# Solving Exponential Situations and Conceptualization

## Resolução de Situações Exponenciais e Conceituação

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## Abstract

*Students have difficulty to use different representational systems. We use the Theory of Conceptual Fields in order to describe and analyze students' responses to situations in various representational system: numerical, first order algebraic, graphic and verbal written. We conducted the study with 59 students (aged between 15-16 years old) of secondary school. From the analysis we show that students do not "translate" the results of a representational system to another. They can solve exponentially in a representational system (RS) but in a different exponential RS they solve it in a non-exponential way. This indicates that students construct operational invariants for each representational system. That is to say, they build and use different schemes for each one of them.*

***Keywords:*** *Schemes; Representational Systems; Education; Secondary School.*

## Resumo

*Os alunos tem dificuldades para utilizar diferentes sistemas de representação. Utilizamos a Teoría dos Campos Conceituais para descrever e analizar as respostas dos alunos em diversas situações, nos sistemas de representação numérico, algébrico de primeira ordem, gráfico e verbal escrito. Realizamos o estudo com 59 alunos (de 15-16 anos) do colégio do ensino medio. A partir da análise verificamos que os alunos não "traduzem" os resultados de um sistema de representação para outro. Eles podem resolver exponencialmente em um sistema de representação (SR) mas em forma não exponencial em um SR diferente. Isto indica que os alunos constroem invariantes operatorios para cada sistema de representação. Quer dizer, constroem e utilizam esquemas diferentes em cada um deles.*

***Palavras-Chave:*** *Esquemas; Sistemas de Representação; Ensino; Ensino medio.*

## Introduction

The enunciation of mathematical concepts in different representational systems (RS), is central to the whole mathematics teaching at schools. Towards the end of secondary school students must be proficient to analysis functions in different representational systems (e.g. write its formula, draw its graphic representation, use them to solve a

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problem, etc.). Mainly, teachers give little importance to the RS as components of concepts and consider them evident, identifying conceptualization with symbolization, when they actually are different and complementary processes, being necessary assume them when teaching.

Although the conceptualization involves much more than the representational systems, it is not possible to conceptualize complex mathematical concepts, without such systems.

## 1. Research related to representational systems

For some time the relevance of the different representational systems in learning and comprehension of mathematics has been emphasized, in particular those related to functions (JANVIER 1978; VERGNAUD, 1990; 2009; 2013; KAPUT, 1987; DUVAL, 1993; 2006).

Among the most influential pioneering works we can mentioned Janvier (1978, 1993), who argues that the study of the functions should not be circumscribed to a single representational framework, instead it should encourage the *translation* of information from one representation to another, or within the same representational systems (JANVIER, 1993; p. 20). According to the author this *translation* is possible because each representation ideally contains the same information as another, but puts into motion different cognitive processes. In his works, Janvier (1978, 1993) has classified the representations associated to each concept, into four classes: Verbal Description, Table, Graph, and Formula Equation.

In the same line, García and Llinares (1994) analyse, with these four categories, the *translations* proposed in the tasks and problems found in textbooks relative to the concept of function. There, the authors state that understanding the notion of function, means being flexible in the process of tasks solving. That is, to be able to *translate* the different perspectives of the concept between different representational systems. Where the *translation* process involves translating the meanings ascribed to different aspects of the concept of function (Ibid., p. 17, author 's italics).

Also Duval (1993) highlights the importance of using different representational systems: *"The (integrative) comprehension of a conceptual content lies in the coordination of at least two registers of representation, and this coordination is manifested by the rapidity and spontaneity of conversion"* (p. 123).

On a different line, Vergnaud (1990; 2007a; 2009; 2013) stresses the importance of the enunciation in different representational systems, but emphasizes the distinction between conceptualization and symbolization.

## 2. Conceptualization and Symbolization in Theory of Conceptual Fields

Conceptualization is the identification of objects of different levels, directly accessible to the perception or not, its properties and relations, and involves a dialectical relations between schemes and situations (VERGNAUD, 1990; 2007a; 2009; 2013). As a greater number of pertinent schemes, the individual may address more complex situations. The scheme is not a behaviour but has the function of generating the activity and the behaviour in situation. For this reason, it is possible to study through the analysis of behaviours, the schemas that guide the responses of the students in a situation, and in particular the operational invariants that make operating the scheme. Thus, the analysis of conceptualization, is make from the students written resolutions when they solve a situation.

The operational invariants are the concepts-in-action and theorems-in-action. The concepts-in-action are pertinent categories, and the theorems-in-action are propositions taken as undeniable truth. The concepts and theorems are built in a solidary way and may be implicit or explicit, more or less formal (see chart 1), and right or wrong. The operational invariants allow to select relevant information and from it -and taking into account the goal to meet-, and it lets infer the most appropriate rules of action to address a situation (VERGNAUD, 1990; 2013). Consequently, the decisions made by a student in a given situation will depend on the activated schema, but more specifically in the concepts-in-action and theorems-in-action that the subject has and therefore can deal with with them the situation.

On the other hand, a Concept is composed of three different sets C(S, I, L) not independent among them, but different (VERGNAUD, 2013; p. 156):

- **The reference [S]:** is the set of situations that give meaning to the concept.
- **The significant [I]:** is the set of operational invariants (concepts-in-action and theorems-in-action) that structure the forms of organization of the activity (schemes) that can be evoked by these situations.

- **The signifier [L]:** is the set of linguistic and symbolic representations (algebraic, graphical, etc.), that allows the representation of concepts and their relationships, and therefore the situations and schemes that evoke.

The concepts are constructed from the action-in-situation, but the linguistic expression and symbolization give it weight and stability, and come to the aid of the implicit conceptualization in action (Ibid., p. 147). Without words and symbols, the experience can not be communicated. And it is this relation that Vergnaud makes between conceptualization and symbolization, that gives relevance to the Representational Systems.

In the field of mathematics, numerical and algebraic notations play an important role in the conceptualization and in the reasoning processes, but they are not a language in itself, but a knowledge: *"Some even consider that the difficulty of mathematics is mainly a linguistic difficulty. This view is wrong, because mathematics is not a language, but knowledge"* (VERGNAUD, 2009; p. 89). Thus, knowledge is expressed in two ways, by the way it is said (predicative form of knowledge), and to what is done in a situation (operational form of knowledg), and therefore the analysis of situations must consider both of them.

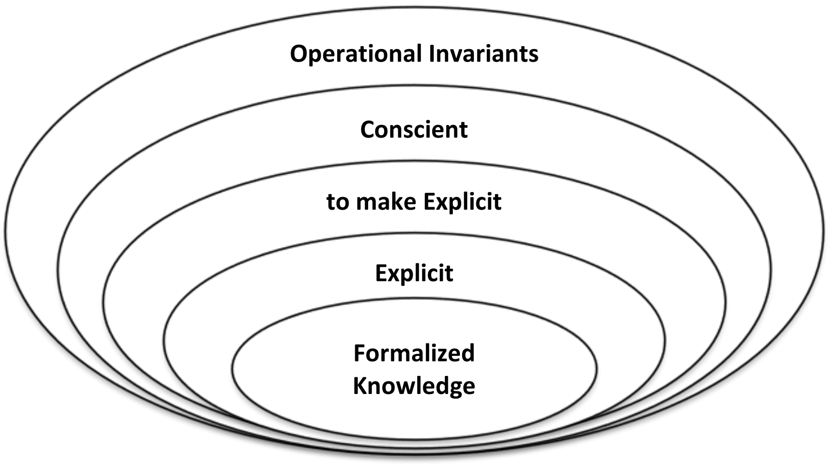



**Chart 1: Classification of Operational Invariants**

## 3. Research Questions

1. Which theorems-in-action lead the students' strategies in each representational system, as they progress in the study of conceptual field?
2. Which features of the conceptualization of the exponential function can be derived from the students' answers?

## 4. Methodology

A set of twelve teaching situations, three sets of tasks and an evaluation was designed. Each set of situations and tasks were designed so that the resolution of the problem required the treatment in at least three systems of representation. Students, whom at the time of implementation had only studied linear functions, should answer with the school knowledge that they had. Each essential features of representational system are described as follows.

**Numeric Representational System [NRS]:** The numerical calculations are added here, as well as tables made by hand or spreadsheets. **First Order Algebraic Equations Representational System [RSA1**]: Refers to the algebraic procedures which parameters are initialized, and its transformations. For example, the transformations that take place during the resolution of an equation also corresponds to this RS. **Second Order Algebraic Representational System [RSA2]:** Corresponds to this RS the treatments with functions family of the type $a.b^x = c$. Within this RS we can frame: the analysis of the conditions of a formula so as to be a function, the deductions of an algebraic expression in other, and the demonstrations. **Graphic – Analytic representational System [RSG]:** All graphic construction, in a system of Cartesian axes corresponding to this RS. The graph can be drawn by hand or by using a software. **Written Verbal Representational System [WVRS]:** Refers to written language forms.

The set of situations were framed in a low-risk saving. Due to the interest of the students by saving money for their senior trip, we proposed a problem about a certain amount of money set in fixed term with compound interest. The students had previously studied linear functions and simple interest. Once the set of situations was designed, a pilot experiment was performed. Then, these were implemented in two fourth year courses (aged between 15-16 years old) of secondary school.

During the class, students studied in groups of four or five members each. For the idea-sharing and comparison of results, there were available a computer and a video projector. Students added at the end of each situation, graphic resolutions and tables made on their own computers as part of the homework. Students always used Excel spreadsheets for building tables, and Graphmatica as to graphic functions.

Classes were recorded in audio and each of them were transcribed class to class, segmenting in episodes. The change of episodes was made according to the teacher talking time and student talking time. The responses of the 59 students to each situation were collected class to class. This produced a total of 885 protocols. We analyzed and identified certain levels (Author 1; Author 2, año). These levels do not keep a strict sequential order, instead they refer to common feature to each type of resolution (Table 1).

**Table 1: Levels present in the Exponential Function Conceptualization**

| Stage | Indicator | Acronym |
|---|---|---|
| **Linear** | Linear answer in all representational systems. | [LR] |
| **Partly Non Linear** | Non Linear answer in at least one representational system. | [PNLR] |
| **Non Linear** | Non Linear answer in all of the representational systems. | [NLR] |
| **Partly Exponentially** | Exponential answer in at least one representational system. | [PER] |
| **Exponential** | Exponential answer in all of the representational systems. | [ER] |

The study was performed again the following year with 56 students. The analysis allowed them to describe five states or levels in the conceptualization of all the groups studied.
In this paper we describe and analyse the strategies used by students to solve the situations in the different systems of representation, and its modification as they proceed in the study of conceptual field. We present the types of resolutions of the five most significant events. For each situation we specify the number of resolutions by level and the analyse them.

## 5. Analysis of the Situations and Results

### 5.1 First Situation

The class begins with a conversation about what is putting money into a fixed term at compound interest. After agreeing that putting a certain amount of money at compound interest, with an interest rate of 1% each month they get 1% more than the previous month, the first situation is proposed.

A group of students has got 12000 pesos for their senior trip, and they want to put the money into a fixed term at compound interest during the term of 30 months (that will be when the senior trip takes places from that moment on). They found out the rates of some bank, and they know that:
The monthly rate of the **Bank 1** is 1,1% and gives them $ 12,132 fulfilled the first month.
The monthly rate of the **Bank 2** is 1,2% and gives them $ 12,144 fulfilled the first month.
The monthly rate of the **Bank 3** is 1,3% and gives them $ 12,156 fulfilled the first month.

- **a)** How did banks calculate that first month?
- **b)** Make an approximate graphic of the variation of money in each bank; calculating at least three values*.
- **c)** To which function corresponds the graphical representation you drew?

Remember that it is very important to write all accounts you do in the sheet, and do not delete anything you write.
* The cartesian axes are given.

With the categories listed in Table 1, the responses of the 59 students were classified (Table 2). The "Total" column indicates the number of students that were in both courses.

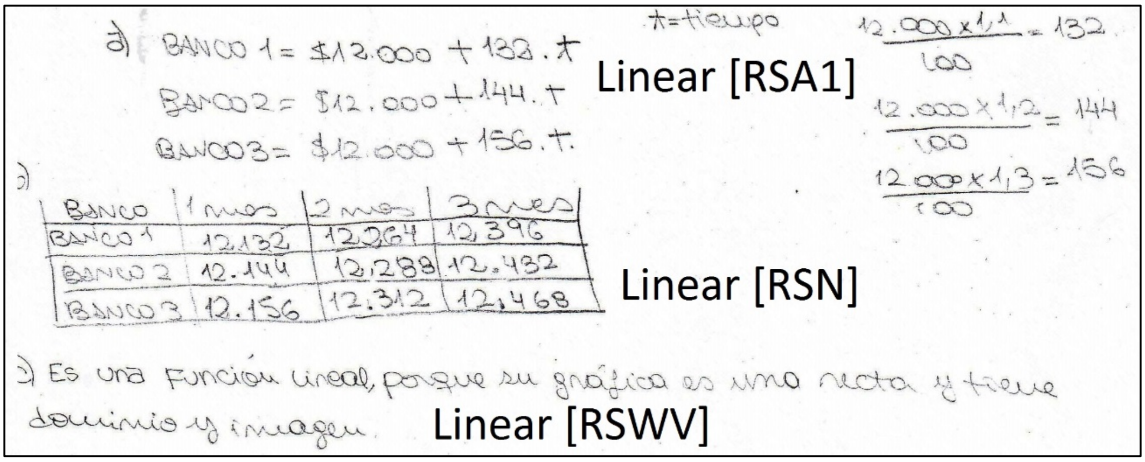
**Table 2: Classification of responses to the first situation**

| [LR] | [PNLR] | [NLR] | [PER] | [ER] | Absents | Total |
|---|---|---|---|---|---|---|
| 12 | 33 | 4 | - | - | 10 | 59 |

Almost all pupils responded in a linear or partially nonlinear way. Also the first non-linear responses appear.

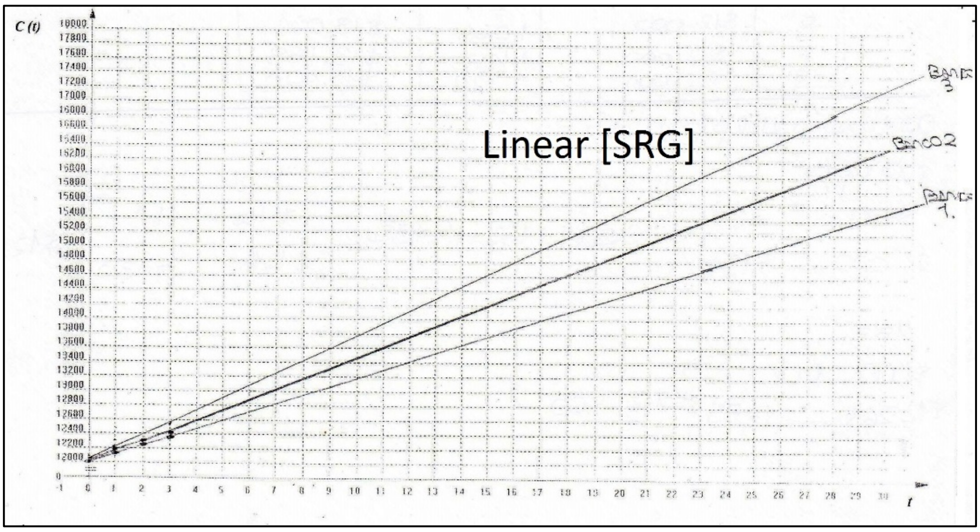
**Linear Response [LR]:** Student A26 calculated in a linear way in all the RS (Fig. 1 and 2).

**Figure 1** - Student A26 Answer to the First Situation, in the First Order Algebraic Equations Representational System, Numeric and Written Verbal Representational System




First calculates the money earned the first month in each bank, and infers the amount of money that will increase monthly and writes the linear expression: $Bank\ 1 = 12000 + 132\,.\,t$ Then he draws three straight lines to display how the quantity of money varies (Figure 2) and states that *"it is a linear function because its graphic is a straight and it has domain and image"*. This proposition that the student considers true is what Vergnaud (1999) calls “theorem-in-action".

**Figure 2 -** Student A26 Answer to the First Situation, in a Graphic Representational System




Since A26 actions are linear in the four RS involved (RSN, RSA1, RSG and RSWV) the following theorems-in-action (Table 3) are inferred.

**Table 3: Operational Invariant involved in the answer to Scenario 1 - Linear Level**

| Representational System | Theorem-in-action | Answer |
|---|---|---|
| **RSN** | "The money set IC increases the same every month" | **[LR]** |
| **RSA1** | "It is possible to calculate the money saved in CI made by a linear algebraic expression" | **[LR]** |
| **RSG** | "The graphic representation of growth of money set in CI, is a straight line" | **[LR]** |
| **RSWV** | "Is a linear function because the graphic is a straight line" | **[LR]** |

The linearity expressed in each RS evidence the mastery of a variety of concepts, schemes and symbolic linear representations, in close connection. But it also shows that the twelve students who solved them in this way, dismissed all the information related to compound interest. Instead, they resolved the situation in the same way that solved the one of simple interest. This is consistent with the theory that claims that the operational invariants available in the schemes, guide the selection of information, goals, and actions to follow (VERGNAUD, 2013; p. 150).

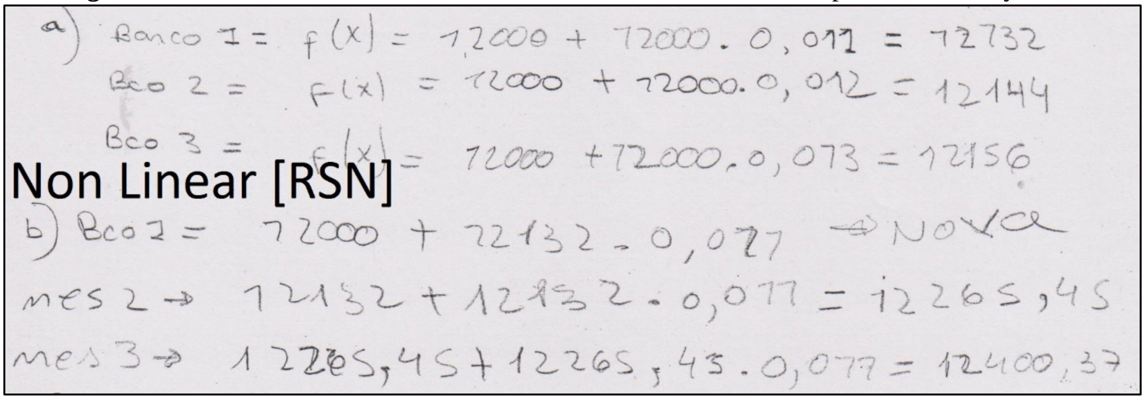
**Partly Nonlinear Response [NLPR]**: In general, this group of students responded in a nonlinearly way in the RSN and linearly in all others. For example, student A36 (Figure 3) calculated recursively the simple interest. This action, origin of an exponential action, allowed to obtain the compound capitalization [RSN]. But in the RS graphic drew three straight lines, and when asked what function had been plotted, said it was a linear function [RSWV] (Figure 4). Thus, the theorems-in-action that seem to guide the action of A36 in each RS are the following (Table 4).

**Table 4: Operational Invariant involved in the answer to Scenario 1 - Level Partly Nonlinear**

| Representational System | Theorem-in-action | Answer |
|---|---|---|
| **RSN** | "Money set CI does not increase the same every month" | **[NLR]** |
| **RSA1** | "The increment of money set CI is not calculated in the same way that the simple interest" | **[NLR]** |
| **RSG** | "The graphic representation of growth of money set IC is a straight line" | **[LR]** |
| **RSWV** | "The increment of money set CI is a straight line" | **[LR]** |

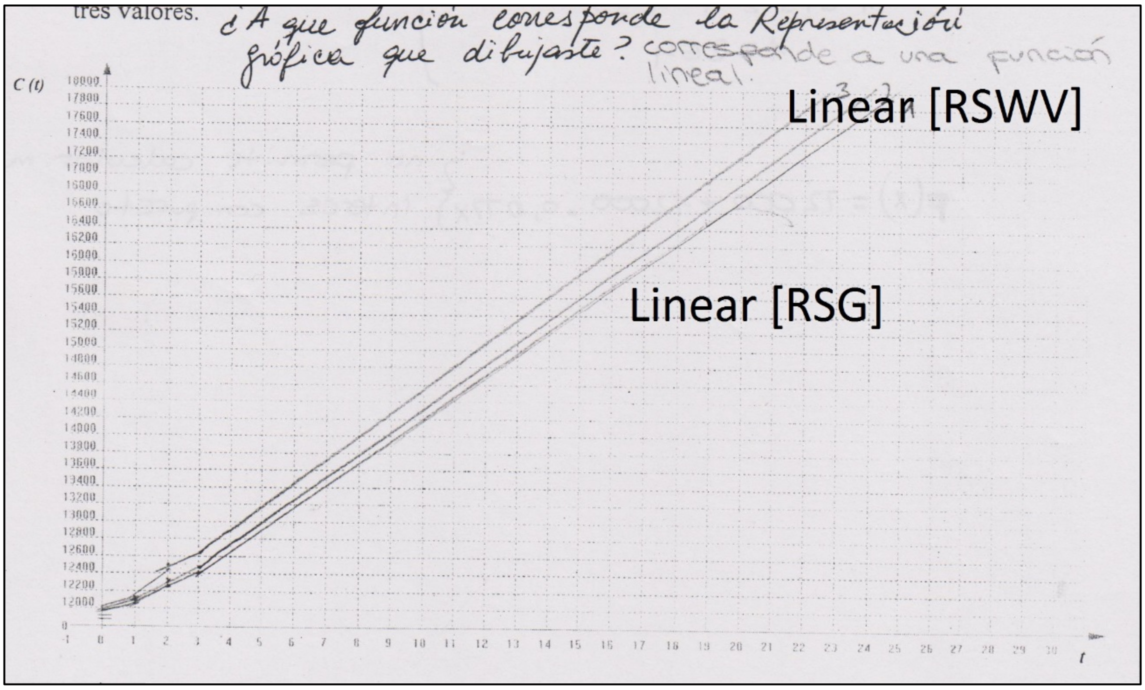
Like A36, a large proportion of students (33) were able to calculate the compound interest (Figure 3), but solved linearly in the others RS (Figure 4).

**Figure 3** - Student A36 answer to the First Situation, in the Numeric Representational System.




This shows that students did not reinterpreted in the others RS the non-linear meaning present in the RSN. That is, the operational invariants that guide the action in the diverse RS, are different. Nonlinear in the RSN and linear in the systems of representation RSWV and RSG (Table 4).

**Figure 4** - Student A36 Response to the First Situation, in the Graphic Representational Systems and Verbal and Written Representational System.




**Nonlinear Response [NLR]:** Students who responded in a nonlinear way were only four. The resolutions show some attempts to formulate an expression that allows them to calculate the amount of money in any month, and although they did not manage to formulate a useful expression, their attempts show a nonlinear procedure RSA1. Finally, they calculated the amount of money for each month in the RSN and built a non-linear graphic, by gathering the points calculated previously. In this way, strategies are guided by nonlinear theorems-in-action in all RSs (Table 5).

**Table 5: Operational Invariant involved in the answer to Situation 1 - Nonlinear Level**

| Representational System | Theorem-in-action | Answer |
|---|---|---|
| **RSN** | "Money does set in IC does not increase the same every month" | **[NLR]** |
| **RSA1** | "Increasing the money put at compound interest is not calculated in the same way that the simple interest" | **[NLR]** |
| **RSG** | "The graph of growth of money set IC is not a straight line" | **[NLR]** |
| **RSWV** | "Increasing the money put at compound interest is not a linear function" | **[NLR]** |

Once finished the situation, each group explained their calculations and procedures on the board. Each resolution was discussed and came to the following conclusion:

> [162: Teacher] *Okay, so we know that in the compound interest the increase of money is not always the same, meaning that its increase is different each month.*

### 5.2 Second Situation

This situation is similar to previously mentioned. We add a table that displays the variation of money in the first bank, for the first 30 months. The table shows that the amount of money does not increase the same way every month. The table has some empty boxes that students must complete. In the first task, the student must complete the empty boxes and propose a formula. In the second task, they had to build similar tables for the other two banks, and give the formulas. In the third task, they had to determine the domain and image as to be functions. Then, they had to graph them on a given Cartesian axis system and explain the difference between this model and the previous one.

The answers to this situation were classified according to Table 6.

**Table 6: Classification of Responses according to the second situation**

| [LR] | [PNLR] | [NLR] | [PER] | [ER] | Absents | Total |
|---|---|---|---|---|---|---|
| - | 36 | 18 | 2 | - | 3 | 59 |

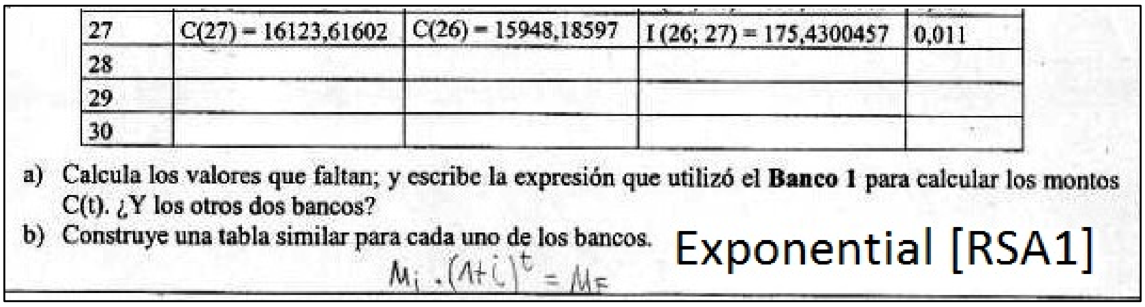
The linear responses disappeared. The amount of partial responses and fully nonlinear increased. The first exponential answers appeared partially.

**Nonlinear Partly Response [NLPR]:** With strategies similar to those used in the previous situation, they calculated recursively the simple interest, and formulate an algebraic expression that aims the algebrization of the calculation procedure. But to show how varying the amount of money placed at interest, they draw three lines. In this way, the actions are guided by theorems in a nonlinear act in the RSN and RSA1, but linear in the RSG.

**Nonlinear Response [NLR]:** With procedures similar to those used in the situation 1, students calculated recursively simple interest for the first three months, and try to make an algebraic calculation procedure. Then they constructed a nonlinear graph. In this way, the actions were guided by theorems-in-nonlinear in all RSs (Table 5).

**Partly Exponential Response [PER]:** Student A14 completed the boxes calculating recursively the simple interest, but in the inferior margin (figure 5) writes the algebraic expression $M_i\ (1+i)^t = Mf$.

**Figure 5-** Student A14 Response to the Second Situation, in the First Order Algebraic Equations Representational System



| | | | | |
|---|---|---|---|---|
| 27 | C(27) = 16123,61602 | C(26) = 15948,18597 | I (26; 27) = 175,4300457 | 0,011 |
| 28 | | | | |
| 29 | | | | |
| 30 | | | | |

a) Calcula los valores que faltan; y escribe la expresión que utilizó el **Banco 1** para calcular los montos C(t). ¿Y los otros dos bancos?

b) Construye una tabla similar para cada uno de los bancos.

$M_i \cdot (1+i)^t = M_F$

Exponential [RSA1]

When the teacher sees this, asks for a representative from each group to explain the procedures on the board. When asking student A14 how the equation was constructed, the following conversation was generated:

[119: A14] *We did the initial amount by one plus the increase of 0,011; where the time (which is the month) is the power.*

[120: Student] *Why one plus I?*

[121: A14] *Because if you multiply the values of the second column by 1,011, it gives you the following value. So we realized that 1,011 is 1 + i.*

[122: Teacher] *Good. Another question: Why t is it the power? What did you think to put it in that way?*

[123: A14] *Is that the previous class we thought something with power. First we multiplied by t, but as multiplying did not work out, then tried with the power.*

...

[163: Teacher] *Of course. It is the same to do 1,011 squared, that multiply 1,011 twice. But you are not sure why t is the power of 1,011.*

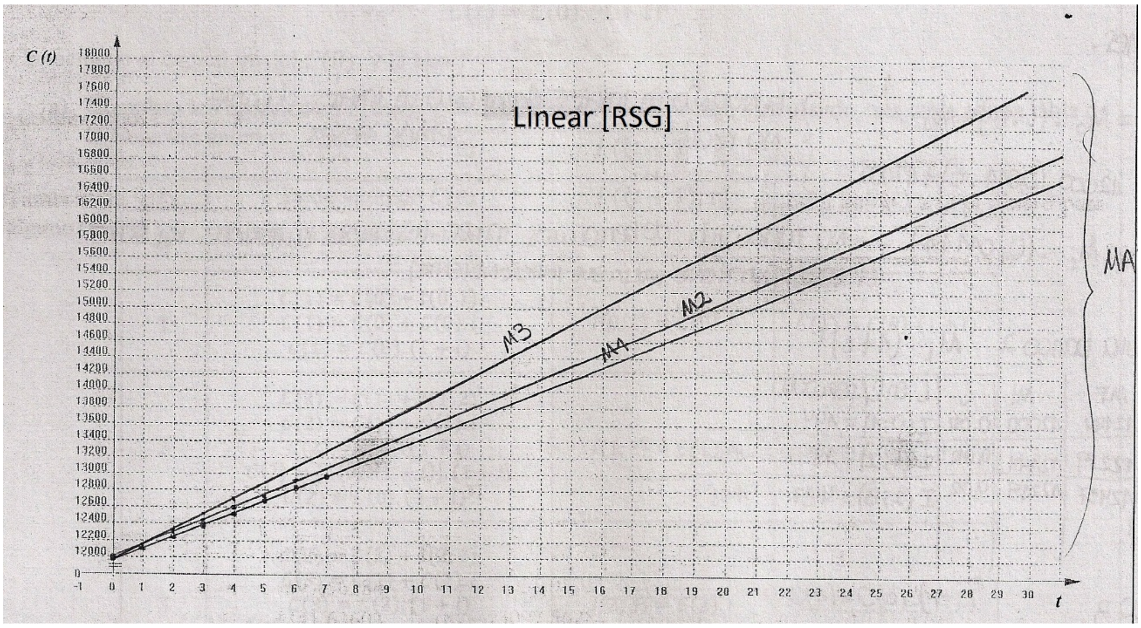
[164: A14] *Because we had discussed that we lacked the time and we did not know where to locate it.*

[165: Teacher] *Of course, as you did not have the independent variable you already knew that it had to be a time. If you multiplied it, it did not work, and you decide to write it as power. We are on the right track.*

During the conversation with the teacher, the students were unable to explain how they came to formulate the algebraic expression. That is, a gap between the operational form and the predicative form of knowledge, can be observed.

After agreeing the formula, they were given ten minutes to graphic. But despite the fact that they had already been spoken that the amount of money did not increase the same, and the fact that they had just determine the exponential algebraic expression, 38 out of the 56 students present in class, drew straight lines. A14 included (Figure 6).

**Figure 6 -** Student A14 Response to the Second Situation, in the Graphic Representational System.




Thus, these pupils solved nonlinearly in RSN, exponentially in the RSA1 and linear in the RSG. The theorems-in-action that seem to guide their actions are described in Table 7.

**Table 7: Operational Invariant involved in the response to Situation 2 - Level Partly Exponential**

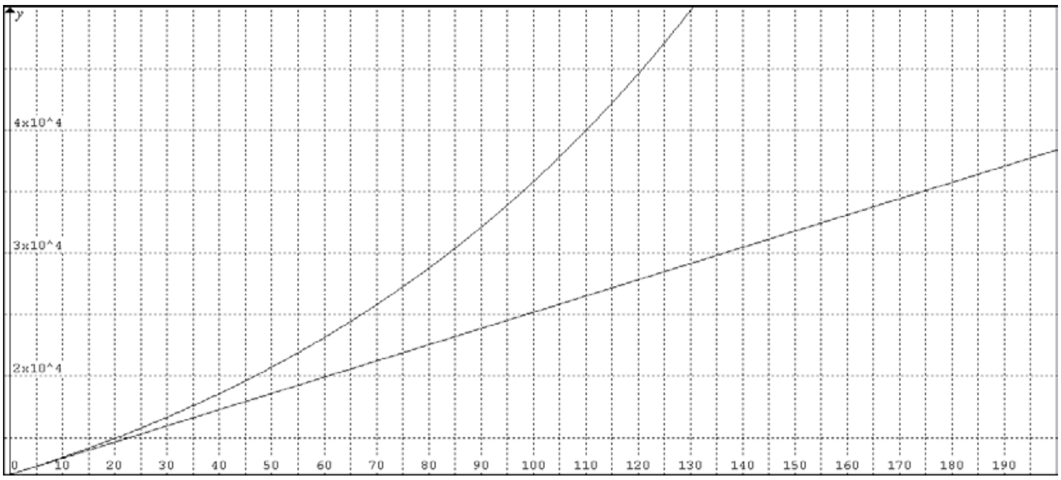
| Representational System | Theorem-in-action | Answer |
|---|---|---|
| **RSN** | “If each amount is multiplied by 1,011 we get the next result” | **[NLR]** |
| **RSA1** | “The algebraic expression is: $Mf = Mi\,.(1+i)^t$. Where *t* is the month for which it is required to know the amount of money ; *i* is the interest rate and *Mi* the initial amount” | **[ER]** |
| **RSG** | “The graphical representation of the increase in money set CI is a straight line” | **[LR]** |

This shows that to understand a RS problem and to solve it, does not implies an understanding in another. That is to say, opposed to what it is believed, the arrangements established in a RS are not immediately reinterpreted into another, at least when the knowledge of the conceptual field is nascent. As to reach the point that the operational invariants used are contradictory among them, but students do not perceive this.

Once the students finished the graphic, idea-sharing is done. One of the student goes to the front of the class and graph a straight line on the board.

> [297: Teacher] *He says that it did not work out well for him because he could not form a straight line, there are some who could make a straight line. I would like to know, which one is correct. If we are going to represent it with a straight line, what interest would be represent? The simple interest or the compound interest?*
> [300: Students] *Simple Interest.*
> [299: Teacher] *Remember that we had found a formula that was for simple interest. What was that formula? Do you have it handy?*

Students dictate the formula and together along with professor they start sketching both formulas with the software Graphmatica. What is being done on the computer is at the same time being projected to the class. The class group also works with scales and finally they reached to the following graph:




> [329: Professor] *Well now you tell me what differences can you find?*
> [330: Student] *That compound interest is always increasing.*
> [331: Teacher] *What about the simple interest?*
> [332: Student] *In simple interest it always increases the same.*
> [333: Teacher] *Right, in the simple interest always increases the same, in the compound interest we know that it does work in that way, but why do they not increase the same? Because if you look in terms of little squares, is very ugly what I 'm going to say to you, but is for the sake of giving you the idea... if you look over here, the first two squares, between month zero and 20, it barely rises half little square, you see? If you see in the following two*

*increases nearly one. In the next two, more than one. You see that they keep on increasing? It increases a little more. However, what 's going on with the simple interest?*

[334: Student] *Always increases the same amount.*

The analysis of the responses shows that students did not reinterpret in the graphical representational system the results obtained in other RS. They begin to link each type of interest with their graphical features only when they talk to the teacher about it. This shows that the construction of operational invariants for RSG does not proceed from other RS, but from the operations that students can do within it.

In order to show that compound interest is a particular case of the exponential functions, after solving a set of exercises, they are proposed a new type of problems.

### 5.3 Fourth Situation

Three people have SIV (swine influenza viruses). Each of them infect five people in the first hour. Then each of those five people infects another five the second hour. And so on. Students must write an algebraic formula that allows them to calculate the amount of new contagious people per hour, as well as its respective graph. The answers to this situation were classified according to Table 8.

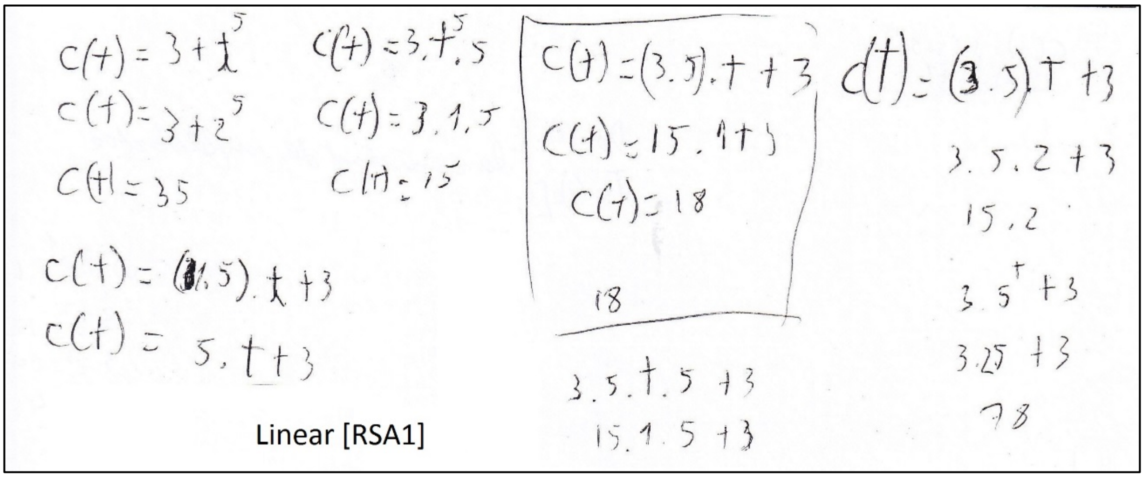
**Table 8 – Classification of responses to situation four**

| [LR] | [PNLR] | [NLR] | [PER] | [ER] | Ausents | Total |
|---|---|---|---|---|---|---|
| - | - | - | 17 | 35 | 7 | 59 |

Partial responses and completely nonlinear responses disappear. The exponential partially answers increase. The first fully exponential responses appear.

**Partly Exponential Response [PER]:** A "new" situation appears, 17 students out of the 52 got back their linear strategies on the RS first order algebraic [RSA1]. For example, in student A6 (Figure 7) can be seen the various resolutions, in which the independent variable is still on the base, instead of being as a power.

**Figure 7** - Student A6 Response to the Fourth Situation, in First Order Algebraic Representational System




Moreover, the expressions partially outlined and inconclusive, in which coexist linear and power resolutions, demonstrate the presence of different operational invariants, that seek to be recombined to give an answer.

Once agreed the algebraic expression, they calculate some values and they built an exponential graph representation (Table 8). We can see A6 difficulties when representing in the coordinate axes, because even if the operational invariants are exponential, the student can not even control it.

**Figure 8** - Student A6 Response to the Fourth Situation, in the Graphic and numeric Representational System.

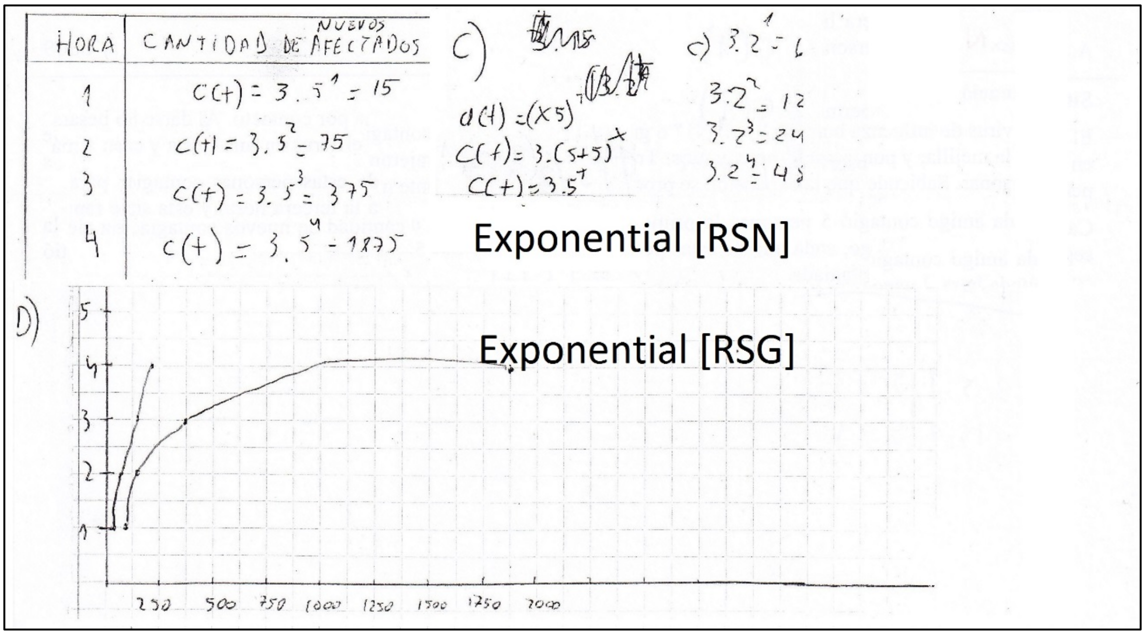



Table 9 theorems-in-action seem to guide its resolution.

**Table 9: Invariant Operatory involved in the response to the Situation 4 - Level Partly Exponential**

| Representational System | Theorem-in-action | Answer |
|---|---|---|
| **RSN** | "The increase in the number of infected is calculated using the formula: $f(x) = k.a^x$" | **[ER]** |
| **RSA1** | "The increase in the number of infected with SIV is calculated with the formula of the linear function " | **[LR]** |
| **RSG** | "Points of the graphical representation of the increase in the amount of infected are joined by a non-straight curve" | **[ER]** |

**Exponential Response [ER]:** The 35 students who solved in a totally exponentially way, made various exponential expressions that allow them to deal with the problem: $f(x) = 3.(5)^t$; $f(x) = 3.\left(1 + {}^{3}\!/_{5}\right)^t$; $f(t) = Mi.(1+i)^t$; $f(t) = 3.(1+5)^t$. Once they agreed formula, they calculated some values, and built the graphic representation. Thus, these decisions seem to be guided by invariant operative exponential (Table 10).

**Table 10: Operational Invariant involved in the response to the situation 4 - Exponential Level**

| Representational System | Theorem-in-action | Answer |
|---|---|---|
| **SRN** | "The increase in the number of infected is calculated using the formula: $f(x) = k.a^x$" | **[ER]** |
| **SRA1** | "The algebraic expression is: $f(t) = k.a^t$. Where *t* is the independent variable ; *a* is the growth rate and *k* the initial amount" | **[ER]** |
| **SRG** | "The graphical representation of the variation of the number of infected is a strictly increasing curve" | **[ER]** |

### 5.4 Fifth Situation

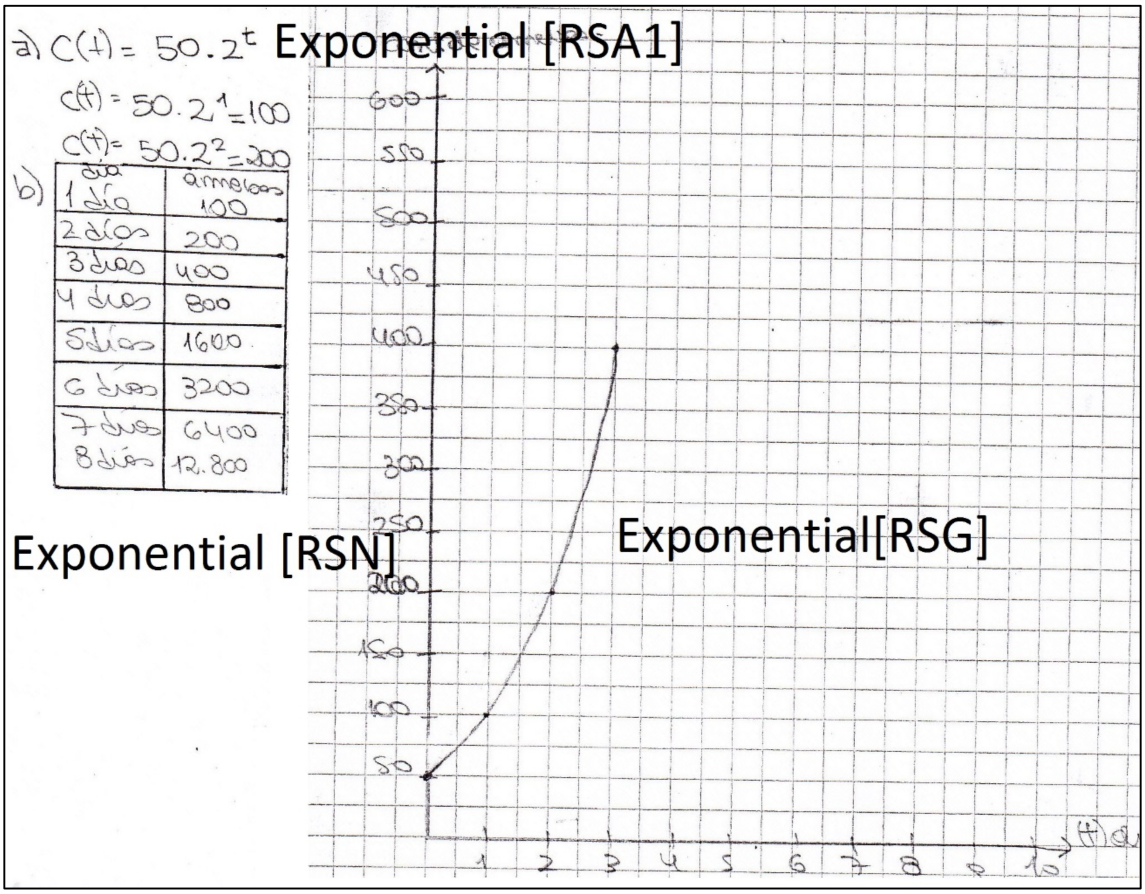
There are 50 amoebae in a laboratory that get duplicated by bipartition day by day. An equation is require, a table with the number of the amoebas for 31-day and its graphics. The answers to this situation were classified according to Table 1.

**Table 11: Classification of responses for the fifth Situation**

| [LR] | [PNLR] | [NLR] | [PER] | [ER] | Ausents | Total |
|---|---|---|---|---|---|---|
| - | - | - | - | 54 | 5 | 59 |

All of the students solved in an exponential way. Figure 9 shows the resolution of A26, who solves exponentially all RS. This shows that in this situation the students were able to extend their linear schemes in an exponential direction, in all of the representational systems.

**Figure 9 -** Student A26 response to the fifth situation, in First Order Algebraic Representational Systems, and Numeric and Graphic Representational Systems




The resolution in the three RS is unique, systematic and organized. There are no longer more than one strategy, there are not unfinished formulations. Thus Student A26 responses show that their actions have been arranged by a unified scheme. Theorems-in-action involved are described in Table 12.

**Table 12: Operational Invariant involved in the response to the situation 5 - Exponential Level**

| **Representational System** | **Theorem-in-action** | **Answer** |
|---|---|---|
| **RSN** | "The increasing in quantities of amoebae is calculated on the immediately preceding number" | **[ER]** |
| **RSA1** | "The algebraic expression to calculate the quantity of amoebae is: $f(t) = k \,.\, a^t$. Where $t$ is the independent variable; *a* is the growth rate, and *k* the initial amount" | **[ER]** |
| **RSG** | "The graphical representation of the variation of the amount of amoeba is a strictly increasing curve" | **[ER]** |

The emergence of resolutions fully exponential in this situation, shows significant progress in the conceptualization of the exponential functions in different RSs. Hence, it emphasizes the importance of the selection of situations.

In the situation six exponential function of the form $f(x) = k.a^x$ is generalized. Then the students solved a set of exercises.

### 5.5 Seventh Situation

Situation seven makes reference to a compound interest problem. Each group of students has 5000 pesos for their senior trip that will be set on a fixed interest, to thirty months. The interest rate of the bank is known. They also have 2000 pesos that will not be settled on interest. A formula is require to calculate the money each month, including the money that was not put at interest; the amount of money for thirty months; the formula for calculating the amount of money without donation, and by a grant of 3200; all graphs; the domain and image of each one of them. The aim of the situation is to generalize the exponential function $f(x) = k.a^x + b$. Answers where classified according to table 13

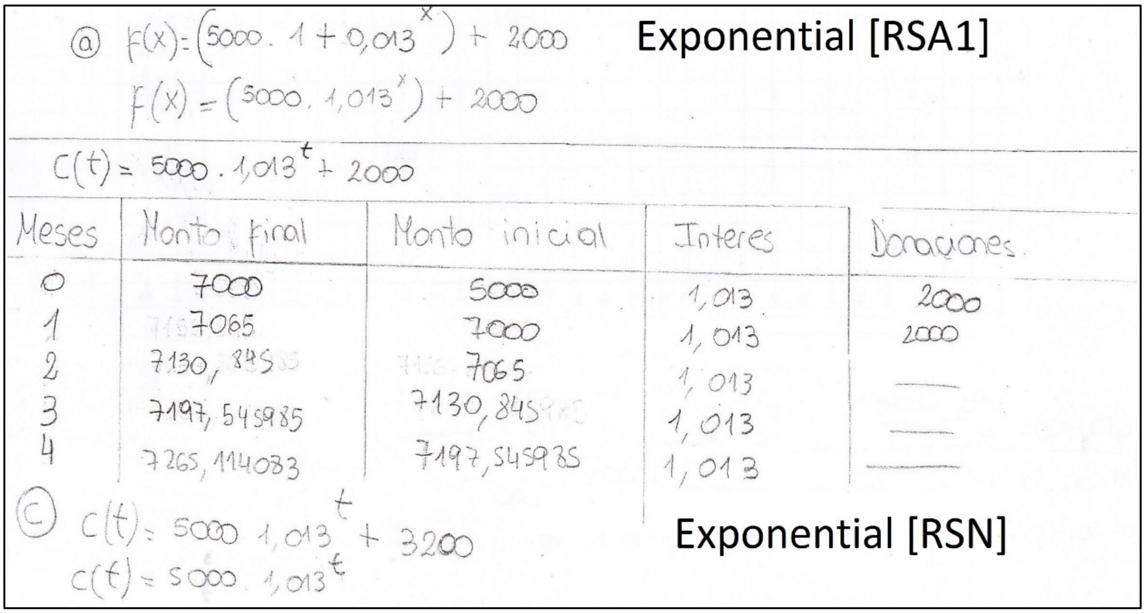
**Table 13 - Classification to responses to the seventh situation.**

| [LR] | [PNLR] | [NLR] | [PER] | [ER] | Ausents | Total |
|---|---|---|---|---|---|---|
| - | - | - | 41 | 13 | 5 | 59 |

Partially exponential responses reappear. Some answers are kept completely exponential.

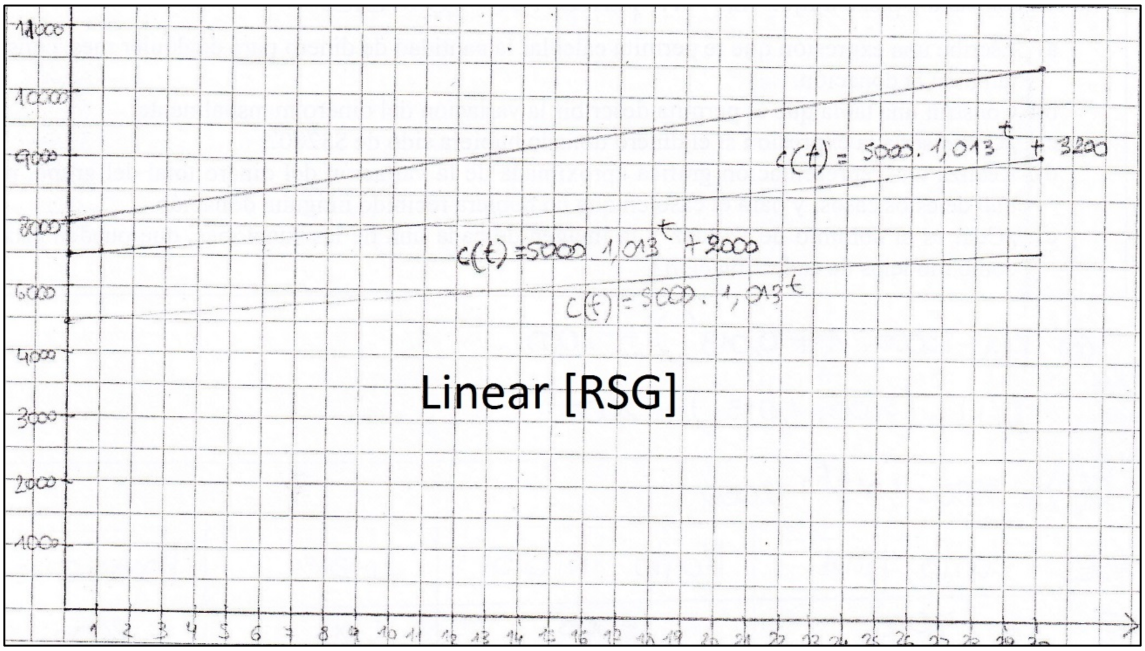
**Partly Exponential Response [PER]:** Students add to their expression of compound interest the amount of the donation (Figure 10). Then they use the expression to calculate. Both strategies are guided by invariant operative exponential.

**Figure 10** - Student A31 Response to the Seventh Situation, in the First Order Algebraic and Numerical Representational Systems.




However when drawing the variation of money, 41 students out of the 54 who solved the situation returned to their nonlinear or linear strategies (Figure 11). So, the answer is exponential in numeric representational systems and algebraic of the first order, but linear in the RSG. These actions show that in front of a "new" situation, students come back to their previous schemes. This also highlights the precariousness of the conceptualization.

**Figure 11 -** Student A31 Response to the Seventh Situation in Graphic Representational System




In Table 14 are described theorems-in-action that seem to guide these students' actions in each RS.

**Table 14: Operational Invariant involved in the response to the situation 7 - Exponential Level**

| Representational System | Theorem-in-action | Answer |
|---|---|---|
| **RSN** | "The variation in the amount of money is calculated by the algebraic expression $k \cdot a^t + b$" | **[ER]** |
| **RSA1** | " The algebraic expression to calculate the amount of money is: $f(t) = k \cdot a^t + b$. Where $t$ is the independent variable; $a$ is the rate of growth, $k$ is the initial amount and $b$ the amount of money that is not put at interest" | **[ER]** |
| **RSG** | "The graphical representation of the variation of money is growing straight " | **[LR]** |

In situation tenth exponential equations are studied, and in situation eleventh the variations of this functions family. Finally a summary is made and evaluated.

## 6. Discussion

At the very beginning of the implementation, students did not translate nor reinterpreted the results obtained from a representational system (RS) to solve the problem in another RS. For instance, while in the first-order algebraic representational system [RSA1], student A14 formulated the algebraic expression of compound interest, in the graphic representational system continues to draw straight lines, even after the whole class agreed that this was indeed the formula and that this variation was not linear. Student A14 is not the only one to proceed along this way, 38 out of the 59 students who solved nonlinearly in numerical RS and algebraic first order, drew straight lines after agreeing with the class group the exponential algebraic expression, and to convey that money does not increase linearly. That is, contrary to what was expected, it is unusual that students translate the conclusions obtain in a representational system to another, at least when knowledge of a conceptual field is emerging. This shows that at the beginning, the operational invariants that make operating the scheme and direct the action in each representational system are different. That is, at the beginning, the schemes that students used in each representational system were different.

The contradictions observed between the formulated responses in different representational systems, and its subsequent review with the class group, allows to infer that the construction of schemes, and in particular the invariants that guide the scheme, was carried out within each representational system. For example, the analysis made by the group class, of linear responses built in the graphic representational system, shows that students had not achieve the constructions of invariants that the help them to deal with nonlinear problems in this RS. However, later on, as the conversation progressed, students used some invariants that allowed them to deal with these situations. Still we do not have to confuse the importance of mediation and explanation, discussion and formalization of knowledge in conceptualizing. For even though Vergnaud (2013: 159-160) recognizes the importance of language and the various tutor acts of mediation -such as calling attention about the relevant information, or take over a part of the actions to be done, as to reduce the space of uncertainty in which the learner must navigate- for him, the most important act of mediation is the choice of the situation. Why? Because even when language intervenes in the conceptualization, is insufficient due to the fact that formation of operational invariants -which is the basis of conceptualization-, is performed during the activity (VERGNAUD, 2013: 159-160). Ie in situation.

Nevertheless, if one wants to prioritize a based teaching situations, one must be willing to overtake the long distance that separates the action from the conceptualization, and from the symbolization.

On the other hand, the fact that students could not put into words the knowledge used in action, evidence that predicative and operative forms of knowledge do not occur in parallel; which corresponds to the classification Vergnaud made (2007) about the operational invariants (Chart 1), where most of the knowledge is unconscious, or implicit consent, showing at the same time that the explicitness and formalization of knowledge happens belatedly.

The students' responses who participated in the implementation are summarized in Table 15.

**Table 15: Development of the Conceptualization along the Situations.**

| [S] | [LR] | [PNLR] | [NLR] | [PER] | [ER] | Absents | Total |
|---|---|---|---|---|---|---|---|
| **1** | 12 | 33 | 4 | - | - | 10 | 59 |
| **2** | - | 36 | 18 | 2 | - | 3 | 59 |
| **4** | - | - | - | 17 | 35 | 7 | 59 |
| **5** | - | - | - | - | 54 | 5 | 59 |
| **7** | - | - | - | 41 | 13 | 5 | 59 |

The classification of responses suggests the existence of a progression in the conceptualization of the exponential function related to RS, that starts from linear schemes, up to the exponential ones (situations from one to five). Highlighting the fact that the fully exponential responses [ER] appear for the first time in situation four and it get generalizes in the situation five. The early appearing of exponential resolutions in all of the RS emphasizes the importance that the design the selection of situations; because in a few situations, students achieved -with mistakes and successes- the conceptualization of a complex knowledge. But, even though there are interesting developments, it is far from supposing an adequate conceptualization in school terms, because the process of homogenization schemes is long and can not be reduced to five situations.

The seventh situation refers to a problem that required of exponential functions of the form $f(x) = k.a^x + b$. This type of function allows the extension of the conceptual field of exponential functions but has the extra complexity that seems linear when is actually nonlinear. The number of responses (41) partially exponential [PE], show the difficulties presented by this situation, and highlights the difficulties in stabilizing this knowledge: because when students -in a given RS- doubted the response, they retook non-exponential strategies in that RS. This also shows that the representational systems do not evolve together. Thus, representational systems are far of being transparent, they have their own complexities, and therefore they require both an extended time of construction, as well as tasks that conducive them.

## 7. Conclusions

It can be observe that while students can solve a problem in a system of representation, they initially fail to do so in another representational system, even within the same situation. That is to say, the results obtained in RS, are not reinterpreted in another, at least when the conceptual knowledge of the field [CC] is incipient. TCF helps explain this by linking the construction of meaning (i.e., the operational invariants), to the set of situations and representational systems used; at the very beginning of the implementation they can use different invariant operative, under the representational system that the situation demands, as they are built for each particular representational system. But they will dominate the conceptual field of exponential functions only when they master a certain variety of concepts, schemes and symbolic representations in close connection. This shows that at the very beginning students use schemes in a disorderly manner, and as they advance the study of field conceptual they starts using the schemes coherently; until they finally succeed in solving exponentially in all of the representational systems. However, it is quite possible that a new situation destabilizes it and generate errors again. Therefore, if one want to improve the teaching of exponential functions, and its conceptualization, it would be appropriate to rethink teaching situations, as well as the treatment that will be made in each representational system.

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