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PHYSICS OF ELEMENTARY PARTICLES AND ATOMIC NUCLEI. THEORY

Charge Dynamics, Majorana Condition and the Topology of the Interacting Electromagnetic Field[1]

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Abstract—In previous works [1, 2], the 2-dimensional charge transport with parallel (in plane) magnetic field was considered from the theoretical point of view showing explicitly that the specific form of the emergent equation enforces the respective field solution to fulfil the Majorana condition. In this paper we review, explain and analyze these important results in the context of the generated physical effects, namely, the quantum ring as spin filter, the quantum Hall effect and a new one of pure topological origin (as the described by the Aharonov–Casher theorems). The link with supersymmetrical models is briefly discussed.

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1. INTRODUCTION AND PREVIOUS RESULTS

Regarding previous works, in 1937 Ettore Majorana propose a new representation to the recently proposed Dirac equation, where the components of the spinor solution are related themselves by complex conjugation [4]. The recent activity in condensed matter physics has focused on the 'Majorana zero modes' [5] i.e. emergent Majorana-like states occurring at exactly zero energy that have a remarkable property of, if they are considered as particles, being their own antiparticles (self-conjugated). As we remarked before, this property is currently expressed as a roughly (sometimes incorrect) equality between the particle's creation and annihilation operators. As we will see below, there exists the general idea that any ordinary fermion can be though of as composed of two Majorana fermions. However, this is an erroneous description because the pair of Majorana fermions have not the correct spinor transformation properties (other points of view see [23–25]). Really, a particular representation were a fermion effectively can be represented as bilinear combination of two states of fractional spin certainly exists [8–13].

From the practical point of view the considered in condensed matter "Majorana zero modes" are believed to exhibit the so called non-Abelian exchange statistics [14, 15] which endows them with a technological potential as building blocks of future quantum memory immune against many sources of decoherence. Recent advances in our understanding of solids with strong spin-orbit coupling, combined with the progress in nanofabrication, put as a possibility the physical realization of these exotic states. In fact, signatures consistent with their existence in quantum wires coupled to conventional superconductors and other type of devices have been reported by several groups [16]. As we have been shown [1, 2], when the magnetic field B is parallel to the plane defined by x, y axis (usually denominated: "B in plane") the Dirac equation takes the following form:

$$\left[\sigma_B \partial_B + \sigma_\perp (\partial_\perp - ieA_\perp) - ie\sigma_z A_z\right]\varphi = 0, \tag{1}$$

here, the subscripts B, $\perp$ and z denote the direction of the B field in the plane, the direction of the component of the potential vector in the plane (obviously, perpendicular to the B direction) and the direction of component of the potential vector coincident with the z axis, respectively.

Defining ω the angle of the magnetic field with respect the x axis in the plane x–y, the transformation takes in this case, the following general form:

$$\psi = e^{i(\alpha\sigma_x + \beta\sigma_y)}\varphi = e^{ie\varphi\sigma_B}\varphi \tag{2}$$

with

$$\alpha = \lambda\cos\omega, \quad \beta = \lambda\sin\omega \tag{3}$$

$$|\varphi|^2 = \lambda^2\left(\cos^2\omega + \sin^2\omega\right) = \lambda^2 \Rightarrow |\phi| = \pm|\lambda| \tag{4}$$

e.g the projection of the field ϕ. Equation (1) explicitly written having account (2), is

$$\begin{gathered}\Big[\sigma_x\partial_x + \sigma_y\partial_y - ieA_\perp \\ \left(\sigma_x\sin^2\omega + \sigma_y\cos^2\omega\right) - ie\sigma_z A_z\Big]\varphi = 0\end{gathered} \tag{5}$$

It is easily seen that, when $\omega = 0$ B coincides with x-axis and when $\omega = \pi/2$, B coincides with the y-axis) Also the Lie algebraic relation holds

$$\begin{gathered}\sigma_B\sigma_\perp = \left(\cos\omega\sigma_x + \sin\omega\sigma_y\right) \\ \times\left(-\sin\omega\sigma_x + \cos\omega\sigma_y\right) = i\sigma_z\end{gathered} \tag{6}$$

as expected. Then, using expression (2) in (1) we obtain explicitly the following non trivial conditions in order to gauge away the magnetic field

$$-\partial_\perp\phi = iA_z \quad \partial_B\phi = -A_\perp\sigma_\perp \tag{7}$$

The first equation is precisely as in the Aharonov-Casher theorem (ACT) case [6] but for the second one the interpretation suggest a complex structure for the field ϕ in a doublet form. Knowing that the doublet can be written as

$$\phi \equiv \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \tag{8}$$

the previous expressions take the following explicit form

$$-\partial_\perp\phi_1 = -\partial_\perp\phi_2 = iA_z \text{ and } \partial_B\phi_1 = -\partial_B\phi_2 = iA_\perp \tag{9}$$

Consequently, the introduction of two real functions u and v show that expression (9) is precisely the Majorana condition over the spinor ϕ

$$\phi \equiv \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} = \begin{pmatrix}\phi_1 \\ \phi_1^*\end{pmatrix} = \begin{pmatrix}u(x_\perp) + iv(x_B) \\ (u(x_\perp) + iv(x_B))^*\end{pmatrix} \tag{10}$$

in such a manner that the the conditions $-\partial_\perp\phi = iA_z$ and $\partial_B\phi = A_\perp$ to remove the magnetic field are Cauchy–Riemann type conditions on the components of ϕ (ϕ_1 and ϕ_2).

Remark 1. Notice that (9) is the Majorana condition over the spinor ϕ that appears as consequence of the magnetic field parallel to the plane of the charge dynamics, in a sharp contrast that in the ACT case.

1.1. Structure of the Magnetic Field: Conditions Over A and ϕ

Being the magnetic field determined from the vector potential $(B = \nabla \times A)$, with the components given by our physical scenario, namely A_z and $A_\perp$, the "in plane" magnetic field is consequently

$$B_B = (\partial_\perp A_z - \partial_z A_\perp), \tag{11}$$

where the simplest possibility was took: $A \neq A(x_B)$ e.g. the vector potential does not depends on the direction of the magnetic field, only on the plane defined by $x_\perp$ and x_z. From (10) follows

$$B = i\partial_\perp^2\phi = \frac{\Phi}{x_\perp} \tag{12}$$

where the total transversal flux to the plane per unit of longitude was used. Having into account the above point, ϕ is given by

$$\phi \cdot \sigma_B = -i(\Phi\sigma_B)x_\perp\left[\ln\left|\frac{x_\perp}{l_0}\right| - \frac{C - x_\perp}{x_\perp}\right] \tag{13}$$

Taking the arbitrary constant $C = 0$ for simplicity, the behaviour of the exponential function in (2) is

$$e^{-ie\phi\cdot\sigma_B} = \left|\frac{l_0}{x_\perp}\right|^{\frac{e\Phi}{l_0}\sigma_B x_\perp} e^{-\frac{e\Phi}{l_0}\sigma_B x_\perp} \tag{14}$$

with l_0 some real constant with units of length (e.g. minimal length). The following condition must be fulfilled in order that φ be normalizable and square integrable

$$\Phi s_B \geq 0 \tag{15}$$

(where s_B is the spin eigenvalue in the B direction) due

$$\varphi = e^{-ie\phi\cdot\sigma_B}\psi(s, w) \tag{16}$$

In the above expression, the function ψ depends on the spin and some complex variable w, to be determined.

1.2. Majorana, Dirac–Weyl States and Discrete Coordinates: Conditions Over $\psi(s,z)$

The simple Dirac–Weyl equation obtained after the elimination of the magnetic field is

$$\left(e^{-ie\phi\cdot\sigma_B}\sigma_B\partial_B + e^{ie\phi\cdot\sigma_B}\sigma_\perp\partial_\perp\right)\psi(s,z) = 0 \qquad (17)$$

to solve the equation (17) a quantization condition should be imposed on the flow (strictly on the product $\phi\cdot\sigma_B$), inducing an automatic discretization over the in plane transverse coordinate $x_\perp$.

$$\phi\cdot\sigma_B = n\pi, \quad n = 0,1,2,,, \qquad (18)$$

If above condition holds, we obtain

$$(\sigma_B\partial_B + \sigma_\perp\partial_\perp)\psi(s,z) = 0 \qquad (19)$$

This expression is very important: this is a simple 2-dimensional free Dirac equation (e.g. without A_μ). The particular phase [7] introduced plus the quantization condition over the flux cancel the effect of the magnetic field, (Dirac equation without interaction).

1.2.1.Analysis of the solution. As was pointed out in [1, 2], looking the specific form of the above equations, there are two possibilities over the spin behaviour of ψ :

(1) $\sigma_B\psi(s,z) = s\psi(s,z)$ (eigenspinor of σ_B)this case is compatible with the assumption that the state is eigenvector of the spin in the magnetic field direction. The Dirac equation is reduced to

$$\left(\partial_B + \frac{i\mathbb{C}}{s}\partial_\perp\right)\psi(s,z) = 0 \qquad (20)$$

with $\mathbb{C}$ the charge conjugation operator. Then, $\psi(s,z)$, and for instance $\varphi(s,z)$ must fulfill the Majorana condition:

$$\mathbb{C}\varphi(s,z) = \pm c\varphi(s,z). \qquad (21)$$

Similarly as in the AC case, $\psi(s,z)$ is an entire function of $z = x_B + \frac{ic}{s}x_\perp$ but the states solution is of Majorana type.

(2) $\sigma_z\psi(s,z) = s\psi(s,z)$ (eigenspinor of σ_Z) in this case the spin remains as in the ACT situation (e.g. in the z direction). Now the Dirac equation is reduced to

$$(\partial_B + is\partial_\perp)\psi(s,z) = 0. \qquad (22)$$

Similarly as in the AC case, $\psi(s,z)$ is an entire function of $z = x_B + isx_\perp$, and the state solution is Dirac–Weyl.

Remark 2. The specific form of the equation (20) shows that the states are Majorana-type. The charge conjugation operator $\mathbb{C}$ enforces the Majorana condition on the states solution of (22). This is a direct consequence of the symmetries of the physical scenario.

1.3. Generalized Momentum Operators and Majorana Conditions

In order to give an interpretation of the equation (1), namely $[\sigma_B\partial_B + \sigma_\perp(\partial_\perp - ieA_\perp) - ie\sigma_zA_z]\varphi = 0$, we rewrite it as follows

$$\left[\sigma_B\underbrace{(\partial_B - ie\sigma_B\sigma_zA_z)}_{\tilde{\Pi}_B} + \sigma_\perp\underbrace{(\partial_\perp - ieA_\perp)}_{\Pi_\perp}\right]\varphi = 0 \qquad (23)$$
$$\Rightarrow \left[\sigma_B\tilde{\Pi}_B + \sigma_\perp\Pi_\perp\right]\varphi = 0$$

then, using the algebra (6) $\sigma_B\sigma_z = i\,\sigma_\perp$and the definition of the charge conjugation operator as function of the sigma matrices, is easy to see that

$$(\partial_B - ie\sigma_B\sigma_zA_z) = (\partial_B + ie\mathbb{C}A_z) \qquad (24)$$

As in ordinary non abelian gauge theories, the operator $\tilde{\Pi}_B$ seems as equipped with a **non abelian** vector potential $\widetilde{A_B} \equiv -\mathbb{C}A_z$. This conceptual interpretation will be utilized in the next section for the analysis of the quantum ring.

2. PHYSICAL EFFECTS AND THE PARALLEL MAGNETIC FIELD

2.1. Quantum Hall Effect

As we know, if the plane where the charges are moving is *finite*, a current transversal to the magnetic field B must appear in the plane (e.g. in the .$x_\perp$ direction). This current will be quantized due the condition (18). This condition explicitly can be written as

$$\phi\cdot\sigma_B = (\Phi\sigma_z)\,\tilde{x}_\perp\left[\ln\left|\frac{x_\perp}{l_0}\right| - 1\right] = n\pi, \qquad (25)$$
$$n = 0,1,2,,,$$

Where $\tilde{x}_\perp = \sigma_\perp x_\perp$ is a new matrix valuated coordinate whose meaning will be analyzed later.

The explicit formula for the Hall current coming from the expression for the surface current

$$n\times B = K_{\text{surface}} \qquad (26)$$

(n: unitary vector normal to the interface surface.) The above current is obviously perpendicular to the magnetic field in the plane(e.g. $x_\perp$ direction). Due the quantization condition, the "emergent" Hall current also is quantized leading the Quantum Hall effect (QHE)

$$\frac{\Phi}{x_\perp}\overset{v}{x}_\perp = \frac{2\pi N\hbar c}{ex_\perp}\overset{v}{x}_\perp = K_{\text{surface}}, \qquad (27)$$

(where $\overset{v}{x}_\perp$ is the unitary vector in the $x_\perp$ direction.)

2.2. The Quantum Ring

Was recently pointed out [17], that the Rashba and Dresselhaus spin-orbit interactions in two dimensions can be regarded as a non-Abelian gauge field reminiscent of the standard Yang–Mills in QFT [18]. The explanation given in such references is that the physical field generated by the gauge field brings to the electron wave function a spin-dependent phase. This phase generally is called the Aharonov–Casher phase. In ref. [17] the authors showed that applying on an AB ring this non-Abelian field plus the usual vector potential, a interference condition completely destructive for one component of the spin while completely constructive for the other component of the spin appears over the full energy range. This enables us to construct a perfect spin filter.

However in [17] the magnetic field was **perpendicular** to the plane of the ring. Now we will proceed analogously but considering the **in plane** magnetic field to see the physical consequences over the physical states and consequently, over the spin control. Now, in order to perform the analysis and to compare with the case described in [17], the same method and definitions will be used remitting to the reader to ref. [17] for more details.

The general Dirac equation with the magnetic field parallel ("in plane") in cylindrical coordinates takes the form

$$\left[\sigma_\rho\partial_\rho + \frac{1}{\rho}\sigma_\varphi\partial_\varphi - ie\underbrace{\left(-\sigma_\rho A_\rho\sin(\omega-\varphi) + \sigma_\varphi A_\varphi\cos(\omega-\varphi)\right)}_{\propto\sigma_\perp A_\perp} - ie\sigma_z A_z\right]\hat{\varphi} \tag{28}$$

$$= \left[\sigma_\rho\partial_\rho + \frac{1}{\rho}\sigma_\varphi\left(\partial_\varphi - ie\sigma_\rho\underbrace{\widetilde{A_z}}_{2iA_z}\right) - ie\left(-\sigma_\rho A_\rho\sin(\omega-\varphi) + \sigma_\varphi A_\varphi\cos(\omega-\varphi)\right)\right]\hat{\varphi} = 0, \tag{29}$$

where in the last equation the properties of the algebra described in the previous paragraph have been used in order to introduce the "non abelian" potential. The explicit form of the Pauli matrices in the configuration that we are interested in are

$$\sigma_\rho = \sigma_x\cos\varphi + \sigma_y\sin\varphi \tag{30}$$

$$\sigma_\varphi = \sigma_y\cos\varphi - \sigma_x\sin\varphi \tag{31}$$

$$\sigma_B = \sigma_\rho\cos(\omega-\varphi) + \sigma_\varphi\sin(\omega-\varphi) \tag{32}$$

$$\sigma_\perp = \sigma_\varphi\cos(\omega-\varphi) - \sigma_\rho\sin(\omega-\varphi) \tag{33}$$

However, φ is the angular cylindrical coordinate, ω is the angle of the magnetic field parallel to the plane measured from the axis x (e.g. $\varphi = 0$) and the state is denoted as $\hat{\varphi}$. For the ring configuration, and having account the condition that the potential doesn't depends on the direction of the magnetic field, the Dirac equation takes this non abelian form

$$\left[\frac{1}{\rho}\sigma_\varphi\left(\partial_\varphi - ie\sigma_\rho\widetilde{A_z}\right)\right]\hat{\varphi} = 0 \tag{34}$$

then, the corresponding second order Hamiltonian for the magnetic field in plane is

$$\mathcal{H}_{\text{ring}} = \frac{1}{\rho^2}\left(\partial_\varphi - ie\sigma_\rho\widetilde{A_z}\right)^2 \tag{35}$$

Notice the non-abelian character of the above equation.

2.2.1. Screening of Rashba term and "in plane" magnetic field. When the Rashba spin-orbit interaction is introduced, the following Hamiltonian is obtained

$$\mathcal{H}_{\text{ring}}\Big|_{B_{n\text{-plane}}} = \frac{\hbar^2}{2m^*R^2}\left(-i\partial_\varphi - \sigma_\rho\left(\underbrace{\widetilde{A_z}}_{\text{potential}} + \underbrace{\frac{\theta R}{2}}_{\text{Rashba}}\right)\right)^2, \tag{36}$$

where $\theta \equiv \frac{2m^*\alpha}{\hbar}$ plays the role of coupling constant of the Rashba term (the same units as in reference [17]). The main point is that the vector potential corresponding to the "in plane" magnetic field (perpendicular to the plane of the ring) is at the same **non-abelian** level that the Rashba term.

In the case treated by the authors in [17], the magnetic field is perpendicular having the Hamiltonian for the ring the following fashion

$$\mathcal{H}_{\text{ring}}\Big|_{B_z} = \frac{\hbar^2}{2m^*R^2}\left(-i\partial_\varphi - \underbrace{\phi_B}_{\frac{e\pi R^2B_z}{h}} - \left(\sigma_\rho\underbrace{\frac{\theta R}{2}}_{\text{Rashba}}\right)\right)^2, \tag{37}$$

where is easily seen that ϕ_B is not at the same "non abelian" level of the Rashba term. Consequently, is this the explanation of the screening of the Rashba interaction by the "in plane" magnetic field.

2.2.2. Physical consequences. We know from previous Sections that we can select solutions that are eigenfunctions of σ_z. Besides this issue, the potential vector A_z must come perpendicularly to the ring plane ($\hat{z}$ direction) in concordance with the ACT situation. Notice that, in sharp contrast with previous references, *the effective Hamiltonian arises from the true*

Dirac equation with minimal coupling. As we can assume in general that we know the magnetic field (B_{pl}) in the plane : $e\widetilde{A_z} \equiv 2ieA_z = 2ieB_{pl}R/h$, then

$$\mathcal{H}_{\text{ring}}\big|_{B_{n\text{-plane}}} = \frac{\hbar}{2m^* R^2}\left[-i\partial_\varphi - \sigma_\rho\left(\frac{(\theta - 4eB_{pl})R}{2}\right)\right]^2 \quad (38)$$

the interplay between the magnetic field parallel to the plane of the ring and the Rashba interaction is clearly seen.

Assuming free interaction into the 2 leads, in response to

$$\mathcal{H}_{\text{lead}} = -\frac{\hbar^2}{2m^*}\partial_x^2 \quad (39)$$

we obtain (same units and notation that in ref. [17]) for the ring Hamiltonian the wave functions

$$\Psi_{\pm\pm} = e^{i(\pm k_\varphi \pm \phi_T)\varphi} e^{-i\beta\sigma_\varphi/2}\chi_\pm \quad (40)$$

($\chi_\pm$ eigenfunctions of σ_z) with the eigenvalues

$$E = \frac{\hbar^2 k_\varphi^2}{2m^* \rho^2} \quad (41)$$

where now the total phase is

$$\phi_T = \sqrt{1 + (\theta - 4eB_{pl})^2 \rho^2} - 1, \quad (42)$$

and

$$\beta = \arctan \xi \quad (43)$$

$$\text{with } (\theta - 4eB_{pl}) \equiv \xi. \quad (44)$$

Notice the important fact that the total phase φ_T is **identically zero** if the following condition holds

$$\theta = 4eB_{pl}. \quad (45)$$

As in [17] the first sign of $\Psi_{\pm\pm}$ denotes the sign of the momentum, and the second one of the spin. In the phase corresponding to Rashba interaction, an small radius ρ of the ring was considered. Following similar task that in [17] in order to realize a perfect spin filter, the wave function (41) at $\varphi = 2\pi$ is

$$\Psi_{\pm\pm}(2\pi, k_\varphi) = e^{\pm 2i\pi k_\varphi} U_{\text{phase}}\chi_\pm; \quad U_{\text{phase}} = e^{\pm 2\pi i\phi_T} e^{-i\beta\sigma_y/2} \quad (46)$$

realizing the spin filter by adjusting parameters:

$$2\pi\phi_T = (2n+1)\pi.$$

At this point, two important cases must be considered:

a. Case (a) Considering (43) and small radius ρ, the above condition is translated to

$$\xi\rho = \sqrt{n + 3/2}, \quad n \in Z = \sqrt{\frac{3}{2}}, \sqrt{\frac{5}{2}}, \sqrt{\frac{7}{2}} \ldots\ldots \quad (47)$$

b. Case (b) Case (a) must be complemented with a condition that only appears as an effect of the existence of the magnetic field in plane that is

$$\xi = 0 \rightarrow \theta = 4eB_{pl}.$$

Both conditions, realize the perfect spin filter being the second condition possible **only** in the case when the magnetic field is "in plane", and it importance will be more evident in the coefficient transmission description, as follows.

The eigenvectors of the phase factor $U_{\text{phase}} = e^{\pm 2\pi i\phi_T} e^{-i\beta\sigma_y/2}$ can be exactly computed,

$$\tilde{\chi}_+ = \begin{pmatrix} \frac{\sqrt{\xi^2 + 1} + 1}{2} \\ \xi/2 \end{pmatrix} \quad (48)$$

$$\tilde{\chi}_- = \begin{pmatrix} \xi/2 \\ -\frac{\sqrt{\xi^2 + 1} + 1}{2} \end{pmatrix} \quad (49)$$

Notice that when the critical value $\theta = 4eB_{pl}$ holds, then $\xi = 0$ consequently the eigenvectors $\tilde{\chi}_\pm$ goes automatically to $\chi_\pm$ (eigenvectors of σ_z) as expected in sharp contrast with similar results in [17] that they correctness is doubtful. Although there are several effective manners to compute the transmission coefficients, we following ref. [17] in order to compare the results with other works involving the similar devices. We first assume the amplitudes of the left-going and right-going wave functions separately for the left lead, the portion $0 < \varphi < \pi$ of the ring, the portion $\pi < \varphi < 2\pi$ of the ring, and the right lead. This amounts to sixteen amplitudes in total when we take the spin degree of freedom into account. The continuation of the wave function at $\varphi = 0$ and $\varphi = \pi$ give eight conditions and the conservation of the generalized momentum at $\varphi = 0$ and $\varphi = 0$ give four conditions. Then, four degrees of freedom finally remain. The S-matrix of the quantum ring is obtained by expressing the four amplitudes of the out-going waves (the left-going wave on the left lead and the right-going wave on the right lead with spin up and down) in terms of the four amplitudes of the incoming waves (the right-going wave on the left lead and the left-going wave on the right lead with spin up and down). The off-diagonal 2×2 block of the 4×4 S–matrix give the transmission coefficients. In our case, the transmission coefficients are proportional to

$$T_{\tilde{\uparrow}\tilde{\uparrow}}, T_{\tilde{\uparrow}\tilde{\downarrow}} \propto \left|1 + e^{2\pi i\phi_T}\right|^2, \quad (50)$$

$$T_{\tilde{\downarrow}\tilde{\uparrow}}, T_{\tilde{\downarrow}\tilde{\downarrow}} \propto \left|1 + e^{-2\pi i\phi_T}\right|^2 \quad (51)$$

where $\tilde{\uparrow}$ and $\tilde{\downarrow}$ denote respectively the spin up (48) and spin down (49) diagonalizing the phase factor $U_{\text{phase}} = e^{\pm 2\pi i \phi_T} e^{-i\beta\sigma_y/2}$.

Summarizing: in the case (a) evidently $T_{\tilde{\uparrow}\tilde{\uparrow}}, T_{\tilde{\uparrow}\tilde{\downarrow}}, T_{\tilde{\downarrow}\tilde{\uparrow}}, T_{\tilde{\downarrow}\tilde{\downarrow}} = 0$, and in the case (b) the transmission coefficients are constant, realizing together the perfect spin filter [17, 19].

2.3. Dirac–Majorana Oscillator: Susy, Algebra and Parastatistics

A relativistic fermion under the action of a linear vector potential usually called the Dirac oscillator [21]. The standard Dirac oscillator can be exactly solved in one, two and three dimensions. It has in the non-relativistic limit the associated Klein–Gordon equations describing a harmonic oscillator in the presence of a strong spin-orbit coupling, and the first experimental realization of this system was reported recently [22]. Motivated by these important reasons plus the possibility to analyze the (super) symmetries into the obtained spectrum, our goal in this Section is to rewrite conveniently the Dirac equation corresponding to the "in plane" magnetic field configuration in the form of the Dirac oscillator.

Our starting point is as follows: in 2 dimensions we have

$$\left[c\sigma_\perp p_\perp + eB\sigma_z X_\perp + mc^2\right]\varphi = E\varphi \equiv H_{2D}\varphi. \quad (52)$$

Introducing the corresponding creation and annihilation operators as

$$H_{2D} = i\left(\frac{eBc\hbar}{2}\right)^{1/2}\sigma_\perp\left(a^+ - a\right) + \left(\frac{eBc\hbar}{2}\right)^{1/2}\sigma_z\left(a^+ + a\right) + mc^2 \quad (53)$$

we can redefine and rearrange the operators in order to put the Hamiltonian in the simpler form:

$$H_{2D} = \frac{i}{\sqrt{2}}\left[a^+(\sigma_\perp - i\sigma_z) - a(\sigma_\perp + i\sigma_z)\right] + \mu, \quad (54)$$

where the energy is given in $(eBc\hbar)^{1/2}$ units and we have defined $\mu = mc\sqrt{\frac{c}{eB\hbar}}$. Explicitly

$$H_{2D} = H_{2D} = \begin{pmatrix} \frac{(a^+ + a)}{\sqrt{2}} + (\mu - E) & \frac{(a^+ - a)}{\sqrt{2}}e^{-i\omega} \\ \frac{(a^+ - a)}{\sqrt{2}}e^{+i\omega} & -\frac{(a^+ + a)}{\sqrt{2}} + (\mu - E) \end{pmatrix}\varphi = 0. \quad (55)$$

The first important observation is that the Hamiltonian (56) has the suggestive fashion of the BHZ phenomenological model [7]. This BHZ model was a "by hand" attempt to explain the topological insulator mechanism. Then, we are able to bring a natural explanation to the topological insulators described in [7] from a pure phenomenological viewpoint. Expanding the state φ in the n basis and taking into account that it must be invariant under $i\mathbb{C}$ $(\equiv -\sigma_2)$ we obtain the following expression

$$\varphi = \begin{pmatrix} 1 \\ e^{i\pi/2} \end{pmatrix} = \sum_{k=0}^{\infty}\left[A_{2k}|2k\rangle + A_{2k+1}|2k+1\rangle\right]. \quad (56)$$

However, the coefficients A_n are not independent. A_{2k} and A_{2k+1} are related to the two first coefficients A_0 and A_1corresponding to the states $|0\rangle$ and $|1\rangle$ respectively, provided again that the following quantization condition over the ω arises:

$$\omega = \pi(k+1), \quad k = 0,1,2\ldots. \quad (57)$$

Consequently, the normalized state solution takes the following form:

$$|\varphi\rangle = \begin{pmatrix} 1 \\ e^{i\pi/2} \end{pmatrix}\sum_{k=0}^{\infty}\left[A_0\underbrace{\frac{\sqrt{(2k-1)!!}}{e^{1/4}}\frac{|2k\rangle}{\sqrt{2k!}}}_{|\Psi_{1/4}\rangle} + A_1\underbrace{\frac{\sqrt{(2k)!!}}{\left(\sqrt{\frac{e\pi}{2}}Erf(1/2)\right)^{1/2}}\frac{|2k+1\rangle}{\sqrt{(2k+1)!}}}_{|\Psi_{3/4}\rangle}\right] \equiv \begin{pmatrix} 1 \\ e^{i\pi/2} \end{pmatrix}\left(A_0|\Psi_{1/4}\rangle + A_1|\Psi_{3/4}\rangle\right). \quad (58)$$

$\left(A_1^2, A_0^2 = \pm 1\right)$ as is easily seen $|\varphi\rangle$ is a coherent state of Klauder–Perelomov/Barut–Girardello type. It can be generated by a suitable displacement operator D and, under normalization, it is eigenstate of the annihilation operator a. The coefficients A_0 and A_1 are arbitrary, in principle, with the property $A_1^2, A_0^2 = \pm 1$. This fact permits us to have two eigenstates of the annihilation operator a with different parity behaviour under such operator: $A_0 = \pm A_1 \Rightarrow |\varphi_\pm\rangle = A_0\begin{pmatrix} 1 \\ e^{i\pi/2} \end{pmatrix}\left(|\Psi_{1/4}\rangle \pm |\Psi_{3/4}\rangle\right)$ then:

$$a|\varphi_\pm\rangle = \pm|\varphi_\pm\rangle \quad (59)$$

Remark 3. The states solution $|\varphi\rangle$ are independent of the energy. It is a characteristic of the Majorana states that commonly appear in quantum transport in nanostructures.

c. Relation with supersymmetric models. The dynamics of the $|\Psi\rangle$ fields were extensively studied in supersymmetric models. In previous references [3], was demonstrated that the analysis of the particular representation that we are interested in can be simplified considering these fields as coherent states in the sense that are eigenstates of a^2 [3]:

$$\begin{aligned} |\Psi_{1/4}(0,\xi,q)\rangle &= \sum_{k=0}^{+\infty} f_{2k}(0,\xi)|2k\rangle \\ &= \sum_{k=0}^{+\infty} f_{2k}(0,\xi)\frac{\left(a^{\dagger}\right)^{2k}}{\sqrt{(2k)!}}|0\rangle, \\ |\Psi_{3/4}(0,\xi,q)\rangle &= \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)|2k+1\rangle \\ &= \sum_{k=0}^{+\infty} f_{2k+1}(0,\xi)\frac{\left(a^{\dagger}\right)^{2k+1}}{\sqrt{(2k+1)!}}|0\rangle. \end{aligned} \tag{60}$$

From a technical point of view these states are a one mode squeezed states constructed by the action of the generators of the SU (1,1) group over the vacuum. For simplicity, we will take all the normalization and the fermionic dependence into the functions $f(\xi)$. Explicitly (supposing in principle no time dependence, e.g. $t = 0$)

$$\begin{aligned} |\Psi_{1/4}(0,\xi,q)\rangle &= f(\xi)|\alpha_+\rangle \\ |\Psi_{3/4}(0,\xi,q)\rangle &= f(\xi)|\alpha_-\rangle, \end{aligned} \tag{61}$$

where $|\alpha_\pm\rangle$ are the CS basic states in the subspaces $\lambda = \frac{1}{4}$ and $\lambda = \frac{3}{4}$ of the full Hilbert space [3]. In the case that the physical states span the full Hilbert space, the Heisenberg–Weyl (HW) realization for the states Ψ must be used:

$$|\varphi\rangle = \frac{f(\xi)}{2}(|\alpha_+\rangle + |\alpha_-\rangle) = f(\xi)|\alpha\rangle. \tag{62}$$

In (48) the linear combination of the states $|\alpha_+\rangle$ and $|\alpha_-\rangle$ corresponding to the $\lambda = 1/2$ CS basis, span now the full Hilbert space (dense). As we will see in a future paper, this particular representation describes perfectly the Majorana fermion behaviour that is phenomenologically obtained [22].

3. CONCLUDING REMARKS AND OUTLOOK

From our previous works [1, 2], we have shown how mathematics can predict physical effects and describe various phenomena with great precision and reliability. Through this specific analysis we have given examples accompanied with new results using as the physical scenario to describe the quantum transport of charged particles, a two-dimensional space with a parallel magnetic field. The mathematical description of the problem presented (related in some points with the techiques given in [26, 27]), the quantum effects that before have been explained through empirical/phenomenological methods are now easily explained as the quantum Hall effect and the Majorana states in low dimensional structures with particular field conditions.

In resume we have:

• the specific form of the arising equation enforce the respective field solution to fulfil the Majorana condition;

• when any physical system is represented by this equation the rise of fields with Majorana type behaviour is immediately explained and predicted;

• there exists a quantized particular phase that removes (gauge away) the action of the vector potential producing non-standard effects being the Quantum Hall Effect (QHE) one of them;

• the interpretation of Dirac equation with a non abelian electromagnetic field appears as a consequence of the conditions imposed on the fields;

• the quantum ring with paralell magnetic field was worked out showing where these new effects must appear.

As was mentioned before, the non-abelian operator was introduced "by hand" in order to reproduce the effects of the interaction of type RD (Rashba–Dresselhaus). In our case, the "non-abelian" term appears due to the presence of the parallel magnetic field parallel to the plane where the charges are transported. Therefore, and as we saw in the problem of the ring in Section 2, there is competition or screening between the effects produced by the interaction RD and from the parallel magnetic field. This competition brings two important consequences, namely:

(1) new spin filter effects (different in escence to [17]), as we have analyzed.

(2) measurable effects of screening that could give a clear explanation of the new effects observed in planar nanostructures described in references [20].

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REFERENCES

1. D. J. Cirilo-Lombardo, Phys. Part. Nucl. Lett. **13**, 26–31 (2016).
2. D. J. Cirilo-Lombardo, Int. J. Geom. Methods Mod. Phys. **12**, 1550088 (2015).

3. D. J. Cirilo-Lombardo, Int. J. Geom. Methods Mod. Phys. **13**, 1650004 (2016).
4. E. Majorana, Nuovo Cim. A **14**, 171 (1937).
5. M. Franz, Nat. Nanotechnol. **8**, 149–152 (2013); arXiv: 1302.3641.
6. Y. Aharonov and A. Casher, Phys. Rev. A **19**, 2461 (1979).
7. Geometric Phases in Physics, Ed. by A. Shapere and F. Wilczek (World Scientific, Singapore, 1989); M. V. Berry, Proc. R. Soc. London, Ser. A **392**, 45 (1984).
8. D. J. Cirilo-Lombardo, Phys. Lett. B **661**, 186–191 (2008).
9. D. J. Cirilo-Lombardo, Found. Phys. **37**, 919–950 (2007).
10. D. J. Cirilo-Lombardo, Found. Phys. **39**, 373–396 (2009).
11. D. J. Cirilo-Lombardo, Eur. Phys. J. C **72**, 2079 (2012).
12. D. J. Cirilo-Lombardo and V. I. Afonso, Phys. Lett. A **376**, 3599–3603 (2012).
13. D. J. Cirilo-Lombardo and T. Prudencio, Int. J. Geom. Methods Mod. Phys. **11**, 1450067 (2014).
14. Y. Aharonov and J. Anandan, Phys. Rev. Lett. **58**, 1593 (1987).
15. Y. Aharonov and A. Casher, Phys. Rev. Lett. **53**, 319 (1984).
16. S. Nadj-Perge et al., Science **346** (6209), 602–607 (2014).
17. N. Hatano et al., Phys. Rev. A **75**, 032107 (2007).
18. T. Fujita et al., J. Appl. Phys. **110**, 121301 (2011).
19. T. C. Cheng et al., Phys. Rev. B **76**, 214423 (2011).
20. M. Valin-Rodriguez and R. G. Nazmitdinov, Phys. Rev. B **73**, 235306 (2006).
21. M. Moshinsky and A. Szczepaniak, "The Dirac oscillator," J. Phys. A **22**, L817 (1989).
22. J. A. Franco-Villafane, E. Sadurni, S. Barkhofen, U. Kuhl, F. Mortessagne, and T. H. Seligman, "First experimental realization of the Dirac oscillator," Phys. Rev. Lett. **111**, 170405 (2013).
23. O. S. Kosmachev, Phys. At. Nucl. **76**, 1157–1160 (2013).
24. O. S. Kosmachev, Phys. At. Nucl. **75**, 885–887 (2012).
25. A. A. Gusev and O. S. Kosmachev, Phys. Part. Nucl. Lett. **5**, 67–71 (2008).
26. A. A. Susko, Phys. At. Nucl. **56**, 674–679 (1993).
27. A. A. Susko and E. Velicheva, Phys. Part. Nucl. **43**, 348–366 (2012).