Published for SISSA by Springer

Received: *December 19, 2009*
Accepted: *December 30, 2009*
Published: *January 21, 2010*

On AGT description of $\mathcal{N} = 2$ SCFT with $N_f = 4$

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Abstract: We consider Alday-Gaiotto-Tachikawa (AGT) realization of the Nekrasov partition function of $\mathcal{N} = 2$ SCFT. We focus our attention on the SU(2) theory with $N_f = 4$ flavor symmetry, whose partition function, according to AGT, is given by the Liouville four-point function on the sphere. The gauge theory with $N_f = 4$ is known to exhibit SO(8) symmetry. We explain how the Weyl symmetry transformations of SO(8) flavor symmetry are realized in the Liouville theory picture. This is associated to functional properties of the Liouville four-point function that are *a priori* unexpected. In turn, this can be thought of as a non-trivial consistency check of AGT conjecture. We also make some comments on elementary surface operators and WZW theory.

Keywords: Supersymmetric gauge theory, Duality in Gauge Field Theories

ArXiv ePrint: 0912.1930

doi:10.1007/JHEP01(2010)097

Contents

1 Introduction

1.1 $\mathcal{N}=2$ theories

In [1], Gaiotto constructed a large family of SU(N_c) $\mathcal{N}=2$ superconformal (quiver) gauge theories by compactifying the (2,0) six-dimensional theory of the type A_{N_c-1} on a two-dimensional Riemann surface. This Riemann surface, which we denote by $\mathcal{C}_{g,n}$, is characterized by its genus g and the weights of its n punctures, and in this way one gets a different $\mathcal{N}=2$ theory for each pair (g,n); we denote such gauge theory by $\mathcal{T}_{g,n}$.

Particular examples of the $\mathcal{N}=2$ theories that can be constructed with this method are the $\mathcal{N}=4$ super Yang-Mills theory and the so-called $\mathcal{N}=2^*$ theory, which correspond to compactifying the six-dimensional theory on $\mathcal{C}_{1,0}$ and on $\mathcal{C}_{1,1}$ respectively. Another simple example is the compactification of the six-dimensional theory on a 4-punctured sphere $\mathcal{C}_{0,4}$; in such case one obtains the $\mathcal{N}=2$ gauge theory with $N_f=4$ flavor symmetry, whose four mass parameters are given by the weights of the four punctures.

In this construction, the modular parameters τ_i of $\mathcal{C}_{g,n}$ give the coupling constant q_i of the corresponding quiver gauge theory $\mathcal{T}_{g,n}$. In other words, the space of parameters of $\mathcal{T}_{g,n}$ is equivalent to the moduli space of complex structures of $\mathcal{C}_{g,n}$. And it turns out that the group of duality transformations of the gauge theory coincides with the mapping class group of the Riemann surface $\mathcal{C}_{g,n}$. This is a very interesting result as it permits to associate different ways of sewing the Riemann surface $\mathcal{C}_{g,n}$ with different coupling limits of the gauge theory $\mathcal{T}_{g,n}$. According to this picture, each way of sewing $\mathcal{C}_{g,n}$ as a set of $2g-2+n$ trinions connected by $3g-3+n$ tubes is in correspondence with each Lagrangian description that the gauge theory $\mathcal{T}_{g,n}$ admits; in the limit where a given tube is thin and long, the gauge theory description becomes weakly coupled.

1.2 The AGT correspondence

In a more recent paper [2], Alday, Gaiotto, and Tachikawa (AGT) reconsidered the $\mathcal{N}=2$ theories of [1] and studied the Nekrasov partition function [3] associated to them. They arrived to the following remarkable observation: In the case of those $\mathcal{N}=2$ theories whose gauge group corresponds to SU(2), it happens that the Nekrasov partition function on $\mathbb{R}^4$ is given by Virasoro conformal blocks of Liouville field theory! This statement was confirmed by explicit computation to several orders in a power expansion and the evidence is quite convincing.

The result of [2] seems to be revealing an intriguing connection between four-dimensional superconformal gauge theories and two-dimensional non-rational conformal theories. This was recently investigated in refs. [4–15], and extended in refs. [16–24] to the case of $\mathrm{SU}(N_c)$ gauge theories, for which the correspondence seems to work with the A_{N_c-1} Toda field theory. The results of [2] also inspired very interesting works in the subject; see for instance the recent [25]. For early work on relations between $\mathcal{N}=2$ gauge theory and Liouville theory see [26] and references therein.

The dictionary between the Nekrasov partition function of the $\mathcal{N}=2$ theory and the conformal blocks of Liouville theory is such that the two deformation parameters $\varepsilon_{1,2}$ of the Nekrasov partition function are given in terms of the Liouville central charge c by the relation $c=1+6Q^2$, with $Q=b+b^{-1}$ and $b=\varepsilon_1=1/\varepsilon_2$. The external Liouville momenta α_i in the n-point conformal blocks are given by mass parameters of the gauge theory (see (1.2)–(1.3) below). On the other hand, the momenta of internal legs in the Liouville conformal blocks are given by the *vev's* a_i of the adjoint scalar fields in the $\mathcal{N}=2$ theory. See [2] for details.

Even more surprising than the relation with Liouville conformal blocks is the fact that the integral of the full Nekrasov partition function of a given $\mathcal{N}=2$ theory $\mathcal{T}_{g,n}$ over a_i turns out to be given by the *full* n-point correlation function of Liouville field theory formulated on a genus-g surface [2]. Schematically,

$$Z_{\text{Nekrasov}} \quad \underset{\text{AGT}}{\longleftrightarrow} \quad \left\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) \right\rangle_{\text{Liouville}} . \tag{1.1}$$

While in Liouville theory the correlation functions are defined by assembling holomorphic and anti-holomorphic conformal blocks together and integrating over the momenta of normalizable states in the intermediate channels, in the gauge theory one integrates the modulus of the partition function over the *vev's* a_i with an appropriate measure. (1.1) somehow generalizes the Pestun's result for the theory on $\mathbb{S}^4$ with $b=1$ [27].

One of the most interesting implications of the observations made in [1, 2] is that the duality transformations in the $\mathcal{N}=2$ theories can be associated to geometric deformations of a Riemann surface, in such a way that duality symmetry of the four-dimensional theory follows from crossing symmetry of a two-dimensional CFT [2]. It happens that trinion decomposition of $\mathcal{C}_{g,n}$ codifies the connection between different Lagrangian descriptions of $\mathcal{T}_{g,n}$ and their weakly coupling regimes. Nevertheless, it turns out that not *all* the symmetries of a given $\mathcal{N}=2$ theory are necessarily associated to crossing symmetry of Liouville correlators. In fact, as we will discuss below, some symmetry transformations in the gauge theory turn out to be captured by non-trivial functional properties of Liouville theory (which are much less evident than crossing symmetry.)

1.3 The theory with $N_f = 4$

In this paper, we will consider the case of SU(2) $\mathcal{N} = 2$ theory with $N_f = 4$ flavor symmetry. This gauge theory possesses global SO(8) symmetry. According to the dictionary of [2], the Nekrasov partition function of this theory is given by the 4-point function in Liouville theory formulated on the sphere topology. The momenta α_i of the four Liouville vertex operators $V_{\alpha_i}(z_i)$ turn out to be given by the mass parameters m_i of the gauge theory by $\alpha_i = Q/2+m_i$, and these parameters are related to the masses μ_i of the four hypermultiplets as follows

$$\mu_1 = m_1 + m_2, \qquad \mu_2 = m_1 - m_2, \tag{1.2}$$

$$\mu_3 = m_3 + m_4, \qquad \mu_4 = m_3 - m_4; \tag{1.3}$$

which already corresponds to a particular trinion decomposition. On the other hand, the gauge coupling constant of the gauge theory is given by the cross-ratio $q = \frac{(z_1-z_3)(z_2-z_4)}{(z_2-z_3)(z_1-z_4)}$, where z_i are the worldsheet locations of the Liouville vertex operators.

The $\mathcal{N} = 2$ theory with $N_f = 4$ has SO(8) global symmetry. The masses μ_i of the hypermultiplets given in (1.2)–(1.3) correspond to the diagonal Cartan elements of SO(8), while m_i correspond to the Cartan elements of its proper subgroup SO(4)×SO(4). From the point of view of a given trinion decomposition, the consideration of the subgroup SO(4)×SO(4) is natural, as one has SO(4)≃SU(2)×SU(2) corresponding to each pair of hypermultiplets in each trinion.

One observes from (1.2)–(1.3) that changing the sign of a mass parameter like $m_2 \to -m_2$ corresponds to exchanging the masses of the hypermultiplets like $\mu_1 \leftrightarrow \mu_2$. In the Liouville theory picture this amounts to perform the reflection $\alpha_2 \to Q-\alpha_2$, which preserves the conformal dimension of the corresponding vertex operator.

On the other hand, it turns out that the SL(2, $\mathbb{Z}$) duality symmetry of the gauge theory mixes with the SO(8) triality in a funny way. The S_3 automorphism of Spin(8) is generated by two transformations that act on the mass μ_i in a non-diagonal way [28]. On of these transformations is

$$\mu_i \to \mu_i - \frac{\mu_1+\mu_2+\mu_3-\mu_4}{2}, \qquad \text{for } i = 1, 2, 3, \tag{1.4}$$

$$\mu_4 \to \frac{\mu_1+\mu_2+\mu_3+\mu_4}{2}, \tag{1.5}$$

and we also have the transformation

$$\mu_i \to \mu_i, \qquad \text{for } i = 1, 2, 3, \tag{1.6}$$

$$\mu_4 \to -\mu_4. \tag{1.7}$$

While (1.6)–(1.7) corresponds to the operation of exchanging the two spinor representation of the group, transformation (1.4)–(1.5) corresponds to the operation of exchanging the vector representation with one of the spinor representations (physically, this corresponds to exchange elementary particles and monopoles in the gauge theory.) These triality transformations have a simple interpretation in terms of the Liouville picture: From (1.2)–(1.3) we observe that performing (1.4)–(1.5) corresponds to exchange $m_1 \leftrightarrow -m_4$ keeping m_2 and m_3 fixed. Since this amounts to exchange the trinion decomposition, this means that triality is in correspondence with crossing symmetry in the Liouville theory. Even simpler is the form that transformation (1.6)–(1.7) adopts in the Liouville side; it corresponds to interchange $m_3 \leftrightarrow m_4$ keeping m_1 and m_2 fixed.

From this we see that some simple transformations of SO(8) can be interpreted in a very natural way in the Liouville theory picture. However, there are some special Weyl transformations whose identification as symmetry in the Liouville theory side is much less evident, and this is what we want to study here. One such a symmetry transformation is given by

$$(\mu_1, \mu_2, \mu_3, \mu_4) \to (-\mu_4, \mu_2, \mu_3, -\mu_1), \tag{1.8}$$

which in terms of the parameters m_i reads

$$m_1 \to \frac{m_1 - m_2 - m_3 + m_4}{2}, \qquad m_2 \to \frac{-m_1 + m_2 - m_3 + m_4}{2}, \tag{1.9}$$

$$m_3 \to \frac{-m_1 - m_2 + m_3 + m_4}{2}, \qquad m_4 \to \frac{m_1 + m_2 + m_3 + m_4}{2}. \tag{1.10}$$

This symmetry is not at all evident in the Liouville theory picture. In fact, it corresponds to a non-diagonal transformation of the momenta α_i (see (2.19)–(2.20) below). In this paper we address the question about how the symmetry under (1.9)–(1.10) can be proven from the Liouville point of view. We will show this symmetry is actually realized in the Liouville theory side by non-trivial functional relations between different four-point correlation functions. These relations are *a priori* unexpected, and this is why this calculation is an interesting consistency check of the proposal in [2].

1.4 Overview

The paper is organized as follows: In section 2, we study the analytic extension of the integral representation of Liouville correlation functions. We use this representation to prove the symmetry under transformations (1.9)–(1.10) from the Liouville point of view. In the 2D CFT language, this symmetry is expressed by eq. (2.21). Because we derive the symmetry relation (2.21) by resorting exclusively to elements of Liouville field theory, we like to think of it as a nice consistency check of AGT conjecture. The full SO(8) is not manifest in the six-dimensional picture, but it emerges in the infrared. Therefore, the fact of proving the Weyl symmetry (1.9)–(1.10) in the Liouville picture is quite interesting. In section 3, we rederive (2.21) in an alternative (less direct) way. The strategy goes as follows: First we consider a five-point function in Liouville theory, which is meant to describe surface operator in the $\mathcal{N} = 2$ gauge theory. Then, we write this Liouville five-point function as a four-point function in the Wess-Zumino-Witten theory (WZW) with affine $\widehat{A}_1$ symmetry. Using functional relations between different solutions to the Knizhnik-Zamolodchikov equation and performing Hamiltonian reduction from WZW to Liouville

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we reobtain the right symmetry relation with the appropriate coefficient. We also make some comments on how to describe the surface operator of the $\mathcal{N}=2$ theory in terms of the $\widehat{A}_1$ WZW theory.

2 Liouville theory and Weyl symmetry

2.1 Integral representation of correlation functions

Liouville theory is defined by the action [29–31]

$$S_L[\varphi]=\frac{1}{4\pi}\int d^2z\left(\partial\varphi\bar{\partial}\varphi+\frac{1}{2\sqrt{2}}QR\varphi+4\pi\mu e^{\sqrt{2}b\varphi}\right) \tag{2.1}$$

where μ is a positive constant. The background charge Q takes the value $Q=b+b^{-1}$ for the Liouville self-potential $\mu e^{\sqrt{2}b\varphi}$ to be a marginal operator. The theory is globally defined after one specifies the boundary conditions. For the theory on the sphere, one imposes the asymptotic behavior $\varphi\sim-2\sqrt{2}Q\log|z|$ for large $|z|$. In the conformal gauge, the linear dilaton term $QR\varphi$ is understood as receiving a contribution from the point at infinity due to the scalar curvature R of the sphere.

Under holomorphic transformations $z\to w$ Liouville field φ transforms like $\varphi\to\varphi-\sqrt{2}Q\log|\frac{dw}{dz}|$. The central charge receives a contribution from the background charge, and it is

$$c=1+6Q^2.$$

Here we are interested in the exponential vertex operators of the theory [32]

$$V_\alpha(z)=e^{\sqrt{2}\alpha\varphi(z)}. \tag{2.2}$$

These are local operators that create primary states of conformal dimension $\Delta_\alpha=\alpha(Q-\alpha)$. Notice that Δ_α remains invariant under $\alpha\to\alpha^*=Q-\alpha$, which means that vertices $V_\alpha(z)=e^{\sqrt{2}\alpha\varphi(z)}$ and $V_{\alpha^*}(z)=e^{\sqrt{2}(Q-\alpha)\varphi(z)}$ have the same conformal dimension.

Correlation functions in Liouville field theory are defined by the expectation value of a product of vertex operators (2.2), namely

$$\Omega^{(n)}(\alpha_1,...\alpha_n|z_1,...z_n)\ \equiv\ \left\langle\prod\nolimits_{i=1}^{n}V_{\alpha_i}(z_i)\right\rangle_{\text{Liouville}}=\int\mathcal{D}\varphi\ e^{-S_L[\varphi]}\prod_{i=1}^{n}e^{\sqrt{2}\alpha_i\varphi(z_i)}.$$

Integrating over the zero mode of the field, one finds the following expression [33, 34]

$$\Omega^{(n)}(\alpha_1,...\alpha_n|z_1,...z_n)=\frac{\Gamma(-s)}{b}\mu^s\int\prod_{r=1}^{s}d^2w_r\ \Omega^{(n+s)}(\alpha_1,...\alpha_n,b,...b|z_1,...z_n,w_1,...w_s)_{|\mu=0}, \tag{2.3}$$

where $s=-b^{-1}(\alpha_1+\alpha_2+...\alpha_n)+1+b^{-2}$, and where the average on the right hand side is defined in terms of the free theory $\mu=0$. The factor $\Gamma(-s)$ in (2.3) arises from the integration over the zero-mode of φ [33, 35], and this also gives a δ-function that completely determines the amount of screening operators $V_b(w)$ that appear in the non-vanishing correlators. In deriving (2.3), the Gauss-Bonnet theorem is used to determine the relation between s, b, and the momenta α_i. For a genus-g n-puncture Riemann surface the relation is

$$bs = Q(1-g) - \sum_{i=1}^{n} \alpha_i. \tag{2.4}$$

Then, (2.3) permits to compute correlators by performing the Wick contraction of the $n+s$ operators and using the free field propagator $\langle \varphi(z_1)\varphi(z_2)\rangle = -2\log|z_1-z_2|$. This yields

$$\Omega^{(n)}(\alpha_1,...\alpha_n|z_1,...z_n) = \Gamma(-s)\Gamma(s+1)b^{-1}\mu^s\, \mathcal{I}_s^{(n)}(\alpha_1,\alpha_2,...\alpha_n|z_1,z_2,...z_n), \tag{2.5}$$

with

$$\mathcal{I}_s^{(n)}(\alpha_1,\alpha_2,...\alpha_n|z_1,z_2,...z_n) = \frac{1}{s!}\prod_{i<j}^{n}|z_i-z_j|^{-4\alpha_i\alpha_j}\int\prod_{r=1}^{s}d^2w_r\prod_{i=1}^{n}\prod_{r=1}^{s}|z_i-w_r|^{-4b\alpha_i}\prod_{r<t}^{s}|w_t-w_r|^{-4b^2}, \tag{2.6}$$

where each integral is over the whole complex plane $\mathbb{C}$.

Expression (2.6) has to be considered just formally. This is because, in general, the amount of screening operators s is not an integer number. Therefore, in order to compute generic correlation functions one has to deal first with the problem of making sense of the integral representation (2.5). In the case of n-point functions with $n \leq 3$ and $s \in \mathbb{Z}_{>0}$, multiple integral (2.6) can be solved using the results of refs. [36, 37]. Generic Liouville correlation functions are thus defined by analytic extension. This analytic extension is accomplished by continuing the multiple integral (2.6) to non-integer (and non-real) values of s. The extension to non-integer s was discussed in the literature, and it plays a crucial rôle in our discussion. To see how it works, let us consider the calculation of the Liouville partition function on the sphere as an illustrative example: This corresponds to $g=0$ and $n=0$. The number of screening operators in this case is $m = s-3 = -2+b^2$, and this is because, in order to compute the genus-zero zero-point function, one has to consider the correlator with three local operators $e^{\sqrt{2}b\varphi(z)}$ inserted at fixed points $z_1=0, z_2=1$, and $z_3=\infty$; this amounts to compensate the volume of the conformal Killing group. In turn, genus-zero Liouville partition function is given by

$$\Omega^{(0)} = \mu^{m+3}b^{-1}\Gamma(-m-3)\Gamma(m+1)\mathcal{I}_m^{(3)}(b,b,b|0,1,\infty),$$

with $m=-2+b^{-2}$. If m was a positive integer number (what happens only if $b^2\in\mathbb{Z}_{>2}$) this integral could be solved by using the Dotsenko-Fateev integral formulas of [36]. However, here we are interested in the case where m is generic enough. The way one circumvents this obstruction is assuming the condition $m\in\mathbb{Z}_{>0}$ through the integration and then analytically extending the final expression. More precisely, one integrates $\mathcal{I}_m^{(3)}(b,b,b|0,1,\infty)$ and obtains

$$\Omega^{(0)} = \frac{\mu^{3+m}}{b}\Gamma(-m-3)\Gamma(m+1)\pi^m\gamma^m(1+b^2)\prod_{r=1}^{m}\gamma(-rb^2)\prod_{r=0}^{m-1}\gamma^2(1-(2+r)b^2)\gamma(-1+(3+r+m)b^2).$$

where $\gamma(x)=\Gamma(x)/\Gamma(1-x)$, which, again, is an expression that seems to make sense only if $m\in\mathbb{Z}_{>0}$. Then, one extends the result to the whole range $b^2\in\mathbb{R}_{>1}$ by rewriting the

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products in the expression above appropriately. This is done by noticing that $\gamma(-1+(3+r+m)b^2) = \gamma((r+1)b^2)$, and using that $m = -2+b^{-2}$ and $1-rb^2 = (m+2-r)b^2$. Rearranging the product of γ-functions and using properties of the Γ-function, one finally finds

$$\Omega^{(0)} = \frac{(1-b^2)\left(\pi\mu\gamma(b^2)\right)^{Q/b}}{\pi^3 Q\gamma(b^2)\gamma(b^{-2})}, \tag{2.7}$$

which is the exact result for the Liouville partition function on the sphere.

The two-point correlation functions can be computed in a similar way, analytically continuing the integral formula for $\mathcal{I}^{(3)}_{b^{-2}-2\alpha b^{-1}}(\alpha,\alpha,b|0,1,\infty)$. The result reads

$$\Omega^{(2)}(\alpha,\alpha|0,1) = \left(\pi\mu\gamma(b^2)\right)^{(Q-2\alpha)/b}\frac{\gamma(2\alpha b-b^2)\gamma(2\alpha b^{-1}-b^{-2})}{\pi(2\alpha-Q)}, \qquad \Omega^{(2)}(\alpha,\alpha^*|0,1) = 1. \tag{2.8}$$

Up to a factor $(Q-2\alpha)/\pi$, the two-point function $\Omega^{(2)}(\alpha,\alpha|0,1)$ coincides with the Liouville reflection coefficient, which yields the functional relation

$$\Omega^{(n)}(\alpha_1,...\alpha_n|z_1,...z_n) = \frac{\pi}{Q-2\alpha_n}\Omega^{(2)}(\alpha_n,\alpha_n|0,1)\Omega^{(n)}(\alpha_1,...,\alpha_{n-1},\alpha_n^*|z_1,...z_n) \tag{2.9}$$

As zero-point and two-point functions, the three-point function can also be computed by performing the appropriate analytic extension of the product of Γ-functions in the Dotsenko-Fateev integral formula. To do this properly, one has first to be reminded of the fact that Liouville theory exhibits self-duality under $b \leftrightarrow b^{-1}$ and that this self-duality is associated to the existence of a second screening operator $V_{1/b}(w) = e^{\sqrt{2}b^{-1}\varphi(w)}$ in the theory. Taking this into account, one has enough information to reconstruct the *full* pole structure of the exact three-point function and then analytically extend integral (2.6). This amounts to introduce the Υ-function [38]

$$\log\Upsilon_b(x) = \int_{\mathbb{R}_{>0}}\frac{dt}{t}\left(\left(\frac{Q}{2}-x\right)^2 e^{-t} - \frac{\sinh^2\left(\left(\frac{Q}{2}-x\right)\frac{t}{2}\right)}{\sinh\left(\frac{tb}{2}\right)\sinh\left(\frac{t}{2b}\right)}\right), \tag{2.10}$$

which presents poles at $x = mb + nb^{-1}$ and $x = -(m+1)b-(n+1)b^{-1}$ with $m,n \in \mathbb{Z}_{>0}$, which manifestly shows the symmetry of the pole structure under $b \leftrightarrow b^{-1}$. This function obeys the self-dual properties

$$\Upsilon_b(x) = \Upsilon_{1/b}(x), \qquad \Upsilon_b(x) = \Upsilon_b(Q-x), \tag{2.11}$$

and the shift properties

$$\Upsilon_b(x+b) = b^{1-2bx}\gamma(bx)\Upsilon_b(x), \qquad \Upsilon_b(x+b^{-1}) = b^{-1+2x/b}\gamma(x/b)\Upsilon_b(x). \tag{2.12}$$

In particular, if we define the function $P(m) = \prod_{r=1}^{m}\gamma(rb^2)$ for $m\in\mathbb{Z}_{>0}$, and $P(0) = 1$, the shift properties imply that

$$P(m) = \frac{\Upsilon_b(mb+b)}{\Upsilon_b(b)}b^{m((m+1)b^2-1)}, \qquad m \in \mathbb{Z}_{\geq 0}. \tag{2.13}$$

This admits the following extension to negative values of m

$$P(m) = b^{-4(m+1)} \frac{\gamma(-m)}{\gamma(-mb^2)} P(-m), \qquad m \in \mathbb{Z}_{<0}; \tag{2.14}$$

see [39] and references therein.

Then, writing the products of Γ-functions that arise in the Dotsenko-Fateev integral formula for $n = 3$ in terms of Υ-functions using (2.13), one finally obtains the exact three-point function [38, 40–42]; namely

$$\Omega^{(3)}(\alpha_1, \alpha_2, \alpha_3|0, 1, \infty) = \left(\pi\mu\gamma(b^2) b^{2-2b^2}\right)^{(Q-\alpha)/b} \frac{\Upsilon_b(b)}{\Upsilon_b(\alpha - Q)} \prod_{i=1}^{3} \frac{\Upsilon_b(2\alpha_i)}{\Upsilon_b(\alpha - 2\alpha_i)}, \tag{2.15}$$

where $\alpha = \alpha_1 + \alpha_2 + \alpha_3$.

The computation of (2.7), (2.8) and (2.15) shows that the method of analytically continuing the integral representation of correlation functions works and it is a powerful tool. We will use this method to derive (2.21) below.

2.2 Four-point function and Weyl symmetry

In this paper, and because we are interested in the AGT description of the $\mathcal{N} = 2$ gauge theory with $N_f = 4$, we are interested in the Liouville four-point function. Four-point function is substantially more complicated than the cases $n = 0, 2, 3$. In that case, the integral representation takes the form

$$\Omega^{(4)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4|0, 1, \infty, q) = \Gamma(-s)\Gamma(s+1) b^{-1} \mu^s \mathcal{I}_s^{(4)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4|q), \tag{2.16}$$

with $s = -b^{-1}(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 - Q)$ and

$$\mathcal{I}_s^{(4)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4|q) = \frac{|q|^{-4\alpha_1\alpha_4}|1-q|^{-4\alpha_2\alpha_4}}{\Gamma(s+1)} \int \prod_{r=1}^{s} d^2 w_r \prod_{r=1}^{s} |w_r|^{-4b\alpha_1}|1-w_r|^{-4b\alpha_2}|w_r - q|^{-4b\alpha_4} \times \prod_{r<t}^{s} |w_t - w_r|^{-4b^2}, \tag{2.17}$$

with $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$, and $z_4 = q$. It was shown in [43] that, if $2\alpha_4/b = -m \in \mathbb{Z}_{<0}$, then the integral (2.17) satisfies the following remarkable property

$$\mathcal{I}_s^{(4)}(\alpha_1, \alpha_2, \alpha_3, \alpha_4|q) = |q|^{4(\widetilde{\alpha}_1\widetilde{\alpha}_4 - \alpha_1\alpha_4)}|1-q|^{4(\widetilde{\alpha}_2\widetilde{\alpha}_4 - \alpha_2\alpha_4)}(-\pi\gamma(1+b))^{s-m} \prod_{r=1}^{s-m} \gamma(2b\alpha_4 - rb^2) \times \prod_{r=0}^{s-m-1} \prod_{i=1}^{3} \gamma(1 - 2b\alpha_i - rb^2) \mathcal{I}_m^{(4)}(\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3, \widetilde{\alpha}_4|q), \tag{2.18}$$

with

$$\widetilde{\alpha}_1 = \frac{Q}{2} + \frac{\alpha_1 - \alpha_2 - \alpha_3 + \alpha_4}{2}, \qquad \widetilde{\alpha}_2 = \frac{Q}{2} - \frac{\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4}{2}, \tag{2.19}$$

$$\widetilde{\alpha}_3 = \frac{Q}{2} - \frac{\alpha_1 + \alpha_2 - \alpha_3 - \alpha_4}{2}, \qquad \widetilde{\alpha}_4 = -\frac{Q}{2} + \frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{2}. \tag{2.20}$$

The proof of (2.18), for $m \in \mathbb{Z}_{>0}$, follows from iterating recursion relations for $\mathcal{I}_s^{(n)}(\alpha_1,...\alpha_n|z_1,...z_n)$; see [43, 44] for details.

Expressions (2.19)–(2.20) anticipate the point we want to make here: Since the mass of the hypermultiplets in the gauge theory μ_i are given by $\alpha_i = \frac{Q}{2} + m_i$, provided with (1.4)–(1.5), then equation (2.18) seems to incarnate some kind of invariance under (1.9)–(1.10) that Liouville four-point function exhibits. To make it precise, what we have to do first is to analytically continue the integral relation (2.18) to complex values of α_4, and then give one such a relation for the *exact* four-point function. The analytic continuation is accomplished by following a recipe: First, we may use expression (2.13); then, we have to remember that a product $\prod_{r=1}^{m} F(r)$ can be extended to the range $m \in \mathbb{Z}_{<0}$ by replacing it by the expression by $\prod_{r=0}^{-m-1} F^{-1}(-r)$, which, in particular, amounts to give the formula (2.14) for $P(m)$ with $m \in \mathbb{Z}_{<0}$. This, together with the convenient use of properties of Γ-functions, leads us to the following remarkable equation

$$|q|^{4\widetilde{\alpha}_1\widetilde{\alpha}_4}|1-q|^{4\widetilde{\alpha}_2\widetilde{\alpha}_4}\frac{\Omega^{(4)}(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\widetilde{\alpha}_3,\widetilde{\alpha}_4|0,1,\infty,q)}{f(\widetilde{\alpha}_1)f(\widetilde{\alpha}_2)f(\widetilde{\alpha}_3)f(\widetilde{\alpha}_4^*)} = |q|^{4\alpha_1\alpha_4}|1-q|^{4\alpha_2\alpha_4}\frac{\Omega^{(4)}(\alpha_1,\alpha_2,\alpha_3,\alpha_4|0,1,\infty,q)}{f(\alpha_1)f(\alpha_2)f(\alpha_3)f(\alpha_4^*)} \tag{2.21}$$

with

$$f(\alpha_i) = \left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{-\alpha_i/2b}\Upsilon_b(2\alpha_i).$$

Notice that $\Upsilon_b(2\alpha_i) = \Upsilon_b(2\alpha_i^*)$. To prove (2.21) we also used the functional properties of the Υ-function (2.11)–(2.12) and the fact that its derivative obeys $\Upsilon_b'(-mb) = (-1)^m\Gamma(m+1)\Gamma(-m)b^{-1}\Upsilon_b(-mb)$. It is important to distinguish between identity (2.18) and (2.21). (2.21) is meant to hold between *exact* four-point functions, and, unlike (2.18), is manifestly symmetric under $\widetilde{\alpha}_i \leftrightarrow \alpha_i$. (2.21) is a remarkable equation: Since the mass parameters m_i in the $\mathcal{N}=2$ gauge theory are given by $\alpha_i = \frac{Q}{2} + m_i$, from (2.19)–(2.20) and (2.21) we see how the symmetry under transformation (1.9)–(1.10) is realized in the Liouville theory picture (cf. eqs. (4.1)–(4.3) of ref. [2]). Indeed, defining

$$X(\alpha_1,\alpha_2,\alpha_3,\alpha_4|q) = |q|^{4\alpha_1\alpha_4}|1-q|^{4\alpha_2\alpha_4}\left(\pi\mu\gamma(b^2)b^{2-2b^2}\right)^{\frac{\alpha_1+\alpha_2+\alpha_3-\alpha_4}{2b}} \times \prod_{i=1}^{4}\Upsilon_b^{-1}(2\alpha_i)\Omega^{(4)}(\alpha_1,\alpha_2,\alpha_3,\alpha_4|0,1,\infty,q). \tag{2.22}$$

we have

$$X(\alpha_1,\alpha_2,\alpha_3,\alpha_4|q) = X(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\widetilde{\alpha}_3,\widetilde{\alpha}_4|q). \tag{2.23}$$

Identity (2.21) was *a priori* unexpected, and the fact this codifies precisely the symmetry under (1.9)–(1.10) in the $\mathcal{N}=2$ gauge theory side is interesting as it permits to understand the SO(8) symmetry of the theory from the Liouville theory point of view (and not only the SO(4)×SO(4) proper subgroup.)

Now, let us make some comments on surface operators in the gauge theory. We will come back to eq. (2.21) later.

3 Surface operator and WZW theory

3.1 Surface operators from WZW correlators

It was argued in [45, 46] that the expectation value of loop and surface operators in the gauge theory can also be described in terms of Liouville correlation functions. While loop operators are described by a monodromy operation performed on a degenerate field $V_{-1/2b} = e^{-\varphi/(\sqrt{2}b)}$, the expectation values of elementary surface operators are given in terms of the correlation function

$$\Omega^{(n+1)}(\alpha_1, ...\alpha_n, -1/(2b)|z_1, ...z_n, x) \ = \left\langle \prod_{i=1}^{n} V_{\alpha_i}(z_i) V_{-\frac{1}{2b}}(x) \right\rangle_{\text{Liouville}}$$

on a $(n+1)$-puncture genus-g Riemann surface, which also involves a degenerate field $V_{-1/2b}$ that is inserted at the point x, and x is related to the parameters that label the corresponding gauge theory configuration.

These surface operators correspond to 1/2 BPS configurations in the $\mathcal{N}=2$ theory. The analogue in the $\mathcal{N}=4$ theory is a singular vortex solution, which is labeled by two real parameters that correspond to its magnetic flux and a θ-angle type parameter. Typically, there are always two real parameters that correspond to physical quantities and label the corresponding vortex-like configuration in the gauge theory, and in the Liouville picture these two real parameters combine to give the complex worldsheet coordinate x where the degenerate operator $V_{-1/2b}(x) = e^{-\varphi(x)/(\sqrt{2}b)}$ is inserted. Surface operators correspond to configurations that are localized on the two-dimensional Riemann surface $\mathcal{C}_{g,n}$; this is discussed in detail in [45–47].

The degenerate state created by the non-normalizable vertex operator $V_{-1/2b}$ is annihilated by the arrange of Virasoro operators

$$\left(L_{-1}^2 + b^2 L_{-2}\right) V_{-\frac{1}{2b}} = 0. \tag{3.1}$$

This expresses the fact that a null state exists in the Verma modulo and it has to be decoupled. Actually, (3.1) can be thought of as a realization of the second order equation of motion

$$\left(\partial_z^2 + T(z)\right) e^{-\varphi/2} = 0 \tag{3.2}$$

at quantum mechanical level. Decoupling equation (3.1) has to be understood as an operator-valued relation which, when implemented on correlation functions, yields a second order differential equations of the Belavin-Polyakov-Zamolodchikov (BPZ) type [48] to be obeyed by those correlators that involve a degenerate field with $\alpha_{n+1} = -1/2b$.

On the other hand, one knows from [49] that given a solution $\Omega^{(5)}(\alpha_1, ...\alpha_4, -1/2b|0, 1, \infty, z, x)$ to the five-point BPZ differential equation one can associate to it a solution $K(j_1, ...j_4|q, x)$ to the four-point Knizhnik-Zamolodchikov (KZ) differential equation [50] with affine symmetry $\widehat{A}_1$ at level $k = b^{-2} + 2$. The exact relation is given by

$$K(j_1, ...j_4|q, x) = \mathcal{N} \frac{|q|^{4(\alpha_1\alpha_4 - b^2 j_1 j_4)}|q-1|^{4(\alpha_2\alpha_4 - b^2 j_2 j_4)}}{|x|^{2\alpha_1/b}|x-1|^{2\alpha_2/b}|x-q|^{2\alpha_4/b}} \Omega^{(5)}(\alpha_1, ...\alpha_4, -1/2b|0, 1, \infty, q, x) \tag{3.3}$$

where

$$\alpha_1 = -\frac{b}{2}(j_1+j_2+j_2+j_4+1), \qquad \alpha_2 = -\frac{b}{2}(-j_1+j_2-j_3+j_4-b^{-2}-1), \quad (3.4)$$

$$\alpha_3 = -\frac{b}{2}(-j_1-j_2+j_3+j_4-b^{-2}-1), \quad \alpha_4 = -\frac{b}{2}(j_1-j_2-j_3+j_4-b^{-2}-1), \quad (3.5)$$

and where $\mathcal{N}$ is a normalization factor that does not depend on (x,q), and where $\alpha_5 = -1/(2b)$, $b^{-2} = k-2$, $z_1 = 0$, $z_2 = 1$, $z_3 = \infty$, and $z_4 = q$. Notice that (3.4)–(3.5) resembles (2.19)–(2.20) and (1.9)–(1.10); we will make this more precise; see (3.9) below.

If the normalization $\mathcal{N}$ is taken to be [51]

$$\mathcal{N} = \mu^{2j_4}\left(\pi\gamma(b^2)b^{2-2b^2}\right)^{2j_4+2\alpha_4/b}\prod_{i=1}^{4}\frac{\Upsilon_b(-2bj_i-b)}{\Upsilon_b(2\alpha_i)} \tag{3.6}$$

then the left hand side of (3.3) can be interpreted as the four-point correlation function of the level-k $\widehat{A}_1$ WZW theory; namely

$$K(j_1,...j_4,|q,x) = \left\langle \prod_{i=1}^4 \Phi_{j_i}(x_i|z_i)\right\rangle_{\mathrm{WZW}}, \tag{3.7}$$

up to an irrelevant b-dependent factor.

It is worth mentioning that an expression similar to (3.3) holds at the level of conformal blocks [51], and not only for the full correlation function. Although (3.3) is usually interpreted as a relation between Liouville theory and $\widehat{sl}(2)_k$ WZW theory, it is pertinent to emphasize that the $\widehat{su}(2)_k$ WZW appears by analytic extending the expressions reversing the sign of k; see [39] and references therein.

Vertex operators $\Phi_j(x|z)$ in (3.7) represent states of the WZW theory. These are given by Kac-Moody primary states with respect to the affine $\widehat{A}_1$ symmetry. These vertices expand $SL(2,\mathbb{R})$-representations of spin j_i, and depend on auxiliary complex variables x_i which allow to organize the representations by means of the following realization

$$J^a(z)\Phi_j(x|w) = -\frac{\mathcal{D}^a_x\Phi_j(x|w)}{(z-w)} + ...$$

with differential operators

$$\mathcal{D}^+_x = x^2\partial_x - 2jx, \quad \mathcal{D}^-_x = \partial_x, \quad \mathcal{D}^3_x = x\partial_x - j,$$

where, as usual, the notation $a = +,-,3$ refers to the indices of the currents $J^{\pm}(z) = J^1(z) \pm iJ^2(z)$ and $J^3(z)$, which generate the affine $\widehat{sl}(2)_k$ algebra.

According to (3.3)–(3.7), the expectation value of elementary surface operators in the $\mathcal{N} = 2$ SCFT with $N_f = 4$ is given by a WZW four-point function. In some sense, the presence of a theory with affine $\widehat{A}_1$ symmetry seems to be natural from the $\mathcal{N} = 2$ theory point of view. In fact, one expects the Riemann surface to have SU(2) structure at the punctures. This suggests that trying to see AGT correspondence from the WZW perspective could be useful to understand the connection between 4D and 2D theories in more detail. We will comment on the WZW model at the end of this section and we will suggest that it could play an important rôle in this story.

3.2 Alternative derivation of (2.21)

What we would like to discuss now is another consequence of relation (2.21). We will see how (2.21), together with (3.3), enable to find a nice relation between correlation functions of Liouville theory and of WZW theory. To see this, first consider the special operator product expansion

$$V_{\alpha_i}(z_i)V_{-1/2b}(x) \underset{z_i\to x}{=} C_- \ |x-z_i|^{2\xi_-} V_{-1/2b+\alpha_i}(z_i) + C_+ |x-z_i|^{2\xi_+} V_{-1/2b-\alpha_i}(z_i), \quad (3.8)$$

with

$$C_+ = (\pi\mu\gamma(b^2))^{b^{-2}} \frac{\gamma(2\alpha_i b^{-1} - 1 - b^{-2})}{b^4 \gamma(2\alpha_i b^{-1})}, \qquad C_- = 1,$$

where $\xi_\pm = (\Delta_{\alpha_i \pm 1/2b} - \Delta_{1/2b} - \Delta_{\alpha_i})$. OPE (3.8) is an important piece of information about the structure of Liouville theory [41, 42]: It permits to write the coincidence limit of the Liouville five-point function $\Omega^{(5)}(\alpha_1,\alpha_2,\alpha_3,\alpha_4,-1/(2b)|0,1,\infty,q,x)$ in terms of only two (i.e. not infinite) four-point contributions $|x-q|^{2\xi_\mp}\,\Omega^{(4)}(\alpha_1,\alpha_2,\alpha_3,-1/(2b)\pm\alpha_4|0,1,\infty,q)$. And here is where equation (2.21) comes to play an important rôle: Using the fact that $\Omega^{(4)}(\alpha_1,\alpha_2,\alpha_3,\alpha_4|0,1,\infty,q)$ and $\Omega^{(4)}(\widetilde{\alpha}_1,\widetilde{\alpha}_2,\widetilde{\alpha}_3,\widetilde{\alpha}_4|0,1,\infty,q)$ are connected in such a direct way, one rapidly finds the remarkable relation

$$K(j_1,...j_4|q,x) \underset{x\to q}{\simeq} \prod_{i=1}^{4} \frac{1}{\gamma(-b^2(2j_i+1))}\Omega^{(4)}(-bj_1,-bj_2,-bj_3,-bj_4|0,1,\infty,q), \quad (3.9)$$

which, in contrast to (3.3), connects Liouville four-point correlation functions to WZW four-point correlation functions. Symbol $\simeq$ stands here because a regularization is needed due to a singular factor $|x-q|^{-2(1+b^{-2})}$ that blows up in the limit $x\to q$. Subleading contribution vanish in the limit, provided the Seiberg bound $\alpha_i > Q/2$ is obeyed [29]. Notice that factors $\gamma(1+b^2(2j_i+1))$ in (3.9) can be absorbed in the normalization of WZW vertices, in such a way that (3.9) induces a natural one-to-one identification between fields $\Phi_j(z_i) \leftrightarrow V_{-bj}(z_i)$ of both theories, cf. (3.3). The fact that (3.9) takes such a simple factorized form is due to remarkable cancellations that occur through the calculation.

Expression (3.9) provides us with a nice realization of the so-called Drinfeld-Sokolov Hamiltonian reduction WZW $\to$ Liouvilleat the level of the four-point correlation functions. From the gauge theory point of view, and according to the interpretation of the degenerate field $V_{-1/2b}$ of Liouville theory as representing the surface operator of the gauge theory [45, 46], expression (3.9) means that the limit $x\to q$ of the expectation value of a surface operator in the $\mathcal{N}=2$ theory is indeed described by a WZW correlation function. Understanding the physical picture behind this fact requires further investigation. In particular, understanding the relation of (3.9) within the context of the Langlands correspondence [52] is matter of further work.

Another interesting aspect about (3.9) is that, even though here we used (2.21) to give a concise proof of it, it is believed to hold independently as it follows from the Hamiltonian reduction realized at the level of correlation functions; see for instance [53]. This means that, without risk of circular arguments, we can consider (3.9) as the starting point and then use it to prove identity (2.21), reversing the story. In fact, once one assumes (3.9), eq. (2.21) simply follows from functional relations between different solutions of the KZ equation that were proven in refs. [54, 55]. It is easy to see that some of the relations discussed in [54], together with Weyl symmetry, lead to identify solutions $K(j_1,...j_4|q,x)$ with solutions $K(\widetilde{j}_1,...\widetilde{j}_4|q,x)$, where $\widetilde{j}_i$ are related to j_i through (2.19)–(2.20) recalling $j_i = -\alpha_i/b$. Nevertheless, the way we proved equation (2.21) in section 2 is more direct and, consequently, is the one we prefer.

3.3 More comments on WZW theory

Before concluding, and since we are already talking about the relation between Liouville and WZW theories, let us make some comments on another connection that exists between these two conformal theories, and which probably may have important implications for AGT.

It was shown in [56, 57] that n-point $\widehat{sl}(2)_k$ WZW correlation functions on the sphere are equivalent to $(2n-2)$-point correlation functions in Liouville theory, where $n-2$ states in the Liouville correlators are degenerate fields $V_{-1/2b}$. In [58] the result of [57] was generalized to genus-g correlation functions. According to the results of [58], any n-point function in WZW theory on a genus-g surface is equivalent to a $(2n+2g-2)$-point function in Liouville theory which involves $n+2g-2$ degenerate fields. This means that Liouville correlation function on a surface $\mathcal{C}_{g,n}$ with one additional field $V_{-1/2b}$ for each trinion that appears in a given way of sewing the surface, is actually equivalent to a n-point WZW correlator. Whether this observation has some deep meaning from the gauge theory point of view is an open question.

The correspondence between Liouville and WZW theories found in [56, 57] was further extended in [59] to the case of $SL(2,\mathbb{R})$ primary states of spectral flowed sectors; this yields a correspondence between n-point WZW correlators and m-point Liouville correlators, where $n-m$ is the total amount of spectral flow units in the WZW observable. A similar relation was proposed between correlators of Liouville theory that involve higher-level degenerate fields $V_{-m/2b}$ and correlators in a yet-to-be explored family of non-rational conformal field theories with central charges given by $c_{(m)} = 3+6(b+b^{-1}(1-m))^2$, [60]. This could lead to describe higher-monodromy loop operators of the $\mathcal{N}=2$ theories in terms of such new family of CFTs. Furthermore, it is a common belief that a generalization of the WZW-Liouville correspondence exists between the $\widehat{sl}(N)_k$ WZW theory and higher-rank Toda field theory (see [61] for recent attempts in this direction). Speculatively, this could lead to a description of SU(N_c) $\mathcal{N}=2$ theories in terms of the WZW with affine symmetry $\widehat{A}_{N_c-1}$. The results of [56–58] permit to describe observables in the $\mathcal{N}=2$ gauge theories in terms of WZW correlators. However, the physical meaning of this still remains elusive. Likely, this picture will eventually permit to extend AGT correspondence and establish a more general connection between gauge theories and two-dimensional conformal theories. In [58] the connection between the Liouville-WZW correspondence and the Langlands correspondence was pointed out; see also [62]. Investigating this from the gauge theory perspective [63] is an interesting project for future investigations.

Acknowledgments

This work was partially supported by University of Buenos Aires, Agencia ANPCyT, and CONICET, through grants UBACyT X861, UBACyT X432, PICT-2007-00849. The author thanks L.F. Alday and D. Gaiotto for helpful discussions, and he specially thanks Y. Tachikawa for pointing out an important mistake in the first version of the preprint. The author also thanks J. Edelstein, Y. Hikida, Yu Nakayama and J. Teschner for very interesting comments, and thanks V. Fateev for pointing out reference [43]. The author is grateful to M. Kleban, M. Porrati, and the members of the Center for Cosmology and Particle Physics of New York University NYU for their hospitality.

References

[1] D. Gaiotto, *N=2 dualities*, arXiv:0904.2715 [SPIRES].

[2] L.F. Alday, D. Gaiotto and Y. Tachikawa, *Liouville Correlation Functions from Four-dimensional Gauge Theories*, arXiv:0906.3219 [SPIRES].

[3] N.A. Nekrasov, *Seiberg-Witten Prepotential from Instanton Counting*, *Adv. Theor. Math. Phys.* **7** (2004) 831 [hep-th/0206161] [SPIRES].

[4] V. Alba and A. Morozov, *Non-conformal limit of AGT relation from the 1-point torus conformal block*, arXiv:0911.0363 [SPIRES].

[5] A. Mironov and A. Morozov, *Nekrasov Functions and Exact Bohr-Zommerfeld Integrals*, arXiv:0910.5670 [SPIRES].

[6] A. Mironov and A. Morozov, *Proving AGT relations in the large-c limit*, *Phys. Lett.* **B 682** (2009) 118 [arXiv:0909.3531] [SPIRES].

[7] A. Marshakov, A. Mironov and A. Morozov, *On non-conformal limit of the AGT relations*, *Phys. Lett.* **B 682** (2009) 125 [arXiv:0909.2052] [SPIRES].

[8] A. Mironov and A. Morozov, *The Power of Nekrasov Functions*, *Phys. Lett.* **B 680** (2009) 188 [arXiv:0908.2190] [SPIRES].

[9] G. Bonelli and A. Tanzini, *Hitchin systems, $N = 2$ gauge theories and W-gravity*, arXiv:0909.4031 [SPIRES].

[10] A. Mironov, S. Mironov, A. Morozov and A. Morozov, *CFT exercises for the needs of AGT*, arXiv:0908.2064 [SPIRES].

[11] A. Marshakov, A. Mironov and A. Morozov, *On Combinatorial Expansions of Conformal Blocks*, arXiv:0907.3946 [SPIRES].

[12] A. Mironov, A. Morozov and S. Shakirov, *Matrix Model Conjecture for Exact BS Periods and Nekrasov Functions*, arXiv:0911.5721 [SPIRES].

[13] H. Itoyama, K. Maruyoshi and T. Oota, *Notes on the Quiver Matrix Model and 2d-4d Conformal Connection*, arXiv:0911.4244 [SPIRES].

[14] D. Gaiotto, *Asymptotically free $N = 2$ theories and irregular conformal blocks*, arXiv:0908.0307 [SPIRES].

[15] V.A. Fateev and A.V. Litvinov, *On AGT conjecture*, arXiv:0912.0504 [SPIRES].

[16] R. Dijkgraaf and C. Vafa, *Toda Theories, Matrix Models, Topological Strings and $N=2$ Gauge Systems*, arXiv:0909.2453 [SPIRES].

[17] T. Eguchi and K. Maruyoshi, *Penner Type Matrix Model and Seiberg-Witten Theory*, arXiv:0911.4797 [SPIRES].

[18] S. Kanno, Y. Matsuo, S. Shiba and Y. Tachikawa, *N=2 gauge theories and degenerate fields of Toda theory*, arXiv:0911.4787 [SPIRES].

[19] L.F. Alday, F. Benini and Y. Tachikawa, *Liouville/Toda central charges from M5-branes*, arXiv:0909.4776 [SPIRES].

[20] A. Mironov and A. Morozov, *Nekrasov Functions from Exact BS Periods: the Case of SU(N)*, arXiv:0911.2396 [SPIRES].

[21] A. Marshakov, A. Mironov and A. Morozov, *Zamolodchikov asymptotic formula and instanton expansion in $N=2$ SUSY $N_f=2N_c$ QCD*, *JHEP* **11** (2009) 048 [arXiv:0909.3338] [SPIRES].

[22] A. Mironov and A. Morozov, *On AGT relation in the case of U(3)*, *Nucl. Phys.* **B 825** (2010) 1 [arXiv:0908.2569] [SPIRES].

[23] N. Wyllard, *A_{N-1} conformal Toda field theory correlation functions from conformal $N=2$ SU(N) quiver gauge theories*, *JHEP* **11** (2009) 002 [arXiv:0907.2189] [SPIRES].

[24] R. Schiappa and N. Wyllard, *An A_r threesome: Matrix models, 2D CFTs and 4D $N=2$ gauge theories*, arXiv:0911.5337 [SPIRES].

[25] A. Gadde, E. Pomoni, L. Rastelli and S.S. Razamat, *S-duality and 2d Topological QFT*, arXiv:0910.2225 [SPIRES].

[26] G. Bertoldi, S. Bolognesi, M. Matone, L. Mazzucato and Y. Nakayama, *The Liouville geometry of $N=2$ instantons and the moduli of punctured spheres*, *JHEP* **05** (2004) 075 [hep-th/0405117] [SPIRES].

[27] V. Pestun, *Localization of the four-dimensional $N=4$ SYM to a two- sphere and 1/8 BPS Wilson loops*, arXiv:0906.0638 [SPIRES].

[28] N. Seiberg and E. Witten, *Monopoles, duality and chiral symmetry breaking in $N=2$ supersymmetric QCD*, *Nucl. Phys.* **B 431** (1994) 484 [hep-th/9408099] [SPIRES].

[29] N. Seiberg, *Notes on quantum Liouville theory and quantum gravity*, *Prog. Theor. Phys. Suppl.* **102** (1990) 319 [SPIRES].

[30] J. Teschner, *Liouville theory revisited*, *Class. Quant. Grav.* **18** (2001) R153 [hep-th/0104158] [SPIRES].

[31] Y. Nakayama, *Liouville field theory: A decade after the revolution*, *Int. J. Mod. Phys.* **A 19** (2004) 2771 [hep-th/0402009] [SPIRES].

[32] J. Teschner, *A lecture on the Liouville vertex operators*, *Int. J. Mod. Phys.* **A 19S2** (2004) 436 [hep-th/0303150] [SPIRES].

[33] M. Goulian and M. Li, *Correlation Functions In Liouville Theory*, *Phys. Rev. Lett.* **66** (1991) 2051.

[34] H. Dorn and H.J. Otto, *On correlation functions for noncritical strings with $c <= 1$ but $d >= 1$*, *Phys. Lett.* **B 291** (1992) 39 [hep-th/9206053] [SPIRES].

[35] P. Di Francesco and D. Kutasov, *World Sheet and Space Time Physics in Two Dimensional (Super) String Theory*, *Nucl. Phys.* **B 375** (1992) 119 [hep-th/9109005] [SPIRES].

[36] V.S. Dotsenko and V.A. Fateev, *Conformal algebra and multipoint correlation functions in 2D statistical models*, Nucl. Phys. **B 240** (1984) 312 [SPIRES].

[37] V.S. Dotsenko and V.A. Fateev, *Four Point Correlation Functions and the Operator Algebra in the Two-Dimensional Conformal Invariant Theories with the Central Charge c ¡ 1*, Nucl. Phys. **B 251** (1985) 691 [SPIRES].

[38] A.B. Zamolodchikov and A.B. Zamolodchikov, *Structure constants and conformal bootstrap in Liouville field theory*, Nucl. Phys. **B 477** (1996) 577 [hep-th/9506136] [SPIRES].

[39] G. Giribet and L. Nicolás, *Comment on three-point function in AdS_3/CFT_2*, J. Math. Phys. **50** (2009) 042304 [arXiv:0812.2732] [SPIRES].

[40] H. Dorn and H.J. Otto, *Two and three point functions in Liouville theory*, Nucl. Phys. **B 429** (1994) 375 [hep-th/9403141] [SPIRES].

[41] J. Teschner, *On the Liouville three point function*, Phys. Lett. **B 363** (1995) 65 [hep-th/9507109] [SPIRES].

[42] A. Pakman, *Liouville theory without an action*, Phys. Lett. **B 642** (2006) 263 [hep-th/0601197] [SPIRES].

[43] V.A. Fateev and A.V. Litvinov, *Coulomb integrals in Liouville theory and Liouville gravity*, JETP Lett. **84** (2007) 531 [SPIRES].

[44] V.A. Fateev and A.V. Litvinov, *Multipoint correlation functions in Liouville field theory and minimal Liouville gravity*, Theor. Math. Phys. **154** (2008) 454 [arXiv:0707.1664] [SPIRES].

[45] L.F. Alday, D. Gaiotto, S. Gukov, Y. Tachikawa and H. Verlinde, *Loop and surface operators in $N = 2$ gauge theory and Liouville modular geometry*, arXiv:0909.0945 [SPIRES].

[46] N. Drukker, J. Gomis, T. Okuda and J. Teschner, *Gauge Theory Loop Operators and Liouville Theory*, arXiv:0909.1105 [SPIRES].

[47] D. Gaiotto, *Surface Operators in $N = 2$ 4D Gauge Theories*, arXiv:0911.1316 [SPIRES].

[48] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nucl. Phys. **B 241** (1984) 333 [SPIRES].

[49] V. Fateev and A.B. Zamolodchikov, *Operator algebra and correlation functions in the two-dimensional $SU(2) \times SU(2)$ chiral Wess-Zumino model*, *Sov. J. Nucl. Phys.* **43** (1987) 657.

[50] V.G. Knizhnik and A.B. Zamolodchikov, *Current algebra and Wess-Zumino model in two dimensions*, Nucl. Phys. **B 247** (1984) 83 [SPIRES].

[51] J. Teschner, *Crossing symmetry in the H_3^+ WZNW model*, Phys. Lett. **B 521** (2001) 127 [hep-th/0108121] [SPIRES].

[52] E. Frenkel, *Lectures on the Langlands program and conformal field theory*, hep-th/0512172 [SPIRES].

[53] J.L. Petersen, J. Rasmussen and M. Yu, *Hamiltonian reduction of SL(2) theories at the level of correlators*, Nucl. Phys. **B 457** (1995) 343 [hep-th/9506180] [SPIRES].

[54] G. Giribet, *On spectral flow symmetry and Knizhnik-Zamolodchikov equation*, Phys. Lett. **B 628** (2005) 148 [hep-th/0508019] [SPIRES].

[55] G.E. Giribet, *A note on Z_2 symmetries of the KZ equation*, J. Math. Phys. **48** (2007) 012304 [hep-th/0608104] [SPIRES].

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[56] A.V. Stoyanovsky, *A relation between the Knizhnik-Zamolodchikov and Belavin-Polyakov-Zamolodchikov systems of partial differential equations*, math-ph/0012013 [SPIRES].

[57] S. Ribault and J. Teschner, *H_3^+ WZNW correlators from Liouville theory*, *JHEP* **06** (2005) 014 [hep-th/0502048] [SPIRES].

[58] Y. Hikida and V. Schomerus, *H_3^+ WZNW model from Liouville field theory*, *JHEP* **10** (2007) 064 [arXiv:0706.1030] [SPIRES].

[59] S. Ribault, *Knizhnik-Zamolodchikov equations and spectral flow in AdS_3 string theory*, *JHEP* **09** (2005) 045 [hep-th/0507114] [SPIRES].

[60] S. Ribault, *A family of solvable non-rational conformal field theories*, *JHEP* **05** (2008) 073 [arXiv:0803.2099] [SPIRES].

[61] S. Ribault, *On sl3 Knizhnik-Zamolodchikov equations and W3 null-vector equations*, *JHEP* **10** (2009) 002 [arXiv:0811.4587] [SPIRES].

[62] G. Giribet, Y. Nakayama and L. Nicolás, *Langlands duality in Liouville-H_3^+ WZNW correspondence*, *Int. J. Mod. Phys.* **A 24** (2009) 3137 [arXiv:0805.1254] [SPIRES].

[63] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, hep-th/0604151 [SPIRES].