Remarks on general monotonic neighbourhood frames

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In this paper we shall discuss some classes of general monotonic neighbourhood frames, or general m-frames. We shall study the classes of point-compact, image compact and replete general m-frames, and the relationships between them. The variety of Boolean algebras with a monotonic modal operator is dually equivalent to two classes of descriptive general m-frames. In this paper we shall clarify this phenomenon showing that there exists a bijective correspondence between these two classes. We shall also prove that the notions of point-compact, and image-compact monotonic frames are preserved by strong bounded morphisms. Also, we will prove some preservation results on general subframes.

Key words: monotonic modal logic, multirelations, neighbourhood frames, lower topology, descriptive monotonic frames.

A monotonic modal algebra is a pair $\langle A, \Diamond \rangle$ where A is a Boolean algebra and $\Diamond$ is a function defined on A such that $\Diamond a \vee \Diamond b \leq \Diamond (a \vee b)$ for all $a, b \in A$. If the operator $\Diamond$ satisfies the identity $\Diamond 0 = 0$, then $\Diamond$ is called a *normal* monotonic operator. Algebras of this type provide algebraic semantics for monotonic modal logics. On the other hand, the monotonic neighbourhood frames are the relational semantics for monotonic modal logics. A

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neighbourhood frame is a structure $\mathcal{F} = \langle X, R \rangle$ where X is a set and R is a multirelation defined on X (see [15]), i.e., R is a relation between X and $\mathcal{P}(X)$. A neighbourhood frame $\mathcal{F}$ is *monotonic* if the set $R(x)$ is closed under supersets for each $x \in X$. If $\mathcal{F}$ is a monotonic neighbourhood frame, then $\langle \mathcal{P}(X), \Diamond_R \rangle$ is a monotonic modal algebra, where the monotonic operator $\Diamond_R$ is defined as $\Diamond_R(U) = \{x \in X \mid \exists Y \in R(x) \text{ such that } Y \subseteq U\}$. It is well known that given a monotonic modal algebra A, there exists a monotonic neighbourhood frame $\mathcal{F} = \langle X, R \rangle$ such that A is isomorphic to a subalgebra of $\langle \mathcal{P}(X), \Diamond_R \rangle$. This representation by means of neighbourhood frames and a topological duality is developed in [10] and [3]. A discrete duality between Boolean algebras with monotone operators and a class of relational structures endowed with multirelations is developed in [5].

In [10], the topological duality is based on the called descriptive monotonic frames. In [3] the duality is based on a more restricted class of descriptive monotonic frames. One of the main objectives of this paper is to analyze the relationship between these two kinds of general monotonic frames, and study some special classes of general frames that are generalizations of the notions of modally saturated monotonic models studied in [2].

The paper is organized in the following fashion. In Section 2 we will recall the principal results of the relational semantics and the algebraic semantics for monotonic modal logics. In Section 3 we will study some special classes of monotonic general frames, like image-compact, point-closed, replete and modally saturated monotonic frames. These notions were first studied by Goldblatt in [6] in the context of Kripke general frames. We will prove that given a general monotonic frame $\langle \mathcal{F}, D \rangle$ such that its underlying topological space $\langle X, \mathcal{T}_D \rangle$ is compact, then $\langle \mathcal{F}, D \rangle$ is replete if and only if it is point-compact. This is a generalization of a similar result is valid for Kripke models (see the Section Topological meaning of $\mathcal{H}$-Closure in [7]). In this section we shall also analyze the relationship between the definition of descriptive general frames given by H. Hansen in [10] and the definition given in [3]. We will see that there exists a bijective correspondence between descriptive monotonic frames and restricted descriptive m-frames.

In [2] it was proved that the concepts of compact and point-compact models are preserved by surjective bounded morphisms between monotonic models. In Section 4 we will extend these results proving that the notions of point-compact and image-compact monotonic frames are preserved by strong bounded morphisms. Also, we will prove some preservation results of general subframes.

1 PRELIMINARIES

Given a set X, we denote by $\mathcal{P}(X)$ the powerset of X, and for a subset Y of X, we write Y^c to denote the complement $X-Y$ of Y in X. Let us recall that a topological *basis* is a collection $D \subseteq \mathcal{P}(X)$ of subsets of a set X such that (1) $\emptyset \in D$, (2) $\bigcup D = X$ and (3) for all $U, V \in D$ and for each $x \in U \cap V$, there exists $W \in D$ such that $x \in W$ and $W \subseteq U \cap V$. A topological basis D generates a topology on X that we will denote by $\mathcal{T}_D$. From now on, let the term *space* stand for a topological space $\langle X, \mathcal{T}_D \rangle$, where the topological basis D is a subalgebra of the Boolean algebra of $\mathcal{P}(X)$. In this case, the elements of D are clopen (closed and open) subsets of X, because D is a Boolean algebra, but an arbitrary clopen set does not need to be an element of D. Given a space $\langle X, \mathcal{T}_D \rangle$ and $Y \subseteq X$, we will use the notation $\mathrm{cl}_D(Y)$, or $\mathrm{cl}(Y)$, to express the *closure* of Y. The set of all closed subsets (compact subsets) of $\langle X, \mathcal{T}_D \rangle$ will be denoted by $\mathcal{C}(X)$ ($\mathcal{K}(X)$). We note that $\mathcal{C}(X)$ and $\mathcal{K}(X)$ are posets under the inclusion relation.

Recall that if $\langle X, \mathcal{T} \rangle$ is a topological space and Y is a subset of X then the family $\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$ of subsets of Y is a topology on Y called the *relative topology* and the topological space $\langle Y, \mathcal{T}_Y \rangle$ is a *subspace* of $\langle X, \mathcal{T} \rangle$.

Let A be a Boolean algebra. The lattice of filters of A will be denoted by $\mathrm{Fi}(A)$. The set of all prime filters or *ultrafilters* of A is denoted by $\mathrm{Ul}(A)$. To each Boolean algebra A we can associate a Stone space $\langle \mathrm{Ul}(A), \mathcal{T}_A \rangle$ whose points are the elements of $\mathrm{Ul}(A)$ with the topology $\mathcal{T}_A = \mathcal{T}_{\beta[A]}$ determined by the basis $\beta[A] = \{\beta(a) \mid a \in A\}$, where $\beta(a) = \{x \in \mathrm{Ul}(A) \mid a \in x\}$.

Some topological properties of a space $\langle X, \mathcal{T}_D \rangle$ can be characterized in terms of the map

$$\varepsilon_D : X \to \mathrm{Ul}(D)$$

defined by $\varepsilon_D(x) = \{U \in D \mid x \in U\}$. For instance,

1. $\langle X, \mathcal{T}_D \rangle$ is Hausdorff iff ε_D is injective, and
2. $\langle X, \mathcal{T}_D \rangle$ is compact iff ε_D is surjective.

A *Stone space*, also called a Boolean space, is a topological space $\langle X, \mathcal{T}_D \rangle$ that is zero-dimensional, T_0 and compact. Equivalently, a Stone space is a totally disconnected and compact space. We note that every Stone space is Hausdorff (see [1] for more details).

If $\langle X, \mathcal{T}_D \rangle$ is a Stone space, then the map ε_D is an homeomorphism between $\langle X, \mathcal{T}_D \rangle$ and $\langle \mathrm{Ul}(D), \mathcal{T}_{\beta[D]} \rangle$. If A is a Boolean algebra, then $A \cong \beta[A]$, by means of the map β. Moreover, it is known that the map $F \to$

$\hat{F} = \{x \in \mathrm{Ul}(A) \mid F \subseteq x\}$ establishes a bijective correspondence between the lattice $\mathrm{Fi}(A)$ of all filters of A and the lattice $\mathcal{C}(\mathrm{Ul}(A))$ of all closed subsets of $\langle \mathrm{Ul}(A), \mathcal{T}_A \rangle$.

Definition 1.1. Let $\langle X, \mathcal{T}_D \rangle$ be a space and let $\mathcal{K} \subseteq \mathcal{P}(X)$. The *lower topology* $\mathcal{L}_D$ on $\mathcal{K}$ is the topology defined on $\mathcal{K}$ taking as sub-basis the collection of all sets of the form

$$L_U = \{Y \in \mathcal{K} \mid Y \cap U \neq \emptyset\}$$

where $U \in D$. The pair $\mathcal{K} = \langle \mathcal{K}, \mathcal{L}_D \rangle$ is called the *lower hyperspace* of $\langle X, \mathcal{T}_D \rangle$ relative to $\mathcal{K}$.

Let $D_U = \{Y \in \mathcal{K} \mid Y \subseteq U\}$ for $U \in D$. We note that $(L_{U^c})^c = D_U$. Recall that if $\langle X, \mathcal{T}_D \rangle$ is a Stone space, then $\langle \mathcal{C}(X), \mathcal{L}_D \rangle$ is a Stone space (see [14] for the details).

2 MONOTONIC GENERAL FRAMES

The algebraic semantic for monotonic modal logics is given by the class of Boolean algebras with a monotonic operator [10].

Recall that a *monotonic algebra* is a pair $A = \langle A, \Diamond \rangle$, where A is a Boolean algebra and $\Diamond : A \to A$ is a monotonic function, i.e., if $a \leq b$ then $\Diamond a \leq \Diamond b$ for all $a, b \in A$. The monotony can be expressed by means of the equation $\Diamond a \vee \Diamond b \leq \Diamond(a \vee b)$ for all $a, b \in A$.

Definition 2.1. *A monotonic neighbourhood frame,* or monotonic frame*,* is a structure $\mathcal{F} = \langle X, R \rangle$ where $R \subseteq X \times \mathcal{P}(X)$, and R is upclosed, i.e., for any $x \in X$ and any $Y \subseteq X$, if $(x, Y) \in R$ and $Y \subseteq Z$, then $(x, Z) \in R$. In other words, the set $R(x) = \{Z \in \mathcal{P}(X) \mid (x, Z) \in R\}$ is an increasing subset of $\langle \mathcal{P}(X), \subseteq \rangle$ for each $x \in X$.

Every monotonic frame $\mathcal{F}$ gives rise to a monotonic algebra of sets in the following way.

Definition 2.2. The *monotonic algebra,* or *complex algebra,* of a monotonic frame $\mathcal{F}$ is the pair

$$A(\mathcal{F}) = \langle \mathcal{P}(X), \Diamond_R \rangle$$

where the monotonic map $\Diamond_R : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by:

$$\Diamond_R(U) = \{x \in X \mid \exists Y \in R(x)\ (Y \subseteq U)\} = \{x \in X \mid R(x) \cap D_U \neq \emptyset\}$$

for each $U \in \mathcal{P}(X)$.

Remark 2.3. Note that by monotonicity we have that

$$\Diamond_R(U) = \{x \in X \mid U \in R(x)\}\,.$$

Let $\mathcal{F}$ be a monotonic frame. The dual map $\Box_R : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by:

$$\Box_R(U) = \{x \in X \mid \forall Y \in R(x)(Y \cap U \neq \emptyset)\} = \{x \in X \mid R\,(x) \subseteq L_U\}$$

for each $U \in \mathcal{P}(X)$.

Remark 2.4. Dually, by monotonicity we have that

$$\Box_R(U) = \{x \in X \mid U^c \notin R(x)\}\,.$$

Now we shall see that each monotonic algebra gives rise to a monotonic frame.

As in Kripke semantics for normal modal logics, the main defect of neighbourhood semantics is the existence of neighbourhood incomplete logics. It can be rectified by equipping neighbourhood frames with an extra structure which restricts the set of possible valuations. This gives rise to the general monotonic frames [10].

Definition 2.5. A *general monotonic neighbourhood frame,* or general monotonic frame, is a structure $\langle \mathcal{F}, D\rangle$ where $\mathcal{F} = \langle X, R\rangle$ is a monotonic frame and D is a collection of admissible subsets of X which contains $\emptyset$ and is closed under finite unions, complementation in X and the modal operator $\Diamond_R$, i.e., $\Diamond_R(U) \in D$ for each $U \in D$.

We note that if $\langle \mathcal{F}, D\rangle$ is a general monotonic frame, then $\langle D, \Diamond_R\rangle$ is a subalgebra of the complex algebra $A(\mathcal{F}) = \langle \mathcal{P}(X), \Diamond_R\rangle$. Moreover, since D is a Boolean subalgebra of $\mathcal{P}(X)$, we have that $\langle X, \mathcal{T}_D\rangle$ is a topological space where D is a basis for the topology $\mathcal{T}_D$. We will refer to $\mathcal{F}$ as the *underlying frame* of $\langle \mathcal{F}, D\rangle$.

Remark 2.6. It is clear that a monotonic frame $\mathcal{F} = \langle X, R\rangle$ is equivalent to the general monotonic frame $\langle \mathcal{F}, \mathcal{P}(X)\rangle$.

Definition 2.7. The *monotonic frame***,** or *ultrafilter frame*, of a monotonic algebra A is a pair

$$\mathcal{F}(A) = \langle \mathrm{Ul}\,(A)\,, R_\Diamond\rangle$$

where the relation $R_\Diamond \subseteq \mathrm{Ul}\,(A) \times \mathcal{P}(\mathrm{Ul}(A))$ is defined by:

$$(x, Y) \in R_\Diamond \quad \text{iff} \quad \exists F \in \mathrm{Fi}(A)\ (\hat{F} \subseteq Y \text{ and } F \subseteq \Diamond^{-1}(x)), \tag{2.1}$$

with $\hat{F} = \{y \in \mathrm{Ul}\,(A) \mid F \subseteq y\}$.

Let A be a monotonic algebra. We note that for any $F \in \mathrm{Fi}(A)$, and for each $x \in \mathrm{Ul}(A)$,

$$(x, \hat{F}) \in R_{\Diamond} \text{ iff } F \subseteq \Diamond^{-1}(x).$$

We also note that $\Diamond_{R_{\Diamond}}(\beta(a)) = \beta(\Diamond(a))$ for all $a \in A$. Thus the map $\beta : A \to \mathcal{P}(\mathrm{Ul}(A))$ is a monomorphism of monotonic algebras (see [2], [3] or [10]). The pair

$$\langle \mathcal{F}(A), \beta[A] \rangle$$

is the general monotonic frame of A.

Definition 2.8. A *bounded morphism* f between the general monotonic frames $\langle \mathcal{F}_1, D_1 \rangle$ and $\langle \mathcal{F}_2, D_2 \rangle$ is a function $f : X_1 \to X_2$ satisfying the following conditions:

1. For all $x \in X_1$ and for every $Y \subseteq X_1$, if $(x, Y) \in R_1$, then $(f(x), f[Y]) \in R_2$.

2. For all $x \in X_1$ and for every $Z \subseteq X_2$, if $(f(x), Z) \in R_2$, then there exists $Y \subseteq X_1$ such that $(x, Y) \in R_1$ and $f[Y] \subseteq Z$.

3. $f^{-1}[U] \in D_1$ for each $U \in D_2$.

We note that by condition 3, f is a continuous function between the topological spaces $\langle X_1, \mathcal{T}_{D_1} \rangle$ and $\langle X_2, \mathcal{T}_{D_2} \rangle$. If f is surjective, then it is called a bounded *epimorphism* and $\langle \mathcal{F}_2, D_2 \rangle$ is called an homomorphic image of $\langle \mathcal{F}_1, D_1 \rangle$. We also note that the conditions 1 and 2 together are equivalent with the following condition:

For all $x \in X$ and for every $Z \subseteq X_2$, $f^{-1}[Z] \in R_1(x)$ iff $Z \in R_2(f(x))$.

We shall say that f is a *strong* bounded morphism if it satisfies the following condition:

(S) If $U \in D_1$, then there exists $V \in D_2$ such that $f[U] = f[X_1] \cap V$, i.e., $U = f^{-1}[V]$.

Definition 2.9. Let $\mathcal{F} = \langle X, R \rangle$ be a monotonic frame. A subset X_1 of X is *R-hereditary* if for every $x \in X_1$ and for every $Y \subseteq X$ such that

$(x, Y) \in R$, there exists $Z \in \mathcal{P}(X_1)$ such that $(x, Z) \in R$ and $Z \subseteq Y$.

Remark 2.10. We note that a subset X_1 of X is *R-hereditary* if and only if the inclusion map $i\colon X_1 \to X$ is a bounded morphism.

Let $\mathcal{F}_2 = \langle X_2, R_2\rangle$ be a monotonic frame. Let X_1 a subset of X_2 and consider $R_1 = R_2 \cap (X_1 \times \mathcal{P}(X_1))$. Then, it is easy to check that $\mathcal{F}_1 = \langle X_1, R_1\rangle$ is a monotonic frame.

Definition 2.11. Let $\langle \mathcal{F}_1, D_1\rangle$ and $\langle \mathcal{F}_2, D_2\rangle$ be two general m-frames where $\mathcal{F}_2 = \langle X_2, R_2\rangle$, $\mathcal{F}_1 = \langle X_1, R_1\rangle$, $X_1 \subseteq X_2$ and $R_1 = R_2 \cap (X_1 \times \mathcal{P}(X_1))$. We shall say that $\mathcal{F}_1$ is a subframe of $\mathcal{F}_2$ if:

SF1 X_1 is an R-hereditary subset of X_2.

We shall say that $\langle \mathcal{F}_1, D_1\rangle$ is a *general subframe* of $\langle \mathcal{F}_2, D_2\rangle$ if $\mathcal{F}_1$ is a subframe of $\mathcal{F}_2$ and $\mathcal{F}_1$ satisfies the following condition:

SF2 $D_1 = \{U \cap X_1 : U \in D_2\}$.

It is easy to see that if $\langle \mathcal{F}_1, D_1\rangle$ is a subframe of $\langle \mathcal{F}_2, D_2\rangle$, then by condition 2 of Definition 2.11, $\mathcal{T}_{D_1} = \{O \cap Y : O \in \mathcal{T}_{D_2}\}$ is a relative topology, and thus $\langle X_1, \mathcal{T}_{D_1}\rangle$ is a subspace of $\langle X_2, \mathcal{T}_{D_2}\rangle$. We note that if SF2 holds, the inclusion map $i\colon X_1 \to X_2$ is a strong bounded morphism.

Lemma 2.12. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two m-frames. If $\mathcal{F}_1$ is a subframe of $\mathcal{F}_2$, then for all $U \in \mathcal{P}(X_2)$:*

1. $\Box_{R_1}(U \cap X_1) = \Box_{R_2}(U) \cap X_1$.

2. $\Diamond_{R_1}(U \cap X_1) = \Diamond_{R_2}(U) \cap X_1$.

Proof. (1) Let $x \in X_1$ and suppose that $x \in \Box_{R_1}(U \cap X_1)$. Let $Y \in \mathcal{P}(X_2)$ such that $(x, Y) \in R_2$. As X_1 is an R-hereditary subset of X_2, there exists $Z \in \mathcal{P}(X_1)$ such that $(x, Z) \in R_1$ and $Z \subseteq Y$. So, $\emptyset \neq Z \cap U \cap X_1 = Z \cap U \subseteq Y \cap U$. Thus, $x \in \Box_{R_2}(U) \cap X_1$. Conversely, suppose that $x \in \Box_{R_2}(U) \cap X_1$. Let $Z \subseteq X_1$ such that $(x, Z) \in R_1$. As $R_1 \subseteq R_2$, we have $(x, Z) \in R_2$, and thus $Z \cap U = Z \cap U \cap X_1 \neq \emptyset$, i.e., $x \in \Box_{R_1}(U \cap X_1)$. (2) follows from (1). $\Box$

3 SOME SPECIAL CLASSES OF GENERAL MONOTONIC FRAMES

Dualities between monotonic algebras and certain classes of general monotonic frames have been developed in detail in [3] and [10]. In the duality theory of monotonic modal logics, Hansen [10] introduced the notion of descriptive monotonic frames to obtain a full duality for the category of Boolean algebras with a monotonic operator. Below we will recall the notion of descriptiveness for general monotonic frames given by H. Hansen.

Definition 3.1. [10] A *descriptive monotonic frame, or* descriptive m-frame, is a general monotonic frame $\langle \mathcal{F}, D\rangle$, where $\langle X, \mathcal{T}_D\rangle$ is a Stone space, and for all $x \in X$, all $C \in \mathcal{C}(X)$ and all $Y \subseteq X$,

PCom $Y \in R(x)$ iff $\exists C \in \mathcal{C}(X)\,[C \subseteq Y \text{ and } C \in R(x)]$,

PClos $C \in R(x)$ iff $\forall U \in D\,[C \subseteq U \to U \in R(x)]$.

Between the class of monotonic general frames and the class of descriptive monotonic frames there exists some interesting classes of general frames that are defined generalizing the properties **PCom** and **PClos** of Definition 3.1. In the following definition we extend certain notions introduced in [2] for monotonic models.

Definition 3.2. Let $\langle \mathcal{F}, D\rangle$ be a general monotonic frame. We shall say that:

1. $\langle \mathcal{F}, D\rangle$ is *compact* if the space $\langle X, \mathcal{T}_D\rangle$ is compact.
2. $\langle \mathcal{F}, D\rangle$ is *image-compact* if for all $x \in X$ and for each $Y \in R(x)$, there exists a compact subset Z of $\langle X, \mathcal{T}_D\rangle$ such that $Z \subseteq Y$ and $Z \in R(x)$.
3. $\langle \mathcal{F}, D\rangle$ is *point-compact in* $\langle \mathcal{K}, \mathcal{L}_D\rangle$, where $\mathcal{K} \subseteq \mathcal{P}(X)$, if $R(x)$ is a compact subset in the topological space $\langle \mathcal{K}, \mathcal{L}_D\rangle$ for each $x \in X$.
4. $\langle \mathcal{F}, D\rangle$ is *replete* if it satisfies the following property:

   (P) For all $x \in X$ and for every $Y \in \mathcal{P}(X)$, if $\bigcap\{\varepsilon_D(y) \mid y \in Y\} \subseteq \Diamond_R^{-1}(\varepsilon_D(x))$, then there exists a subset $Z \subseteq X$ such that $Z \in R(x)$ and $Z \subseteq \mathrm{cl}(Y)$, where $\mathrm{cl}(Y)$ is the closure of Y in the space $\langle X, \mathcal{T}_D\rangle$.
5. $\langle \mathcal{F}, D\rangle$ is *modally saturated in* $\langle \mathcal{K}, \mathcal{L}_D\rangle$, where $\mathcal{K} \subseteq \mathcal{P}(X)$, if it is image-compact and point-compact in $\langle \mathcal{K}, \mathcal{L}_D\rangle$.

Remark 3.3. Let $\langle \mathcal{F}, D\rangle$ be a general monotonic frame. The notion of image-compact is an adaptation of Condition **PCom** of Definition 3.1. Later we will show that the notion of point-compact is related to Condition **PClos** of Definition 3.1.

The notion of replete monotonic general frame is a generalization of the notion of $\mathcal{H}$-*closed* Kripke model introduced by R. Goldblatt in [7] page 112, and it is related to one of the conditions used to define the notion of replete general frame in normal modal logics (see condition **VI** of Definition 1.19.1 of [6]).

In [7] R. Goldblatt also introduces the notion of $\mathcal{H}$-*compact Kripke model*. This notion is equivalent to our definition of compactness given in the item (1) of Definition 3.2. Moreover, Goldblatt also defines an $\mathcal{H}$-*saturated structure* (Kripke models, for us) as an $\mathcal{H}$-compact and $\mathcal{H}$-closed structure. In our terminology, an $\mathcal{H}$-compact and $\mathcal{H}$-closed monotonic model $\mathcal{M}$ is a compact, and point-compact model, or from Proposition 10 of [2], $\mathcal{M}$ is compact and replete.

Let $\langle \mathcal{F}, D \rangle$ be a general monotonic frame. Let us consider the set

$$\operatorname{ran} R = \mathcal{K}_R = \{Y \subseteq X \mid \exists x \in X \, ((x, Y) \in R)\} .$$

Then, we can consider the *hyperspace* $\langle \mathcal{K}_R, \mathcal{L}_D \rangle$ of $\langle X, \mathcal{T}_D \rangle$ relative to $\mathcal{K}_R$ (see definition 1.1).

The following results were proved in Propositions 9 and 10 of [2] for monotonic models. Now we will extend these results for general monotonic frames. Although the proof is similar to the proofs given in [2], we decided to include it due to a matter of completeness.

Proposition 3.4. *Let $\langle \mathcal{F}, D \rangle$ be a general monotonic frame.*

1. *If $\langle \mathcal{F}, D \rangle$ is point-compact in $\langle \mathcal{K}_R, \mathcal{L}_D \rangle$, then $\langle \mathcal{F}, D \rangle$ is replete.*
2. *If $\langle \mathcal{F}, D \rangle$ is compact, then $\langle \mathcal{F}, D \rangle$ is replete iff it is point-compact in $\langle \mathcal{K}_R, \mathcal{L}_D \rangle$.*

Proof. (1) Let $x \in X$ and let $Y \in \mathcal{P}(X)$. Assume that $\bigcap \{\varepsilon_D(y) \mid y \in Y\} \subseteq \Diamond_R^{-1}(\varepsilon_D(x))$. Suppose that for all $Z_i \in R\,(x)$, $Z_i \nsubseteq \operatorname{cl}(Y)$. As D is a basis for $\langle X, \mathcal{T}_D \rangle$, we have that for each $Z_i \in R\,(x)$ there exists $U_i \in D$ such that $Y \subseteq U_i$ and $Z_i \nsubseteq U_i$, i.e., $Z_i \cap U_i^c \neq \emptyset$. Thus, $R\,(x) \subseteq \bigcup \left\{L_{U_i^c} \mid Y \subseteq U_i\right\}$. Since $R(x)$ is a compact subset of $\langle \mathcal{K}_R, \mathcal{T}_D \rangle$, there exists a finite set $\{U_1, ..., U_n\}$ such that $R\,(x) \subseteq L_{U_1^c} \cup \cdots \cup L_{U_n^c}$. Then, $x \notin \Diamond_R\,(U_1 \cap \ldots \cap U_n)$, and $Y \subseteq U_1 \cap \ldots \cap U_n$. So, $\Diamond_R\,(U_1 \cap \ldots \cap U_n) \notin \varepsilon_D(x)$ and $U_1 \cap \ldots \cap U_n \in \bigcap \{\varepsilon_D(y) \mid y \in Y\}$, which is a contradiction. Therefore, there exists $Z \in R\,(x)$ such that $Z \subseteq \operatorname{cl}(Y)$.
(2) Suppose that $\langle \mathcal{F}, D \rangle$ is replete. Consider $W \subseteq D$ such that

$$R\,(x) \subseteq \bigcup \{L_U \mid U \in W\} .$$

Suppose that for every finite subset W_0 of W,

$$R(x) \nsubseteq \bigcup \{L_U \mid U \in W_0\}. \tag{3.1}$$

First, we prove that $\bigcap \{U^c : U \in W\} \neq \emptyset$. On the contrary, suppose that $X = \bigcup \{U : U \in W\}$. As $\langle \mathcal{F}, D\rangle$ is compact, $X = U_1 \cup \ldots \cup U_n$, for some finite subset $\{U_1, ..., U_n\}$ of W. From (3.1), there exists $Y \in R(x)$ such that $Y \cap U_1 = \emptyset$,...,$Y \cap U_n = \emptyset$, i.e., $Y \cap (U_1 \cup \ldots \cup U_n) = Y \cap X = \emptyset$, which is impossible. Thus, $Z = \bigcap \{U^c \mid U \in W\} \neq \emptyset$. It is evident that Z is a closed subset of $\langle X, \mathcal{T}_D\rangle$. We now prove that

$$\bigcap \{\varepsilon_D(z) \mid z \in Z\} \subseteq \Diamond_R^{-1}(\varepsilon_D(x)). \tag{3.2}$$

If $V \in \bigcap \{\varepsilon_D(z) : z \in Z\}$, then $Z = \bigcap \{U^c \mid U \in W\} \subseteq V$. It is clear that V^c is a closed subset of $\langle X, \mathcal{T}_D\rangle$, and as $\langle X, \mathcal{T}_D\rangle$ is compact, V^c is compact. It follows that there exists a finite set $\{U_1, ..., U_n\}$ such that $U_1^c \cap \ldots \cap U_n^c \subseteq V$. Then, $\Diamond_R(U_1^c \cap \ldots \cap U_n^c) \subseteq \Diamond_R(V)$. From (3.1), there exists $T \in R(x)$ such that $T \cap (U_1 \cup \ldots \cup U_n) = \emptyset$, i.e., $T \subseteq U_1^c \cap \ldots \cap U_n^c \subseteq V$. Thus, $x \in \Diamond_R(V)$. So, (3.2) is valid. By hypothesis, there exists $Y \in R(x)$ such that $Y \subseteq \mathrm{cl}(Z) = Z$. Then $Y \cap U = \emptyset$, for every $U \in W$. Thus, $R(x) \nsubseteq \bigcup \{L_U \mid U \in W\}$. The other direction follows from (1). □

Lemma 3.5. *Let $\langle \mathcal{F}, D\rangle$ be a general monotonic frame such that $\langle X, \mathcal{T}_D\rangle$ is a Stone space. Then $\langle \mathcal{F}, D\rangle$ is image-compact iff it satisfies Condition* $\mathbf{PCom}$ *of Definition 3.1.*

Proof. We recall that in a Stone space $\langle X, \mathcal{T}_D\rangle$ a subset Y of X is compact iff it is closed. Thus, the result follows. □

3.1 *Descriptive m-frames and restricted descriptive m-frames*

In this subsection we will analyze the relationship between the definition of descriptive m-frame given by H. Hansen in [10] and the definition given in [3].

First, we recall the definition introduced in [3]. In order to differentiate it from Hansen´s Definition 3.1, we shall write these general monotonic frames as restricted descriptive m-frames. So, we will prove that this notion is equivalent to the definition given by H. Hansen.

Definition 3.6. [3] A *restricted descriptive m-frame* is a triple $\langle X, R, \mathcal{T}_D\rangle$ where

1. $\langle X, \mathcal{T}_D\rangle$ is a Stone space,

2. $R \subseteq X \times \mathcal{C}(X)$,

3. $\Box_R(U) \in D$, and

4. $R(x) = \bigcap \{L_U \mid x \in \Box_R(U)\}$ for all $x \in X$.

We now prove that Condition 4 of Definition 3.6 is equivalent to saying that the relation $R \subseteq X \times \mathcal{C}(X)$ is point-compact in $\langle \mathcal{C}(X), \mathcal{L}_D \rangle$. We note that every restricted descriptive m-frame $\langle X, R, \mathcal{T}_D \rangle$ is image-compact, because $\operatorname{ran} R = \mathcal{K}_R \subseteq \mathcal{C}(X)$.

Lemma 3.7. *Let $\langle X, R, D \rangle$ be a triple such that $\langle X, \mathcal{T}_D \rangle$ is Stone space, $R \subseteq X \times \mathcal{C}(X)$ such that $R(x)$ is an increasing subset of $\langle \mathcal{C}(X), \subseteq \rangle$ for each $x \in X$, and $\Box_R(U) \in D$ for all $U \in D$. Then the following conditions are equivalent:*

1. *R is point-compact in $\langle \mathcal{C}(X), \mathcal{L}_D \rangle$.*

2. *$R(x) = \bigcap \{L_U \mid x \in \Box_R(U)\}$ for all $x \in X$.*

3. *$\langle X, R, D \rangle$ is a replete general monotonic frame.*

4. *For all $x \in X$ and for every $Y \in \mathcal{C}(X)$, if $(\varepsilon_D(x), \varepsilon_D[Y]) \in R_{\Diamond_R}$, then $(x, Y) \in R$.*

Proof. (1) $\Rightarrow$ (2) Let $x \in X$. The inclusion $R(x) \subseteq \bigcap \{L_U \mid x \in \Box_R(U)\}$ is clear. Let $Z \in \mathcal{C}(X)$ such that $Z \in \bigcap \{L_U : x \in \Box_R(U)\}$ and suppose that $Z \notin R(x)$. Then, we have that for every $K \in R(x)$, $K \nsubseteq Z$. Since $Z \in \mathcal{C}(X)$ and the elements of D are clopen, for each $K \in R(x)$ there exists $U_K \in D$ such that $Z \subseteq U_K$ and $K \nsubseteq U_K$, i.e., $Z \subseteq U_K$ and $K \cap U_K^c \neq \emptyset$. In consequence, $R(x) \subseteq \{L_{U^c} \mid Z \subseteq U\}$. As $R(x)$ is compact in $\langle \mathcal{C}(X), \mathcal{L}_D \rangle$, there exist $U_1, \ldots, U_n \in D$ such that

$$R(x) \subseteq L_{U_1^c} \cup \ldots \cup L_{U_n^c} = L_U$$

where $U = U_1^c \cup \ldots \cup U_n^c \in D$. So, $x \in \Box_R(U)$ and we get that $Z \in L_U$. On the other hand, $Z \subseteq U_i$ for $1 \leq i \leq n$. Then, $Z \subseteq U_1 \cap \ldots \cap U_n$ and $Z \cap U = \emptyset$ which is a contradiction. Thus, $Z \in R(x)$ and $R(x) = \bigcap \{L_U \mid x \in \Box_R(U)\}$.

(2) $\Rightarrow$ (3) Let $x \in X$ and let $Y \in \mathcal{P}(X)$ such that $\bigcap \{\varepsilon_D(y) \mid y \in Y\} \subseteq \Diamond_R^{-1}(\varepsilon_D(x))$. We prove that $\operatorname{cl}(Y) \in R(x)$. Suppose that $\operatorname{cl}(Y) \notin R(x) = \bigcap \{L_U \mid x \in \Box_R(U)\}$. Then, there exists $U \in D$ such that $R(x) \subseteq L_U$,

$x \in \Box_R(U)$ and $\text{cl}(Y) \subseteq U^c$. So $Y \subseteq U^c$, then $U^c \in \bigcap\{\varepsilon_D(y) \mid y \in Y\} \subseteq \Diamond_R^{-1}(\varepsilon_D(x))$, and thus $x \in \Diamond_R(U^c) = \Box_R(U)^c$, which is a contradiction. So $\text{cl}(Y) \in R(x)$, and $\langle X, R, D\rangle$ is replete.

(3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1) see [3]. $\Box$

Let $\langle X, \mathcal{T}_D\rangle$ be a Stone space with a basis D. Consider a relation $R \subseteq X \times \mathcal{P}(X)$. We define the *restriction* R_r of R to $\mathcal{C}(X)$ as

$$(x, Z) \in R_r \text{ iff } Z \in \mathcal{C}(X) \text{ and } Z \in R(x).$$

Given a relation $S \subseteq X \times \mathcal{C}(X)$ we define the relation S^e, called the *extension* of S to $\mathcal{P}(X)$ as

$$(x, Y) \in S^e \text{ iff there exists } Z \in \mathcal{C}(X) \text{ such that } Z \subseteq Y \text{ and } Z \in S(x).$$

Proposition 3.8. *Let $\langle \mathcal{F}, D\rangle$ be a descriptive monotonic frame. Then:*

1. $\Box_R(U) = \Box_{R_r}(U)$ *for all* $U \in D$.
2. *The relation $R_r \subseteq X \times \mathcal{C}(X)$ is point-compact in $\langle \mathcal{C}(X), \mathcal{L}_D\rangle$.*
3. $(R_r)^e = R$.

Thus, $\langle X, R_r, \mathcal{T}_D\rangle$ is a restricted descriptive m-frame.

Proof. We note that $\langle \mathcal{C}(X), \mathcal{L}_D\rangle$ is compact because $\langle X, \mathcal{T}_D\rangle$ is a Stone space (see [14]).

(1) Let $U \in D$. As $R_r \subseteq R$, we get $\Box_R(U) \subseteq \Box_{R_r}(U)$. Let $x \in \Box_{R_r}(U)$ and let $Y \in R(x)$. By Property **PCom** of Definition 3.1, there exists $C \in \mathcal{C}(X)$ such that $C \subseteq Y$ and $C \in R(x)$. It is clear that $C \in R_r(x)$, and as $x \in \Box_{R_r}(U)$, we have $C \cap U \neq \emptyset$. So, $Y \cap U \neq \emptyset$. Thus, $x \in \Box_R(U)$.

(2) By Lemma 3.7 and (1), it is enough to show that

$$R_r(x) = \bigcap \{L_U : x \in \Box_R(U)\}. \tag{3.3}$$

It is clear that the inclusion $R_r(x) \subseteq \bigcap\{L_U \mid x \in \Box_R(U)\}$ holds. Let $Z \in \mathcal{C}(X)$ such that $Z \in \bigcap\{L_U \mid x \in \Box_R(U)\}$ and suppose that $Z \notin R(x)$. By Property **PClos** of Definition 3.1, there exists $U \in D$ such that $Z \subseteq U$ and $U \notin R(x)$. By remark 2.3, $x \notin \Diamond_R(U)$, i.e., $x \in \Box_R(U^c)$. Since $Z \in L_{U^c}$, we have that $Z \cap U^c \neq \emptyset$, which is a contradiction. Thus, the identity (3.3) is valid.

(3) Let $(x, Y) \in (R_r)^e$. Then there exists $Z \in \mathcal{C}(X)$ such that $Z \subseteq Y$ and $Z \in R_r(x)$. Since $R_r \subseteq R$, we have $Z \in R(x)$, and as R is monotonic, $Y \in R(x)$. Let $(x, Y) \in R$. By Property **PCom** of Definition 3.1, there exists $C \in \mathcal{C}(X)$ such that $C \subseteq Y$ and $C \in R(x)$. So, $C \in R_r(x)$. Since there exists $C \in \mathcal{C}(X)$ such that $C \subseteq Y$ and $C \in R_r(x)$, we get that $(x, Y) \in (R_r)^e$. □

Proposition 3.9. *Let $\langle X, R, D\rangle$ be a triple such that $\langle X, \mathcal{T}_D\rangle$ is a Stone space, $R \subseteq X \times \mathcal{C}(X)$ and $\Box_R(U) \in D$ for all $U \in D$. If R is point-compact in $\langle \mathcal{C}(X), \mathcal{L}_D\rangle$, then*

1. *$\langle X, R^e, D\rangle$ is a descriptive m-frame, and*
2. *$(R^e)_r = R$.*

Proof. (1) By the definition of the relation R^e, $R^e(x)$ is an increasing set for each $x \in X$. We prove that $\Box_R(U) = \Box_{R^e}(U)$ for all $U \in D$. Let $U \in D$. As $R \subseteq R^e$, we have that $\Box_{R^e}(U) \subseteq \Box_R(U)$. Suppose that $x \in \Box_R(U)$. Let $Y \in R^e(x)$. Then there exists $Z \in \mathcal{C}(X)$ such that $Z \subseteq Y$ and $Z \in R(x)$. So, $Z \cap U \neq \emptyset$ because $x \in \Box_R(U)$. So, $Y \cap U \neq \emptyset$, i.e., $x \in \Box_{R^e}(U)$. Condition **PCom** is immediate from the definition of R^e. We also note that by definition of R^e we have that if $C \in R(x)$ then $C \in R^e(x)$ whenever $C \in \mathcal{C}(X)$. We prove Condition **PClos**. Let $C \in \mathcal{C}(X)$ such that $C \in R^e(x)$ and let $U \in D$ such that $C \subseteq U$. From $C \in R^e(x)$ we have that there exists $Y \in \mathcal{C}(X)$ such that $Y \in R(x)$ and $Y \subseteq C \subseteq U$. Then, by definition of R^e, we get that $U \in R^e(x)$. Let $C \in \mathcal{C}(X)$. Assume that for all $U \in D$, $C \subseteq U$ implies $U \in R^e(x)$. Suppose that $C \notin R^e(x)$, i.e., $C \notin R(x)$. As $R(x)$ is point-compact in the Stone space $\langle \mathcal{C}(X), \mathcal{L}_D\rangle$, we have that $R(x)$ is closed in $\langle \mathcal{C}(X), \mathcal{L}_D\rangle$. So, there exists $U \in D$ such that $x \in \Box_R(U)$, $R(x) \subseteq L_U$ and $C \notin L_U$. Then, $x \in \Box_R(U)$ and $C \cap U = \emptyset$. As $C \subseteq U^c \in D$, we get $U^c \in R^e(x)$, i.e., $x \in \Diamond_{R^e}(U^c) = \Box_{R^e}(U)^c = \Box_R(U)^c$, which is a contradiction. Thus, $\langle X, R^e, D\rangle$ is a descriptive m-frame.

(2) It is easy to check that the equality $(R^e)_r = R$ holds. □

Let $\langle X_1, R_1, \mathcal{T}_{D_1}\rangle$ and $\langle X_2, R_2, \mathcal{T}_{D_2}\rangle$ be two restricted descriptive m-frames. Consider the monotonic general frames $\langle \mathcal{F}_1, D_1\rangle$ and $\langle \mathcal{F}_2, D_2\rangle$ where $\mathcal{F}_1 = \langle X_1, R_1\rangle$ and $\mathcal{F}_2 = \langle X_2, R_2\rangle$. Recall that for all $Y \in \mathcal{K}(X_1) = \mathcal{C}(X_1)$ and for every f continuos function, $f[Y] \in \mathcal{K}(X_2) = \mathcal{C}(X_2)$. Then, a function $f\colon X_1 \to X_2$ is called a bounded morphism between the restricted descriptive m-frames $\langle X_1, R_1, \mathcal{T}_{D_1}\rangle$ and $\langle X_2, R_2, \mathcal{T}_{D_2}\rangle$ if and only if f is a bounded morphism between the monotonic general frames $\langle \mathcal{F}_1, D_1 \rangle$ and $\langle \mathcal{F}_2, D_2 \rangle$.

Proposition 3.10. *Let $f \colon X_1 \to X_2$ be a bounded morphism between the restricted descriptive m-frames $\langle X_1, R_1, \mathcal{T}_{D_1} \rangle$ and $\langle X_2, R_2, \mathcal{T}_{D_2} \rangle$. Then,*

1. *For all $x \in X_1$ and for every $Y \subseteq X_1$, if $(x, Y) \in R_1^e$, then $(f(x), f[Y]) \in R_2^e$.*
2. *For all $x \in X_1$ and for every $Z \subseteq X_2$, if $(f(x), Z) \in R_2^e$, then there exists $Y \subseteq X_1$ such that $(x, Y) \in R_1^e$ and $f[Y] \subseteq Z$.*

Thus, $f \colon X_1 \to X_2$ is a bounded morphism between the descriptive m-frames $\langle X_1, R_1^e, D_1 \rangle$ and $\langle X_2, R_2^e, D_2 \rangle$. In addition, if f is a strong bounded morphism between the restricted descriptive m-frames $\langle X_1, R_1, \mathcal{T}_{D_1} \rangle$ and $\langle X_2, R_2, \mathcal{T}_{D_2} \rangle$, then f is a strong bounded morphism between the descriptive m-frames $\langle X_1, R_1^e, D_1 \rangle$ and $\langle X_2, R_2^e, D_2 \rangle$.

Proof. It follows from the inclusions $R_1 \subseteq R_1^e$ and $R_2 \subseteq R_2^e$. □

Proposition 3.11. *Let $f \colon X_1 \to X_2$ be a bounded morphism between the descriptive m-frames $\langle X_1, R_1, D_1 \rangle$ and $\langle X_2, R_2, D_2 \rangle$. Then,*

1. *For all $x \in X_1$ and for every $Y \in \mathcal{C}(X_1)$, if $(x, Y) \in (R_1)_r$, then $(f(x), f[Y]) \in (R_2)_r$.*
2. *For all $x \in X_1$ and for every $Z \subseteq \mathcal{C}(X_2)$, if $(f(x), Z) \in (R_2)_r$, then there exists $C \in \mathcal{C}(X_1)$ such that $(x, C) \in (R_1)_r$ and $f[C] \subseteq Z$.*

Thus, $f \colon X_1 \to X_2$ is a bounded morphism between the restricted descriptive m-frames $\langle X_1, (R_1)_r, \mathcal{T}_{D_1} \rangle$ and $\langle X_2, (R_2)_r, \mathcal{T}_{D_2} \rangle$. In addition, if f is a strong bounded morphism between the descriptive m-frames $\langle X_1, R_1, \mathcal{T}_{D_1} \rangle$ and $\langle X_2, R_2, \mathcal{T}_{D_2} \rangle$, then f is a strong bounded morphism between the restricted descriptive m-frames $\langle X_1, (R_1)_r, \mathcal{T}_{D_1} \rangle$ and $\langle X_2, (R_2)_r, \mathcal{T}_{D_2} \rangle$.

Proof. (1) Let $x \in X_1$ and $Y \in \mathcal{C}(X_1)$ such that $(x, Y) \in (R_1)_r$. Then, $(x, Y) \in R_1$. As f is a bounded morphism, it follows that $(x, f[Y]) \in R_2$. Since $\langle X, \mathcal{T}_{D_1} \rangle$ is a Stone space, we get that Y is a compact subset of $\langle X, \mathcal{T}_{D_1} \rangle$. As f is a continuous function between the topological spaces $\langle X, \mathcal{T}_{D_1} \rangle$ and $\langle X, \mathcal{T}_{D_2} \rangle$, we have that $f[Y]$ is a compact subset of $\langle X, \mathcal{T}_{D_2} \rangle$ and therefore $f[Y] \in \mathcal{C}(X_2)$. Thus, $(x, f[Y]) \in (R_2)_r$.

(2) Let $x \in X_1$ and $Z \in \mathcal{C}(X_2)$ such that $(f(x), Z) \in (R_2)_r$. Then, $(f(x), Z) \in R_2$. As f is a bounded morphism, it follows that there exists

$Y \subseteq X_1$ such that $(x, Y) \in R_1$ and $f[Y] \subseteq Z$. By Property **PCom** of Definition 3.1, there exists $C \in \mathcal{C}(X_1)$ such that $C \subseteq Y$ and $(x, C) \in R_1$. It is easy to check that $(x, C) \in (R_1)_r$ and $f[C] \subseteq f[Y] \subseteq Z$. □

Corollary 3.12. *There exists a bijective correspondence between descriptive monotonic frames and restricted descriptive* m*-frames. Strong bounded morphisms between two descriptive monotonic frames are strong bounded morphisms between their corresponding restricted descriptive* m*-frames and viceversa.*

4 PRESERVATION PROPERTIES

In this section we study some properties that are preserved by means of surjective bounded morphisms, and some valid properties in a general monotonic frame that are preserved in its subframes. Just to be clear, we say that a class $\mathbf{K}$ reflects a construction if its complement $\mathbf{K^c}$, which is the class of all frames that are not in $\mathbf{K}$, is closed under that construction.

Proposition 4.1. *Let* $f : \langle \mathcal{F}_1, D_1 \rangle \to \langle \mathcal{F}_2, D_2 \rangle$ *be a surjective bounded morphism between the general monotonic frames* $\langle \mathcal{F}_1, D_1 \rangle$ *and* $\langle \mathcal{F}_2, D_2 \rangle$*. Then,*

1. *If* $\langle \mathcal{F}_1, D_1 \rangle$ *is point-compact in* $\langle \mathcal{K}_{R_1}, \mathcal{L}_{D_1} \rangle$*, then* $\langle \mathcal{F}_2, D_2 \rangle$ *is point-compact in* $\langle \mathcal{K}_{R_2}, \mathcal{L}_{D_2} \rangle$*.*
2. *If* $\langle \mathcal{F}_1, D_1 \rangle$ *is image-compact, then* $\langle \mathcal{F}_2, D_2 \rangle$ *is image-compact.*

Proof. (1) Let $b \in X_2$ and let $F \subseteq D_2$. We prove that $R_2(b)$ is a compact subset of the hyperspace $\langle \mathcal{K}_{R_2}, \mathcal{L}_{D_2} \rangle$. Suppose that for any finite subset F_j of F we get

$$R_2(b) \cap \bigcap \{L_V^c \mid V \in F_j\} \neq \emptyset. \tag{4.1}$$

We prove that $R_2(b) \cap \bigcap \{L_V^c \mid V \in F\} \neq \emptyset$. Since f is surjective, there exists $a \in X_1$ such that $f(a) = b$. We prove that

$$R_1(a) \cap \bigcap \left\{L_{f^{-1}[V]}^c \mid V \in F_j\right\} \neq \emptyset \tag{4.2}$$

for any finite subset F_j of F. Suppose that there exists a finite subset F_0 of F such that

$$R_1(a) \cap \bigcap \left\{L_{f^{-1}[V]}^c \mid V \in F_0\right\} = \emptyset. \tag{4.3}$$

By (4.1), $R_2(f(a)) \cap \bigcap \{L_V^c \mid V \in F_0\} \neq \emptyset$. So, there exists $Y \in R_2(b)$ such that $Y \cap V = \emptyset$ for any $V \in F_0$. As f is a bounded morphism, there exists $Z \subseteq X_1$ such that $(a, Z) \in R_1$ and $f[Z] \subseteq Y$. Then $f[Z] \cap V = \emptyset$ for every $V \in F_0$. So, $Z \cap f^{-1}[V] = \emptyset$ for all $V \in F_0$. Thus, $Z \in R_1(a) \cap \bigcap \left\{L_{f^{-1}[V]}^c \mid V \in F_0\right\}$, which contradicts (4.3). Thus, (4.2) is valid. As $\langle \mathcal{F}_1, D_1\rangle$ is point-compact, there exists $Y \subseteq X_1$ such that $Y \in R_1(a) \cap \bigcap \left\{L_{f^{-1}[V]}^c \mid V \in F\right\}$. As f is a bounded morphism, we have that $(f(a), f[Y]) \in R_2$ and $f[Y] \in \bigcap \{L_V^c \mid V \in F\}$. Therefore, $\langle \mathcal{F}_2, D_2\rangle$ is point-compact.

(2) Assume that $\langle \mathcal{F}_1, D_1\rangle$ is image-compact. Let $b \in X_2$ and $Y \subseteq X_2$ such that $(b, Y) \in R_2$. Since f is surjective, there exists $a \in X_1$ such that $f(a) = b$. So, $(f(a), Y) \in R_2$, and as f is a bounded morphism, there exists $Z \subseteq X_1$ such that $(a, Z) \in R_1$ and $f[Z] \subseteq Y$. Since $\langle \mathcal{F}_1, D_1\rangle$ is image-compact, there exists a compact subset $H \subseteq X_1$ such that $(a, H) \in R_1$ and $H \subseteq Z$. It is easy to check that $f[H]$ is a compact subset of X_2, and as $(f(a), f[H]) \in R_2$ and $f[H] \subseteq f[Z] \subseteq Y$, we get that $\langle \mathcal{F}_2, D_2\rangle$ is image-compact. □

Proposition 4.2. *Let $f : \langle \mathcal{F}_1, D_1\rangle \to \langle \mathcal{F}_2, D_2\rangle$ be a strong bounded morphism between the general monotonic frames $\langle \mathcal{F}_1, D_1\rangle$ and $\langle \mathcal{F}_2, D_2\rangle$. Then,*

1. *If $\langle \mathcal{F}_2, D_2\rangle$ is point-compact in $\langle \mathcal{K}_{R_2}, \mathcal{L}_{D_2}\rangle$, then $\langle \mathcal{F}_1, D_1\rangle$ is point-compact in $\langle \mathcal{K}_{R_1}, \mathcal{L}_{D_1}\rangle$.*
2. *If f is surjective and $\langle \mathcal{F}_1, D_1\rangle$ is replete, then $\langle \mathcal{F}_2, D_2\rangle$ is replete.*
3. *If f is injective and $\langle \mathcal{F}_2, D_2\rangle$ is image-compact, then $\langle \mathcal{F}_1, D_1\rangle$ is image-compact.*
4. *If $\langle \mathcal{F}_2, D_2\rangle$ is replete, then $\langle \mathcal{F}_1, D_1\rangle$ is replete.*

Proof. (1) Let $a \in X_1$ and let $W \subseteq D_1$. We prove that $R_1(a)$ is a compact subset of the hyperspace $\langle \mathcal{K}_{R_1}, \mathcal{L}_{D_1}\rangle$. Suppose that for any finite subset W_j of W we get

$$R_1(a) \cap \bigcap \{L_U^c \mid U \in W_j\} \neq \emptyset. \tag{4.4}$$

As f is strong, for each $U \in D_1$ there exists $V_U \in D_2$ such that $U = f^{-1}[V_U]$. We prove that $R_2(f(a)) \cap \bigcap \left\{L_{V_U}^c \mid U \in W_j\right\} \neq \emptyset$ for any finite subset W_j of W. Suppose that there exists a finite subset W_0 of W such that $R_2(f(a)) \cap \bigcap \left\{L_{V_U}^c \mid U \in W_0\right\} = \emptyset$. By (4.4), $R_1(a) \cap \bigcap \{L_U^c \mid U \in W_0\} \neq \emptyset$, i.e., there exists $Y \in R_1(a)$ such that $Y \cap U = Y \cap f^{-1}[V_U] = \emptyset$ for all

$U \in W_0$. Since f is a bounded morphism, $f[Y] \in R_2(f(a))$ and $f[Y] \cap V_U = \emptyset$ for all $U \in W_0$, i.e., $f[Y] \in R_2(f(a)) \cap \bigcap \{L^c_{V_U} \mid U \in W_0\} = \emptyset$, which is impossible. Then, as $\langle \mathcal{F}_2, D_2 \rangle$ is point-compact, there exists $Z \subseteq X_2$ such that $Z \in R_2(f(a))$ and $Z \cap V_U = \emptyset$ for all $U \in W$. As f is a bounded morphism, there exists $Z' \subseteq X_1$ such that $Z' \in R_1(a)$ and $f[Z'] \subseteq Z$. It is clear that $Z' \cap f^{-1}[V_U] = Z' \cap U = \emptyset$ for all $U \in W$. Thus, $Z' \in R_1(a) \cap \bigcap \{L^c_U \mid U \in W\}$, i.e., $\langle \mathcal{F}_1, D_1 \rangle$ is point-compact.

(2) Assume that $\langle \mathcal{F}_1, D_1 \rangle$ is replete. Let $\varepsilon_i = \varepsilon_{D_i}$ for $i = 1, 2$. Let $b \in X_2$ and $Y \subseteq X_2$ such that

$$\bigcap \{\varepsilon_2(y) \mid y \in Y\} \subseteq \Diamond^{-1}_{R_2}((\varepsilon_2(b)). \tag{4.5}$$

We prove that there exists $Y' \subseteq X_2$ such that $(b, Y') \in R_2$ and $Y' \subseteq \mathrm{cl}(Y)$. Since f is surjective, there exists $a \in X_1$ such that $f(a) = b$. We prove that

$$\bigcap \{\varepsilon_1(x) \mid x \in f^{-1}[Y]\} \subseteq \Diamond^{-1}_{R_1}((\varepsilon_1(a)). \tag{4.6}$$

Let $U \in \bigcap \{\varepsilon_1(x) \mid x \in f^{-1}[Y]\}$. Then $f^{-1}[Y] \subseteq U$. As f is strong, there exists $V_U \in D_2$ such that $U = f^{-1}[V_U]$. So, $f^{-1}[Y] \subseteq f^{-1}[V_U]$. We prove that $Y \subseteq V_U$. Let $y \in Y$. Since f is surjective, there exists $x \in X_1$ such that $f(x) = y$. So, $x \in f^{-1}[Y] \subseteq f^{-1}[V_U]$. Consequently, $x \in f^{-1}[V_U]$, i.e., $f(x) = y \in V_U$. Then, $V_U \in \bigcap \{\varepsilon_2(y) \mid y \in Y\} \subseteq \Diamond^{-1}_{R_2}((\varepsilon_2(b))$, and it follows that $b = f(a) \in \Diamond_{R_2}(V_U)$. Thus, $a \in f^{-1}[\Diamond_{R_2}(V_U)] = \Diamond_{R_1}(f^{-1}[V_U]) = \Diamond_{R_1}(U)$, i.e., $U \in \Diamond^{-1}_{R_1}((\varepsilon_1(a))$. Therefore (4.6) is valid. Since $\langle \mathcal{F}_1, D_1 \rangle$ is replete, there exists $Z \subseteq X_1$ such that

$$(a, Z) \in R_1 \text{ and } Z \subseteq \mathrm{cl}(f^{-1}[Y]). \tag{4.7}$$

As $f(a) = b$, and f is a bounded morphism, $(b, f[Z]) \in R_2$. Since f is a strong bounded morphism, it is easy to see that from $Z \subseteq \mathrm{cl}(f^{-1}[Y])$ it follows that $f[Z] \subseteq \mathrm{cl}(Y)$. Thus, $\langle \mathcal{F}_2, D_2 \rangle$ is replete.

(3) Let $a \in X_1$ and let $Y \in R_1(a)$. Then $f[Y] \in R_2(f(a))$. As $\langle \mathcal{F}_2, D_2 \rangle$ is image-compact, then there exists a compact subset H of X_2 such that $H \subseteq f[Y]$ and $H \in R_2(f(a))$. As f is a bounded morphism, there exists $Z \subseteq X_1$ such that $(a, Z) \in R_1$ and $f[Z] \subseteq H$. So, $Z \subseteq f^{-1}[H]$, and as $\langle \mathcal{F}_1, D_1 \rangle$ is monotonic, $(a, f^{-1}[H]) \in R_1$. We prove that $f^{-1}[H]$ is compact. Let $W \subseteq D_1$ such that $f^{-1}[H] \subseteq \bigcup \{U : U \in W\}$. As f is strong, for each $U \in W$ there exists $V_U \in D_2$ such that $U = f^{-1}[V_U]$. We prove that $H \subseteq \bigcup \{V_U : U \in W\}$. Let $y \in H$. As $H \subseteq f[Y]$, there exists $x \in X_1$ such that $f(x) = y$. So, $x \in f^{-1}[H]$, and consequently there exists $U \in W$

such that $x \in U = f^{-1}[V_U]$, i.e., $f(x) = y \in \bigcup \{V_U : U \in W\}$. As H is compact, there exist $U_1, \ldots, U_n \in W$ such that $H \subseteq V_{U_1} \cup \ldots \cup V_{U_n}$. So, $f^{-1}[H] \subseteq U_1 \cup \ldots \cup U_n$, and thus $f^{-1}[H]$ is compact. Since f is injective, it follows that $f^{-1}[H] \subseteq Y$, and we get that $\langle \mathcal{F}_1, D_1 \rangle$ is image-compact.

(4) Suppose that $\langle \mathcal{F}_2, D_2 \rangle$ is replete. Let $a \in X_1$ and $Y \in \mathcal{P}(X_1)$ such that

$$\bigcap \{\varepsilon_{D_1}(y) \mid y \in Y\} \subseteq \Diamond^{-1}_{R_1}(\varepsilon_{D_1}(a)).$$

We prove that

$$\bigcap \{\varepsilon_{D_2}(x) \mid x \in f[Y]\} \subseteq \Diamond^{-1}_{R_2}(\varepsilon_{D_2}(f(a))). \tag{4.8}$$

Let $U \in D_2$ such that $U \in \bigcap \{\varepsilon_{D_2}(x) \mid x \in f[Y]\}$. We have that $f[Y] \subseteq U$. So, $Y \subseteq f^{-1}(f[Y]) \subseteq f^{-1}(U) \in D_1$ and it implies that $f^{-1}(U) \in \bigcap \{\varepsilon_{D_1}(y) \mid y \in Y\}$. It follows that $f^{-1}(U) \in \Diamond^{-1}_{R_1}(\varepsilon_{D_2}(a))$, i.e., $a \in \Diamond_{R_1}\left(f^{-1}(U)\right)$. Thus, there exists $Z \in R_1(a)$ such that $Z \subseteq f^{-1}(U)$. Since f is a bounded morphism, $f[Z] \in R_2(f(a))$ and $f[Z] \subseteq U$. Then, $U \in \Diamond^{-1}_{R_2}(\varepsilon_{D_2}(f(a)))$ and (4.8) is valid. As $\langle \mathcal{F}_2, D_2 \rangle$ is replete, there exists $Z \subseteq X_2$ such that $(f(a), Z) \in R_2$ and $Z \subseteq \mathrm{cl}_{D_2}(f[Y])$. Again, since f is a bounded morphism, there exists $V \subseteq X_1$ such that $(a, V) \in R_1$ and $f[V] \subseteq Z \subseteq \mathrm{cl}_{D_2}(f[Y])$. Now, we prove that $V \subseteq \mathrm{cl}_{D_1}(Y)$. Let $x \in V$ and $U \in D_1$ such that $x \in U$. As f is a strong bounded morphism, there exists $W \in D_2$ such that $x \in U = f^{-1}(W)$. So, $f(x) \in W$ and from $f[V] \subseteq \mathrm{cl}_{D_2}(f[Y])$, we get that $f(x) \in \mathrm{cl}_{D_2}(f[Y])$. Since W is a basis element, $W \cap f[Y] \neq \emptyset$, i.e., there exists $y \in Y$ such that $f(y) \in W$. Then, $y \in f^{-1}(W) = U$ and it follows that $U \cap Y \neq \emptyset$. Therefore, $x \in \mathrm{cl}_{D_1}(Y)$ and $\langle \mathcal{F}_1, D_1 \rangle$ is replete. □

Corollary 4.3. *Let $f : \langle \mathcal{F}_1, D_1 \rangle \to \langle \mathcal{F}_2, D_2 \rangle$ be a surjective strong bounded morphism between the general monotonic frames $\langle \mathcal{F}_1, D_1 \rangle$ and $\langle \mathcal{F}_2, D_2 \rangle$. Then,*

1. *$\langle \mathcal{F}_1, D_1 \rangle$ is point-compact in $\langle \mathcal{K}_{R_1}, \mathcal{L}_{D_1} \rangle$ iff $\langle \mathcal{F}_2, D_2 \rangle$ is point-compact in $\langle \mathcal{K}_{R_2}, \mathcal{L}_{D_2} \rangle$.*
2. *$\langle \mathcal{F}_1, D_1 \rangle$ is replete iff $\langle \mathcal{F}_2, D_2 \rangle$ is replete.*
3. *If f is injective, then $\langle \mathcal{F}_1, D_1 \rangle$ is image-compact iff $\langle \mathcal{F}_2, D_2 \rangle$ is image-compact.*

Proposition 4.4. *Let $\langle \mathcal{F}_1, D_1 \rangle$ and $\langle \mathcal{F}_2, D_2 \rangle$ be two general monotonic frames. Suppose that $\langle \mathcal{F}_1, D_1 \rangle$ is a general subframe of $\langle \mathcal{F}_2, D_2 \rangle$. Then,*

1. *If $\langle \mathcal{F}_2, D_2 \rangle$ is point-compact in $\langle \mathcal{K}_{R_2}, \mathcal{L}_{D_2} \rangle$, then $\langle \mathcal{F}_1, D_1 \rangle$ is point-compact in $\langle \mathcal{K}_{R_1}, \mathcal{L}_{D_1} \rangle$.*
2. *If $\langle \mathcal{F}_2, D_2 \rangle$ is image-compact, then $\langle \mathcal{F}_1, D_1 \rangle$ is image-compact.*
3. *If $\langle \mathcal{F}_2, D_2 \rangle$ is replete, then $\langle \mathcal{F}_1, D_1 \rangle$ is replete.*

Proof. It follows from 4.2 and from the fact that if $\langle \mathcal{F}_1, D_1 \rangle$ is a general subframe of $\langle \mathcal{F}_2, D_2 \rangle$, then the inclusion map $i\colon X_1 \to X_2$ is a strong bounded morphism. $\square$

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