Comment on ‘Lorentz transformations and the wave equation’

2017 Eur. J. Phys. 38 018001

(http://iopscience.iop.org/0143-0807/38/1/018001)

IOP Publishing European Journal of Physics

Eur. J. Phys. **38** (2017) 018001 (4pp) doi:10.1088/0143-0807/38/1/018001

Comment

Comment on ‘Lorentz transformations and the wave equation’

Héctor O Di Rocco

Facultad de Ciencias Exactas, Universidad Nacional del Centro and IFAS-CIFICEN (CONICET-UNCPBA), Pinto 399, 7000 Tandil, Argentina

Received 13 April 2016, revised 9 September 2016
Accepted for publication 17 October 2016
Published 9 November 2016

Abstract

In this comment we make some clarifications with respect to certain asumptions and demands required by Ricardo Heras in his paper entitled ‘Lorentz transformations and the wave equation’ (2016 *Eur. J. Phys.* **37** 025603).

1. Introduction

In [1] the author obtains the Lorentz transformations (LT) from the form invariance of the wave equation. His method consisted in finding the LT after finding the transformations laws for operators $\partial/\partial x$ and $\partial/\partial t$ in terms of operators $\partial/\partial x'$ and $\partial/\partial t'$. Heras is competent in his job and therefore this comment does not indicate errors in his paper. Our remarks are related to two words used in his text: assuming and demand. Let us examine the first one: *assuming*. Immediately before his equation (3), Heras wrote: ‘By *assuming* linearity for the involved transformations of operators, we can write...’. But, is it possible to assume other transformation law? Can the undergraduate think that the assumed linearity is, in a way, arbitrary? It must be clear that the LT between the pairs (x, t) and (x', ct') *must be* linear. This was properly emphasized by Einstein [2] and disclosed by Resnick [3]. Effectively, the length of a rod cannot depend on its position in space. Similarly, the time interval between two events cannot depend on the number indicated by the hands of the watch. An alternative paragraph could be: ‘Considering the *neccesary linearity* for the involved transformations of operators, we write...’.

The other concept is *demand*. To determine the value of his factor A (which we call A^{Heras} to avoid confusion with our proper notation below), Heras writes: ‘we *demand* ... should appropriately reduce to the corresponding Galilean transformation $\partial/\partial t = \partial/\partial t' - v\partial/\partial x'$. From this... if $\partial/\partial t = 0$ then $\partial/\partial t' = v\partial/\partial x'$’. Although this is plausible from the point of view of physics, this assumption might seem necessary. It must be clear that it is not necessary to demand the Galilean limit, because $\partial/\partial t' = v\partial/\partial x'$ every time $\partial/\partial t = 0$, as we will see below.

In this work we start from the fact that transformations between pairs (x, t) and (x', ct') *must be linear*, and with no other assumptions we arrive at the LT. Consider the standard configuration in which two intertial frames S and S' are in relative motion with speed v along direction xx', as in [1]. Additionally, the transverse coordinates satisfy $y = y'$, $z = z'$.

The first of the equations, $x' = Ax + Bt \equiv A[x + (B/A)t]$, implies that $B = -vA$. Indeed, for an observer located at the origin of S', the proposition $x' = 0$ must be identical to the proposition $x = vt$ because for observers in S the origins of the two frames are separated by a distance vt [3]. Then

$$x' = A[x - vt], \quad t' = Cx + Dt \tag{1}$$

with the inverse transformations

$$x = \frac{Dx' + vAt'}{A(D + vC)}, \quad t = \frac{-Cx' + At'}{A(D + vC)}. \tag{2}$$

As we know, the Galilean and Lorentzian values are:

	Galilean	*Lorentzian*
A	1	γ
B	$-v$	$-\gamma v$
C	0	$-\gamma v/c^2$
D	1	γ

(3)

with

$$\gamma = \frac{1}{\sqrt{1 - (v/c)^2}}, \tag{4}$$

but this is not relevant at this point since all the work will be made in terms of A, C and D.

To correlate the quantities in the system S with those of the system S' we transform the operators of the unidimensional wave equation

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} \tag{5}$$

in the form

$$\partial/\partial x = A\partial/\partial x' + C\partial/\partial t', \tag{6}$$

and

$$\partial/\partial t = -vA\partial/\partial x' + D\partial/\partial t' \tag{7}$$

so that

$$\frac{\partial^2}{\partial x^2} = A^2\frac{\partial^2}{\partial x'^2} + 2AC\frac{\partial^2}{\partial x'\partial t'} + C^2\frac{\partial^2}{\partial t'^2} \tag{8}$$

and

$$\frac{\partial^2}{\partial t^2} = v^2A^2\frac{\partial^2}{\partial x'^2} - 2vAD\frac{\partial^2}{\partial x'\partial t'} + D^2\frac{\partial^2}{\partial t'^2}. \tag{9}$$

Replacing the right-hand side of equations (8) and (9) in equation (5), it results that, in general

$$\left(A^2 - \frac{v^2A^2}{c^2}\right)\frac{\partial^2\Psi}{\partial x'^2} + 2\left(AC + \frac{vAD}{c^2}\right)\frac{\partial^2\Psi}{\partial x'\partial t'} = \left(\frac{D^2}{c^2} - C^2\right)\frac{\partial^2\Psi}{\partial t'^2}. \tag{10}$$

If now we assume the postulate of relativity (form invariance for the wave equation), then the second term of the left-hand side must be canceled and, necessarily

$$C = \frac{-vD}{c^2}. \tag{11}$$

The condition given by equation (11) is impossible for the Galilean transformations because $C = 0$ but $D \neq 0$ (see the table (3)). Replacing equation (11) in equation (10), and canceling the common factor $(1 - (v/c)^2)$, we obtain

$$A^2 \frac{\partial^2 \Psi}{\partial x'^2} = \frac{D^2}{c^2} \frac{\partial^2 \Psi}{\partial t'^2};$$

and requiring the form invariance $\partial^2\Psi/\partial x'^2 = (1/c^2)\partial^2\Psi/\partial t'^2$ (postulate of the constance of the speed of light), $D = A$ and $C = -vA/c^2$. Therefore, it is clear that in equation (7) when $\partial/\partial t = 0$ then $\partial/\partial t' = v\partial/\partial x'$ with no assumption of a Galilean limit.

With the values of C and D in terms of A, equations (1) take the form

$$x' = A(x - vt) \tag{12}$$

and

$$t' = A\left(-\frac{v}{c^2}x + t\right) \tag{13}$$

and (2)

$$x = \frac{(x' + vt')}{A(1 - v^2/c^2)} \tag{14}$$

and

$$t = \frac{\left(\frac{v}{c^2}x' + t'\right)}{A(1 - v^2/c^2)}. \tag{15}$$

To keep the form between pairs x' and x as well as between t' and t, it is required that

$$A = \frac{1}{A(1 - v^2/c^2)}$$

so

$$A = \frac{1}{\sqrt{1 - v^2/c^2}} = \gamma, \tag{16}$$

and therefore $D = \gamma$ and $C = -\gamma v/c^2$, as shown in the table in (3).

If the above argument (keep the form) looks a little artificial to the reader, we can calculate the difference in the coordinates of two events corresponding to the ends of a rigid rule, measured simultaneously in the system S (therefore $t_1 \equiv t_2$):

$$x_2' - x_1' = A(x_2 - x_1)$$

whereas for the analogous situation in S' (therefore $t_1' \equiv t_2'$):

$$x_2 - x_1 = \frac{x_2' - x_1'}{A(1 - v^2/c^2)}.$$

Then, if in each system, the lengths $L = x_2 - x_1$ (in S) and $L' = x_2' - x_1'$ (in S') are equal (i.e.: $1m$), the length contraction is reciprocal when $A = \gamma$.

Finally, it is interesting to note that the expresion for A^{Heras}, which with the known notation $\beta = v/c$, is

$$A^{\mathrm{Heras}} = \gamma(1 + \beta) = \sqrt{\frac{1 + \beta}{1 - \beta}}.$$

In addition to having been obtained by Parker and Schmieg, as mentioned by Heras in his [5], this was also obtained by Moriconi [4] and Di Rocco [5] in their treatments of special relativity using the frequency as the essential concept, without using the LT, which are obtained *a posteriori.*

Acknowledgments

Above all, the author acknowledges the generous exchange of ideas with his younger colleague Ricardo Heras (Department of Physics and Astronomy, University College London). Additionally, the author is deeply grateful for the support of the Universidad Nacional del Centro and the Consejo Nacional de Investigaciones Científicas y Técnicas (all from Argentina). He is also very grateful for the assistance offered by Mariana Di Rocco in the translation of the paper into English.

References

[1] Heras R 2016 Lorentz transformations and the wave equation *Eur. J. Phys.* **37** 025603
[2] Einstein A 1953 *The Meaning of Relativity* (Princeton, NJ: Princeton University Press)
[3] Resnick R 1968 *Introduction to Special Relativity* (New York: Wiley)
[4] Moriconi M 2006 Special theory of relativity through the Doppler effect *Eur. J. Phys* **27** 1409–23
[5] Di Rocco H O 2009 Entendiendo la relatividad especial usando la frecuencia como concepto esencial *Rev. Mex. Física* E **55** 92–6 (http://rmf.smf.mx/page/rmf-e)