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Dynamical symmetries, coherent states and nonlinear realizations: The $SO(2,4)$ case

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Nonlinear realizations of the $SO(4,2)$ group are discussed from the point of view of symmetries. Dynamical symmetry breaking is introduced. One linear and one quadratic model in curvature are constructed. Coherent states of the Klauder–Perelomov type are defined for both cases taking into account the coset geometry. A new spontaneous compactification mechanism is defined in the subspace invariant under the stability subgroup. The physical implications of the symmetry rupture in the context of nonlinear realizations and direct gauging are analyzed and briefly discussed.

Keywords: Group theory; conformal symmetry; coherent states; nonlinear realizations.

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1. Introduction

Studies of higher-dimension theories that involve (spontaneously) broken symmetries and noncommutativity in the quantum case are motivated by searches for a unified theory. Dimensional reduction of such theories is not unique and becomes extremely involved when gravity is included. We believe that the guiding principles

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for the reduction are provided by the observed (or desirable) physical field content and by the group theoretical structure itself.

From the technical point of view, we have to extend physical fields into an extra (internal) space with preserving the general noncommutative quantum structure. However, the development of a mechanism that permit us to display the set of physical fields in interaction with the corresponding four-dimensional world implies that some of the original symmetries of the higher-dimension manifold have been broken. There exist many theoretical attempts to realize the above ideas such as string and brane theories but none of them can be treated as the final answer: formulation of such theories contain serious problems that are still not solved. In spite of the fact that in these theories the solution seems to include a non-commutative structure [1, 2], the concrete implementation of these symmetries in a substructure of any (super) manifold seems to be very complicated from the technical and geometrical viewpoints.

However, there exists another way to attack the unification problem that is in the context of gauge theories of gravity [3–5]. The first model of gauge gravitation theory was suggested by Utiyama [6] in 1956 just two years after the birth of gauge theory itself. He was the first who generalized the original $SU(2)$ gauge model of Yang and Mills to an arbitrary symmetry Lie group and, in particular, to the Lorentz group in order to describe gravity. However, he met the problem of treating general covariant transformations and a pseudo-Riemannian metric which had no partner in the Yang–Mills gauge theory. To eliminate this drawback, representing a tetrad gravitational field as a gauge field of a translation subgroup of the Poincaré group was attempted because, by analogy with gauge potentials in the Yang–Mills gauge theory, the indices of a tetrad field μ were treated as those of a translation group, see [3, 4, 7–11] and references therein. Since the Poincaré group comes from the Wigner–Inonu contraction of de Sitter groups $SO(2, 3)$ and $SO(1, 4)$ and it is a subgroup of the conformal group, gauge theories on fiber bundles with these structure groups were also considered [12–18]. Because these fiber bundles fail to be natural, the lift of the group Diff(X) of diffeomorphisms of the fiber onto the base should be defined [19, 20]. However, these gauging approaches contain the problem with a nonlinear (translation) summand of an affine connection being a soldering form, but neither a frame (vierbein) field nor a tetrad field. Thus, the latter does not have the status of a gauge field [21–23]. At the same time, a gauge theory in the case of spontaneous symmetry breaking also contains classical Higgs fields, besides the gauge and matter ones [24–32]. Therefore, based on the mathematical definition of a pseudo-Riemannian metric, some authors formulated gravitation theory as a gauge theory with a reduced Lorentz structure where a metric gravitational field is treated as a Higgs field [33–37].

The most satisfactory answer to the formulation of gravity as a gauge theory was developed in the pure geometrical context in the works of Volkov *et al.* [38, 39]; in the context of supergravity by Arnowitt and Pran Nath [40]; and finally by

Mansouri [41] who was able to solve some of the problems listed before by means of a principal fiber bundle imposing a condition of orthogonality of the generators of the fiber and base manifold. Such conditions that break the symmetry of the original group are implemented by means of a particular choice of the metric tensor. This approach was implemented in a supergroup structure obtaining a gauge theory of supergravity. Note that the underlying geometry must be reductive (in the Cartan sense) or weakly reductive in the case of supergravity.

As always, even the problem to determine which fields transform as gauge fields and which not, as well as which fields are physical ones and which are redundant, nonetheless remains. Also the relation between the coset factorization (as in the case of the nonlinear realization approach [47–49]) and the specific breaking of the symmetry in the pure topological theories of grand unification (GUT) is still unclear.

2. Coset Coherent States

Let us remind the definition of coset coherent states

$$H_0 = \{g \in G \mid \mathcal{U}(g)V_0 = V_0\} \subset G. \tag{1}$$

Consequently, the orbit is isomorphic to the coset, e.g.

$$\mathcal{O}(V_0) \simeq G/H_0. \tag{2}$$

Analogously, if we remit to the operators, e.g.

$$|V_0\rangle\langle V_0| \equiv \rho_0 \tag{3}$$

then the orbit

$$\mathcal{O}(V_0) \simeq G/H \tag{4}$$

with

$$H = \{g \in G \mid \mathcal{U}(g)V_0 = \theta V_0\} = \{g \in G \mid \mathcal{U}(g)\rho_0\mathcal{U}^\dagger(g) = \rho_0\} \subset G. \tag{5}$$

The orbits are identified with coset spaces of G with respect to the corresponding stability subgroups H_0 and H being the vectors V_0 in the second case defined within a phase. From the quantum viewpoint $|V_0\rangle \in \mathcal{H}$ (the Hilbert space) and $\rho_0 \in \mathcal{F}$ (the Fock space) are V_0 normalized fiducial vectors (an embedded unit sphere in $\mathcal{H}$).

3. Symmetry Breaking Mechanism: The $SO(4, 2)$ Case

3.1. *General considerations*

(i) Let $a, b, c = 1, 2, 3, 4, 5$ and $i, j, k = 1, 2, 3, 4$ (in the six-matrix representation) then the Lie algebra of $SO(2, 4)$ is

$$i[J_{ij}, J_{kl}] = \eta_{ik}J_{jl} + \eta_{jl}J_{ik} - \eta_{il}J_{jk} - \eta_{jk}J_{il}, \tag{6}$$

$$i[J_{5i}, J_{jk}] = \eta_{ik} J_{5j} - \eta_{ij} J_{5k}, \tag{7}$$

$$i[J_{5i}, J_{5j}] = -J_{ij}, \tag{8}$$

$$i[J_{6a}, J_{bc}] = \eta_{ac} J_{6b} - \eta_{ab} J_{6c}, \tag{9}$$

$$i[J_{6a}, J_{6b}] = -J_{ab}. \tag{10}$$

(ii) Identifying the first set of commutation relations (6) as the Lie algebra of the $SO(1,3)$ with generators $J_{ik} = -J_{ki}$.

(iii) The commutation relations (6) plus (7) and (8) are identified as the Lie algebra $SO(2,3)$ with the additional generators J_{5i} and $\eta_{ij} = (1,-1,-1,-1)$.

(iv) The commutation relations (6)–(10) are the Lie algebra $SO(2,4)$ written in terms of the Lorentz group $SO(1,3)$ with the additional generators J_{5i}, J_{6b}, and $J_{ab} = -J_{ba}$, where $\eta_{ab} = (1,-1,-1,-1,1)$. It follows that the embedding is given by the chain $SO(1,3) \subset SO(2,3) \subset SO(2,4)$.

From the six-dimensional matrix representation, we know from that parameterizing the coset $\mathcal{C} = \frac{SO(2,4)}{SO(2,3)}$ and $\mathcal{P} = \frac{SO(2,3)}{SO(1,3)}$, then any element G of $SO(2,4)$ is written as

$$SO(2,4) \approx \frac{SO(2,4)}{SO(2,3)} \times \frac{SO(2,3)}{SO(1,3)} \times SO(1,3), \tag{11}$$

explicitly

$$\begin{aligned} G &= e^{-iz^a(x)J_a} G(H) \\ &= e^{-iz^a(x)J_a} e^{-i\varepsilon^k(x)P_k} H(\Lambda). \end{aligned} \tag{12}$$

Consequently, we have $G(H) : H \to G$ is an embedding of an element of $SO(2,3)$ into $SO(2,4)$ where $J_a \equiv \frac{1}{\lambda} J_{6a}$ and $H(\Lambda) : \Lambda \to H$ is an embedding of an element of $SO(1,3)$ into $SO(2,3)$ where $P_k \equiv \frac{1}{m} J_{5k}$ as follows

$$G = e^{-iz^a(x)J_a} \underbrace{e^{-i\varepsilon^k(x)P_k} \underbrace{\begin{pmatrix} \boxed{SO(3,1)} & \mathbf{0} \\ \mathbf{0} & \boxed{I_{2x2}} \end{pmatrix}}_{H(\Lambda)}}_{G(H)} \tag{13}$$

then any element G of $SO(2,4)$ is written as the product of an $SO(2,4)$ boost, an ADS boost, and a Lorentz rotation.

4. Goldstone Fields and Symmetries

(i) Our starting point is to introduce two six-dimensional vectors V_1 and V_2 being invariant under $SO(3,1)$ in a canonical form. Explicitly

$$\left.\underbrace{\begin{pmatrix}0\\0\\0\\0\\A\\0\end{pmatrix}}_{V_1}+\underbrace{\begin{pmatrix}0\\0\\0\\0\\0\\-B\end{pmatrix}}_{V_2}=\underbrace{\begin{pmatrix}0\\0\\0\\0\\A\\-B\end{pmatrix}}_{V_0}\right\}\text{ invariant under } SO(3,1). \tag{14}$$

(ii) Now we take an element of $Sp(2)\subset Mp(2)$ embedded in the six-dimensional matrix representation operating over V as follows

$$\mathcal{M}V\to\underbrace{\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&a&b\\0&0&0&0&c&d\end{pmatrix}}_{Sp(2)\subset Mp(2)}\underbrace{\begin{pmatrix}0\\0\\0\\0\\A\\-B\end{pmatrix}}_{V_0}=\begin{pmatrix}0\\0\\0\\0\\A'\\-B'\end{pmatrix}=V', \tag{15}$$

where

$$A'=aA-bB,\quad -B'=cA-dB \tag{16}$$

consequently we obtain a *Klauder–Perelomov generalized coherent state* with the fiducial vector V_0.

(iii) The specific task to be made by the vectors is to perform the symmetry breakdown to $SO(3,1)$. Using the transformed vectors above ($Sp(2)\sim Mp(2)$ CS) the symmetry of G can be extended to an internal symmetry as $SU(1,1)$ given by $\widetilde{G}$ below (note that $|\lambda|^2-|\mu|^2=1$):

$$\widetilde{G}V'=\underbrace{e^{-iz^a(x)J_a}e^{-i\varepsilon^k(x)P_k}\underbrace{\begin{pmatrix}\boxed{SO(3,1)}&\mathbf{0}\\\mathbf{0}&\boxed{\begin{matrix}\lambda&\mu\\\mu^*&\lambda^*\end{matrix}}\end{pmatrix}}_{\widetilde{H}(\Lambda)}}_{\widetilde{G}(H)}V' \tag{17}$$

$$= e^{-iz^a(x)J_a} \underbrace{e^{-i\varepsilon^k(x)P_k} \underbrace{\begin{pmatrix} \boxed{\quad SO(3,1) \quad} & \mathbf{0} \\ \mathbf{0} & \boxed{\begin{matrix} \alpha & 0 \\ 0 & \beta \end{matrix}} \end{pmatrix}}_{H(\Lambda)}}_{G(H)} V_0 = GV_0, \tag{18}$$

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda^*\alpha & -\mu\beta \\ 0 & 0 & 0 & 0 & -\mu^*\alpha & \lambda\beta \end{pmatrix} \tag{19}$$

and if we also ask for $\mathrm{Det}\mathcal{M} = 1$ then $\alpha\beta = 1$, e.g. the additional phase: it will bring us the 10th Goldstone field. The other nine are given by $z^a(x)$ and $\varepsilon^k(x)$ $(a, b, c = 1, 2, 3, 4, 5$ and $i, j, k = 1, 2, 3, 4)$ coming from the parameterization of the cosets $\mathcal{C} = \frac{SO(2,4)}{SO(2,3)}$ and $\mathcal{P} = \frac{SO(2,3)}{SO(1,3)}$.

5. Invariant $SO(2,4)$ Action and Breakdown Mechanism

5.1. *Linear in* R^{AB}

$$S = \int \mu_{AB} \wedge R^{AB} \tag{20}$$

in this case we note first, that the $SO(2,4)$-valuated tensor μ_{AB} acts as multiplier in S (without any role in dynamics, generally speaking). Having this fact in mind, let us consider the following points.

(i) If we have two diffeomorphic (or gauge) nonequivalent $SO(2,4)$-valuated connections, namely Γ^{AB} and $\widetilde{\Gamma}^{AB}$, their difference transforms as a second rank six-tensor under the action of $SO(2,4)$

$$\kappa^{AB} = G^A_{\ C} G^B_{\ D} \kappa^{CD}, \tag{21}$$

$$\kappa^{AB} \equiv \widetilde{\Gamma}^{AB} - \Gamma^{AB}. \tag{22}$$

(ii) If we now calculate the curvature from $\widetilde{\Gamma}^{AB}$, we obtain

$$\widetilde{R}^{AB} = R^{AB} + \mathcal{D}\kappa^{AB}, \tag{23}$$

where the $SO(2,4)$ covariant derivative is defined in the usual way

$$\mathcal{D}\kappa^{AB} = d\kappa^{AB} + \Gamma^A_C \wedge \kappa^{CB} + \Gamma^B_D \wedge \kappa^{AD}. \tag{24}$$

(iii) Redefining the $SO(2,4)$ six vectors as $V_2^A \equiv \psi^A$ and $V_1^B \equiv \varphi^B$ (in order to put all in the standard notation), the 2-form κ^{AB} can be constructed as

$$\kappa^{AB} \to \psi^{[A}\varphi^{B]}dU. \tag{25}$$

Then we introduce all into the $\widetilde{R}^{AB}$ (U scalar function) and get

$$\begin{aligned}\widetilde{R}^{AB} &= R^{AB} + \mathcal{D}(\psi^{[A}\varphi^{B]}dU)\\ &= R^{AB} + (\psi^{[A}\mathcal{D}\varphi^{B]} - \varphi^{[A}\mathcal{D}\psi^{B]}) \wedge dU.\end{aligned} \tag{26}$$

The next step is to find the specific form of μ_{AB} such that $\widetilde{\mu}_{AB} = \mu_{AB}$ (invariant under tilde transformation) in order to make the splitting of the transformed action $\widetilde{S}$ reductive as follows.

(iv) Let us define

$$\widetilde{\theta}^A = \widetilde{\mathcal{D}}\varphi^A \tag{27}$$

with the connection $\widetilde{\Gamma}^{AB} = \Gamma^{AB} + \kappa^{AB}$, then

$$\begin{aligned}\widetilde{\theta}^A &= \underbrace{\mathcal{D}\varphi^A}_{\theta^A} + \kappa^A_{\ B}\varphi^B,\\ \widetilde{\theta}^A &= \theta^A + [\psi^A(\varphi^B)^2 - \varphi^A(\psi\cdot\varphi)] \wedge dU,\end{aligned} \tag{28}$$

where $(\varphi^B)^2 = (\varphi_B\varphi^B)$ and $(\psi\cdot\varphi) = \psi_B\varphi^B$.

In the same manner, we also define

$$\begin{aligned}\widetilde{\eta}^A &= \widetilde{\mathcal{D}}\psi^A,\\ \widetilde{\eta}^A &= \eta^A + [\psi_2^A(\psi\cdot\varphi) - \varphi^A(\psi^B)^2] \wedge dU.\end{aligned} \tag{29}$$

(v) To determine μ_{AB}, we propose to cast it in the form

$$\mu_{AB} \propto \rho_s[a\psi^F\varphi^E\epsilon_{ABCDEF}(\theta^C \wedge \eta^D + \theta^C \wedge \theta^D + \eta^C \wedge \eta^D) + b\kappa^{AB}] \tag{30}$$

with ρ_s, a, b scalar functions in particular contractions of vectors and bivectors $SO(2,4)$-valuated with ϵ_{ABCDEF}) to be determined. The behavior under the tilde transformation is

$$\widetilde{\mu}_{AB} \propto \mu_{AB} - \frac{1}{2}\rho_s a\psi^F\varphi^E\epsilon_{ABEF}d\xi \wedge dU, \tag{31}$$

where $\xi = (\psi^A)^2(\varphi^B)^2 - (\psi\cdot\varphi)^2$.

(vi) Finally, we have to look at the behavior of the transformed action

$$\widetilde{S} = \int \widetilde{\mu}_{AB} \wedge \widetilde{R}^{AB} = S + \int \frac{1}{2}\rho_s a\kappa_{AB} \wedge R^{AB} \wedge d\xi + \int \mu_{AB} \wedge \mathcal{D}\kappa^{AB}. \tag{32}$$

We see that till this point, the $SO(2,4)$-valuated six-vectors ψ^F and φ^E are in principle arbitrary. However, under the conditions as discussed in Sec. 1, the vectors go to the fiducial ones modulo a phase. Consequently

$$\xi \to A^2 B^2 \tag{33}$$

and the bivector comes to

$$\kappa^{AB} \to \psi^{[A}\varphi^{B]}dU \to \Delta(AB)\epsilon^{\alpha\beta} = \alpha\beta AB\epsilon^{\alpha\beta} = AB\epsilon^{\alpha\beta}, \quad \alpha,\beta : 5, 6, \tag{34}$$

where we define the 2nd rank antisymmetric tensor $\epsilon^{\alpha\beta}$ and

$$\Delta = \mathrm{Det}\begin{pmatrix} \lambda^*\alpha & -\mu\beta \\ -\mu^*\alpha & \lambda\beta \end{pmatrix} = \alpha\beta = 1(\text{unitary transformation}) \tag{35}$$

Below we consider two important cases with respect to the components m and λ.

5.2. $A = m$ *and* $B = \lambda$

(1) If the coefficients $A = m$ and $B = \lambda$ play the role of *constant parameters* we have

$$d\xi \to d(\lambda^2 m^2) = 0 \tag{36}$$

and

$$\mathcal{D}\kappa^{AB} \to d(\lambda m)\epsilon^{\alpha\beta} \wedge dU = 0 \tag{37}$$

making the original action S invariant, e.g.

$$\widetilde{S}|_{V_0} = \int \widetilde{\mu}_{AB} \wedge \widetilde{R}^{AB} = \int \mu_{AB} \wedge R^{AB} = S \tag{38}$$

being $\widetilde{S}|_{V_0}$ the restriction of $\widetilde{S}$ under the subspace generated by V_0 and consequently breaking the symmetry from $SO(2, 4) \to SO(1, 3)$.

(2) The connections after the symmetry breaking (when the mentioned conditions with λ and m constants are fulfilled) become

$$\widetilde{\Gamma}^{AB} = \Gamma^{AB} + \kappa^{AB} \Rightarrow \text{b.o.s.} \to \widetilde{\Gamma}^{ij} = \Gamma^{ij}; \quad \widetilde{\Gamma}^{i5} = \Gamma^{i5}, \quad \widetilde{\Gamma}^{i6} = \Gamma^{i6}, \tag{39}$$

$$\text{but} \quad \widetilde{\Gamma}^{56} = \Gamma^{56} - (\lambda m)dU. \tag{40}$$

(3) Vectors $\widetilde{\theta}^A$ and $\widetilde{\eta}^A$ after the symmetry breaking and under the same conditions become

$$\begin{aligned}
\widetilde{\theta}^A &= \underbrace{d\varphi^A + \Gamma^A_{\ C} \wedge \varphi^C}_{\theta^A} + \kappa^A_{\ B}\varphi^B \Rightarrow \text{b.o.s.},\\
\widetilde{\theta}^i &= \theta^i = 0 + \Gamma^i_{\ 5}m + 0 \Rightarrow \theta^i = \Gamma^i_{\ 5}m,\\
\widetilde{\theta}^5 &= 0 = 0 + 0 = 0,\\
\widetilde{\eta}^A &= \underbrace{d\psi^A + \Gamma^A_{\ C} \wedge \psi^C}_{\theta^A} + \kappa^A_{\ B}\psi^B \Rightarrow \text{b.o.s.},\\
\widetilde{\eta}^i &= \eta^i = 0 - \Gamma^i_{\ 6}\lambda + 0 \Rightarrow \eta^i = -\Gamma^i_{\ 6}\lambda,\\
\widetilde{\eta}^6 &= \eta^6 = 0
\end{aligned}$$

and evidently $\mu_{i5} = \mu_{i6} = 0$.

(4) Consequently from the last points, curvatures become

$$R^{ij} = R^{ij}_{\{\}} + m^{-2}\theta^i \wedge \theta^j + \lambda^{-2}\eta^i \wedge \eta^j, \tag{41}$$

$$R^{i5} = m^{-1}\left[\overbrace{d\theta^i + \omega^i{}_j \wedge \theta^j}^{D\theta^i} + \left(\frac{m}{\lambda}\right)\eta^i \wedge \Gamma^{65}\right] = m^{-1}\left[D\theta^i - \frac{m}{\lambda}\eta^i \wedge \Gamma^{65}\right], \tag{42}$$

$$R^{i6} = -\lambda^{-1}\left[D\eta^i - \left(\frac{m}{\lambda}\right)^{-1}\theta^i \wedge \Gamma^{56}\right], \tag{43}$$

$$R^{56} = d\Gamma^{56} + (m\lambda)^{-1}\theta_i \wedge \eta^i, \tag{44}$$

where D is the $SO(1,3)$ covariant derivative.

(5) The tensor responsible for the symmetry breaking becomes

$$\mu_{ij} = -2\rho_s a\lambda m\epsilon_{ijkl}(\theta^k \wedge \eta^l + \theta^k \wedge \theta^l + \eta^k \wedge \eta^l) \tag{45}$$

$$\mu_{56} = -\rho_s b\epsilon_{56}\lambda m dU. \tag{46}$$

(6) Consequently, with all ingredients at hand, the action will be

$$S = \int \mu_{AB} \wedge R^{AB} = \underbrace{\int \mu_{ij} \wedge R^{ij}}_{S_1} + \underbrace{\int \mu_{56} \wedge R^{56}}_{S_2}, \tag{47}$$

where

$$\begin{aligned} S_1 &= -2\int \rho_s a\epsilon_{ijkl}(\theta^k \wedge \eta^l + \theta^k \wedge \theta^l + \eta^k \wedge \eta^l) \wedge \left(\lambda m R^{ij}_{\{\}} + \frac{\lambda}{m}\theta^i \wedge \theta^j + \frac{m}{\lambda}\eta^i \wedge \eta^j\right) \\ &= -2\int \rho_s a\epsilon_{ijkl}\left(\theta^k \wedge \eta^l \wedge \lambda m R^{ij}_{\{\}} + \theta^k \wedge \theta^l \wedge \lambda m R^{ij}_{\{\}} + \eta^k \wedge \eta^l \wedge \lambda m R^{ij}_{\{\}}\right) \\ &\quad -2\int \rho_s a\epsilon_{ijkl}\left(\theta^k \wedge \eta^l \wedge \frac{\lambda}{m}\theta^i \wedge \theta^j + \theta^k \wedge \theta^l \wedge \frac{\lambda}{m}\theta^i \wedge \theta^j + \eta^k \wedge \eta^l \wedge \frac{\lambda}{m}\theta^i \wedge \theta^j\right) \\ &\quad -2\int \rho_s a\epsilon_{ijkl}\left(\theta^k \wedge \eta^l \wedge \frac{m}{\lambda}\eta^i \wedge \eta^j + \theta^k \wedge \theta^l \wedge \frac{m}{\lambda}\eta^i \wedge \eta^j + \eta^k \wedge \eta^l \wedge \frac{m}{\lambda}\eta^i \wedge \eta^j\right) \end{aligned}$$

and

$$S_2 = -\lambda m \int \rho_s b\epsilon_{56} \wedge (d\Gamma^{56} + (m\lambda)^{-1}\theta_i \wedge \eta^i).$$

(7) At this point (the mathematical justification will come later) we can naturally associate the tetrad field with the θ-form

$$\theta^k \sim e^k_a \omega^a \tag{48}$$

consequently a metric can be induced in M_4:

$$\eta_{ab} = g_{jk}e^j_a e^k_b, \quad g_{jk} = \eta_{ab}e^a_j e^b_k, \quad e^k_a e^b_k = \delta^a_b, \ldots \tag{49}$$

where η_{jk} is the Minkowski metric. That allows us to lift up and to lower down indices, and η^i with the following symmetry typical of a $SU(2,2)$ Clifford structure

$$\eta^k \sim f_a^k \omega^a, \tag{50}$$

$$e_j^a f_a^k g_{lk} = f_{lj} = -f_{jl} \tag{51}$$

that consequently allows us to introduce into the model an electromagnetic field (that will be proportional to f_{lj}).

(8) So we can re-write the action as

$$\begin{aligned} S_1 &= -2\int \rho_s a \epsilon_{ijkl}(\theta^k \wedge \eta^l + \theta^k \wedge \theta^l + \eta^k \wedge \eta^l) \wedge \left(\lambda m R^{ij}_{\{\}} + \frac{\lambda}{m}\theta^i \wedge \theta^j + \frac{m}{\lambda}\eta^i \wedge \eta^j\right) \\ &= -2\int \rho_s a \left[\lambda m\Big(f_{ij}R^{ij}_{\{\}} + (g_{ij} + f_i^k f_{kj})R^{ij}_{\{\}}\Big) + \left(\frac{\lambda}{m} + \frac{m}{\lambda}\right) f^{kj} f_{kj} \right. \\ &\quad \left. + \left(\frac{\lambda}{m}\sqrt{g} + \frac{m}{\lambda}\sqrt{f}\right)\right] d^4x. \end{aligned} \tag{52}$$

In the above expression, we have taken into account the following:

(i) terms $\sim \eta \wedge \eta \wedge \eta \wedge \theta$ and $\eta \wedge \theta \wedge \theta \wedge \theta$ vanish;
(ii) terms $\sim \eta \wedge \eta \wedge \theta \wedge \theta$ and $\eta \wedge \eta \wedge \theta \wedge \theta$ lead to $\rightarrow f^{kj} f_{kj}$;
(iii) term $\sim \epsilon_{ijkl}\theta^k \wedge \eta^l \wedge R^{ij}_{\{\}}$ leads $\rightarrow f_{ij}R^{ij}_{\{\}}$ picking the antisymmetric part of the generalized Ricci tensor (containing torsion);
(iv) term $\sim \epsilon_{ijkl}(\theta^k \wedge \theta^l + \eta^k \wedge \eta^l)R^{ij}_{\{\}}$ leads to $\rightarrow (g_{ij} + f_i^k f_{kj})R^{ij}_{\{\}}$ picking the symmetric part of the generalized Ricci tensor (containing Einstein–Hilbert plus quadratic torsion term);
(v) terms $\sim \eta \wedge \eta \wedge \eta \wedge \eta$ and $\theta \wedge \theta \wedge \theta \wedge \theta$ lead to the volume elements $\sqrt{f}$ and $\sqrt{g}$, respectively, where we defined as usual $g \equiv \mathrm{Det}(g_{lk})$ and $f \equiv \mathrm{Det}(f_{lk}) = (f^*_{lk} f^{lk})^2$.

5.3. $A = m(x)$ *and* $B = \lambda(x)$: *Spontaneous subspace*

If the coefficients $A = m(x)$ and $B = \lambda(x)$ are not *constant* but functions of coordinates, we have

$$d\xi \rightarrow d(\lambda^2 m^2) = 2d(\lambda m) \tag{53}$$

and

$$\mathcal{D}\kappa^{AB} \rightarrow d(\lambda m)\epsilon^{\alpha\beta} \wedge dU. \tag{54}$$

Consequently from the following explicit computations

$$\begin{aligned} \widetilde{S} &= \int \widetilde{\mu}_{AB} \wedge \widetilde{R}^{AB} \\ &= S + \int \frac{1}{2}\rho_s a \kappa_{AB} \wedge R^{AB} \wedge d\xi + \int \mu_{AB} \wedge \mathcal{D}\kappa^{AB} \end{aligned}$$

$$
\begin{aligned}
&= S - \int \frac{1}{2}\rho_s a R^{AB} \wedge \kappa_{AB} \wedge d\xi + \int \mu_{AB} \wedge \mathcal{D}\kappa^{AB} \\
&= S - \int \frac{1}{2}\rho_s a R_{\alpha\beta}\epsilon^{\alpha\beta}\lambda m dU \wedge 2d(\lambda m) + \int \mu_{\alpha\beta}\epsilon^{\alpha\beta} d(\lambda m) \wedge dU \\
&= S + \int \frac{1}{2}\rho_s a R_{\alpha\beta}\epsilon^{\alpha\beta}\lambda m 2d(\lambda m) \wedge dU + \int \mu_{\alpha\beta}\epsilon^{\alpha\beta} d(\lambda m) \wedge dU, \\
\widetilde{S} &= S + \int [\mu_{\alpha\beta} + \rho_s a R_{\alpha\beta}\lambda m]\epsilon^{\alpha\beta} d(\lambda m) \wedge dU.
\end{aligned}
\tag{55}
$$

we obtain the required condition:

$$
\begin{gathered}
\widetilde{S} = S \quad \text{if} \\
\mu_{\alpha\beta} = -\rho_s a R_{\alpha\beta}\lambda m,
\end{gathered}
\tag{56}
$$

then we see that μ_{AB} takes the place of an induced metric and it is proportional to the curvature

$$R_{\alpha\beta} = \Lambda \mu_{\alpha\beta} \tag{57}$$

$$\text{with} \quad \Lambda = -(\rho_s a \lambda m)^{-1}. \tag{58}$$

Note that we have now a four-dimensional space-time plus the above "internal" space of a constant curvature. This point is very important as a new compactification-like mechanism.

6. Supergravity as a Gauge Theory and Topological QFT

In previous works [51, 52] we have shown, by means of a toy model, that there exists a supersymmetric analog of the above symmetry breaking mechanism coming from the topological QFT. Here, we recall some of the above ideas in order to see clearly the analogy between the group structures of the simplest supersymmetric case, $Osp(4)$, and of the classical conformal group $SO(2,4)$.

The starting point is the super $SL(2C)$ algebra (strictly speaking $Osp(4)$)

$$
\begin{gathered}
[M_{AB}, M_{CD}] = \epsilon_C({}_A M_B)_D + \epsilon_D({}_A M_B)_C, \\
[M_{AB}, Q_C] = \epsilon_C({}_A Q_B), \quad \{Q_A, Q_B\} = 2M_{AB}.
\end{gathered}
\tag{59}
$$

Here, the indices $A, B, C\ldots$ stand for $\alpha, \beta, \gamma\ldots(\dot\alpha, \dot\beta, \dot\gamma\ldots)$ where spinor indices: $\alpha, \beta(\dot\alpha, \dot\beta) = 1, 2(\dot 1, \dot 2)$ in the Van der Werden spinor notation. We define the super-connection A due to the following "gauging"

$$A^p T_p \equiv \omega^{\alpha\dot\beta} M_{\alpha\dot\beta} + \omega^{\alpha\beta} M_{\alpha\beta} + \omega^{\dot\alpha\dot\beta} M_{\dot\alpha\dot\beta} + \omega^{\alpha} Q_\alpha - \omega^{\dot\alpha}\overline{Q}_{\dot\alpha}, \tag{60}$$

where (ωM) defines a 10-dimensional[a] bosonic manifold and $p \equiv$ multi-index, as usual. Analogically, the super-curvature is defined by $F \equiv F^p T_p$ with the following

[a]Corresponding to the number of generators of $SO(4,1)$ or $SO(3,2)$ that define the group manifold.

detailed structure

$$F(M)^{AB} = d\omega^{AB} + \omega^{A}_{\ C} \wedge \omega^{CB} + \omega^{A} \wedge \omega^{B}, \tag{61}$$

$$F(Q)^{A} = d\omega^{A} + \omega^{A}_{\ C} \wedge \omega^{C}. \tag{62}$$

From (60), it is easy to see that there are a bosonic part and a fermionic one associated with the even and odd generators of the superalgebra. Our proposal for the "toy" action was (as before for $SO(2,4)$) as follows:

$$S = \int F^{p} \wedge \mu_{p}, \tag{63}$$

where the tensor μ_p (that plays the role of a $Osp(4)$ diagonal metric as in the Mansouri proposal)[b] is *defined* as

$$\mu_{\alpha\dot{\beta}} = \zeta_{\alpha} \wedge \overline{\zeta}_{\dot{\beta}}, \quad \mu_{\alpha\beta} = \zeta_{\alpha} \wedge \zeta_{\beta}, \quad \mu_{\alpha} = \nu\zeta_{\alpha}, \ldots \tag{64}$$

with $\zeta_\alpha(\overline{\zeta}_{\dot{\beta}})$ anti-commuting spinors (suitable basis) and ν the parameter of the breaking of super $SL(2C)$ ($Osp(4)$) to $SL(2C)$ symmetry of μ_p. Note that the introduction of the parameter ν means that we do not take care of the particular dynamics to break the symmetry.

In order to obtain dynamical equations of the theory, we proceed to perform variation of the proposed action (63)

$$\delta S = \int \delta F^{p} \wedge \mu_{p} + F^{p} \wedge \delta\mu_{p} = \int d_{A}\mu_{p} \wedge \delta A^{p} + F^{p} \wedge \delta\mu_{p}, \tag{65}$$

where d_A is the exterior derivative with respect to the super-$SL(2C)$ connection and $\delta F = d_A \delta A$ have been used. Then, as the result, the dynamics is described by

$$d_{A}\mu = 0, \quad F = 0. \tag{66}$$

The first equation claims that μ is covariantly constant with respect to the super $SL(2C)$ connection. This fact will be very important when the super $SL(2C)$ symmetry breaks down to $SL(2C)$ because $d_A\mu = d_A\mu_{AB} + d_A\mu_A = 0$, a soldering form will appear. The second equation gives the condition for a super Cartan connection $A = \omega^{AB} + \omega^{A}$ to be flat, as it is easy to see from the reductive components of above expressions

$$\begin{aligned} F(M)^{AB} &= R^{AB} + \omega^{A} \wedge \omega^{B} = 0, \\ F(Q)^{A} &= d\omega^{A} + \omega^{A}_{\ C} \wedge \omega^{C} = d_{\omega}\omega^{A} = 0, \end{aligned} \tag{67}$$

where now d_ω is the exterior derivative with respect to the $SL(2C)$ connection and $R^{AB} \equiv d\omega^{AB} + \omega^{A}_{\ C} \wedge \omega^{CB}$ is the $SL(2C)$ curvature. Then

$$F = 0 \Leftrightarrow R^{AB} + \omega^{A} \wedge \omega^{B} = 0 \quad \text{and} \quad d_{\omega}\omega^{A} = 0, \tag{68}$$

the second condition says that the $SL(2C)$ connection is super-torsion free. The first does not say that the $SL(2C)$ connection is flat, but it claims that it is homogeneous

[b]In general this tensor has the same structure as the Cartan–Killing metric of the group under consideration.

with a cosmological constant related to the explicit structure of the Cartan forms ω^A, as we will see when the super $SL(2C)$ action is reduced to the Volkov–Pashnev model [42].

6.1. *The geometrical reduction: Extended symplectic super-metrics*

6.1.1. *Example: Volkov–Pashnev metric*

The super-metric under consideration, proposed by Volkov and Pashnev in [42], is the simplest example of symplectic (super) metrics induced by the symmetry breaking from a pure topological first-order action. It can be obtained from the $Osp(4)(superSL(2C))$ action via the following procedure.

(i) The Inönu–Wigner contraction [43] in order to pass from $SL(2C)$ to the super-Poincare algebra (corresponding to the original symmetry of the model of [42, 44]) then, the even part of the curvature is split into a $\mathbb{R}^{3,1}$ part $R^{\alpha\dot{\beta}}$ and a $SO(3,1)$ part $R^{\alpha\beta}(R^{\dot{\alpha}\dot{\beta}})$ associated with the remaining six generators of the original five-dimensional $SL(2C)$ group. This fact is easily realized by knowing that the underlying geometry is reductive: $SL(2C) \sim SO(4,1) \to SO(3,1) + \mathbb{R}^{3,1}$. Then, we rewrite the superalgebra (59) as

$$\begin{array}{lll} [M,M] \sim M & [M,\Pi] \sim \Pi & [\Pi,\Pi] \sim M \\ [M,S] \sim S & [\Pi,S] \sim S & \{S,S\} \sim M+\Pi \end{array} \tag{69}$$

with $\Pi \sim M_{\alpha\dot{\beta}}$, $M \sim M_{\alpha\beta}(M_{\dot{\alpha}\dot{\beta}})$, and re-scale $m^2\Pi = P$ and $mS = Q$. In the limit $m \to 0$, one recovers the super Poincare algebra. Note that one does not re-scale M since one wants to keep $[M,M] \sim M$ Lorentz algebra, that also is a symmetry of (1).

(ii) The *spontaneous* breaking of the super $SL(2C)$ down to the $SL(2C)$ symmetry of μ_p (e.g. $\nu \to 0$ in μ_p) certainly exists in such a manner that the even part of the super $SL(2C)$ action, namely $F(M)^{AB}$ remains.

After these evaluations, it has been explicitly realized that the even part of the original super $SL(2C)$ action (now a super-Poincare invariant) can be related with the original metric (1) as follows:

$$R(M) + R(P) + \omega^\alpha\omega_\alpha - \omega^{\dot{\alpha}}\omega_{\dot{\alpha}} \to \omega^\mu\omega_\mu + \mathbf{a}\omega^\alpha\omega_\alpha - \mathbf{a}^*\omega^{\dot{\alpha}}\omega_{\dot{\alpha}}\ |_{VP}\,. \tag{70}$$

Note that there is mapping $R(M) + R(P) \to \omega^\mu\omega_\mu\ |_{VP}$ that is well defined and can be realized in different forms, and the map of interest here $\omega^\alpha\omega_\alpha - \omega^{\dot{\alpha}}\omega_{\dot{\alpha}} \to \mathbf{a}\omega^\alpha\omega_\alpha - \mathbf{a}^*\omega^{\dot{\alpha}}\omega_{\dot{\alpha}}\ |_{VP}$ that associate the Cartan forms of the original super $SL(2C)$ action (63) with the Cartan forms of the Volkov–Pashnev supermodel: $\omega^\alpha = (\mathbf{a})^{1/2}\omega^\alpha\ |_{VP}$, $\omega^{\dot{\alpha}} = (\mathbf{a}^*)^{1/2}\omega^{\dot{\alpha}}\ |_{VP}$. Then, the origin of the coefficients $\mathbf{a}$ and $\mathbf{a}^*$ becomes clear from the geometrical point of view.

From the first condition in (68) and the association (70) it is not difficult to see that, as in the case of the space-time cosmological constant $\Lambda : R = \frac{\Lambda}{3} e \wedge e$

($e \equiv$ space$-$time tetrad), there is a cosmological term from the superspace related to the complex parameters $\mathbf{a}$ and $\mathbf{a}^*$: $R = -(\mathbf{a}\omega^\alpha\omega_\alpha - \mathbf{a}^*\omega^{\dot\alpha}\omega_{\dot\alpha})$ and it is easy to see from the minus sign in above expression, why for supersymmetric (supergravity) models it is more natural to use $SO(3,2)$ instead of $SO(4,1)$.

Note that the role of the associated spinorial action in (63) is constrained by the nature of $\nu\zeta_\alpha$ in μ_p as follows.

(i) If they are of the same nature of the ω^α, this term is a total derivative and has no influence onto the equations of motion, then the action proposed by Volkov and Pashnev in [42] has the correct fermionic form.

(ii) If they are not of the same $SL(2C)$ invariance that the ω^α, the symmetry of the original model is modified. In this direction, a relativistic supersymmetric model for particles was proposed in [45] considering an N-extended Minkowsky superspace and introducing central charges to the superalgebra. Hence the underlying rigid symmetry gets enlarged to N-extended super-Poincare algebra. Considering for our case similar to superextension that in [45], we can introduce the following new action:

$$\begin{aligned} S &= -m\int_{\tau 1}^{\tau 2} d\tau\sqrt{\overset{\circ}{\omega}_\mu\overset{\circ}{\omega}{}^\mu + a\dot\theta^\alpha\dot\theta_\alpha - a^*\dot{\overline{\theta}}{}^{\dot\alpha}\dot{\overline{\theta}}_{\dot\alpha} + i(\theta^{\alpha i}A_{ij}\dot\theta^j_\alpha - \overline{\theta}{}^{\dot\alpha i}A_{ij}\dot{\overline{\theta}}{}^j_{\dot\alpha})} \\ &= \int_{\tau 1}^{\tau 2} d\tau L(x,\theta,\overline{\theta}) \end{aligned} \tag{71}$$

that is the super-extended version of the superparticle model proposed in [42] with the addition of a first-order fermionic part. The matrix tensor A_{ij} introduces the symplectic structure of such manner that now $\zeta_{\alpha i} \sim A_{ij}\theta^j_\alpha$ is not covariantly constant under d_ω. Note that the "Dirac-like" (fermionic) part is naturally *inside* the square root because it is a part of the full curvature, fact that was not published by the authors in [45] (the authors did not realize the existence of this term, see also [29]) that it does not take into account the geometrical origin of the action. An interesting point is to perform the same quantization as in the first part of the research given in [44] in order to obtain and compare the spectrum of physical states with the one obtained in [45]. This issue will be presented elsewhere [46].

The spontaneous symmetry breaking happens here because the parameter does not have any dynamics. But this does not happen in the nonlinear realization approach where the parameters have a particular dynamics associated with the space-time coordinates.

7. Quadratic in R^{AB}

The previous action, linear in the generalized curvature, has some drawbacks that make necessary introduction of additional "subsidiary conditions" due to the fact that the curvatures R^{i5} and R^{i6} do not play any role in the linear-/first-order action.

Such curvatures have a very important information about the dynamics of θ and η fields. In order to simplify the equations of motion we define

$$\Gamma^{56} \equiv A, \tag{72}$$

$$m^{-1}\theta^i \equiv \widetilde{\theta}^i, \tag{73}$$

$$\lambda^{-1}\eta^i \equiv \widetilde{\eta}^i, \tag{74}$$

and as always

$$R^{ij} = R^{ij}_{\{\}} + m^{-2}\theta^i \wedge \theta^j + \lambda^{-2}\eta^i \wedge \eta^j \tag{75}$$

with the $SO(1,3)$ curvature $R^{ij}_{\{\}} = d\omega^{ij} + \omega^i{}_\lambda \wedge \omega^{\lambda j}$. Consequently from the quadratic Lagrangian density

$$S = \int R_{AB} \wedge R^{AB} \tag{76}$$

we obtain the following equations of motion:

$$\frac{\delta(R_{AB} \wedge R^{AB})}{\delta\theta^i} \to D(D\widetilde{\theta}_j) + 2R_{ij} \wedge \widetilde{\theta}^i - \widetilde{\theta}^i \wedge \widetilde{\eta}_i \wedge \widetilde{\eta}_j + \widetilde{\theta}_j \wedge A \wedge A = 0, \tag{77}$$

$$\frac{\delta(R_{AB} \wedge R^{AB})}{\delta\eta^i} \to D(D\widetilde{\eta}_j) + 2R_{jk} \wedge \widetilde{\eta}^k - \widetilde{\theta}^i \wedge \widetilde{\eta}_i \wedge \widetilde{\theta}_j + \widetilde{\eta}_j \wedge A \wedge A = 0, \tag{78}$$

$$\frac{\delta(R_{AB} \wedge R^{AB})}{\delta\Gamma^{56}} \to \widetilde{\theta}^i \wedge \widetilde{\theta}_i = \widetilde{\eta}^i \wedge \widetilde{\eta}_i, \tag{79}$$

$$\frac{\delta(R_{AB} \wedge R^{AB})}{\delta\omega^i_j} \to -DR_{kl} + D\widetilde{\theta}_k \wedge \widetilde{\theta}_l + D\widetilde{\eta}_k \wedge \widetilde{\eta}_l + \widetilde{\theta}_k \wedge \widetilde{\eta}_l \wedge A = 0. \tag{80}$$

7.1. *Maxwell equations and the electromagnetic field*

As we claimed before we can identify

$$\theta^i \equiv e^i_\mu dx^\mu, \tag{81}$$

$$\eta^i \equiv f^i_\mu dx^\mu \tag{82}$$

with the symmetries

$$e^i_\mu e^\nu_i = \delta^\nu_\mu, \quad e^i_\mu e_{i\nu} = g_{\mu\nu} = g_{\nu\mu} \tag{83}$$

and

$$f^i_\mu f^\nu_i = \delta^\nu_\mu, \quad e_{i\nu} f^i_\mu = f_{\mu\nu} = -f_{\nu\mu} \tag{84}$$

such that the geometrical (Bianchi) condition

$$\nabla_{[\rho} f_{\mu\nu]} = \nabla^*_\rho f^{\rho\nu} = 0 \tag{85}$$

or in the language of differential forms

$$D(\widetilde{\theta}^i \wedge \widetilde{\eta}_i) = 0 \tag{86}$$

holds, thus the curvatures R^{i6} and R^{i5} are enforced to be null. And conversely if R^{i6} and R^{i5} are zero then $D(\widetilde{\theta}^i \wedge \widetilde{\eta}_i) = 0$ or equivalently $\nabla_{[\rho} f_{\mu\nu]} = \nabla^*_\rho f^{\rho\nu} = 0$.

Proof. From expressions (42), (43), namely: $R^{i5} = [D\widetilde{\theta}^i - \widetilde{\eta}^i \wedge \Gamma^{65}]$ and $R^{i6} = [-D\widetilde{\eta}^i + \widetilde{\theta}^i \wedge \Gamma^{56}]$ we make

$$R^{i5} \wedge \widetilde{\eta}_i + \widetilde{\theta}_i \wedge R^{i6} = D(\widetilde{\theta}^i \wedge \widetilde{\eta}_i) + (\widetilde{\eta}^i \wedge \Gamma^{56}) \wedge \widetilde{\eta}_i + \widetilde{\theta}_i \wedge (\widetilde{\theta}^i \wedge \Gamma^{56}), \tag{87}$$

$$R^{i5} \wedge \widetilde{\eta}_i + \widetilde{\theta}_i \wedge R^{i6} = D(\widetilde{\theta}^i \wedge \widetilde{\eta}_i). \tag{88}$$

In the last line we used the constraint given by Eq. (79) Consequently if R^{i6} and R^{i5} are zero, then $D(\widetilde{\theta}^i \wedge \widetilde{\eta}_i) = 0$ or equivalently $\nabla_{[\rho} f_{\mu\nu]} = \nabla^*_\rho f^{\rho\nu} = 0$ and vice versa. □

Corollary 1. *Note that the vanishing of the R^{56} curvature (that transforms as a Lorentz scalar) does not modify the equation of motion for Γ^{56} and simultaneously defines the electromagnetic field as*

$$R^{56} = d\Gamma^{56} + (m\lambda)^{-1}\theta_i \wedge \eta^i = 0, \tag{89}$$

$$\Rightarrow dA - F = 0. \tag{90}$$

7.2. *Equations of motion in components and symmetries*

Let us define

$$R^{ij}_{\{\}\mu\nu} = \partial_\mu \omega^{ij}_\nu - \partial_\nu \omega^{ij}_\mu + \omega^i_{\mu k}\omega^{kj}_\nu - \omega^{kj}_\mu \omega^i_{\nu k}, \tag{91}$$

$$T^i_{\mu\nu} = \partial_\mu e^i_\nu - \partial_\nu e^i_\mu + \omega_\mu{}^i{}_k e^k_\nu - \omega_\nu{}^i{}_k e^k_\mu, \tag{92}$$

$$S^i_{\mu\nu} = \partial_\mu f^i_\nu - \partial_\nu f^i_\mu + \omega_\mu{}^i{}_k f^k_\nu - \omega_\nu{}^i{}_k f^k_\mu. \tag{93}$$

Note that $S^i_{\mu\nu}$ is a totally antisymmetric torsion field due to the symmetry of $f^i_\nu dx^\nu \equiv \eta^i$. Consequently the equations of motion in components become

$$\begin{aligned}
&\nabla_\mu[\sqrt{|g|}R^{ij\mu\nu}] + \sqrt{|g|}(-m^{-2}T^{ji\nu} + \lambda^{-2}S^{ji\nu}) - \sqrt{|g|}(\lambda m)^{-1} f^{[i\nu}A^{i]} = 0,\\
&\nabla_\mu[\sqrt{|g|}(R^{ij\mu\nu}_{\{\}} - m^{-2}e^{[i\mu}e^{j]\nu} + \lambda^{-2}f^{[i\mu}f^{j]\nu})]\\
&\quad + \sqrt{|g|}(-m^{-2}T^{ji\nu} + \lambda^{-2}S^{ji\nu}) - \sqrt{|g|}(\lambda m)^{-1} f^{[i\nu}A^{i]} = 0,\\
&\nabla_\mu(\sqrt{|g|}T^{j\mu v}) + \sqrt{|g|}\Big(R^{j\nu}_{\{\}} - m^{-2}e^{j\nu} + A^i A^\nu\Big) = 0,\\
&\nabla_\mu(\sqrt{|g|}S^{j\mu i}) + \sqrt{|g|}\Big(R^{ij}_{\{\}} - \lambda^{-2}f^{ij} + A^{[i}A^{j]}\Big) = 0,\\
&\nabla_{[\mu}A_{\nu]} = F_{\mu\nu} = (\lambda m)^{-1}F_{\mu\nu},\\
&\nabla_{[\rho}F_{\mu\nu]} = 0.
\end{aligned} \tag{94}$$

8. Nonlinear Realizations Viewpoint

Note that in our case, Eqs. (81) and (82) identify $\theta^i \sim e^i$ and $\eta^i \sim f^i$ making the table below completely clear. Note that Γ^{65} is identified with the **g** of Ivanov and Niederle [14, 15].

	this work	[14, 15]
R^{ij}	$R^{ij}_{\{\}} + m^{-2}\theta^i \wedge \theta^j + \lambda^{-2}\eta^i \wedge \eta^j$	$R^{ij}_{\{\}} + 4ge^i \wedge f^j$
R^{i5}	$m^{-1}\left[D\theta^i - \frac{m}{\lambda}\eta^i \wedge \Gamma^{65}\right]$	$De^i + 2ge^i \wedge \mathbf{g}$
R^{i6}	$-\lambda^{-1}\left[D\eta^i - \left(\frac{m}{\lambda}\right)^{-1}\theta^i \wedge \Gamma^{56}\right]$	$Df^i - 2gf^i \wedge \mathbf{g}$
R^{56}	$d\Gamma^{56} + (m\lambda)^{-1}\theta_i \wedge \eta^i$	$d\mathbf{g} + 4ge_i \wedge f^i$
DS/ADS reduction	Yes	No

Algebra and transformations in the case of the work of Ivanov and Niederle are different due to the different definitions of the generators of the $SO(2,4)$ algebra, however, the meaning of $\mathbf{g}$ which is associated to the connection Γ^{65} remains obscure for us because of the second Cartan structure equations R^{i5} and R^{i6}. Note that, although the group theoretical viewpoint in the case of the simultaneous nonlinear realization of the affine and conformal group [50] to obtain Einstein gravity is more or less clear, the pure geometrical picture is still hard to recognize due to the factorization problem and the orthogonality between coset elements and the corresponding elements of the stability subgroup.

9. Discussion

In this work, we introduced two geometrical models: one linear and another one quadratic in curvature. Both models are based on the $SO(2,4)$ group. Dynamical breaking of this symmetry was considered. In both cases, we introduced coherent states of the Klauder–Perelomov type, which as defined by the action of a group (generally a Lie group) are invariant with respect to the stability subgroup of the corresponding coset being related to the possible extension of the connection which maintains the proposed action invariant.

The linear action, unlike the cases of West, Kerrick or even McDowell and Mansouri [41], uses a symmetry breaking tensor that is dynamic and unrelated to a particular metric. Such a tensor depends on the introduced vectors (i.e. the coherent states) that intervene in the extension of the permissible symmetries of the original connection. Only some components of the curvature, defined by the second structure equation of Cartan, are involved in the action, leaving the remaining ones as a system of independent or ignorable equations in the final dynamics. The quadratic action, however, is independent of any additional structure or geometric artifacts and all the curvatures (e.g. all the geometrical equations for the fields) play a role in the final action (Lagrangian of the theory).

With regard to the parameters that come into play λ and m (they play the role of a cosmological constant and a mass, respectively), we saw that in the case of linear action if they are taken dependent on the coordinates and under the conditions of the action invariance, a new spontaneous compactification mechanism is defined in the subspace invariant under the stability subgroup.

Following this line of research with respect to possible physical applications, we consider scenarios of the Grand Unified Theory, derivation of the symmetries of the Standard Model together with the gravitational ones. The general aim is to obtain in a precisely established way the underlying fundamental theory. This will be important, in particular, to solve the problem of hierarchies and fundamental constants, the masses of physical states, and their interaction.

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