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Non-Riemannian generalizations of the Born–Infeld model and the meaning of the cosmological term

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Theory of gravitation based on a non-Riemannian geometry with dynamical torsion field is geometrically analyzed. To this end, the simplest Lagrangian density is introduced as a measure (reminiscent of a sigma model) and the dynamical equations are derived. Our goal is to rewrite this generalized affine action in a suitable form similar to the standard Born–Infeld (BI) Lagrangian. As soon as the functional action is rewritten in the BI form, the dynamical equations lead the trace-free GR-type equation and the field equations for the torsion, respectively: both equations *emerge* from the model in a sharp contrast with other attempts where additional assumptions were heuristically introduced. In this theoretical context, the Einstein κ, Newton G and the analog to the absolute b-field into the standard BI theory all arise from the same geometry through geometrical invariant quantities (as from the curvature R). They can be clearly identified and correctly interpreted both physical and geometrically. Interesting theoretical and physical aspects of the proposed theory are given as clear examples that show the viability of this approach to explain several problems of actual interest. Some of them are the dynamo effect and geometrical origin of $\alpha\Omega$ term, origin of primordial magnetic fields and the role of the torsion in the actual symmetry of the standard model. The relation with gauge theories, conserved currents, and other problems of astrophysical character is discussed with some detail.

Keywords: Non-Riemannian geometry; Born–Infeld; fundamental constants; Anomalous MHD; magnetogenesis.

1. Introduction

A theme that is repeatedly discussed in theoretical backgrounds is if the existent energy–matter of our natural world emerge from the same geometry of the space-time. As is well known, the common standard approaches based only on General

Relativity (GR) are not based on symmetry principles: if this were the case, a harmonious interplay "matter–energy ↔ spacetime" must be automatically fulfilled, from the dynamical viewpoint. This was precisely the drawback of the Einstein GR equations in the right-hand side, namely:

$$R_{\alpha\beta} - \frac{g_{\alpha\beta}}{2} R = \kappa T_{\alpha\beta}$$

with $\kappa T_{\alpha\beta}$ (energy–momentum distribution) added "by hand" according to the physical scenario under consideration. It is heuristically implemented in GR by the additive energy–momentum tensor that contains the matter fields and the energy distribution. The "additive" fact breaks the characteristic nonlinearity of the gravity theories.

Other current attempt that contains similar drawbacks as before is, for example, the unimodular gravity. It is well known that the unimodular gravity is obtained from Einstein–Hilbert action in which the *unimodular condition*

$$\sqrt{-\det g_{\mu\nu}} = 1$$

is heuristically imposed from the very beginning [1–3]. The resulting field equations correspond to the traceless Einstein equations and it can be shown that they are equivalent to the full Einstein equations with the cosmological constant term Λ, where Λ enters as an integration constant. We see that the equivalence between unimodular gravity and GR is given by the arbitrariness to select a suitable value for lambda. On the other hand the idea that the cosmological term arises as an integration constant is one of the motivations for the study of the unimodular gravity, for recent study, see [5, 4] in the context of supergravity. The fact that the determinant of the metric is fixed has clearly profound consequences on the structure of the given theory. First of all, it reduces the full group of diffeomorphism to invariance under the group of unimodular general coordinate transformations which are transformations that leave the determinant of the metric unchanged.

In the non-Riemannian case, as we have pointed out previously [12–16], the corresponding affine geometrical structure induces naturally the following constraint:

$$\frac{K}{g} = \text{constant}$$

that imposes a condition (ratio) between both basic tensors through their determinants: the metric g and the fundamental one K (in the sense of a non-symmetric theory), independently of the precise functional form of K or g. In this work, our starting point will be precisely the last one, where a metric affine structure in the spacetime manifold will be considered as described in Sec. 2, with the Lagrangian function or geometrical action is taken as a measure or the square root of the determinant of a particular combination of the fundamental tensors of the geometry:

$$\sqrt{|\det f(g_{\mu\nu}, f_{\mu\nu}, R_{\mu\nu})|}$$

with the $(0,2)$ tensors $g_{\mu\nu}, f_{\mu\nu}, R_{\mu\nu}$: the symmetric metric, the antisymmetric (that acts as potential of the torsion field) and the generalized Ricci tensor (proper of the non-Riemannian geometry). The three tensors are related with a Clifford structure of the tangent space (for details see [17]) and the explicit choice for $f(g_{\mu\nu}, f_{\mu\nu}, R_{\mu\nu})$ is given in Sec. 3 and the geometrical explanation in Sec. 4. This type of Lagrangians, reminiscent of the nonlinear sigma model, are geometrical non-Riemannian generalizations of Nambu–Goto and Born–Infeld (BI) ones: this similarity is that we use for the physical analysis of our model. Due to the basic structure of the theory, induced energy–momentum tensors and fundamental constants (as G now functions in reality) emerge naturally from the same geometry: this issue, that allows the physical realization of the Mach principle, is treated in some detail in Sec. 8 after the obtention of the (trace-free) dynamical equations in Sec. 5. In Sec. 6, the trace-free gravitational equations and the meaning of a cosmological-like term as integration constant are discussed from the physical viewpoint. The important role played by the dual of the torsion field as energy–matter carrier is given in Sec. 7 and some physical consequences of the model, as the geometrical origin of the $\alpha\Omega$-dynamo, are presented in Sec. 9. In Sec. 10, the direct relation between the torsion (geometry) with axion electrodynamics and Chern–Simons (CS) theory is discussed considering the structure of the dual vector of the torsion field. Finally, in Sec. 11, an explanation about the magnetogenesis in FRW scenario, the structure of the GUT where the standard model is derived and the role of the axion in the dynamics of the cosmic magnetic field is presented with some concluding remarks in Sec. 12.

2. Basis of the Metrical-Affine Geometry

The starting point is a hypercomplex construction of the (metric compatible) spacetime manifold [17, 18]

$$M, g_{\mu\nu} \equiv e_\mu \cdot e_\nu, \tag{1}$$

where for each point $p \in M$ there exists a local affine space A. The connection over A, $\widetilde{\Gamma}$, defines a generalized affine connection Γ on M, specified by (∇, K), where K is an invertible $(1,1)$ tensor over M. We will demand for the connection to be compatible and rectilinear, that is,

$$\nabla K = KT, \quad \nabla g = 0, \tag{2}$$

where T is the torsion, and g the spacetime metric (used to raise and lower the indices and determining the geodesics), that is preserved under parallel transport. This generalized compatibility condition ensures that the generalized affine connection Γ maps autoparallels of Γ on M into straight lines over the affine space A (locally). The first equation above is equal to the condition determining the connection in terms of the fundamental field in the UFT non-symmetric. Hence, K can be identified with the fundamental tensor in the non-symmetric fundamental theory. This fact gives us the possibility to restrict the connection to a (anti-)Hermitian theory.

The covariant derivative of a vector with respect to the generalized affine connection is given by

$$A^{\mu}{}_{;\nu} \equiv A^{\mu}{}_{,\nu} + \Gamma^{\mu}{}_{\alpha\nu} A^{\alpha}, \tag{3}$$

$$A_{\mu;\nu} \equiv A_{\mu}{}_{,\nu} - \Gamma^{\alpha}{}_{\mu\nu} A_{\alpha}. \tag{4}$$

The generalized compatibility condition (2) determines the 64 components of the connection by the 64 equations

$$K_{\mu\nu;\alpha} = K_{\mu\rho} T^{\rho}{}_{\nu\alpha} \quad \text{where } T^{\rho}{}_{\nu\alpha} \equiv 2\Gamma^{\rho}{}_{[\alpha\nu]}. \tag{5}$$

Note that by contracting indices ν and α in the first equation above, an additional condition over this hypothetic fundamental (non-symmetric) tensor K is obtained

$$K_{\mu\alpha;}{}^{\alpha} = 0$$

that, geometrically speaking, reads

$$d^{*}K = 0.$$

This is a current-free condition over the tensor K that can be exemplified in the simplest case with the prototype of non-symmetric fundamental tensor: $K_{\mu\nu} = g_{\mu\nu} + f_{\mu\nu}$

$$d^{*}K = d^{*}g + d^{*}f \Rightarrow d^{*}f = 0 \quad \text{(current-free e.o.m.)},$$

where *usually* $g_{\mu\nu}$ plays the role of the spacetime metric and $f_{\mu\nu}$ the role of electromagnetic field.

The metric is uniquely determined by the metricity condition, which puts 40 restrictions on the derivatives of the metric

$$g_{\mu\nu,\rho} = 2\Gamma_{(\mu\nu)\rho}. \tag{6}$$

The spacetime curvature tensor, that is defined in the usual way, has two possible contractions: the Ricci tensor $R^{\lambda}_{\mu\lambda\nu} = R_{\mu\nu}$, and the second contraction $R^{\lambda}_{\lambda\mu\nu} = 2\Gamma^{\lambda}{}_{\lambda[\nu,\mu]}$, which is identically zero due to the metricity condition (2).

In order to find a symmetry of the torsion tensor, let us denote the inverse of K by $\widehat{K}$. Therefore, $\widehat{K}$ is uniquely specified by condition $\widehat{K}^{\alpha\rho}\, K_{\alpha\sigma} = K^{\alpha\rho}\widehat{K}_{\alpha\sigma} = \delta^{\rho}_{\sigma}$.

As it was pointed out in [12–16], inserting explicitly the torsion tensor as the antisymmetric part of the connection in (5), and multiplying by $\frac{1}{2}\widehat{K}^{\alpha\nu}$, results, after straightforward computations, in

$$(Ln\sqrt{-K})_{,\mu} - \Gamma^{\nu}_{(\mu\nu)} = 0, \tag{7}$$

where $K = \det\,(K_{\mu\rho})$. Note that from expression (7) we arrive at the relation between the determinants K and g:

$$\frac{K}{g} = \text{constant}$$

(strictly a constant scalar function of the coordinates). Now we can write

$$\Gamma^{\nu}{}_{\alpha\nu,\beta} - \Gamma^{\nu}{}_{\beta\nu,\alpha} = \Gamma^{\nu}{}_{\nu\beta,\alpha} - \Gamma^{\nu}{}_{\nu\alpha,\beta}, \tag{8}$$

as the first term of (7) is the derivative of a scalar. Then, the torsion tensor has the symmetry

$$T^{\nu}{}_{\nu[\beta,\alpha]} = T^{\nu}{}_{\nu[\alpha,\beta]} = 0. \tag{9}$$

This implies that the trace of the torsion tensor, defined as $T^{\nu}{}_{\nu\alpha}$, is the gradient of a scalar field

$$T_\alpha = \nabla_\alpha \phi. \tag{10}$$

In [18], an interesting geometrical analysis is presented of non-symmetric field structures. There, expressions precisely as (1) and (2) ensure that the basic non-symmetric field structures (i.e. K) take on a definite geometrical meaning when interpreted in terms of affine geometry.

3. Geometrical Lagrangians: Determinantal to Generalized Born–Infeld Action

Let us to start with the geometrical Lagrangian introduced in [12–16]

$$\mathcal{L}_g = \sqrt{\det[\lambda(g_{\alpha\beta} + F_{\alpha\beta}) + R_{\alpha\beta}]}. \tag{11}$$

It can be rewritten as

$$\mathcal{L}_g = \sqrt{\det(G_{\alpha\beta} + \mathcal{F}_{\alpha\beta})} \tag{12}$$

with the following redefinitions

$$G_{\alpha\beta} = \lambda g_{\alpha\beta} + R_{(\alpha\beta)} \quad \text{and} \quad \mathcal{F}_{\alpha\beta} = \lambda F_{\alpha\beta} + R_{[\alpha\beta]}, \tag{13}$$

where a totally antisymmetric torsion tensor $T^{\alpha}_{\gamma\beta} = \varepsilon^{\alpha}_{\gamma\beta\delta} h^{\delta}$ is assumed (h^{δ} its dual a vector field). Consequently the generalized Ricci tensor splits into symmetric and antisymmetric parts, namely:

$$R_{\mu\nu} = \overbrace{\overset{\circ}{R}_{\mu\nu} - T_{\mu\rho}{}^{\alpha} T_{\alpha\nu}{}^{\rho}}^{R_{(\mu\nu)}} + \overbrace{\overset{\circ}{\nabla}_{\alpha} T_{\mu\nu}{}^{\alpha}}^{R_{[\mu\nu]}},$$

where $\overset{\circ}{R}_{\mu\nu}$ is the general relativistic Ricci tensor constructed with the Christoffel connection. The expansion of the determinant leads to the BI generalization in the usual form:

$$\mathcal{L}_g = \sqrt{|G|}\sqrt{1 + \frac{1}{2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{1}{16}(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}^{\mu\nu})^2} \tag{14}$$

$$= \Lambda^2 \sqrt{|g|}\sqrt{1 + \frac{1}{2}\Lambda_1^2 F_{\mu\nu}F^{\mu\nu} - \frac{1}{16b^4}(\Lambda_2^2 F_{\mu\nu}\widetilde{F}^{\mu\nu})^2}, \tag{15}$$

where

$$\Lambda = \lambda + \frac{g_{\alpha\beta}R^{(\alpha\beta)}}{4}, \tag{16}$$

$$\Lambda_1^2 = \lambda^2 \left(1 + \frac{2}{\lambda} \frac{F_{\mu\nu} R^{[\mu\nu]}}{F_{\mu\nu} F^{\mu\nu}} + \frac{1}{\lambda^2} \frac{R_{[\mu\nu]} R^{[\mu\nu]}}{F_{\mu\nu} F^{\mu\nu}} \right), \tag{17}$$

$$\Lambda_2^2 = \lambda^2 \left(1 + \frac{2}{\lambda} \frac{F_{\mu\nu} \widetilde{R}^{[\mu\nu]}}{F_{\mu\nu} \widetilde{F}^{\mu\nu}} + \frac{1}{\lambda^2} \frac{R_{[\mu\nu]} \widetilde{R}^{[\mu\nu]}}{F_{\mu\nu} \widetilde{F}^{\mu\nu}} \right). \tag{18}$$

Although the action is exact and has the correct limit, the analysis can be simplest and substantially improved using the following action:

$$\mathcal{L}_{gs} = \sqrt{\det \left[\lambda g_{\alpha\beta} \left(1 + \frac{R_s}{4\lambda} \right) + \lambda F_{\alpha\beta} \left(1 + \frac{R_A}{\lambda} \right) \right]}, \tag{19}$$

$$R_s \equiv g^{\alpha\beta} R_{(\alpha\beta)}; \quad R_A \equiv f^{\alpha\beta} R_{[\alpha\beta]} \tag{20}$$

(*with* $f^{\alpha\beta} \equiv \frac{\partial \ln(\det F_{\mu\nu})}{\partial F_{\alpha\beta}}, \det F_{\mu\nu} = 2F_{\mu\nu}\widetilde{F}^{\mu\nu}$) that contains all necessary information and is more suitable to manage. In the next section, we will give the exact justification about these choices and its simplification.

4. Choice of the Functional Action and Physics Geometrization

Let us give now the explanation about the specific choice of the Lagrangian (19) for the simplest subsequent analysis: geometrical and dynamical. The quantum aspects as supersymmetrical extensions, however, will be not treated here.

Symplectic geometry started from the study of phase spaces for mechanical systems but, with the subsequent seminal works of Cartan that introduce the symplectic structure into the geometry of the spacetime calculus, that thinking changed radically.

The existence of a symplectic structure on a manifold is a very significant constraint and many simple and natural constructions in symplectic geometry lead to manifolds which cannot possess a symplectic structure (or to spaces which cannot possess a manifold structure). However, these spaces often inherit a bracket of functions from the Poisson bracket on the original symplectic manifold. It is a (semi-)classical limit of quantum theory and also is the theory dual to Lie algebra theory and, more generally, to Lie algebroid theory.

From the point of view of the Poisson structure associated to the differential forms induced by the unitary transformation from the G-valuated tangent space implies automatically the existence of an *even non-degenerate (super)metric*. If the induced structure from the tangent space (via Ambrose–Singer theorem) is intrinsically related to a (super)manifold structure. From the structure of the tangent space $T_p(M)$ we have seen that [10, 17]

$$\begin{aligned} U_A^B(P) &= \delta_A^B + \mathcal{R}_{A\mu\nu}^B dx^\mu \wedge dx^\nu \\ &= \delta_A^B + \omega^k (\mathcal{T}_k)_A^B \end{aligned}$$

(with $A, B, \ldots$ generally a multi-index) where the Poisson structure is evident (as the dual of the Lie algebra of the group manifold) in our case leading to the identification

$$\mathcal{R}^{B}_{A\mu\nu} dx^{\mu} \wedge dx^{\nu} \equiv \omega^{k} (\mathcal{T}_{k})^{B}_{A}.$$

Then, because we have in the general case a (matrix) automorphic structure, the blocks inside the (sigma model-inspired) Lagrangian, namely:

$$\lambda g_{\alpha\beta} \left(1 + \frac{R_s}{4\lambda}\right) + \lambda F_{\alpha\beta} \left(1 + \frac{R_A}{\lambda}\right)$$

in our case, are justified. That means that above blocks contain all the field dynamics for a preliminary theoretical analysis.

Remark 1. Note that:

(1) the curvatures, the differential forms and the other geometrical operators depend also on the field where they are defined: $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.
(ii) typical example is the quaternionic $\mathbb{H}$-case (e.g.: SU(2)-structure of the UFT of Borchsenius theory) the metric is quaternion valuated with the property $g^{\dagger}_{[ij]} = -g_{[ji]}$.

5. Field Equations

The variational process is a crucial point both from the mathematical and from the physical viewpoint. As we have been analyzed before, there exist several difficulties concerning the starting physical assumptions involving the variational procedure. The geometry of the spacetime manifold is to be determined by the Noether symmetries

$$\frac{\delta L_G}{\delta g^{\mu\nu}} = 0, \quad \frac{\delta L_G}{\delta f^{\mu\nu}} = 0, \tag{21}$$

where the functional (Hamiltonian) derivatives in the sense of Palatini (in this case with respect to the potentials) are understood. The choice "measure-like" form for the geometrical Lagrangian L_G (reminiscent of a nonlinear sigma model), as is evident, satisfies the following principles:

(i) *the principle of the natural extension of the Lagrangian density as square root of the fundamental line element containing also* $F_{\mu\nu}$.
(ii) *the symmetry principle between* $g_{\mu\nu}$ *and* $F_{\mu\nu}$ *(e.g.* $g_{\mu\nu}$ *and* $F_{\mu\nu}$ *should enter into* L_G *symmetrically).*
(iii) *the principle that the spinorial symmetry, namely*

$$\nabla_{\mu} g_{\lambda\nu} = 0, \tag{22}$$

$$\nabla_{\mu} \sigma_{\lambda\nu} = 0 \tag{23}$$

with

$$g_{\lambda\nu} = \gamma_\lambda \cdot \gamma_\nu, \tag{24}$$

$$\sigma_{\lambda\nu} = \gamma_\lambda \wedge \gamma_\nu \sim *F_{\lambda\nu} \tag{25}$$

should be derivable from L_G (21).

The last principle is key because it states that the spinorial invariance of the fundamental spacetime structure (kinematic symmetry of the world picture) should be derivable from the dynamical symmetries given by (21). The fact that the L_G satisfies the three principles will be demonstrated below showing also that it has the simpler form.

Note that the action density proposed by Einstein in [27] in his non-symmetric field theory satisfies (i) and (ii) but not (iii) [11].

Remark 2. Due to the totally antisymmetric character of the torsion field it is completely determined by the fundamental (structural 2-form) antisymmetric tensor, and consequently the variations must acquire the form given by expression (21): it means that *metric and torsion have their respective potentials*.

5.1. $\delta_g L_G$

The starting point for the metrical variational procedure is in the same way as in the standard BI theory: from the following factorization of the geometrical Lagrangian:

$$\mathcal{L} = \sqrt{|g|}\sqrt{\det(\alpha\lambda)}\sqrt{1 + \frac{1}{2b^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16b^4}(F_{\mu\nu}\widetilde{F}^{\mu\nu})^2} \equiv \sqrt{|g|}\sqrt{\det(\alpha\lambda)}\mathbb{R}, \tag{26}$$

where

$$b = \frac{\alpha}{\beta} = \frac{1 + (R_S/4\lambda)}{1 + (R_A/4\lambda)}, \tag{27}$$

$$R_S = g^{\alpha\beta}R_{\alpha\beta}, \tag{28}$$

$$R_A = f^{\alpha\beta}R_{\alpha\beta}, \tag{29}$$

and λ an arbitrary constant, we perform the variational metric procedure with the following result (for details see Appendix A):

$$\delta_g\mathcal{L} = 0 \Rightarrow R_{(\alpha\beta)} - \frac{g_{\alpha\beta}}{4}R_s = \frac{R_s}{2\mathbb{R}^2\alpha^2}\left[F_{\alpha\lambda}F_\beta{}^\lambda - F_{\mu\nu}F^{\mu\nu}\frac{R_{(\alpha\beta)}}{R_s}\right] \tag{30}$$

$$+ \frac{R_s}{4\mathbb{R}^2\alpha^2 b^2}\left[F_{\mu\nu}\widetilde{F}^{\mu\nu}\left(\frac{F_{\eta\rho}\widetilde{F}^{\eta\rho}}{8}g_{\alpha\beta} - F_{\alpha\lambda}\widetilde{F}_\beta{}^\lambda\right) + \frac{F_{\eta\rho}\widetilde{F}^{\eta\rho}}{2}\frac{R_{(\alpha\beta)}}{R_s}\right]$$

$$+ 2\lambda\left[g_{\alpha\beta} + \frac{1}{\mathbb{R}^2\alpha^2}\left(F_{\alpha\lambda}F_\beta{}^\lambda + \frac{F_{\mu\nu}\widetilde{F}^{\mu\nu}}{2b^2}\left(\frac{F_{\eta\rho}\widetilde{F}^{\eta\rho}}{8}g_{\alpha\beta} - F_{\alpha\lambda}\widetilde{F}_\beta{}^\lambda\right)\right)\right], \tag{31}$$

Remark 3. Note that:

(1) Equation (31) is *trace-free* type, consequently the trace of the third term of the above equation (that is the cosmological one) is equal to zero. This happens trivially if $\lambda = 0$ or $4\mathbb{R}^2\alpha^2 = -(F_{\alpha\lambda}F^{\alpha\lambda} - \frac{(F_{\mu\nu}\widetilde{F}^{\mu\nu})^2}{4b^2})$. In terms of the Maxwell Lagrangian we have $(\mathbb{R}\alpha)^2 = (L_{\text{Maxwell}} + \frac{(F_{\mu\nu}\widetilde{F}^{\mu\nu})^2}{16b^2}) \equiv \mathcal{W}(I_S, I_P, b)$ that allow us to simplify Eq. (31) once more as follows:

$$
\begin{aligned}
R_{(\alpha\beta)} - \frac{g_{\alpha\beta}}{4}R_s = \frac{R_s}{2\mathcal{W}}&\left[F_{\alpha\lambda}F_\beta{}^{\lambda} - F_{\mu\nu}F^{\mu\nu}\frac{R_{(\alpha\beta)}}{R_s}\right] \\
&+ \frac{R_s}{4\mathcal{W}b^2}\left[F_{\mu\nu}\widetilde{F}^{\mu\nu}\left(\frac{F_{\eta\rho}\widetilde{F}^{\eta\rho}}{8}g_{\alpha\beta} - F_{\alpha\lambda}\widetilde{F}_\beta{}^{\lambda}\right)\right. \\
&\left.+ \frac{F_{\eta\rho}\widetilde{F}^{\eta\rho}}{2}\frac{R_{(\alpha\beta)}}{R_s}\right] + 2\lambda\left[g_{\alpha\beta} + \frac{1}{\mathcal{W}}\left(F_{\alpha\lambda}F_\beta{}^{\lambda}\right.\right. \\
&\left.\left.+ \frac{F_{\mu\nu}\widetilde{F}^{\mu\nu}}{2b^2}\left(\frac{F_{\eta\rho}\widetilde{F}^{\eta\rho}}{8}g_{\alpha\beta} - F_{\alpha\lambda}\widetilde{F}_\beta{}^{\lambda}\right)\right)\right].
\end{aligned}
$$

(2) b takes the place of limiting parameter (maximum value) for the electromagnetic field strength.

(3) b is not a constant, in general, in sharp contrast with the BI or string theory cases.

(4) Because b is the ratio $\frac{\alpha}{\beta} = \frac{1+(R_S/4\lambda)}{1+(R_A/\lambda 4)}$ involving both curvature scalars from the contractions of the generalized Ricci tensor: it is preponderant when the symmetrical contraction of $R_{\alpha\beta}$ is greater than the skew one.

(5) The fact pointed out in (ii), namely that the curvature scalar plays the role as some limiting parameter of the field strength, was conjectured by Mansouri in [19] in the context of gravity theory over group manifold (generally with symmetry breaking). In such a case, this limit was stablished after the explicit integration of the internal group-valuated variables that is not our case here.

(6) In similar form that the Eddington conjecture: $R_{(\alpha\beta)} \propto g_{\alpha\beta}$, we have a condition over the ratios as follows:

$$\frac{R_{(\alpha\beta)}}{R_s} \propto \frac{g_{\alpha\beta}}{D} \tag{32}$$

that seems to be universal.

(7) The equations are the simplest ones when $b^{-2} = 0 (\beta = 0)$, taking the exact "quasilinear" form

$$R_{(\alpha\beta)} - \frac{g_{\alpha\beta}}{4}R_s = \underbrace{\frac{R_s}{2\alpha^2}\left[F_{\alpha\lambda}F_\beta{}^{\lambda} - F_{\mu\nu}F^{\mu\nu}\frac{R_{(\alpha\beta)}}{R_s}\right]}_{\text{Maxwell-like}} + 2\lambda\underbrace{\left[g_{\alpha\beta} + \frac{1}{\mathcal{W}}F_{\alpha\lambda}F_\beta{}^{\lambda}\right]}_{\widetilde{g}_{\text{eff}}}, \tag{33}$$

this particular case (e.g. projective invariant) will be used through this work.

5.2. $\delta_f L_G$

Let us take as starting point the geometrical Lagrangian (19)

$$\mathcal{L}_{gs} = \sqrt{\det\left[\lambda g_{\alpha\beta}\left(1+\frac{R_s}{4\lambda}\right)+\lambda F_{\alpha\beta}\left(1+\frac{R_A}{4\lambda}\right)\right]} \tag{34}$$

$$= \sqrt{|g|}\lambda^2\alpha^2\left(\sqrt{1+\frac{1}{2}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}-\frac{1}{16}(\mathcal{F}_{\mu\nu}\widetilde{\mathcal{F}}^{\mu\nu})^2}\right) \tag{35}$$

then, having into account that. $R_A = f^{\mu\nu}R_{\mu\nu}$ and $\frac{\partial \ln(\det F_{\mu\nu})}{\partial F_{\alpha\beta}} = f^{\alpha\beta}$ we obtain

$$\frac{\delta L_G}{\delta F_{\sigma\omega}} = 0 \to \left(\frac{\sqrt{|g|}\lambda\beta}{2\mathbb{R}b}\right)\left[\mathbb{F}^{\sigma\omega}\beta - \frac{\mathbb{F}}{4\lambda}R_{[\mu\nu]}\chi^{\mu\nu\sigma\omega}\right] = 0, \tag{36}$$

where $\mathbb{F} \equiv [F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}b^{-2}(F_{\mu\nu}\widetilde{F}^{\mu\nu})^2]$, $\mathbb{F}^{\sigma\alpha} \equiv [F^{\sigma\alpha} - \frac{1}{4}b^{-2}(F_{\mu\nu}\widetilde{F}^{\mu\nu})\widetilde{F}^{\sigma\alpha}]$ and $\chi^{\mu\nu\sigma\omega} \equiv f^{\mu\omega}f^{\sigma\nu} - f^{\mu\sigma}f^{\omega\nu}$.

Contracting (36) with $F_{\alpha\beta}$, a condition over the curvature and the electromagnetic field invariants is obtained as

$$\left(\frac{\sqrt{|g|}\lambda\beta}{\mathbb{R}b}\right)\mathbb{F}\left[\beta - \frac{R_A}{2\lambda}\right] = 0.$$

The condition satisfied for $R_A = -4\lambda$ is the exact projective invariant case (that corresponds with $\beta = 0$), and for $R_A = 2\lambda$.

6. Emergent Trace-Free Gravitational Equations: The Meaning of Λ

Starting from the trace-free equation (31)

$$\underbrace{\overset{\circ}{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2}\overset{\circ}{R}}_{\equiv G_{\alpha\beta}} = \underbrace{6\left(-h_\alpha h_\beta + \frac{g_{\alpha\beta}}{2}h_\gamma h^\gamma\right)}_{\equiv T^h_{\alpha\beta}} + \frac{g_{\alpha\beta}}{2}R_s + T^F_{\alpha\beta} + 2\lambda\rho_{\alpha\beta}, \tag{37}$$

$$\rho_{\alpha\beta} \equiv g_{\alpha\beta} + \frac{1}{\mathcal{W}}\left(F_{\alpha\lambda}F_\beta{}^\lambda + \frac{F_{\mu\nu}\widetilde{F}^{\mu\nu}}{2b^2}\left(\frac{F_{\eta\rho}\widetilde{F}^{\eta\rho}}{8}g_{\alpha\beta} - F_{\alpha\lambda}\widetilde{F}_\beta{}^\lambda\right)\right), \tag{38}$$

$$T^F_{\alpha\beta} \equiv \frac{R_s}{2\mathcal{W}}\left\{\left(F_{\alpha\lambda}F_\beta{}^\lambda - F_{\mu\nu}F^{\mu\nu}\frac{R_{(\alpha\beta)}}{R_s}\right)\right.$$
$$\left. + \frac{1}{2b^2}\left[F_{\mu\nu}\widetilde{F}^{\mu\nu}\left(\frac{F_{\eta\rho}\widetilde{F}^{\eta\rho}}{8}g_{\alpha\beta} - F_{\alpha\lambda}\widetilde{F}_\beta{}^\lambda\right) + \frac{(F_{\eta\rho}\widetilde{F}^{\eta\rho})^2}{2}\frac{R_{(\alpha\beta)}}{R_s}\right]\right\}. \tag{39}$$

Then, as $\overset{\circ}{\nabla}{}^{\alpha} G_{\alpha\beta} = \overset{\circ}{\nabla}{}^{\alpha} (T^h_{\alpha\beta} + T^F_{\alpha\beta}) = 0$ consequently

$$\nabla^{\alpha}\left(\frac{g_{\alpha\beta}}{2}R_s + 2\lambda\rho_{\alpha\beta}\right) = 0$$
$$\Rightarrow \left(\frac{g_{\alpha\beta}}{2}R_s + 2\lambda\rho_{\alpha\beta}\right) = \Lambda g_{\alpha\beta} \to R_s = 2\Lambda \tag{40}$$

come back to the original trace-free expressions we have the expected formula

$$\underbrace{\overset{\circ}{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2}\overset{\circ}{R}}_{\equiv G_{\alpha\beta}} = \underbrace{6\left(-h_{\alpha}h_{\beta} + \frac{g_{\alpha\beta}}{2}h_{\gamma}h^{\gamma}\right)}_{\equiv T^h_{\alpha\beta}} + T^F_{\alpha\beta} + \Lambda g_{\alpha\beta}. \tag{41}$$

Remark 4. Tracing the first expression in (40) we have $R_s = 2\Lambda = \overset{\circ}{R} + 6h_{\mu}h^{\mu}$ linking the value of the curvature and the norm of the torsion vector field. Consequently, if the dual of the torsion field has the role of the energy–matter carrier, the meaning of lambda as the vacuum energy is immediately stablished.

7. The Vector h_μ and the Energy–Matter Interpretation

One of the main characteristics in unified field theoretical models is the possibility to introduce the energy and matter through its geometrical structure. In our case the torsion field takes the role of right-hand side of the standard GR gravity equation by means of its dual, namely h_μ.

Consequently, in order to explain the physical role of h_μ we know (due to the Hodge–de Rham decomposition (Appendix B) that it can be decomposed as:

$$h_{\alpha} = \nabla_{\alpha}\Omega + \varepsilon_{\alpha}^{\beta\gamma\delta}\nabla_{\beta}A_{\gamma\delta} + \gamma_1\overbrace{\varepsilon_{\alpha}^{\beta\gamma\delta}M_{\beta\gamma\delta}}^{\text{axial vector}} + \gamma_2\overbrace{P_{\alpha}}^{\text{polar vector}}, \tag{42}$$

where γ_1 and γ_2 can be phenomenologically related to physical constants (e.g. $\gamma_1 = \gamma_2 = \gamma = \frac{8\pi}{c}\sqrt{G}$ is a physical constant related to the Blackett formula [20]). The arguments in favor of this type of theories and from the decomposition (42) can be resumed as follows:

(i) the existence of an angular momentum Helmholtz theorem [21, 22]: the theorem in analysis is exactly as in E_3, but in the four-dimensional case M_4 there exists an additional *axial vector*;

(iii) the concept of *chirality* is achieved in the model by the existence of polar and axial vectors in expression (42).

(iv) if $\Omega, A_{\gamma\delta}$ are the wave tensors and $\varepsilon_{\alpha}^{\beta\gamma\delta}M_{\beta\gamma\delta}, P_{\alpha}$ the particle vectors (vector and axial part), the concept of an inertial-wave vector is introduced in Eq. (42).

Consequently, from the equation of motion for the torsion namely: $\nabla_{\alpha}T^{\alpha\beta\gamma} = -\lambda F^{\beta\gamma}$ and back to (42) we obtain the following important equation:

$$\overset{\circ}{\Box}A_{\gamma\delta} - \gamma[\nabla_{\alpha}M^{\alpha}{}_{\gamma\delta} + (\nabla_{\gamma}P_{\delta} - \nabla_{\delta}P_{\gamma})] = -\lambda F_{\gamma\delta}. \tag{43}$$

Consider, in particular, the case when $\lambda F\ _{\gamma\delta} \to 0$:

$$\mathring{\Box} A_{\gamma\delta} = \gamma[\nabla_\alpha M^\alpha{}_{\gamma\delta} + (\nabla_\gamma P_\delta - \nabla_\delta P_\gamma)]. \tag{44}$$

We can immediately see that, if $M^\alpha{}_{\gamma\delta}$ is identified with the intrinsic spin angular momentum of the ponderable matter, P_δ is its lineal momentum vector and $A_{\gamma\delta}$ is the gravitational radiation tensor, then Eq. (4) states that the sum of the intrinsic spin angular momentum and the orbital angular momentum of ponderable matter is conserved if the gravitational radiation is absent.

8. Is the G Really Constant?

At this level, no assertion can state with respect to G or even with respect to c. The link with the general relativistic case is given by the identification of electromagnetic energy–momentum tensor with the term analogous $T^F_{\alpha\beta}$ in our metric variational equations:

$$\frac{8\pi G}{c^4}\left(F_{\alpha\lambda}F_\beta{}^\lambda - F_{\mu\nu}F^{\mu\nu}\frac{g_{\alpha\beta}}{4}\right) \to \frac{R_s}{2\mathcal{W}}\left(F_{\alpha\lambda}F_\beta{}^\lambda - F_{\mu\nu}F^{\mu\nu}\frac{R_{(\alpha\beta)}}{R_s}\right).$$

Consequently, we have

$$\kappa = \frac{8\pi G}{c^4} \to \frac{R_s}{2\mathbb{R}^2\alpha^2} \quad \text{and} \quad \frac{g_{\alpha\beta}}{4} = \frac{R_{(\alpha\beta)}}{R_s} \tag{45}$$

the above expression says that the ratio must remain constant due to the Noether symmetries and conservation laws of the field equations. Note that (as in the case of b) there exists a limit for all the physical fields coming from the *geometrical invariants quantities*.

9. Physical Consequences

9.1. *Electrodynamic structure in* $3 + 1$

The starting point will be the line element in $3 + 1$ splitting [6, 7] (Appendix C): the four-dimensional spacetime is split into three-dimensional space and one-dimensional time to form a foliation of three-dimensional spacelike hypersurfaces. The metric of the spacetime is consequently given by $ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$, where γ_{ij} is the metric of the three-dimensional hypersurface, α is the lapse function, and β^i is the shift function (see Appendix C for details). For any nonlinear Lagrangian, in sharp contrast with the Einstein–Maxwell case, the field equations $d * \mathbb{F} = *\mathbb{J}$ and the Bianchi-geometrical condition $dF = 0$ (where we have defined the Hodge dual $*$ and $\mathbb{F} = \frac{\partial\mathcal{L}}{\partial F}$) are expressed by the vector fields

$$E, B, \mathbb{E} = \frac{\partial\mathcal{L}}{\partial E}, \quad \mathbb{B} = \frac{\partial\mathcal{L}}{\partial B} \tag{46}$$

In our case given by the geometrical Lagrangian $\mathcal{L}_g$ (not be confused with the Lie derivative $\mathcal{L}_\beta$!)

$$\nabla \cdot \mathbb{E} = -\overline{h} \cdot \mathbb{B} + 4\pi\rho_e, \tag{47}$$

$$\nabla \cdot B = 0, \tag{48}$$

$$\begin{aligned} \nabla \times (\alpha E) &= -(\partial_t - \mathcal{L}_\beta)B \\ &= -\partial_0 B + (\beta \cdot \nabla)B - (B \cdot \nabla)\beta, \end{aligned} \tag{49}$$

$$\begin{aligned} \nabla \times (\alpha \mathbb{B}) + h_0\mathbb{B} - \overline{h} \times \mathbb{E} &= -(\partial_t - \mathcal{L}_\beta)\mathbb{E} + 4\pi\alpha j \\ &= \partial_0 \mathbb{E} - (\beta \cdot \nabla)\mathbb{E} + (\mathbb{E} \cdot \nabla)\beta + 4\pi\alpha j, \end{aligned} \tag{50}$$

where h^μ is the axial torsion vector.

9.2. *Dynamo effect and geometrical origin of $\alpha\Omega$ term*

In the case of weak field approximation and $(F^{01} \to E^i, F^{jk} \to B^i)$ the electromagnetic Maxwell-type equations in 3+1 take the form

$$\nabla_\nu F^{\nu\mu} = T^{\mu\nu\rho} F_{\nu\rho} = \varepsilon^{\mu\nu\rho}{}_\delta h^\delta F_{\nu\rho} \quad (d^*F = ^* J), \tag{51}$$

$$\overline{\nabla} \cdot \overline{E} = -\overline{h} \cdot \overline{B}, \tag{52}$$

$$\partial_t \overline{E} - \overline{\nabla} \times \overline{B} = h^0 \overline{B} - \overline{h} \times \overline{E} \tag{53}$$

and

$$\nabla_\nu {}^*F^{\nu\mu} = 0 \quad (dF = 0), \tag{54}$$

$$\overline{\nabla} \cdot \overline{B} = 0, \tag{55}$$

$$\partial_t \overline{B} = -\overline{\nabla} \times \overline{E}. \tag{56}$$

Putting all together, the set of equations is

$$\overline{\nabla} \cdot \overline{E} + \overline{h} \cdot \overline{B} = \rho_{\text{ext}}, \tag{57}$$

$$\partial_t \overline{E} - \overline{\nabla} \times \overline{B} = h^0 \overline{B} - \overline{h} \times \overline{E} - \sigma_{\text{ext}}[\overline{E} + \overline{v} \times \overline{B}], \tag{58}$$

$$\overline{\nabla} \cdot \overline{B} = 0, \tag{59}$$

$$\partial_t \overline{B} = -\overline{\nabla} \times \overline{E}, \tag{60}$$

where we have introduced external charge density and current. Following the standard procedure we take the rotational to the second equation above, obtaining straightforwardly the modified dynamo equation

$$\begin{aligned} \overline{\nabla} \times \partial_t \overline{E} + \overline{\nabla}^2 \overline{B} &= \overline{\nabla} \times (h^0 \overline{B}) + (\overline{h} \cdot \overline{B} - \rho_{\text{ext}})\overline{h} \\ &\quad + (\overline{\nabla} \cdot \overline{h})\overline{E} - \sigma_{\text{ext}}[\partial_t \overline{B} + (\overline{\nabla} \cdot \overline{v})\overline{B}], \end{aligned} \tag{61}$$

where the standard identities of the vector calculus plus the first, the third and the fourth equations above have been introduced. Note that in the case of the standard approximation and (in the spirit of this research) without any external or additional ingredients, we have

$$\overline{\nabla}^2\overline{B} = h^0(\overline{\nabla} \times \overline{B}) + (\overline{h} \cdot \overline{B})\overline{h} + (\overline{\nabla} \cdot \overline{h})\overline{E}. \tag{62}$$

Here we can see that there exists an α-term with a pure geometrical origin, and not only a turbulent one that is given by h^0 (the zero component of the dual of the torsion tensor).

9.3. *Comparison with the mean field formalism*

Now we compare the above equations with respect to the mean field formalism [23]. Starting from expressions (51)–(56) as before, we have

$$\eta\overline{\nabla}^2\overline{B} + \overline{\nabla} \times (\overline{v} \times \overline{B}) - \partial_t\overline{B} \underbrace{+\eta\overline{\nabla} \times (-h^0\overline{B} + \overline{h} \times \overline{E})}_{\mathcal{E}_{\mathrm{Geom}}} = 0. \tag{63}$$

$\mathcal{E}_{\mathrm{Geom}}$ takes the place of electromotive force due to the torsion field with full analogy as $\mathcal{E} = \langle u \times b\rangle$ is the mean electromotive force due to fluctuations. Also, as in the mean field case that there are the splitting

$$\mathcal{E} = \mathcal{E}^{\langle 0\rangle} + \mathcal{E}^{\langle\overline{B}\rangle} \tag{64}$$

with $\mathcal{E}^{\langle 0\rangle}$ independent of $\langle\overline{B}\rangle$ and $\mathcal{E}^{\langle\overline{B}\rangle}$ linear and homogeneous in $\overline{B}$, we have in the torsion case the following correspondence:

$$-h^0\overline{B} \longleftrightarrow \mathcal{E}^{\langle\overline{B}\rangle},$$

$$\overline{h} \times \overline{E} \longleftrightarrow \mathcal{E}^{\langle 0\rangle},$$

$$\text{geometrical} \longleftrightarrow \text{turbulent}.$$

Consequently, the problems of mean-field dynamo theory that are concerned with the generation of a mean EMF by turbulence have in this model a pure geometric counterpart. In the past years, attention has shifted from kinematic calculations, akin to those familiar from quasilinear theory for plasmas, to self-consistent theories which account for the effects of small-scale magnetic fields (including their back-reaction on the dynamics) and for the constraints imposed by the topological conservation laws, such as that for magnetic helicity. Here the torsion vector generalizes (as we can see from above set of equations) the concept of helicity. The consequence of this role of the dual torsion field is that the traditionally invoked mean-field dynamo mechanism (i.e. the so-called alpha effect) may be severely quenched or increased at modest fields and magnetic Reynolds numbers, and that spatial transport of this generalized magnetic helicity is crucial to mitigating this quench. Thus, the dynamo problem is seen in our model as one of the generalized helicity transports, and so may be tackled like other problems in turbulent transport. A key element in this approach is to understand the evolution of the torsion vector field besides that of the turbulence energy and the generalized helicity profiles in spacetime. This forces us to confront the problem of spreading of strong MHD turbulence, and a spatial variant or analogue of the selective decay problem with the dynamics of the torsion field.

10. Torsion, Axion Electrodynamics or Chern–Simons Theory?

Let us briefly review the electromagnetic sector of the QCD theory based on a gauge symmetry $\mathrm{SU}(3) \times U(1)$

$$L_{\mathrm{QCD/QED}} = \sum \overline{\psi}_f[\gamma^\mu(\partial_\mu - ig_f t^\alpha A^\alpha_\mu - iq_f A_\mu) - m_f]\psi_f$$
$$- \frac{G^\alpha_{\mu\nu}G^{\alpha\mu\nu}}{4} - \frac{F_{\mu\nu}F^{\mu\nu}}{4} - \frac{g^2\theta G^\alpha_{\mu\nu}\widetilde{G}^{\alpha\mu\nu}}{32\pi^2} - \frac{g^2\theta F_{\mu\nu}\widetilde{F}^{\mu\nu}}{32\pi^2}. \tag{65}$$

As is well known, electromagnetic fields will couple to the electromagnetic currents, namely: $J_\mu = \sum_f q_f \overline{\psi}_f \gamma_\mu \psi_f$ consequently, there appear terms that will induce through the quark loop the coupling of $F_{\mu\nu}\widetilde{F}^{\mu\nu}$ (the anomaly) to the QCD topological charge. The effective Lagrangian can be written as

$$L_{\mathrm{MCS}} = -\frac{F_{\mu\nu}F^{\mu\nu}}{4} - A_\mu J^\mu - \frac{c}{4}\theta F_{\mu\nu}\widetilde{F}^{\mu\nu}, \tag{66}$$

where a pseudoscalar field $\theta = \theta(\overline{x}, t)$ (playing the role of the axion field) is introduced and $c = \sum_f \frac{(q_f e)^2}{2\pi^2}$. This is the CS Lagrangian where, if θ is constant, the last term is a total divergence: $F_{\mu\nu}\widetilde{F}^{\mu\nu} = \partial_\mu J^\mu_{\mathrm{CS}}$. The question appears if θ is not a constant $\theta F_{\mu\nu}\widetilde{F}^{\mu\nu} = \theta\partial_\mu J^\mu_{\mathrm{CS}} = \partial_\mu(\theta J^\mu_{\mathrm{CS}}) - J^\mu_{\mathrm{CS}}\partial_\mu\theta$.

Now we can see from the previous section that if, from the general decomposition of the four-dimensional dual of the torsion field via the Hodge–de Rham theorem, we retain b_α as gradient of a pseudoscalar (e.g. axion) these equations coincide in form with the respective equation for MCS theory. Precisely because under this condition $h_\alpha = \nabla_\alpha\theta$, in flat space (curvature = 0 but torsion $\neq 0$) the equations become the same as in [8], namely

$$\overline{\nabla} \cdot \overline{E} - c\overline{P} \cdot \overline{B} = \rho_{\mathrm{ext}}, \tag{67}$$

$$\partial_t\overline{E} - \overline{\nabla} \times \overline{B} = -c\dot{\theta}\overline{B} + c\overline{P} \times \overline{E} - \sigma_{\mathrm{ext}}[\overline{E} + \overline{v} \times \overline{B}], \tag{68}$$

$$\overline{\nabla} \cdot \overline{B} = 0, \tag{69}$$

$$\partial_t\overline{B} = -\overline{\nabla} \times \overline{E} \tag{70}$$

provided:

$$h^0 \to -c\dot{\theta}, \tag{71}$$

$$\overline{h} \to -c\overline{P}, \tag{72}$$

where from QCD the constant c is determined as $c = \frac{e^2}{2\pi}$ and the $\partial_\mu\theta = (\dot{\theta}, \overline{P})$ in the [8] notation. The main difference is that while in the case of photons in axion

ED was given by [9] the Lagrangian where the above equations are derived as

$$L_{\mathrm{MCS}} = -\frac{F_{\mu\nu}F^{\mu\nu}}{4} - A_\mu J^\mu + \frac{c}{4}P_\mu J^\mu_{\mathrm{CS}}, \quad J^\mu_{\mathrm{CS}} \equiv \varepsilon^{\mu\sigma\rho\nu}A_\sigma F_{\rho\nu} \tag{73}$$

in our case is the dual of the torsion field (that we take as the gradient of a pseudoscalar) responsible for the structure of the set of equations.

11. Magnetic Helicity Generation and Cosmic Torsion Field

Here we consider the projective invariant case $\beta = 0$ ($R_A = -4\lambda$) where the gravitational and field equations are considerably simplified because $\mathbb{R} = 1$ and $b^{-1} = 0$. Scalar curvature R and the torsion 2-form field $T^a_{\mu\nu}$ with a SU(2) Yang–Mills structure are defined in terms of the affine connection $\Gamma^\lambda_{\mu\nu}$ and the SU(2) valuated (structural torsion potential) ${f^a}_\mu$ by

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu}, \quad R_{\mu\nu} = R^\lambda_{\mu\lambda\nu}, \\ R^\lambda_{\mu\lambda\nu} &= \partial_\nu\Gamma^\lambda_{\mu\rho} - \partial_\rho\Gamma^\lambda_{\mu\nu} + \cdots, \\ {T^a}_{\mu\nu} &= \partial_\mu {f^a}_\nu - \partial_\nu {f^a}_\mu + \varepsilon^a_{bc}{f^b}_\mu {f^c}_\nu. \end{aligned} \tag{74}$$

G and Λ are the geometrically induced Newton gravitational constant (as we have been discussed before) and the integration cosmological constant, respectively. From the last equation for the totally antisymmetric Torsion 2-form, the potential ${f^a}_\mu$ defines the affine connection $\Gamma^\lambda_{\mu\nu}$. Similar to the case of Einstein–Yang–Mills systems, for our new UFT model it can be interpreted as a prototype of gauge theories interacting with gravity (e.g. QCD, GUTs, etc.). We stress here the important fact that all the fundamental constants are really geometrically induced as required by the Mach principle. After varying the action, we obtain the gravitational equation (41), namely

$$\overset{\circ}{R}_{\alpha\beta} - \frac{g_{\alpha\beta}}{2}\overset{\circ}{R} = 6\left(-h_\alpha h_\beta + \frac{g_{\alpha\beta}}{2}h_\gamma h^\gamma\right) + \kappa_{\mathrm{geom}}\left[F_{\alpha\lambda}{F_\beta}^{\lambda} - F_{\mu\nu}F^{\mu\nu}\frac{g_{\alpha\beta}}{4}\right] + \Lambda g_{\alpha\beta} \tag{75}$$

with the "gravitational constant" geometrically induced as

$$\kappa_g \equiv \frac{R_s}{2\mathcal{W}} = \left.\frac{8\pi G}{c^4}\right|_{\mathrm{today}} \tag{76}$$

and the field equation for the torsion 2-form in differential form

$$d^*T^a + \frac{1}{2}\varepsilon^{abc}(f_b \wedge^* T_c -^* T_b \wedge f_c) = -\lambda^* f^a. \tag{77}$$

Note that κ_g (76) and Λ are not independent, but related by $R_s = 2\Lambda$. In the case $\beta = 0$ we have the simplest expression:

$$\kappa_g \equiv \frac{R_s}{2\left(1 + \frac{R_s}{4\lambda}\right)^2} = \frac{\Lambda}{\left(1 + \frac{2\Lambda}{4\lambda}\right)^2}$$

in consequence, generalizing the conjecture of Markov in [24], if Λ is proportional to the energy, κ goes as Λ if $|\Lambda| \le 1$, and as Λ^{-1} in other case.

We are going to seek for a classical solution of (75) and (77) with the following spherically symmetric ansatz for the metric and gauge connection:

$$ds^2 = d\tau^2 + a^2(\tau)\sigma^i \otimes \sigma^i \equiv d\tau^2 + e^i \otimes e^i. \tag{78}$$

Here τ is the Euclidean time and the dreibein is defined by $e^i \equiv a(\tau)\sigma^i$. The gauge connection is

$$f^a \equiv f^a_\mu dx^\mu = f\sigma^a, \tag{79}$$

for $a, b, c = 1, 2, 3$, and for $a, b, c = 0$, we have

$$f^0 \equiv f^0_\mu dx^\mu = s\sigma^0. \tag{80}$$

This choice for the potential torsion is according to the symmetries involved in the problem.

The σ^i 1-form satisfies the SU(2) Maurer-Cartan structure equation

$$d\sigma^a + \varepsilon^a{}_{bc}\sigma^b \wedge \sigma^c = 0 \tag{81}$$

Note that in the ansatz the frame and SU(2) (isospin-like) indices are identified (as for the case with the non-abelian-Born–Infeld (NBI) Lagrangian of [30]). The torsion 2-form

$$T^\gamma = \frac{1}{2}T^\gamma{}_{\mu\nu}dx^\mu \wedge dx^\nu \tag{82}$$

becomes

$$\begin{aligned} T^a &= df^a + \frac{1}{2}\varepsilon^a{}_{bc}f^b \wedge f^c \\ &= \left(-f + \frac{1}{2}f^2\right)\varepsilon^a{}_{bc}\sigma^b \wedge \sigma^c, \end{aligned} \tag{83}$$

$$\begin{aligned} d^*T^a + \tfrac{1}{2}\varepsilon^{abc}(f_b \wedge^* T_c -^* T_b \wedge f_c) &= -2\lambda^* f^a, \\ (-2f + f^2)(1-f)d\tau \wedge e^b \wedge e^c &= -2\lambda d\tau \wedge e^b \wedge e^c, \end{aligned} \tag{84}$$

$${}^*T^a \equiv h(-2f + f^2)d\tau \wedge \frac{e^a}{a^2}, \tag{85}$$

$${}^*f^a = -f\frac{d\tau \wedge e^b \wedge e^c}{a^3}. \tag{86}$$

Note that to be complete in our description of the possible physical scenarios, we include f^0 as a $U(1)$ component of the torsion potential (although does not belong to the space SU(2)/$U(1)$). Taking all the above issues into account, the expression for the torsion is analogous to the non-abelian 2-form strength field of [30].

Inserting T^a from (83) into the dynamical equation (77), we obtain

$$(-2f + f^2)(1-f)d\tau \wedge e^b \wedge e^c = -\lambda d\tau \wedge e^b \wedge e^c, \tag{87}$$

and from expression (87), we have an algebraic cubic equation for f

$$(-2f + f^2)(1-f) + \lambda = 0. \tag{88}$$

We can see that, in contrast with our previous work with a dualistic theory [30] where the NBI energy–momentum tensor of BI was considered, there exist three non-trivial solutions for f, depending on the cosmological constant λ. In this preliminary analysis of the problem, only the values of f make the quantity $(-f+\frac{1}{2}f^2)\in\mathbb{R}$. Consequently, for $\lambda=2$, we find $f=2.35$, then

$$T^{a}_{bc}=\frac{2}{5}\frac{\varepsilon^{a}_{bc}}{a^2};\quad T^{a}_{0c}=0. \tag{89}$$

That is, only spatial torsion field is non-vanishing while cosmic time torsion field vanishes (an analogous feature with magnetic and electric Yang–Mills can be seen in the solution of Giddings and Strominger and in [30]). Substituting the expression for the torsion 2-form (89)[a] into the symmetric part of the variational equation we reduce Eq. (41) or (75) to an ordinary differential equation for the scale factor a,

$$3\left[\left(\frac{\dot{a}}{a}\right)^2-\frac{1}{a^2}\right]-\Lambda=\frac{3\kappa_g}{4a^2}(f^2+s^2)+\frac{3}{2a^4}f^2(f-2)^2 \tag{90}$$

which in the case for the computed value for $f\sim 2.35$ with $s=10$ and $\Lambda\lesssim 1$ the scale factor is described in Fig. 1 and the scale factor goes as:

$$a(\tau)=\Lambda^{-1/2}\sqrt{\left(1-\frac{12\kappa_g^2\Lambda}{\alpha}\right)^{1/2}\sinh\left(\sqrt{\Lambda/3}(\tau-\tau_0)\right)-1+\kappa_g(f^2+s^2)/4}, \tag{91}$$

where we define the geometrically induced fine structure function $\alpha\equiv\kappa_g(f^2+s^2)/4$.

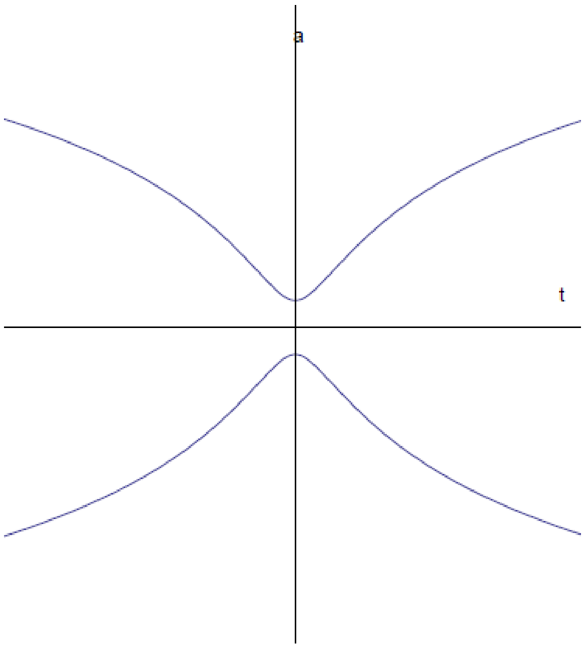

Fig. 1. Wormhole solution (a versus t) for $f=2.35$ and constants $k=\pi<1$.

[a] In the tetrad: $\overset{\circ}{R}_{00}=-3\frac{\ddot{a}}{a},\overset{\circ}{R}_{ab}=-[\frac{\ddot{a}}{a}+2(\frac{\dot{a}}{a})^2-\frac{2}{a^2}]$.

11.1. *Primordial symmetries of standard model and torsion field*

In [25], the cross-section for neutrino helicity spin flip obtained from this type of $f(R;T)$ model of gravitation with dynamic torsion field introduced by one of the authors in [16] was phenomenologically analyzed using the relation with the axion decay constant f_a (Peccei–Quinn parameter) due to the (partially logarithmical) energy dependence of the cross-section, Consequently, the link with the phenomenological energy/mass window was found from the astrophysical and high energy viewpoints. The important point is that, in relation with the torsion vector interaction Lagrangian, the f_a parameter gives an estimate of the torsion field strength that can variate with time within cosmological scenarios as the described above, potentially capable of modifying the overall leptogenesis picture, beside the magnetogenesis, the bariogenesis and also to obtain some indication about the original (super) symmetry of the early universe.

In FRW scenario given here, we saw that the torsion through its dual vector, namely:

$$h^0 = \frac{2}{5}\frac{\delta_a^0 C_\tau}{a^2} d\tau \wedge e^a \tag{92}$$

goes as $\sim a^{-2}$ with C_τ a covariantly constant vector field (e.g. $\overset{\circ}{\nabla} C_\tau = 0$) that we take of the form $C_\tau \sim (\dot{\theta} + q_\tau)$ (due to the Hodge–de Rham decomposition of h_μ, expression (42)) where θ is a pseudoscalar field playing the obvious role of axion and q_τ: vector field linking h^0 with the magnetic field via the equation of motion for the torsion. Consequently, the torsion dual vector h^i has the maximum value when the radius of the universe is $a_{\min}$, e.g. $a_{\min} = a(\tau_0)$ increases to the maximum value of the spin-flip neutrino cross-section and, for instance, the quantity of right neutrinos compensating consequently the actual asymmetry of the electroweak sector of the standard model (see the behavior of a in Fig. 1). This fact indicates that the original symmetry group contains naturally $\mathrm{SU}_R(2) \times \mathrm{SU}_L(2) \times U(1)$ typically inside GUTs structurally based generally on SO(10), SU(5) or some exceptional groups as $E(6)$, $E(7)$, etc.

Also, it is interesting to note that from the FRW line element written in terms of the cosmic time the Hubble flow electromagnetic fields $E_\mu \equiv (0, E_i) = a^{-2}(0, \partial_\tau A_i)$ and $B_\mu \equiv (0, B_i) = a^{-2}(0, \varepsilon_{ijk}\partial_j A_k)$

$$\overline{\nabla} \cdot \overline{E} + \left(\frac{\alpha}{f}\overline{\nabla}\theta + \overline{\Pi}\right) \cdot (a^2\overline{B}) = 0, \tag{93}$$

$$\partial_\tau(a^2\overline{E}) - \overline{\nabla} \times (a^2\overline{B}) = \left(\frac{\alpha}{f}\partial_\tau\theta + \Pi_0\right)(a^2\overline{B}) - \left(\frac{\alpha}{f}\overline{\nabla}\theta + \overline{\Pi}\right) \times \overline{E}, \tag{94}$$

$$\overline{\nabla} \cdot \overline{B} = 0, \tag{95}$$

$$\partial_t\overline{B} = -\overline{\nabla} \times \overline{E}, \tag{96}$$

where $\Pi_\mu \equiv f_\mu(u_\mu, \gamma^5 b_\mu, eA_\mu, \dots,)$ is a vector function of physical entities as potential vector, vorticity, angular velocity, axial vector, etc. as described by expression (42). In principle, we can suppose that it is zero (low back reaction [28]) then

$$\overline{h} = \frac{\alpha}{f}\overline{\nabla}\theta, \quad h^0 = \frac{\alpha}{f}\partial_\tau\theta \tag{97}$$

being $[\partial_\tau^2 - \overline{\nabla}^2 - \frac{\alpha}{f}\partial_\tau\theta\overline{\nabla}\times](a^2\overline{B}) = 0$ the second-order equation for the magnetic field that shows the chiral character of the plasma particles.

11.2. *Magnetogenesis and cosmic helicity*

Now we pass to see which role is played by the torsion field in the magnetic field generation in a FRW cosmology. Taking as the starting point the (hyper)electrodynamic equations [29] and introducing a Fourier mode decomposition $\overline{B}(\overline{x}) = \int d^3\overline{k}\overline{B}(\overline{k})e^{-i\overline{k}\cdot\overline{x}}$ with $\overline{B}(\overline{k}) = h_i\overrightarrow{e}_i$ where $i = 1, 2$, $\overrightarrow{e}_i^2 = 1$, $\overrightarrow{e}_i \cdot \overrightarrow{k} = \overrightarrow{e}_1 \cdot \overrightarrow{e}_2 = 0$, the torsion-modified dynamical equations for the expanding FRW become

$$\dot{\overline{z}} + \left[\left(2\dot{a} + \frac{k^2}{\sigma}\right) + \frac{ah^0|k|}{\sigma}\right]\overline{z} = 0, \tag{98}$$

$$\dot{z} + \left[\left(2\dot{a} + \frac{k^2}{\sigma}\right) - \frac{ah^0|k|}{\sigma}\right]z = 0, \tag{99}$$

where the magnetic field is written in terms of complex variable $z(\overline{z})$ as

$$z = h_1 + ih_2, \tag{100}$$

$$\overline{z} = h_1 - ih_2. \tag{101}$$

From Eq. (99) we see that the solution for z namely:

$$z = z_0 e^{-\left(2a + \frac{k^2}{\sigma}\tau\right) + \int \frac{ah^0|k|}{\sigma}d\tau} \tag{102}$$

contains the instable mode in the sense of [29] $\frac{k}{\sigma}\tau < \int \frac{ah^0}{\sigma}d\tau$. Consequently, a defined polarization of the magnetic field appears and from the dynamical equation for the torsion field: $\nabla_{[\mu}h_{\nu]} = -\lambda\widetilde{F}_{\mu\nu}$ that in this case is

$$\nabla_{[i}h_{\tau]} = \nabla_i(a^{-2}q_\tau) = -\lambda B_i, \tag{103}$$

which implies a relation between the vector part of the h^0 (namely q_τ) with the vector potential A^k of the magnetic field as follows:

$$\nabla_i q_\tau \approx -\lambda\varepsilon_{ijk}\nabla^j A^k. \tag{104}$$

Consequently, the primordial magnetic field (or seed) would be connected in a self-consistent way with the torsion field by means of the dual vector h_0. It (h_μ) in turn, would be connected phenomenologically with the physical fields (matter) of theory through Hodge–de Rham decomposition expression (42). We note from expression (102) that the pseudoscalar (axion) controls the stability, growth and dynamo effect but not the generation of the magnetic field (primordial or seed) as is clear from expression (104) where the (pseudo) vector part of h_0 contributes directly to the generation of the magnetic field as clearly given by Eq. (103).

12. Discussion and Perspectives

In this paper, we have introduced a simple geometric Lagrangian in the context of an unified theory based in a non-Riemannian geometry. From the functional action proposed, that is, as square root or measure (Nambu–Goto–Born–Infeld type), the dynamic fundamental equations were derived: an equation analogous to trace-free Einstein equations and a dynamic equation for the torsion (which was taken totally antisymmetric). Although the aim of this paper is to introduce the theoretical basis of the model, from this starting point, we bring some results and possible explanations about a few problems in the current research [26]. Some of them are as follows:

(1) From the geometrical viewpoint:

- (i) The cosmological term appears as integration constant of a natural manner and is linked with the curvature and fundamental fields.
- (ii) The fundamental constants (as G) are really functions of the spacetime coordinates geometrically induced and linked between them.
- (iii) There are a geometrical origin (not turbulent) of the α-term and the dynamo effect given by the torsion field.

(2) From an FRW cosmological scenario:

- (iv) A new wormhole solution in cosmological spacetime with torsion field is presented and analyzed.
- (v) We show that primordial cosmic magnetic fields can be originated by the dual torsion field h_μ.
- (vi) The axion field, that is contained in h_μ, controls the dynamics and stability of the cosmic magnetic field, but is not responsible for the magnetogenesis itself.
- (vii) The dynamic torsion field h_μ acts as a mechanism for the reduction of an original GUT symmetry of the universe containing $\sim \mathrm{SU}(3) \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_L \times U(1)$ to $\mathrm{SU}(3) \times \mathrm{SU}(2)_L \times U(1)$ today.
- (viii) From (vii) the GUT candidates are SO(10), SU(5) or some exceptional groups as $E(6)$, $E(7)$, for example.

Appendix A.

g-variation

$$\mathcal{L} = \sqrt{|g|}\sqrt{\det(\alpha\lambda)}\sqrt{1 + \frac{1}{2b^2}F_{\mu\nu}F^{\mu\nu} - \frac{1}{16b^4}(F_{\mu\nu}\widetilde{F}^{\mu\nu})^2} \equiv \sqrt{|g|}\sqrt{\det(\alpha\lambda)}\mathbb{R}, \tag{A.1}$$

where

$$b = \frac{\alpha}{\beta} = \frac{1 + (R_S/4\lambda)}{1 + (R_A/4\lambda)}, \tag{A.2}$$

$$R_S = g^{\alpha\beta} R_{\alpha\beta}, \tag{A.3}$$

$$R_A = f^{\alpha\beta} R_{\alpha\beta}, \tag{A.4}$$

and λ is arbitrary constant. Knowing that in the metrical case we have as usual procedure:

$$\delta_g \mathcal{L} = [\delta(\sqrt{|g|}\sqrt{\det(\alpha\lambda)})\mathbb{R} + \sqrt{|g|}\sqrt{\det(\alpha\lambda)}\delta\mathbb{R}], \tag{A.5}$$

$$\delta(F_{\mu\nu}F^{\mu\nu}) = 2F_{\mu\lambda}F_{\nu}{}^{\lambda}\delta g^{\mu\nu}, \tag{A.6}$$

$$\delta(\widetilde{F}_{\mu\nu}F^{\mu\nu}) = \left(-\frac{1}{2}\widetilde{F}_{\eta\rho}F^{\eta\rho}g_{\mu\nu} + 4\widetilde{F}_{\mu\rho}F_{\nu}{}^{\rho}\right)\delta g^{\mu\nu}, \tag{A.7}$$

then

$$\begin{aligned}\left[2R_{(\alpha\beta)} - \frac{g_{\alpha\beta}}{2}R_s\right]\mathbb{R} = {} & \frac{R_s}{\mathbb{R}\alpha^2}\left[F_{\alpha\lambda}F_{\beta}{}^{\lambda} + \frac{1}{2b^2}F_{\mu\nu}\widetilde{F}^{\mu\nu}\left(\frac{F_{\eta\rho}\widetilde{F}^{\eta\rho}}{8}g_{\alpha\beta} - F_{\alpha\lambda}\widetilde{F}_{\beta}{}^{\lambda}\right)\right] \\ & - F_{\mu\nu}F^{\mu\nu}R_{(\alpha\beta)} + \frac{R_{(\alpha\beta)}}{\mathbb{R}\alpha^2}\left[\frac{\left(F_{\eta\rho}\widetilde{F}^{\eta\rho}\right)^2}{4\mathbb{R}^2b^2} - F_{\mu\nu}F^{\mu\nu}\right] \\ & + 4\lambda\left[g_{\alpha\beta} + \frac{1}{\mathbb{R}^2\alpha^2}\left(F_{\alpha\lambda}F_{\beta}{}^{\lambda} + \frac{F_{\mu\nu}\widetilde{F}^{\mu\nu}}{2b^2}\right.\right. \\ & \left.\left.\times\left(\frac{F_{\eta\rho}\widetilde{F}^{\eta\rho}}{8}g_{\alpha\beta} - F_{\alpha\lambda}\widetilde{F}_{\beta}{}^{\lambda}\right)\right)\right],\end{aligned} \tag{A.8}$$

$$\begin{aligned}R_{(\alpha\beta)} - \frac{g_{\alpha\beta}}{4}R_s = {} & \frac{R_s}{2\mathbb{R}^2\alpha^2}\left[F_{\alpha\lambda}F_{\beta}{}^{\lambda} - F_{\mu\nu}F^{\mu\nu}\frac{R_{(\alpha\beta)}}{R_s}\right] \\ & + \frac{R_s}{4\mathbb{R}^2\alpha^2b^2}\left[F_{\mu\nu}\widetilde{F}^{\mu\nu}\left(\frac{F_{\eta\rho}\widetilde{F}^{\eta\rho}}{8}g_{\alpha\beta} - F_{\alpha\lambda}\widetilde{F}_{\beta}{}^{\lambda}\right)\right. \\ & \left. + \frac{F_{\eta\rho}\widetilde{F}^{\eta\rho}}{2}\frac{R_{(\alpha\beta)}}{R_s}\right] + 2\lambda\left[g_{\alpha\beta} + \frac{1}{\mathbb{R}^2\alpha^2}\left(F_{\alpha\lambda}F_{\beta}{}^{\lambda} + \frac{F_{\mu\nu}\widetilde{F}^{\mu\nu}}{2b^2}\right.\right. \\ & \left.\left.\times\left(\frac{F_{\eta\rho}\widetilde{F}^{\eta\rho}}{8}g_{\alpha\beta} - F_{\alpha\lambda}\widetilde{F}_{\beta}{}^{\lambda}\right)\right)\right],\end{aligned} \tag{A.9}$$

Appendix B.

Some remarks on the general Hodge–de Rham decomposition of $h = h_\alpha dx^\alpha$.

Theorem B.1. *If $h = h_\alpha dx^\alpha \notin F'(M)$ is a* 1*-form on M, then there exist a zero-form Ω, a* 2*-form $\alpha = A_{[\mu\nu]} dx^\mu \wedge dx^\nu$ and a harmonic* 1*-form $q = q_\alpha dx^\alpha$ on M such that*

$$h = d\Omega + \delta\alpha + q \to h_\alpha = \nabla_\alpha \Omega + \varepsilon_\alpha^{\beta\gamma\delta} \nabla_\beta A_{\gamma\delta} + q_\alpha. \tag{B.1}$$

Note that even if it is not harmonic, and assuming that q_α is a polar vector, an axial vector can be added such that the above expression takes the form

$$h_\alpha = \nabla_\alpha \Omega + \varepsilon_\alpha^{\beta\gamma\delta} \nabla_\beta A_{\gamma\delta} + \varepsilon_\alpha^{\beta\gamma\delta} M_{\beta\gamma\delta} + q_\alpha, \tag{B.2}$$

where $M_{\beta\gamma\delta}$ is a completely antisymmetric tensor (of such a manner that $\varepsilon_\alpha^{\beta\gamma\delta} M_{\beta\gamma\delta} \equiv \gamma^5 b_\alpha$ is an axial vector).

Consequently, we know that in unified theories where we are not able to deal with energy–momentum tensor, the fields and their interactions are effectively restricted due to the same geometrical framework: the spacetime itself. This fact permits us to rewrite (14) considering the physical quantities of interest:

$$h_\alpha = \nabla_\alpha \Omega + \varepsilon_\alpha^{\beta\gamma\delta} \nabla_\beta A_{\gamma\delta} + \gamma^5 b_\alpha + (P_\alpha - eA_\alpha).$$

Appendix C.

C.1. *Electrodynamical equations in* 3 + 1

The starting point will be the line element in 3 + 1 splitting [6, 7]: the four-dimensional spacetime is split into three-dimensional space and one-dimensional time to form a foliation of three-dimensional spacelike hypersurfaces. The metric of the spacetime is consequently, given by

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

where γ_{ij} is the metric of the three-dimensional hypersurface, α is the lapse function, and β^i is the shift function. At every spacetime point, a fiducial observer (FIDO) is introduced in such a way that his corresponding world-line is perpendicular to the hypersurface where he is stationary.

His FIDO 4-vector velocity is then given by

$$U^\mu = \frac{1}{\alpha}(1, -\beta^i), \quad U_\mu = (-\alpha, 0, 0, 0),$$

one deals with the physical quantities defined on the three-dimensional hypersurface as measured by the FIDO. For example, the electric field and the magnetic field are defined with the help of the U^μ respectively, by

$$E^\mu = F^{\mu\nu} U_\nu,$$
$$B^\mu = -\frac{1}{2\sqrt{-g}} \varepsilon^{\mu\nu\rho\sigma} U_\nu F_{\mu\nu},$$

note that the zero components are null: $E^0 = B^0 = 0$. Also, the 4-current J^μ can be similarly decomposed as

$$J^\mu = \rho_e U^\mu + j^\mu,$$

where we defined

$$\rho_e = -J^\mu U_\mu,$$
$$j^\mu = J^\mu + J^\nu U_\nu U^\mu$$

then $j^0 = 0$. So that j, E and B can be treated as 3-vectors in spacelike hypersurfaces. In terms of these 3-vectors the Maxwell equations can be written as

$$\begin{aligned}
\nabla \cdot E &= 4\pi\rho_e, \\
\nabla \cdot B &= 0, \\
\nabla \times (\alpha E) &= -(\partial_t - \mathcal{L}_\beta)B \\
&= -\partial_0 B + (\beta \cdot \nabla)B - (B \cdot \nabla)\beta, \\
\nabla \times (\alpha B) &= -(\partial_t - \mathcal{L}_\beta)E + 4\pi\alpha j \\
&= \partial_0 E - (\beta \cdot \nabla)E + (E \cdot \nabla)\beta + 4\pi\alpha j.
\end{aligned}$$

The derivatives in these equations are covariant derivatives with respect to the metric of the absolute space γ_{ij} being $\mathcal{L}_\beta$ the Lie derivative operator geometrically defined as $\mathcal{L}_\beta V = d(i_\beta \cdot V)$ with V a vector field.

ZAMOs observers

$$U = \frac{1}{\alpha}(\partial_t - \beta^i e_i)$$

in the Boyer–Lindquist coordinates we have e_r, e_θ and $e_\varphi = \frac{1}{\sqrt{g_{\varphi\varphi}}}\partial_\varphi$. The plasma 4-velocity (medium) u can be expressed as $u = \gamma(U + \overline{v})$ where $\overline{v}$ is the plasma 3-velocity with respect to the ZAMOs.

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