Delta}(E_\gamma^{N_\gamma})$ can be written as a sum of terms where the three elements of $U^+(V)^{\otimes 3}$ are products of $E_{\gamma_j}^{n_j N_{\gamma_j}}$ and $Y(n_2, \dots, n_{j-1})$, $1 \leq j \leq k$, so they are annihilated by $E_i \rightharpoonup'$. In particular the middle term have terms E_i^n only for $n \in \mathbb{N}N_i$, so the remaining terms in $\iota_{i,V}\mathcal{S}\pi_{i,V}(x_{(2)})$ are of the form $E_i^{nN_i}$, $n \in \mathbb{N}_0$. As $\Delta_i(E_\gamma^{N_\gamma}) = (\mathrm{id} \otimes m)(\mathrm{id} \otimes \iota_{i,V}\mathcal{S}\pi_{i,V} \otimes \mathrm{id})(\underline{\Delta} \otimes \mathrm{id})\underline{\Delta}(E_\gamma^{N_\gamma})$, there exist $Z(n_2, \dots, n_{k-1}) \in U^+(V)$ such that $E_i \rightharpoonup' Z(n_2, \dots, n_{k-1}) = 0$ and

$$\Delta_i(E_\gamma^{N_\gamma}) = E_\gamma^{N_\gamma} \otimes 1 + 1 \otimes E_\gamma^{N_\gamma}$$
$$+ \sum_{n_i \in \mathbb{N}} E_{\gamma_{k-1}}^{n_{k-1} N_{\gamma_{k-1}}} \dots E_{\gamma_2}^{n_2 N_{\gamma_2}} \otimes Z(n_2, \dots, n_{k-1}).$$

By Theorem 4.8,

$$\underline{\Delta}(E_\beta^{N_\beta}) = \underline{\Delta} \circ T_i(E_\gamma^{N_\gamma}) = \mathfrak{R}_i^V \circ (T_i \otimes T_i) \circ \Delta_i(E_\gamma^{N_\gamma})$$

$$= \mathfrak{R}_i^V \left(E_\beta^{N_\beta} \otimes 1 + 1 \otimes E_\beta^{N_\beta} \right)$$

$$+ \mathfrak{R}_i^V \left(\sum_{n_i \in \mathbb{N}} E_{\beta_{k-1}}^{n_{k-1} N_{\beta_{k-1}}} \dots E_{\beta_2}^{n_2 N_{\beta_2}} \otimes T_i(Z(n_2, \dots, n_{k-1})) \right).$$

If n is not a multiple of N_i, then

$$(T_i(Z(n_2, \dots, n_{k-1}))) \triangleleft \mathfrak{E}_i^{(n)} \overset{(26)}{=} (n)_{q_{ii}}^{-1} \left(\partial_i^L \circ T_i(Z(n_2, \dots, n_{k-1})) \right) \triangleleft \mathfrak{E}_i^{(n-1)}$$

$$\overset{(33)}{=} -\chi(\delta, \alpha_i)^{-1} q_{ii}^{-1}(n)_{q_{ii}}^{-1} \left(T_i(E_i \rightharpoonup' Z(n_2, \dots, n_{k-1})) \right) \triangleleft \mathfrak{E}_i^{(n-1)} = 0,$$

where $\delta = N_\gamma \gamma - \sum_{j=2}^{k-1} n_j N_{\gamma_j} \gamma_j$ is the degree of $Z(n_2, \dots, n_{k-1})$. As also $\delta_i^L(E_\beta^{N_\beta}) = 0$, we have that $E_\beta^{N_\beta} \triangleleft \mathfrak{E}_i^{(n)} = 0$. Therefore

$$\underline{\Delta}(E_\beta^{N_\beta}) = E_\beta^{N_\beta} \otimes 1 + \sum_{n \in \mathbb{N}_0} E_i^{nN_i} \otimes E_\beta^{N_\beta} \triangleleft \mathfrak{E}_i^{(nN_i)}$$

$$+ \sum_{n_i \in \mathbb{N}} \sum_{n \in \mathbb{N}_0} E_{\beta_{k-1}}^{n_{k-1} N_{\beta_{k-1}}} \dots E_{\beta_2}^{n_2 N_{\beta_2}} E_i^{nN_i} \otimes T_i(Z(n_2, \dots, n_{k-1})) \triangleleft \mathfrak{E}_i^{(nN_i)},$$

which concludes the inductive step. $\qquad\square$

Theorem 4.10. $Z^+(V)$ *is a normal right coideal subalgebra of* $U^+(V)$.
Moreover $Z^+(V) =^{co\pi_V|_{U^+(V)}} U^+(V)$.

Proof. By Proposition 4.9 $Z^+(V)$ is a right coideal subalgebra, and is normal by Theorem 4.3. Then the right coideal $U^+(V)(Z^+(V))^+$ is a Hopf ideal, and $Z^+(V) =^{co\pi_V|_{U^+(V)}} U^+(V)$ by Proposition 2.2 (ii), since $\mathfrak{u}^{\geq 0}(V) = U^+(V)/U^+(V)(Z^+(V))^+$. $\qquad\square$

We consider the following subalgebras of $U^+(V)$:

$$\mathcal{K}^K(V) = \bigcap_{i=1}^{\theta} \ker \partial_i^K, \quad \mathcal{K}^L(V) = \bigcap_{i=1}^{\theta} \ker \partial_i^L, \quad \mathcal{K}(V) = \mathcal{K}^K(V) \cap \mathcal{K}^L(V).$$

Remark 4.11. $\mathcal{K}^L(V)$ is a right coideal subalgebra since it is the intersection of right coideal subalgebras. Similarly $\mathcal{K}^K(V)$ is a left coideal subalgebra.

Lemma 4.12. (i) *For all* $x \in U^+(V)$, $i \in \mathbb{I}$, $\partial_i^L(\mathcal{S}(x)) = -\mathcal{S}(\partial_i^K(x))$.
(ii) *The restriction of* $\mathcal{S}$ *gives bijective* $\mathbb{Z}^\theta$-*graded antialgebra maps*

$$\ker \partial_i^K \overset{\sim}{\longrightarrow} \ker \partial_i^L, \quad i \in \mathbb{I}, \qquad \mathcal{K}^K(V) \longrightarrow \mathcal{K}^L(V).$$

Proof. (i) Let $x \in U^+(V)_n$, $n \in \mathbb{N}_0$. If $n = 0$, then both sides are 0. Let $n \geq 1$. As $\mathcal{S}$ is an anticoalgebra graded map and $\mathcal{S}(E_i) = -E_i$ for all $i \in \mathbb{I}$,

$$\sum_{i=1}^{\theta} \partial_i^L(\mathcal{S}(x)) \otimes E_i = \underline{\Delta}_{1,n-1}^{\mathrm{cop}} \circ \mathcal{S}(x) = (\mathcal{S} \otimes \mathcal{S}) \circ \underline{\Delta}_{n-1,1}(x)$$

$$= -\sum_{i=1}^{\theta} \mathcal{S}(\partial_i^K(x)) \otimes E_i.$$

(ii) From the identity in (i) , $\mathcal{S}(\ker \partial_i^K) \subseteq \ker \partial_i^L$, $\mathcal{S}(\mathcal{K}^K(V)) \subseteq \mathcal{K}^L(V)$. Also $\mathcal{S}$ is a $\mathbb{Z}^\theta$-graded, and bijective since $U^+(V)$ is connected. $\qquad\square$

We obtain a different characterization of $Z^+(V)$.

Theorem 4.13. *The subalgebra $Z^+(V)$ coincides with $\mathcal{K}^L(V)$, $\mathcal{K}^K(V)$. In particular $Z^+(V)$ is a braided Hopf subalgebra of $U^+(V)$.*

Proof. By Corollary 4.2, $Z^+(V) \subseteq \mathcal{K}^L(V)$. If $x \in \mathcal{K}^L(V)$ is an homogeneous element of degree $n \geq 2$, then $\pi_V(x)$ is an homogeneous element of the same degree, annihilated by all the corresponding derivations ∂_i^L of the Nichols algebra $\mathfrak{u}^+(V)$. But the intersection of the kernels of ∂_i^L in $\mathfrak{u}^+(V)$ is $\mathbf{k}1$ by [MiS, Proposition 2.4], so $\pi_V(x) = 0$. Then $U^+(V)N(\mathcal{K}^L(V))^+ \subseteq \ker \pi_V$. As $\ker \pi_V = U^+(V)(Z^+(V))^+$ by Theorem 4.10, both Hopf ideals coincide, so by Proposition 2.2 $Z^+(V) = N(\mathcal{K}^L(V))$. Therefore $Z^+(V) = \mathcal{K}^L(V)$.

Also $Z^+(V) \subseteq \mathcal{K}^K(V)$ by Corollary 4.2, and by Lemma 4.12(ii) the Hilbert series of $\mathcal{K}^K(V)$ and $\mathcal{K}^L(V)$ coincide. As $Z^+(V) = \mathcal{K}^L(V)$, we have that $Z^+(V) = \mathcal{K}^K(V)$. Therefore $Z^+(V)$ is a braided Hopf subalgebra of $U^+(V)$ since it is simultaneously a right and a left coideal subalgebra. $\qquad\square$

Corollary 4.14. *The subalgebras $Z^\pm(V)$, $Z^0(V)$, $Z(V)$ do not depend on the choice of the element of maximal length.*

Proof. By Theorem 4.13, $Z^+(V) = \mathcal{K}(V)$, and $\mathcal{K}(V)$ does not depend on that element of maximal length; the proof for $Z^-(V)$ is analogous. As $Z^0(V)$ is the group algebra of the subgroup generated by $K_\beta^{N_\beta}$, $L_\beta^{N_\beta}$, $\beta \in \mathcal{O}(V)$, the proof for $Z(V)$ follows since it is generated by $Z^\pm(V)$, $Z^0(V)$. $\qquad\square$

We now deduce the corresponding statement for $Z(V)$ and $U(V)$.

Theorem 4.15. *$Z(V)$ is a Hopf subalgebra of $U(V)$.*

Proof. Let $\beta \in \mathcal{O}(V)$. The $U^{+0}(V)$-coaction is $\lambda(E_\beta^{N_\gamma}) = K_\gamma^{N_\gamma} \otimes E_\gamma^{N_\gamma}$, $\gamma \in \mathcal{O}(V)$, and $\underline{\Delta}(Z^+(V)) \subset Z^+(V) \otimes Z^+(V)$ by Theorem 4.13, so $\Delta(Z^+(V)) \subset Z^+(V)\Gamma(V) \otimes Z^+(V)$. Analogously, $\Delta(Z^-(V)) \subset Z^-(V) \otimes Z^-(V)\Gamma(V)$. $\quad\square$

Remark 4.16. Assume that χ is such that $\chi(N_\beta\beta, \alpha_j)=1$ for all $\beta \in \mathcal{O}(V)$, $1 \leq j \leq \theta$. Then $Z(V)$ is a central Hopf subalgebra and $\Gamma(V)$ acts trivially on $U^\pm(V)$, $\mathfrak{u}^\pm(V)$ so the quotient

$$\widehat{\mathfrak{u}}(V) = \mathfrak{u}(V)/<K^{N_\beta\beta} - 1, L^{N_\beta\beta} - 1 : \beta \in \mathcal{O}(V) >$$

is well defined. The triangular decomposition on $\mathfrak{u}(V)$ induces a triangular decomposition on $\widehat{\mathfrak{u}}(V)$, where $\widehat{\mathfrak{u}}^{\pm}(V)$ is canonically identified with $\mathfrak{u}^{\pm}(V)$, and the zero part is $\widehat{\mathfrak{u}}^0(V) \simeq \mathbf{k}(\mathbb{Z}^{2\theta}/\Gamma(V))$. The algebra $\widehat{\mathfrak{u}}(V)$ can be seen as a quantum double of the bosonizations of $\mathfrak{u}^{\pm}(V)$ with $\mathbf{k}(\mathbb{Z}^{\theta}/ < K^{N_\beta \beta} - 1 : \beta \in \mathcal{O}(V) >)$. We have an extension of Hopf algebras [AD]:

$$\mathbf{k} \to Z(V) \hookrightarrow U(V) \twoheadrightarrow \widehat{\mathfrak{u}}(V) \to \mathbf{k},$$

which generalizes the case of quantum groups $U_q(\mathfrak{g})$ at roots of unity and the corresponding small quantum groups $\mathfrak{u}_q(\mathfrak{g})$ obtained as a quotient.

Remark 4.17. The graded dual $\mathcal{L}(V) = U^+(V)^*$ is a braided Hopf algebra containing a copy of $\mathfrak{u}^+(V)^* \simeq \mathfrak{u}^+(V)$; moreover is a finite module over this subalgebra. $\mathcal{L}(V)$ was called the *Lusztig algebra* [AAGTV] and resembles the q-divided power algebra [L2].

5. SOME EXAMPLES

5.1. **Braidings of super type** A. We apply previous results to braidings of super type A, see [AAY]. Fix $N \in \mathbb{N}$, $q \in \mathbf{k}$ a root of unity of order $N \geq 3$. There exists a braiding $C(\theta, q; i_1, \dots, i_j)$ of super type A_θ for each ordered subset $1 \leq i_1 < \cdots < i_k \leq \theta$. Its matrix $(q_{ij})_{1 \leq i,j \leq \theta}$ satisfies

- $q_{ij}q_{ji} = 1$ if $1 \leq i, j < \theta$, $|i - j| > 1$,
- if $i = i_\ell$ for some $1 \leq \ell \leq k$, then $q_{ii} = -1$, $q_{i-1,i}q_{i,i-1}q_{i+1,i}q_{i,i+1} = 1$,
- if $i \neq i_\ell$ for all $1 \leq \ell \leq k$, then $q_{ii}q_{i-1,i}q_{i,i-1} = q_{ii}q_{i+1,i}q_{i,i+1} = 1$,

where $q = q_{11}^2 q_{12}q_{21}$. The notation is close to the simple chains in [H2].

The positive roots corresponding to this braiding are

$$\Delta_+^V = \{\alpha_{j,k} : 1 \leq j \leq k \leq \theta\}, \qquad \text{where } \alpha_{j,k} = \alpha_j + \alpha_{j+1} + \cdots + \alpha_k.$$

Set $t_k^V = s_1^V s_2 \cdots s_k$, $w = t_\theta^V t_{\theta-1} \ldots t_1$, where we use for the t's the same convention for concatenation as for the s's. The expression of w in s_i's is reduced and this is the element of maximal length by Lemma 2.3 since

$$(47) \qquad t_\theta t_{\theta-1} \ldots t_{\theta-j+2} s_1 \ldots s_{k-j-1}(\alpha_{k-j}) = \alpha_{j,k} \qquad 1 \leq j \leq k \leq \theta.$$

Let $| | : \mathbb{Z}^\theta \to \mathbb{Z}_2$ be the group map such that $|\alpha_i| = 1$ if $i \in \{i_1, \dots, i_k\}$ and $|\alpha_i| = 0$ otherwise, which defines the parity of the roots. $\mathcal{O}(V)$ is the set of even roots, $\chi(\beta, \beta) = q^{\pm 1}$ if $\beta \in \mathcal{O}(V)$, and $\chi(\beta, \beta) = -1$ otherwise.

The algebra $U^+(V)$ is presented by generators E_i, $1 \leq i \leq \theta$, and relations

$$(48) \qquad E_i E_j = q_{ij} E_j E_i, \qquad\qquad\qquad j - i \geq 2,$$

$$(49) \qquad [(\mathrm{ad}_c E_{i-1})(\mathrm{ad}_c E_i)E_{i+1}, E_i]_c, \quad E_i^2, \qquad i \in \{i_1, \dots, i_k\},$$

$$(50) \qquad (\mathrm{ad}_c E_i)^2 E_{i\pm 1}, \qquad\qquad\qquad i \notin \{i_1, \dots, i_k\}.$$

Let $E_{j,k} = (\mathrm{ad}_c E_j) \cdots (\mathrm{ad}_c E_{k-1})E_k = T_j \cdots T_{k-1}(E_k)$, $1 \leq j \leq k \leq \theta$. They are the generators of the PBW basis corresponding to the previous expression of w, ordered lexicographically:

$$\alpha_{1,1} < \alpha_{1,2} < \cdots < \alpha_{1,\theta} < \alpha_{2,2} < \alpha_{2,3} < \cdots < \alpha_{\theta,\theta}.$$

By Remark 3.5 $E_{j,k}^2 = 0$ for each odd root $\alpha_{j,k}$.

The computation of $\underline{\Delta}(E_{j,k})$ follows as for braidings of Cartan type with matrix A_θ, see [AS1, Lemma 6.5].

Lemma 5.1. *Let $1 \le j \le k \le \theta$. Then*

$$(51) \qquad \underline{\Delta}(E_{j,k}) = E_{j,k} \otimes 1 + 1 \otimes E_{j,k} + \sum_{j \le \ell < k} (1 - \widetilde{q_{l,l+1}}) E_{j,\ell} \otimes E_{\ell+1,k}.$$

Proof. The proof is by induction on $k - j$, the case $k = j$ is trivial. Assume that (51) holds for $j+1, k$. Notice that

$$\chi(\alpha_j, \alpha_{j+1,k})\chi(\alpha_{j+1,k}, \alpha_j) = \widetilde{q_{j,j+1}}, \quad \chi(\alpha_j, \alpha_{\ell+1,k})\chi(\alpha_{\ell+1,k}, \alpha_j) = 1 \text{ if } \ell > j.$$

Then we compute:

$$\underline{\Delta}(E_{j,k}) = \underline{\Delta}(E_j)\underline{\Delta}(E_{j+1,k}) - \chi(\alpha_j, \alpha_{j+1,k})\underline{\Delta}(E_{j+1,k})\underline{\Delta}(E_j)$$
$$= E_{j,k} \otimes 1 + 1 \otimes E_{j,k} + (1 - \widetilde{q_{j,j+1}})E_j \otimes E_{j+1,\ell}$$
$$+ \sum_{j+1 \le \ell < k} (1 - \widetilde{q_{l,l+1}}) E_{j,\ell} \otimes E_{\ell+1,k}$$
$$+ \sum_{j+1 \le \ell < k} (1 - \widetilde{q_{l,l+1}})\chi(\alpha_j, \alpha_{j+1,k})E_{j+1,\ell} \otimes [E_j, E_{\ell+1,k}]_c,$$

From (48), $[E_j, E_{\ell+1,k}]_c = 0$ if $\ell > j$, which completes the proof. $\qquad\square$

We compute now $\underline{\Delta}(E_{j,k}^N)$ for each even root $\alpha_{j,k}$. Compare this with the case of Cartan braidings of type A [AS1, Lemma 6.9].

Proposition 5.2. *Let $1 \le j \le k \le \theta$ be such that $\alpha_{j,k} \in \mathcal{O}(V)$. Then*

$$\underline{\Delta}(E_{j,k}^N) = E_{j,k}^N \otimes 1 + 1 \otimes E_{j,k}^N$$
$$(52) \qquad + \sum_{\ell:\, j \le \ell < k,\, \alpha_{j,\ell} \in \mathcal{O}(V)} (1 - \widetilde{q_{\ell,\ell+1}})^N \chi(\alpha_{j,\ell}, \alpha_{\ell+1,k})^{\frac{N(N-1)}{2}} E_{j,\ell}^N \otimes E_{\ell+1,k}^N.$$

Proof. By induction on $k - j$. The case $k = j$ is trivial. Assume that

$$\underline{\Delta}(E_{j+1,k}^N) = E_{j+1,k}^N \otimes 1 + 1 \otimes E_{j+1,k}^N$$
$$+ \sum_{\substack{\ell:\, j+1 \le \ell < k, \\ \alpha_{j+1,\ell} \in \mathcal{O}(\rho_j(V))}} (1 - \widetilde{q_{\ell,\ell+1}})^N s_j^* \chi(\alpha_{j+1,\ell}, \alpha_{\ell+1,k})^{\frac{N(N-1)}{2}} E_{j+1,\ell}^N \otimes E_{\ell+1,k}^N.$$

Then $\Delta_j(E_{j+1,k}^N) = \underline{\Delta}(E_{j+1,k}^N)$ since $(\underline{\Delta} \otimes \mathrm{id})\underline{\Delta}(E_{j+1,k}^N)$ does not have terms E_j^n, $n \in \mathbb{N}$. If $\ell > j$, then $\widetilde{q_{\ell,\ell+1}} = \widetilde{q_{\ell,\ell+1}}$, and

- $\alpha_{j+1,\ell} \in \mathcal{O}(\rho_j(V))$ if and only if $s_j(\alpha_{j+1,\ell}) = \alpha_{j,\ell} \in \mathcal{O}(V)$,
- $s_j^* \chi(\alpha_{j+1,\ell}, \alpha_{\ell+1,k}) = \chi(s_j(\alpha_{j+1,\ell}), s_j(\alpha_{\ell+1,k})) = \chi(\alpha_{j,\ell}, \alpha_{\ell+1,k})$.

We apply now Theorem 4.8:

$$\underline{\Delta}(E_{j,k}^N) = \underline{\Delta} \circ T_j(E_{j+1,k}^N) = \mathfrak{R}_i^{\rho_j(V)} \circ (T_j \otimes T_j) \circ \Delta_j(E_{j+1,k}^N)$$

$$= \mathfrak{R}_i^{\rho_j(V)} \left(E_{j,k}^N \otimes 1 + 1 \otimes E_{j,k}^N \right)$$

$$+ \sum_{\substack{\ell: j+1 \leq \ell < k, \\ \alpha_{j,\ell} \in \mathcal{O}(V)}} (1 - \widetilde{q_{\ell,\ell+1}})^N \chi(\alpha_{j,\ell}, \alpha_{\ell+1,k})^{\frac{N(N-1)}{2}} \mathfrak{R}_i^{\rho_j(V)} \left(E_{j,\ell}^N \otimes E_{\ell+1,k}^N \right).$$

$\mathfrak{R}_i^{\rho_j(V)} \left(E_{j,\ell}^N \otimes E_{\ell+1,k}^N \right) = \left(E_{j,\ell}^N \otimes E_{\ell+1,k}^N \right)$ since $E_{\ell+1,k}^N \triangleleft \mathfrak{E}_j^{(n)} = 0$ for all $n \geq 1$, $\ell > j$. Also $\mathfrak{R}_i^{\rho_j(V)} \left(E_{j,k}^N \otimes 1 \right) = E_{j,k}^N \otimes 1$. By Theorem 4.13 it suffices to compute $E_{j,k}^N \triangleleft \mathfrak{E}_j^{(nN)}$. As $\underline{\Delta}$ is $\mathbb{N}_0$-graded, $E_{j,k}^N \triangleleft \mathfrak{E}_j^{(nN)} = 0$ if $n > 1$, so we compute X_N, where $E_j^N \otimes X_n$ is a summand in the expression of $\underline{\Delta}(E_{j,k}^N) = \underline{\Delta}(E_{j,k})^N$. By Lemma 5.1,

$$E_j^N \otimes X_n = (1 - \widetilde{q_{j,j+1}})^N (E_j \otimes E_{j+1,k})^N$$

$$= (1 - \widetilde{q_{j,j+1}})^N \chi(\alpha_j, \alpha_{j+1,k})^{\frac{N(N-1)}{2}} E_j^N \otimes E_{j+1,k}^N,$$

so we have

$$\mathfrak{R}_i^{\rho_j(V)} \left(1 \otimes E_{j,k}^N \right) = 1 \otimes E_{j,k}^N + (1 - \widetilde{q_{j,j+1}})^N \chi(\alpha_j, \alpha_{j+1,k})^{\frac{N(N-1)}{2}} E_j^N \otimes E_{j+1,k}^N,$$

which completes the proof. $\square$

5.2. **Braidings of type** $\mathfrak{br}(2;5)$. Now fix $\theta = 2$, ζ a root of unity of order 5, $(q_{ij})_{i,j\in\mathbb{I}}$, $(r_{ij})_{i,j\in\mathbb{I}}$ two matrices such that

$$q_{11} = \zeta, \quad \widetilde{q_{12}} = \zeta^2, \quad q_{22} = r_{22} = -1, \quad r_{11} = -\zeta^3, \quad \widetilde{r_{12}} = \zeta^3.$$

Let $\chi, \psi : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbf{k}^\times$ be the bicharacters defined by $(q_{ij})_{i,j\in\mathbb{I}}$, $(r_{ij})_{i,j\in\mathbb{I}}$, respectively, and V, W the corresponding braided vector spaces. They correspond to [H2, Table 1, row 13] and are related with the Lie superalgebra $\mathfrak{br}(2;5)$ over a field of characteristic 5 [AA]. Their generalized Cartan matrices are $C^V = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$, $C^W = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$, and $\rho_1(V) = V$, $\rho_1(W) = W$, $\rho_2(V) = W$. The elements of maximal length are $(\sigma_1^V \sigma_2)^4$, $(\sigma_1^W \sigma_2)^4$, and the positive roots Δ_+^V, Δ_+^W are, respectively,

$$\{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 5\alpha_1 + 3\alpha_2, 3\alpha_1 + 2\alpha_2, 4\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2, \alpha_2\},$$

$$\{\alpha_1, 4\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 5\alpha_1 + 2\alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2, \alpha_2\}.$$

The sets of Cartan roots $\mathcal{O}(V)$, $\mathcal{O}(W)$ are, respectively,

$$\{\alpha_1, 2\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2, \alpha_1 + \alpha_2\}, \quad \{\alpha_1, 3\alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, \alpha_1 + \alpha_2\}.$$

The algebra $U^+(V)$ is presented by generators E_1, E_2 and relations

$$E_2^2, \quad (\mathrm{ad}_c E_1)^4 E_2, \quad [E_{1112}, E_{112}]_c, \quad [E_{4\alpha_1 + 3\alpha_2}, E_{12}]_c.$$

The algebra $U^+(W)$ is presented by generators $\underline{E}_1, \underline{E}_2$ and relations

$$\underline{E}_2^2, \quad (\mathrm{ad}_c \underline{E}_1)^5 \underline{E}_2, \quad \left[\underline{E}_1, \underline{E}_{3\alpha_1+2\alpha_2}\right]_c + r_{12}\underline{E}_{2\alpha_1+\alpha_2}^2, \quad \left[\underline{E}_{3\alpha_1+2\alpha_2}, \underline{E}_{12}\right]_c.$$

Here $E_{i_1 \cdots i_k j} = (\mathrm{ad}_c E_{i_1}) \ldots (\mathrm{ad}_c E_{i_k}) E_j$, and analogous notation with $\underline{E}$.

Lemma 5.3. *The root vectors of $\mathcal{O}(V)$, $\mathcal{O}(W)$ are, respectively,*

$$E_{\alpha_1} = E_1, \qquad\qquad\qquad E_{3\alpha_1+2\alpha_2} = q_{12}^3 \zeta^3 (1-\zeta^3)^5 (1+\zeta)[E_{112}, E_{12}]_c,$$
$$E_{2\alpha_1+\alpha_2} = q_{21}(1-\zeta^3)E_{112}, \qquad E_{\alpha_1+\alpha_2} = q_{21}^2 \zeta(1-\zeta^3)^7(1+\zeta)^3 E_{12},$$
$$\underline{E}_{\alpha_1} = \underline{E}_1, \qquad\qquad\qquad \underline{E}_{2\alpha_1+\alpha_2} = -r_{12}^2 \zeta^2 (1+\zeta)^2 (1-\zeta^3)^4 \underline{E}_{112},$$
$$\underline{E}_{3\alpha_1+\alpha_2} = r_{21}(1-\zeta^2)\underline{E}_{1112}, \qquad \underline{E}_{\alpha_1+\alpha_2} = r_{12}^2 \zeta(1+\zeta)^3 (1-\zeta^3)^{10} \underline{E}_{12}.$$

Proof. First we compute

$$E_{2\alpha_1+\alpha_2} = T_1^V T_2(\underline{E}_1) = T_1^V(E_{21}) = -q_{21}\zeta \partial_1^L(E_{1112}) = q_{21}(1-\zeta^3)E_{112}.$$

A similar computation gives the expression of $\underline{E}_{3\alpha_1+\alpha_2}$. Using this,

$$\begin{aligned} E_{3\alpha_1+2\alpha_2} &= T_1^V T_2(\underline{E}_{3\alpha_1+\alpha_2}) = (1-\zeta^2)^2 T_1^V([[E_{21}, [E_{21}, E_1]_c]_c) \\ &= [-L_1 \cdot \partial_1^L(E_{112}), L_1 \cdot \partial_1^L(L_1 \cdot \partial_1^L(E_{112}))]_c \\ &= q_{12}^3 \zeta^3 (1-\zeta^3)^5 (1+\zeta)[E_{112}, E_{12}]_c. \end{aligned}$$

The proof for the remaining root vectors is analogous by iterating compositions of $T_1^V T_2$ and $T_1^W T_2$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Box$

Now we compute the coproduct of powers of root vectors.

Lemma 5.4. $E_{\alpha_1}^5, E_{\alpha_1+\alpha_2}^{10}, \underline{E}_{\alpha_1}^{10}, \underline{E}_{\alpha_1+\alpha_2}^5$ *are primitive, and*

$$\underline{\Delta}(E_{2\alpha_1+\alpha_2}^{10}) = E_{2\alpha_1+\alpha_2}^{10} \otimes 1 + 1 \otimes E_{2\alpha_1+\alpha_2}^{10} + \frac{q_{21}^{30} E_{\alpha_1}^{10} \otimes E_{\alpha_1+\alpha_2}^{10}}{(1-\zeta^3)^{55}(1+\zeta)^{25}}$$
$$+ \frac{q_{21}^{30}}{(1-\zeta^3)^{10}} E_{\alpha_1}^5 \otimes E_{3\alpha_1+2\alpha_2}^5,$$

$$\underline{\Delta}(E_{3\alpha_1+2\alpha_2}^5) = E_{3\alpha_1+2\alpha_2}^5 \otimes 1 + 1 \otimes E_{3\alpha_1+2\alpha_2}^5 + \frac{q_{12}^{15} E_{\alpha_1}^5 \otimes E_{\alpha_1+\alpha_2}^{10}}{(1-\zeta^3)^{40}(1+\zeta)^{15}},$$

$$\underline{\Delta}(\underline{E}_{3\alpha_1+\alpha_2}^5) = \underline{E}_{3\alpha_1+\alpha_2}^5 \otimes 1 + 1 \otimes \underline{E}_{3\alpha_1+\alpha_2}^5 + \frac{r_{21}^{35} \zeta^2 \underline{E}_{\alpha_1}^{10} \otimes \underline{E}_{\alpha_1+\alpha_2}^5}{(1-\zeta^3)^{40}(1+\zeta)^5},$$

$$\underline{\Delta}(\underline{E}_{2\alpha_1+\alpha_2}^{10}) = \underline{E}_{2\alpha_1+\alpha_2}^{10} \otimes 1 + 1 \otimes \underline{E}_{2\alpha_1+\alpha_2}^{10} - \frac{r_{21}^{45} \underline{E}_{\alpha_1}^{10} \otimes \underline{E}_{\alpha_1+\alpha_2}^{10}}{(1-\zeta^3)^{50}}$$
$$- \frac{r_{12}^5 (1+\zeta)^{10}}{(1-\zeta^3)^{10}} \underline{E}_{3\alpha_1+\alpha_2}^5 \otimes \underline{E}_{\alpha_1+\alpha_2}^5.$$

Proof. We know that $E_{\alpha_1}^5$, $\underline{E}_{\alpha_1}^{10}$ are primitive. Now $E_{\alpha_1+\alpha_2}^{10}$, $\underline{E}_{\alpha_1+\alpha_2}^5$ are also primitive since $\underline{\Delta}$ is $\mathbb{N}_0^\theta$-graded and $Z^+(V)$ is a Hopf subalgebra by Theorem 4.13. Also $\underline{\Delta}(E_{3\alpha_1+2\alpha_2}^5)$ is the sum of the two terms $\underline{E}_{3\alpha_1+\alpha_2}^5 \otimes 1$, $1 \otimes \underline{E}_{3\alpha_1+\alpha_2}^5$ plus $c\,E_{\alpha_1}^5 \otimes E_{\alpha_1+\alpha_2}^{10}$ for some $c \in \mathbf{k}$ since the first term of the

coproduct should be a power of previous roots by Proposition 4.9 and then we use again Theorem 4.13. As

$$\underline{\Delta}([E_{112}, E_{12}]_c^5) = (1 \otimes [E_{112}, E_{12}]_c + E_1 \otimes \partial_1^L([E_{112}, E_{12}]_c) + ...)^5,$$

and $\partial_1^L([E_{112}, E_{12}]_c) = -\zeta^3(1 - \zeta^3)(1 + \zeta)^2 E_{12}$, such c is computed from $(E_1 \otimes E_{12}^2)^5 = q_{21}^{10} E_1^5 \otimes E_{12}^{20}$ using Lemma 5.3. Similarly $\underline{\Delta}(E_{2\alpha_1+\alpha_2}^{10})$ can have two extra summands $aE_{\alpha_1}^{10} \otimes E_{\alpha_1+\alpha_2}^{10}$ and $bE_{\alpha_1}^5 \otimes E_{3\alpha_1+2\alpha_2}^5$. As $\partial_i^L(E_{112}) = (1 + \zeta)(1 - \zeta^3)E_{12}$, the first appears from

$$(E_1 \otimes E_{12})^{10} = q_{21}^{45} E_1^{10} \otimes E_{12},$$

while the second appears from the product

$$((1 \otimes E_{112})(E_1 \otimes E_{12}))^5 = q_{21}^{15} E_1^5 \otimes (E_{112} E_{12})^5,$$

The calculus of coproducts of the other two expressions is similar. $\qquad\square$

Remark 5.5. The coproduct of the power root vectors in Lemma 5.4 seems close to the algebra of functions of B_2, and the Lie algebra $\mathfrak{g}_0$ corresponding to the Lie superalgebra $\mathfrak{br}(2; 5)$ in characteristic 5 is of type B_2. The PBW bases of $U(V)$ and $\mathfrak{u}(V)$ resemble the PBW bases of the enveloping and the restricted enveloping algebras of the Lie superalgebra of $\mathfrak{br}(2; 5)$ respectively, since the generators have the same degree and the corresponding height.

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FAMAF-CIEM (CONICET), UNIVERSIDAD NACIONAL DE CÓRDOBA, MEDINA ALLENDE S/N, CIUDAD UNIVERSITARIA, 5000 CÓRDOBA, REPÚBLICA ARGENTINA.

E-mail address: angiono@famaf.unc.edu.ar