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Membrane Solutions to Hořava Gravity

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Abstract—We have investigated purely gravitational membrane solutions to the Hořava nonrelativistic theory of gravity with detailed balance in $3+1$ dimensions. We find that for arbitrary values of the running parameter $\lambda > 1/3$ there exist two branches of membrane solutions, and that in the special case $\lambda = 1$ one of them is degenerate, the lapse function being undetermined. For negative values of the cosmological constant, the solution contains a single membrane sitting at the center of space, which extends infinitely in the transverse direction, approaching a Lifshitz metric. For positive values of the cosmological constant, the solution represents a space that is bounded in the transverse direction, with two parallel membrane-like or point-like singularities sitting at each of the boundaries.

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1. INTRODUCTION

The power-counting renormalizable nonrelativistic theory of gravity recently proposed by Hořava [1] is a theory of gravity in which general covariance is partly abandoned in favor of renormalizability. A state of the theory is defined by a four-dimensional manifold $\mathcal{M}$ equipped with a three-dimensional foliation $\mathcal{F}$, with a pseudo-Riemannian structure defined by an Euclidean three-dimensional metric in each slice of the foliation $g_{ij}(\vec{x},t)$, a shift vector $N^i(\vec{x},t)$, and a lapse function $N(\vec{x},t)$. This structure can be encoded in the ADM-decomposed metric

$$ds^2 = -N^2(\vec{x},t)dt^2 + g_{ij}(\vec{x},t)\left(dx^i + N^i(\vec{x},t)dt\right) \times \left(dx^j + N^j(\vec{x},t)dt\right). \quad (1)$$

The dynamics for the set $(\mathcal{M}, \mathcal{F}, g_{ij}, N_i, N)$ is defined as being gauge-invariant with respect to foliation-preserving diffeomorfisms, and having a UV fixed point at $\mathbf{z} = 3$, where the dynamical critical exponent $\mathbf{z}$ is defined as the scaling dimension of time as compared to that of space directions $[\vec{x}] = -1$, $[t] = -\mathbf{z}$. This choice leads to power-counting renormalizability of the theory in the UV. To the resulting action one may add relevant deformations given by operators of lower dimensions, that lead the theory to a IR fixed point with $\mathbf{z} = 1$, in which the symmetry between space and time is restored, and thus a generally covariant theory may emerge.

To have control on the number of terms arising as possible potential terms, one may impose the so-called detailed balance condition: the potential term in the action for $(3+1)$-dimensional nonrelativistic gravity is built from the square of the functional derivative of a suitable action for Euclidean three-dimensional gravity (here three-dimensional indices are contracted with the inverse DeWitt metric). Condensed matter experience on this kind of construction tells us that the higher-dimensional theory satisfying the detailed balance condition inherits the quantum properties of the lower-dimensional one. It has to be noted that the theory is still well defined even when the detailed balance condition is broken softly, in the sense of adding relevant operators of dimension lower than that of the operators appearing at the short distance fixed point $\mathbf{z} = 3$. With such a deformation, in the UV the theory satisfies the detailed balance, while in the IR the theory flows to a $\mathbf{z} = 1$ fixed point.

We will not go through the above described steps in more detail, but state the resulting action that will be relevant to our purposes. The interested reader can refer to the original paper [1]. The action for nonrelativistic gravity satisfying the detailed balance condition can be written as

$$S = \int \sqrt{g}N\left[\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2\mu^2(\Lambda_W R - 3\Lambda_W^2)}{8(1-3\lambda)} + \frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}R^2\right.$$

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$$- \frac{\kappa^2}{2w^4}\left(C_{ij} - \frac{\mu w^2}{2}R_{ij}\right)\left(C^{ij} - \frac{\mu w^2}{2}R^{ij}\right)\Bigg]. \quad (2)$$

Here Λ_W, κ, λ, μ, and w are arbitrary couplings, R, R_{ij}, C_{ij}, and K_{ij} are the scalar curvature, the Ricci tensor, the Cotton-York tensor and the extrinsic curvature, respectively, of the three-dimensional sections of the foliation. The dynamics in the infrared is controlled by the first two terms, and then, if $\lambda = 1$, general relativity is recovered. On the other hand, in the UV the third and fourth terms become dominant, and the anisotropy between space and time is explicit.

This action is invariant under foliation preserving diffeomorphisms, namely under changes of the coordinates of the form

$$x^{i'} = x^{i'}(x^j, t), \quad t' = t'(t), \quad (3)$$

under which the spatial metric, shift vector and lapse function transform as

$$\begin{aligned} g'_{ij}(x^{i'}, t') &= \partial_{i'}x^k\partial_{j'}x^l g_{kl}(x^r, t), \\ N^{i'}(x^{i'}, t') &= \partial_k x^{i'}\partial_{t'}tN^k(x^j, t), \\ N'(x^{i'}, t') &= \partial_{t'}tN(x^j, t). \end{aligned} \quad (4)$$

Equations (3) and (4) ensure that if the lapse function is initially chosen to be independent of the space coordinates in a given coordinate system, it cannot be turned into a space-dependent form by a change of coordinates. In other words, the spatial independence of the lapse function is a covariant statement. This implies the existence of two possible versions of Hořava gravity, a "projectable theory" in which the lapse function is space-independent, $N = N(t)$, and a "non-projectable theory" in which the lapse function is allowed to depend on space, $N = N(x^i, t)$.

In the non-projectable case, the equations of motion obtained by varying the above action are

$$\begin{aligned} &\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) - \frac{\kappa^2\mu^2(\Lambda_W R - 3\Lambda_W^2)}{8(1-3\lambda)} \\ &\quad - \frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}R^2 + \frac{\kappa^2}{2w^4}Z_{ij}Z^{ij} = 0, \end{aligned} \quad (5)$$

$$\nabla_k(K^{k\ell} - \lambda K g^{k\ell}) = 0, \quad (6)$$

$$\begin{aligned} &\frac{2}{\kappa^2}E_{ij}^{(1)} - \frac{2\lambda}{\kappa^2}E_{ij}^{(2)} + \frac{\kappa^2\mu^2\Lambda_W}{8(1-3\lambda)}E_{ij}^{(3)} \\ &\quad + \frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}E_{ij}^{(4)} - \frac{\mu\kappa^2}{4w^2}E_{ij}^{(5)} \\ &\quad - \frac{\kappa^2}{2w^4}E_{ij}^{(6)} = 0, \end{aligned} \quad (7)$$

where

$$Z_{ij} \equiv C_{ij} - \frac{\mu w^2}{2}R_{ij}, \quad (8)$$

$$\begin{aligned} E_{ij}^{(1)} &= N_i\nabla_k K^k{}_j + N_j\nabla_k K^k{}_i - K^k{}_i\nabla_j N_k \\ &\quad - K^k{}_j\nabla_i N_k - N^k\nabla_k K_{ij} \\ &\quad - 2NK_{ik}K_j{}^k - \frac{1}{2}NK^{k\ell}K_{k\ell}g_{ij} + NKK_{ij} + \dot{K}_{ij}, \\ E_{ij}^{(2)} &= \frac{1}{2}NK^2 g_{ij} + N_i\partial_j K + N_j\partial_i K \\ &\quad - N^k(\partial_k K)g_{ij} + \dot{K}g_{ij}, \\ E_{ij}^{(3)} &= N(R_{ij} - \frac{1}{2}Rg_{ij} + \frac{3}{2}\Lambda_W g_{ij}) \\ &\quad - (\nabla_i\nabla_j - g_{ij}\nabla_k\nabla^k)N, \\ E_{ij}^{(4)} &= NR(2R_{ij} - \frac{1}{2}Rg_{ij}) \\ &\quad - 2\big(\nabla_i\nabla_j - g_{ij}\nabla_k\nabla^k\big)(NR), \\ E_{ij}^{(5)} &= \nabla_k\big[\nabla_j(NZ^k{}_i) + \nabla_i(NZ^k{}_j)\big] \\ &\quad - \nabla_k\nabla^k(NZ_{ij}) - \nabla_k\nabla_\ell(NZ^{k\ell})g_{ij}, \\ E_{ij}^{(6)} &= -\frac{1}{2}NZ_{k\ell}Z^{k\ell}g_{ij} + 2NZ_{ik}Z_j{}^k \\ &\quad - N(Z_{ik}C_j{}^k + Z_{jk}C_i{}^k) + NZ_{k\ell}C^{k\ell}g_{ij} \\ &\quad - \frac{1}{2}\nabla_k\big[N\epsilon^{mk\ell}(Z_{mi}R_{j\ell} + Z_{mj}R_{i\ell})\big] \\ &\quad + \frac{1}{2}R^n{}_\ell\nabla_n\big[N\epsilon^{mk\ell}(Z_{mi}g_{kj} + Z_{mj}g_{ki})\big] \\ &\quad - \frac{1}{2}\nabla_n\big[NZ_m^n\epsilon^{mk\ell}(g_{ki}R_{j\ell} + g_{kj}R_{i\ell})\big] \\ &\quad - \frac{1}{2}\nabla_n\nabla^n\nabla_k\big[N\epsilon^{mk\ell}(Z_{mi}g_{j\ell} + Z_{mj}g_{i\ell})\big] \\ &\quad + \frac{1}{2}\nabla_n\big[\nabla_i\nabla_k(NZ_m^n\epsilon^{mk\ell})g_{j\ell} \\ &\quad + \nabla_j\nabla_k(NZ_m^n\epsilon^{mk\ell})g_{i\ell}\big] \\ &\quad + \frac{1}{2}\nabla_\ell\big[\nabla_i\nabla_k(NZ_{mj}\epsilon^{mk\ell}) + \nabla_j\nabla_k(NZ_{mi}\epsilon^{mk\ell})\big] \\ &\quad - \nabla_n\nabla_\ell\nabla_k(NZ_m^n\epsilon^{mk\ell})g_{ij}. \end{aligned} \quad (9)$$

In the projectable case, Eq. (5) is replaced by its spatial integral.

Since the original proposal of [1], there has been a growing number of research papers in the area. Formal developments were presented in [2–9], some spherically symmetric solutions were presented in [10–16], rotating solutions were studied in [17], string-like ansätze were investigated in [18, 19], toroidal solutions were found in [20], gravitational waves were studied in [21] and [22], cosmological implications were investigated in [23–35], and interesting features of field theory in curved space and black hole physics were presented in [36–42]. In [43–46], potentially harmful instabilities were pointed out, originating in the additional scalar graviton mode that propagates by virtue of the reduced gauge symmetry. Moreover, it was shown that the extra mode becomes strongly coupled in the infrared in nontrivial backgrounds. To cure these problems, a so-called "healthy extension" of the non-projectable theory was proposed, in which additional terms containing derivatives of the lapse function were included in the action, which has the effect of eliminating the instabilities [47]. Alternatively, a covariant theory whose partly gauge fixed version reproduces the non-projectable Hořava dynamics, was developed, in which the extra mode was shown to be harmless [48]. In [49], a reinterpretation of a secondary constraint that appears when $\lambda \neq 1$ in the infrared limit of the non-projectable theory, leads to a new counting of degrees of freedom in which the extra mode is not present. Finally, in [50], an additional $U(1)$ gauge symmetry was introduced, that kills the scalar graviton, avoiding the aforementioned problems.

In Einstein gravity, it is easy to prove that no nontrivial solution with the symmetry of a domain wall exist in the absence of matter. Indeed, the only solution of the equations of motion compatible with a smooth and flat domain wall ansatz is that of an AdS/Mikowski space-time, depending on the cosmological constant. In Hořava gravity, on the other hand, the terms containing higher spatial derivatives could in principle play the role of a matter contribution, allowing for the existence of nontrivial domain wall solutions in vacuum. This is one of the motivations of our work.

In Einstein gravity, the knowledge of the cosmological solution corresponding to a given kind of matter can be used to obtain a domain wall solution through the so-called domain wall/cosmology correspondence [51]. Indeed, given a metric with the Friedmann-Lemaitre-Robertson-Walker (FLRW) form, it can be mapped to a domain wall ansatz via a suitably defined Wick rotations of the coordinates. Moreover, such a transformation maps the Friedman equations into the equation of motion corresponding to the domain wall, ensuring that cosmological solutions are mapped into domain-wall solutions. In Hořava gravity, on the other hand, the situation is very different. The essential anisotropy between space and time, present in the theory, is an obstacle for the domain wall/cosmology correspondence to work. First, a cosmological ansatz has a single independent function that can be identified with the scale factor, while, as we will see below in further detail, a domain wall ansatz has in principle two independent functions. Moreover, the anisotropy between space and time implies that the equations of motion for a domain wall are not mapped under a Wick rotation into Friedmann-like equations for a cosmological ansatz. In consequence, a large amount of research regarding cosmological solutions of Hořava gravity [23–35] gives no clue to the form of the domain wall solutions of the theory, making a study of such solutions a subject of independent inquiry. This is a second motivation for the present paper.

In this paper, we start the investigation of domain wall solutions of Hořava theory. The simplest possible setup being that of a purely gravitational theory, we will not include any matter degree of freedom in our equations. In attention to the the fact that all solutions to the aforementioned controversy about the extra scalar mode that were proposed in the literature, correspond to modifications of the non-projectable theory, we will restrict our investigation to that case. Since in the absence of matter there is no possible $\mathbb{Z}_2$ symmetry, to be broken differently at each side of the wall forming "domains," we will sometimes use the somewhat more accurate name "membrane" for our solutions.

2. MEMBRANE SOLUTIONS

This paper deals with the issue of membrane solutions of Hořava theory. In the present nonrelativistic context, a flat domain wall solution is defined as a solution having translational and rotational symmetry in two dimensions, i.e., being invariant with respect to the $ISO(2) \times \mathbb{R}$ group of transformations. The symmetry of the solution is reduced with respect to that of a relativistic domain wall $ISO(2,1)$ because of the non-relativistic nature of Hořava theory. An ansatz that preserves such symmetry can be easily written as

$$ds^2 = -e^{V(z)}dt^2 + e^{U(z)}(dx^2 + dy^2) + dz^2. \quad (10)$$

Note that, by virtue of the reduced symmetry of the theory, there is no set of coordinates in which $U(z) = V(z)$ as it would happen in Einstein's theory, and thus the ansatz has two independent functions to be determined by the equations of motion. The variables x, y can be chosen as describing a two-torus T_2 of volume V_{xy} or a two-plane $\mathbb{R}^2$ (that can be considered as the infinite V_{xy} limit). In what follows we will call the"lapse function" to $e^{V(z)}$ and the "spatial volume function" to $e^{U(z)}$.

Notice that the ansatz (10) has a conformally flat spatial metric. This implies that the terms of the action (2) containing the Cotton-York tensor does not contribute to the equations of motion. By inspecting Eqs. (5)–(9) we can anticipate that the resulting equations of motion will be lower than sixth order. Indeed, by replacing the ansatz in the above equations (5)–(7) we get the following two independent equations of motion:

$$(4\Lambda_W + U'^2)[3(4\Lambda_W + U'^2) + 8U'']$$

$$- 8(\lambda - 1)U''^2 = 0, \quad (11)$$

$$48\Lambda_W^2 + \left[(4V' - U')U'^2 + 8\Lambda_W(U' + 2V')\right] U' + 8(\lambda - 1)\Big[(U'' - 2U'^2 - U'V')U'' - 2U'U^{(3)}\Big] = 0. \quad (12)$$

Note that in the above equations the constants μ, κ, and w do not appear. This can be traced back to the action (2) or to the equations of motion (5)–(7) in which, when the extrinsic curvature K_{ij} and the Cotton-York tensor C_{ij} vanish, as it happens for our ansatz, the constant w cancels, and the product $\kappa^2\mu^2$ can be factored out. As can be seen in the second equation, the case $\lambda = 1$ is special in the sense that the total differential order of the system is reduced. As we will see, this property manifests itself in a non-analyticity of the solutions as functions of the parameter λ.

2.1. Solutions with $\Lambda_W < 0$

2.1.1. Solutions with $\lambda = 1$. We first fix our attention on the case $\lambda = 1$. Solving Eq. (11) for $U(z)$ and inserting the solution into Eq. (12) to get $V(z)$, we obtain the corresponding solution. It reads

$$e^{U_o(z)} = \cosh\left(\frac{3\sqrt{-\Lambda_W}}{4}(z - z_o)\right)^{8/3},$$
$$e^{V_o(z)} = \cosh\left(\frac{3\sqrt{-\Lambda_W}}{4}(z - z_o)\right)^{2/3} \times \sinh\left(\frac{3\sqrt{-\Lambda_W}}{4}(z - z_o)\right)^{2}, \quad (13)$$

where z_o is an integration constant, other two integration constants have been reabsorbed in the definitions of t and of x, y. We see that the solution is $\mathbb{Z}_2$-symmetric, and that its asymptotic form as $z \to \pm\infty$ is that of AdS space-time:

$$e^{U_o(z)} \propto e^{2\sqrt{-\Lambda_W}|z - z_o|},$$
$$e^{V_o(z)} \propto e^{2\sqrt{-\Lambda_W}|z - z_o|}. \quad (14)$$

To have a physical interpretation of this solution, we evaluate some of the observable scalars of the theory. We start with the space-time curvature that reads

$$R^{(4)} = \frac{3\Lambda_W}{4}\left[11 + 5\tanh\left(\frac{3}{4}\sqrt{-\Lambda_W}(z - z_o)\right)^2\right] \underset{z\to\pm\infty}{\longrightarrow} 12\Lambda_W. \quad (15)$$

We see that it approaches a constant value as $z \to \pm\infty$, as may have been expected from is AdS asymptotic form. More interestingly, it shows a peak at $z = z_o$. A low-energy observer, armed only with the tools of Einstein gravity, would conclude that some kind of matter with positive energy density must be sitting there, to partly cancel the contribution of the cosmological constant. By observing that the total spatial volume of the $z = z_o$ slice is proportional to V_{xy}, he/she would identify the matter distribution as a "membrane" located at $z = z_o$. Nevertheless, from our privileged high-energy point of view, we know that what we have is a purely gravitational soliton since no additional matter has been added to Hořava theory. With this at hand, we will refer to our solution as a purely gravitational membrane located at $z = z_o$.

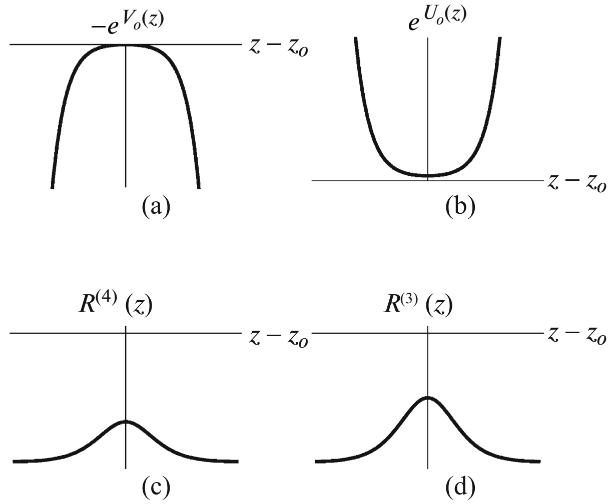

Fig. 1. Plots of the lapse function $g_{00} = -e^{V_o(z)}$ (a), the spatial volume function $g_{xx} = g_{yy} = e^{U_o(z)}$ (b), the space-time curvature $R^{(4)}(z)$ (c), and the space curvature $R^{(3)}(z)$ (d), for $\Lambda_W < 0$ and $\lambda = 1$.

Since the Hořava theory explicitly distinguishes time from space, a separate invariant of interest is the space curvature which in our case is given by

$$R^{(3)} = 3\Lambda_W\left[1 + \tanh\left(\frac{3}{4}\sqrt{-\Lambda_W}(z - z_o)\right)^2\right] \underset{z\to\pm\infty}{\longrightarrow} 6\Lambda_W. \quad (16)$$

Again it presents a peak at $z = z_o$, which reinforces the identification of that point as the location of a membrane. Plots of the space and space-time curvatures are included in Fig. 1.

2.1.2. Solutions with $\lambda \neq 1$. Let us next assume $\lambda \neq 1$. Then, solving Eq. (11) for $U(z)$ and inserting the solution into Eq. (12) to get $V(z)$, we have

$$e^{U_\pm(z)} = \cosh\left(\frac{\sqrt{-\Lambda_W}}{p_\pm(\lambda)}(z - z_o)\right)^{2p_\pm(\lambda)},$$

$$e^{V_\pm(z)} = \left[\cosh\left(\frac{\sqrt{-\Lambda_W}}{p_\pm(\lambda)}(z-z_o)\right)^{\frac{5p_\pm(\lambda)-4\lambda-2}{3p_\pm(\lambda)-2}},\right.$$
$$\left.\sinh\left(\frac{\sqrt{-\Lambda_W}}{p_\pm(\lambda)}(z-z_o)\right)\right]^2 = N^2, \quad (17)$$

with

$$p_\pm(\lambda) = \frac{2(\lambda-1)}{-2\pm\sqrt{6\lambda-2}}, \quad (18)$$

where again z_o is an integration constant and other two constants have been absorbed in the definition of t and of x, y. Here the subindex $\pm$ indicates that we have two different branches of solutions according to the choice of the sign in $p_\pm(\lambda)$. Note that for $\lambda < 1/3$ no real solution exists, so in the remainder of this subsection we will focus on the parameter range $1 \neq \lambda \geq 1/3$.

Both $\pm$ solutions are $\mathbb{Z}_2$ symmetric and centered at z_o. To explore the asymptotic behavior we take $z \to \pm\infty$ to have

$$e^{U_\pm(z)} \propto \exp\left[2\sqrt{-\Lambda_W}\,\mathrm{sgn}(p_\pm(\lambda))|z-z_o|\right],$$
$$e^{V_\pm(z)} \propto \exp\left[2\mathbf{z}_\pm(\lambda)\sqrt{-\Lambda_W}\right.$$
$$\left.\times\,\mathrm{sgn}(p_\pm(\lambda))|z-z_o|\right], \quad (19)$$

where

$$\mathbf{z}_\pm(\lambda) = \frac{4(2p_\pm(\lambda)-\lambda-1)}{p_\pm(\lambda)(3p_\pm(\lambda)-2)}. \quad (20)$$

Here we see that our metric corresponds to an asymptotically Lifshitz space-time, similar to those studied in [52], whose scaling exponent is given by $\mathbf{z}_\pm(\lambda)$. At this point, it is convenient to stress that such a scaling exponent is not in principle related to the dynamical critical exponent $\mathbf{z} = 3$ of Hořava theory.

In the solution with the $+$ sign, $p_+(\lambda)$ is always positive, and thus the spatial volume function is exponentially growing as $z \to \pm\infty$ independently of the value of the parameter λ. The lapse function instead grows exponentially whenever $\mathbf{z}_+(\lambda) > 0$ ($\lambda < 3$) and decreases exponentially for $\mathbf{z}_+(\lambda) < 0$ ($\lambda > 3$). In the limiting case $\mathbf{z}_+(\lambda) = 0$ ($\lambda = 3$), the function asymptotes a constant value. Plots of the corresponding solutions for different values of λ can be seen in Fig. 2.

On the other hand, in the solution with the minus sign, the spatial volume function is exponentially decreasing as $z \to \pm\infty$ when $p_-(\lambda) < 0$ ($\lambda > 1$), and exponentially growing if $p_-(\lambda) > 0$ ($\lambda < 1$). Additionally, $\mathbf{z}_-(\lambda)$ is always positive, which implies that the lapse function grows exponentially as $z \to \pm\infty$ for any value of λ. Plots of the corresponding solutions for different values of λ can be seen in Fig. 3.

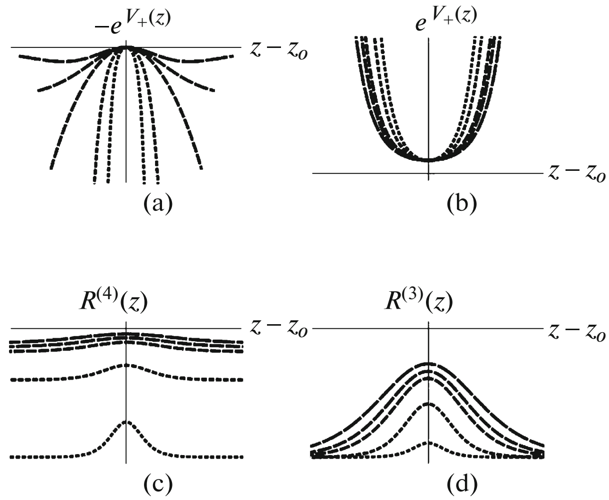

Fig. 2. Plots of the lapse function $g_{00} = -e^{V_+(z)}$ (a) and the spatial volume function $g_{xx} = g_{yy} = e^{U_+(z)}$ (b), space-time curvature $R^{(4)}(z)$ (c) and space curvature $R^{(3)}(z)$ (d) for $\Lambda_W < 0$ and different values of λ. Notice the change in the asymptotic behavior of $e^{V_+(z)}$ at $\lambda = 3$ in plot (a).

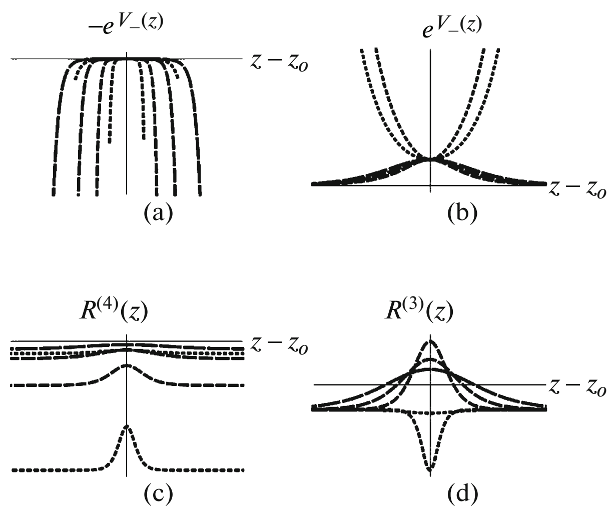

Fig. 3. Plots of the lapse function $g_{00} = -e^{V_-(z)}$ (a) and the spatial volume function $g_{xx} = g_{yy} = e^{U_-(z)}$ (b), space-time curvature $R^{(4)}(z)$ (c) and space curvature $R^{(3)}(z)$ (d) for $\Lambda_W < 0$ and different values of λ. Notice the change in the asymptotic behavior of $e^{U_-(z)}$ at $\lambda = 1$ in plot (b).

Again, to have a physical interpretation, we evaluate the space-time curvature scalar for this solution. It reads

$$R^{(4)} = R^{(4)}_\infty(\lambda) + (R^{(4)}_{z_o}(\lambda)$$
$$- R^{(4)}_\infty(\lambda))\mathrm{sech}\left(\frac{\sqrt{-\Lambda_W}(z-z_o)}{p_\pm(\lambda)}\right)^2. \quad (21)$$

Its asymptotic value is consistent with Lifshitz space-time and it is given by

$$R^{(4)}_{\infty}(\lambda) = \frac{-2\Lambda_W}{(2-3p_\pm(\lambda))^2 p_\pm(\lambda)^2} \times \Big[p_\pm(\lambda)(48(1+\lambda) - p_\pm(\lambda)(20 - 24\lambda + 3p_\pm(\lambda) \times (4+9p_\pm(\lambda)))) - 16(1+\lambda)^2\Big]. \quad (22)$$

As before, a peak at $z = z_o$ appears, with its value given by

$$R^{(4)}_{z_o}(\lambda) = R^{(4)}_{\infty}(\lambda) - \frac{2\Lambda_W}{(2-3p_\pm(\lambda))^2 p_\pm(\lambda)^2} \times \Big[8\lambda(1+2\lambda) + p_\pm(\lambda)(-4(1+3\lambda) + p_\pm(\lambda)(14 - 24\lambda + 3p_\pm(\lambda)(-8+9p_\pm(\lambda))))\Big]. \quad (23)$$

Such a peak, together with the observation that the spatial volume of the $z = z_o$ slice is proportional to V_{xy}, allows us to make the same interpretation as before, referring to our solution as a purely gravitational membrane sitting at $z = z_o$. This interpretation is reinforced by evaluating the space curvature scalar, which reads

$$R^{(3)} = 2\Lambda_W \left[3 + \frac{2-3p_\pm(\lambda)}{p_\pm(\lambda)} \times \operatorname{sech}\left(\frac{\sqrt{-\Lambda_W}(z-z_o)}{p_\pm(\lambda)}\right)^2\right] \underset{z\to\infty}{\longrightarrow} 6\Lambda_W, \quad (24)$$

and also shows a peak at $z = z_o$. As can be easily seen in the above equation, the space curvature at infinity is independent of λ. On the other hand, its sign at z_o depends on λ, for the solution with the + sign it is always positive, while for the solution with the − sign it is positive for $\lambda > 1$ and negative otherwise.

Plots of both space-time and space curvatures for each kind of the ± solutions are shown in Figs. 2 and 3.

Notice that these solutions are analytic in the parameter λ in its whole range except $\lambda = 1$. There, the solution with the + sign is analytic and approaches the solution for $\lambda = 1$ given in (13). On the other hand, the solution with the − sign is non-analytic at that point, since $p_-(\lambda)$ vanishes there.

2.1.3. Degenerate solutions. In the present $\Lambda_W < 0$ case, Eqs. (11), (12) allow for an additional (degenerate) branch of solutions for any value of λ. It is given by

$$e^{U_d(z)} = e^{2\sqrt{-\Lambda_W}(z-z_o)}, \qquad e^{V_d(z)} = \text{arbitrary function}. \quad (25)$$

The existence of such an infinite branch of solutions for which the lapse function is not determined was already pointed out in the spherically symmetric case in [10] and in the warped BTZ string context in [18]. As shown in [10], this degeneracy is closely related to the detailed balance condition and is lifted by its small violation.

2.2. Solutions with $\Lambda_W > 0$

2.2.1. Solutions with $\lambda = 1$. Proceeding as in the previous sections, we first assume that $\lambda = 1$. The corresponding solution reads

$$e^{U_o(z)} = \cos\left(\frac{3\sqrt{\Lambda_W}}{4}(z-z_o)\right)^{8/3}, \qquad e^{V_o(z)} = \cos\left(\frac{3\sqrt{\Lambda_W}}{4}(z-z_o)\right)^{2/3} \times \sin\left(\frac{3\sqrt{\Lambda_W}}{4}(z-z_o)\right)^{2}, \quad (26)$$

where, as before, z_o is an integration constant, the solution being $\mathbb{Z}_2$-symmetric around $z = z_o$. The lapse function is real in the region $|z - z_o| \le 2\pi/3\sqrt{\Lambda_W}$ and imaginary otherwise. Thus the surfaces $z = z_o \pm 2\pi/3\sqrt{\Lambda_W}$ define the boundaries of space-time. At those boundaries, the lapse function and the function $e^{U_o(z)}$ vanish.

To have a physical interpretation of the solution, we evaluate the curvature scalar which takes the form

$$R^{(4)} = \frac{3}{4}\Lambda_W \left[(11 - 5\tan\left(\frac{3}{4}\sqrt{\Lambda_W}(z-z_o)\right)^2\right] \underset{z\to z_o\pm 2\pi/(3\sqrt{\Lambda_W})}{\longrightarrow} -\infty. \quad (27)$$

It takes a finite positive value proportional to Λ_W at $z = z_o$, and it blows up to negative values at the boundaries. From the point of view of a low-energy observer who interprets the solution in the light of Einstein's theory, some kind of negative energy density must be localized close to each boundary in order to cancel the contribution of the positive cosmological constant, resulting in a negative space-time curvature. From our high-energy point of view, on the other hand, we know that we are in the presence of a purely gravitational soliton. Since $e^{U_o(z)}$ vanishes at the boundaries, the spatial volume of the slices $z = z_o \pm 2\pi/(3\sqrt{\Lambda_W})$ is zero, implying that the singularities are pointlike. On the other hand, all the remaining slices $z \ne z_o \pm 2\pi/(3\sqrt{\Lambda_W})$ have a finite nonzero volume proportional to V_{xy}. Thus the interpretation of this solution is that of two pointlike singularities sitting at the poles of the geometry $z =$

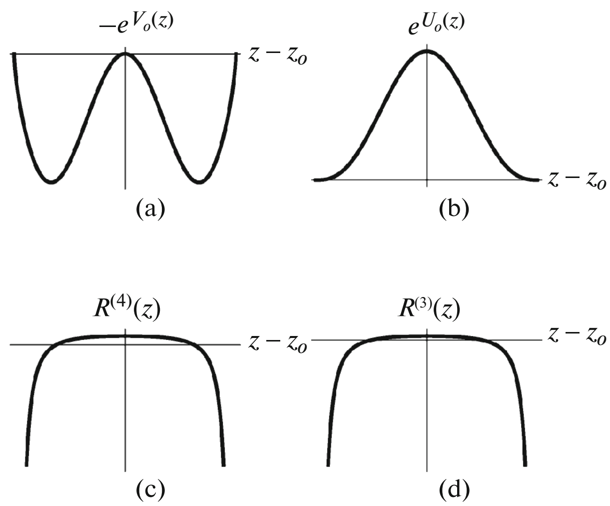

Fig. 4. Plots of the lapse function $g_{00} = -e^{V_o(z)}$ (a) and the spatial volume function $g_{xx} = g_{yy} = e^{U_o(z)}$ (b), space-time curvature $R^{(4)}(z)$ (c) and space curvature $R^{(3)}(z)$ (d) for $\Lambda_W > 0$ and $\lambda = 1$.

$z_o \pm 2\pi/(3\sqrt{\Lambda_W})$ and separated by a finite physical distance $4\pi/(3\sqrt{\Lambda_W})$. The space curvature confirm such a view:

$$R^{(3)} = -3\Lambda_W\left[-1+\tan\left(\frac{3}{4}\sqrt{\Lambda_W}(z-z_o)\right)^2\right] \underset{z\to z_o\pm 2\pi/(3\sqrt{\Lambda_W})}{\longrightarrow} -\infty, \tag{28}$$

being again singular at the boundaries, while at the center it is positive and finite, proportional to Λ_W.

Plots of the solutions together with their spatial and space-time curvatures can be seen in Fig. 4.

2.2.2. Solutions with $\lambda \neq 1$. Next, we move to the case $\lambda \neq 1$. Then, solving Eq. (11) for $U(z)$ and inserting the solution into Eq. (12) to get $V(z)$, we have

$$e^{U_\pm(z)} = \cos\left(\frac{\sqrt{\Lambda_W}}{p_\pm(\lambda)}(z-z_o)\right)^{2p_\pm(\lambda)},$$
$$e^{V_\pm(z)} = \left[\cos\left(\frac{\sqrt{\Lambda_W}}{p_\pm(\lambda)}(z-z_o)\right)^{\frac{5p_\pm(\lambda)-4\lambda-2}{3p_\pm(\lambda)-2}} \times \sin\left(\frac{\sqrt{\Lambda_W}}{p_\pm(\lambda)}(z-z_o)\right)\right]^2, \tag{29}$$

where, as before, z_o is an integration constant, and $p_\pm(\lambda)$ is given by the expression (18). Again the solution does not exists for $\lambda < 1/3$, while for $\lambda > 1/3$ we get two branches of solutions identified with the $\pm$ subindices according to the choice of the sign in $p_\pm(\lambda)$.

In both branches, the lapse function is real in the region $|z-z_o| < \pi p_\pm(\lambda)/(2\sqrt{\Lambda_W})$ and imaginary otherwise. Thus the surfaces $z = z_o \pm \pi p_\pm(\lambda)/(2\sqrt{\Lambda_W})$ determine the boundaries of space. At the boundary, the behavior of the solution depends on the value of λ. In the solution with the $+$ sign, the exponent of the cosine in the lapse function changes its sign at $\lambda = 3$, implying that at $\lambda < 3$ the lapse function vanishes at the boundary, while at $\lambda > 3$ it diverges. On the other hand, in the solution with the minus sign, the exponent of the cosine in the spatial volume function changes its sign at $\lambda = 1$, resulting in a solution whose spatial volume vanishes at the boundary at $\lambda < 1$ and diverges at $\lambda > 1$.

The space-time curvature in the above solutions reads

$$R^{(4)} = \tilde{R}^{(4)}(\lambda) + \left(R^{(4)}_{z_o}(\lambda) - \tilde{R}^{(4)}(\lambda)\right) \times \sec\left(\frac{\sqrt{\Lambda_W}(z-z_o)}{p_\pm(\lambda)}\right)^2, \tag{30}$$

where

$$\tilde{R}^{(4)}(\lambda) = \frac{-2\Lambda_W}{(2-3p_\pm(\lambda))^2 p_\pm(\lambda)^2} \times\Big[-16(1+\lambda)^2 + p_\pm(\lambda)(48(1+\lambda) - p_\pm(\lambda)(20 - 24\lambda + 3p_\pm(\lambda)(4+9p_\pm(\lambda))))\Big], \tag{31}$$

and

$$R^{(4)}_{z_o}(\lambda) = \tilde{R}^{(4)}(\lambda) - \frac{2\Lambda_W}{(2-3p_\pm(\lambda))^2 p_\pm(\lambda)^2} \times\Big[8\lambda(1+2\lambda) + p_\pm(\lambda)(-4(1+3\lambda) + p_\pm(\lambda)(14 - 24\lambda + 3p_\pm(\lambda)(-8+9p_\pm(\lambda))))\Big]. \tag{32}$$

Again it is clear that the solution is singular at the boundaries, where the function

$$\sec\left(\sqrt{-\Lambda_W}(z-z_o)/p_\pm(\lambda)\right)$$

diverges. These singularities are separated by a finite distance $\pi p_\pm(\lambda)/\sqrt{\Lambda_W}$.

Since the spatial volume function vanishes at the boundaries whenever $p_\pm(\lambda)$ is positive, in such a case the singularities are pointlike. This happens for any λ in the $+$ branch of solutions, and for $\lambda < 1$ in the $-$ branch. In the rest of the $-$ branch, that is, at $\lambda > 1$, the volume of the boundaries diverges, implying that the singularities can be interpreted as two parallel purely gravitational membranes, sitting at the boundaries of space-time. In this last case, an alternative interpretation arises from compactifying the variable z into a cylindrical topology, resulting in

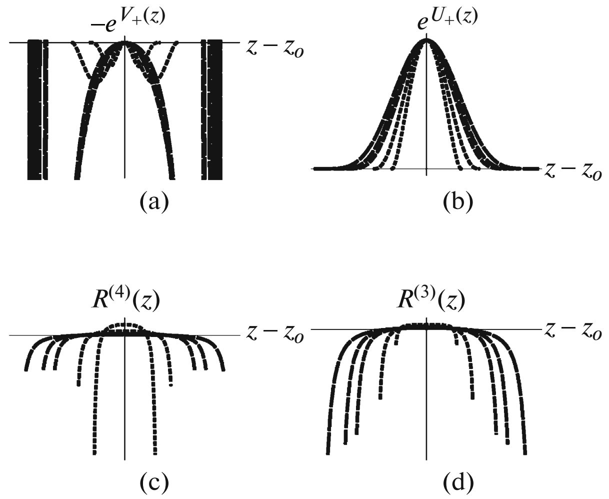

Fig. 5. Plots of the lapse function $g_{00} = -e^{V_+(z)}$ (a) and the spatial volume $g_{xx} = g_{yy} = e^{U_+(z)}$ (b), space-time curvature $R^{(4)}(z)$ (c) and space curvature $R^{(3)}(z)$ (d) for $\Lambda_W > 0$ and different values of λ. Notice the changed behavior of the lapse function at $\lambda = 3$, in figure (a).

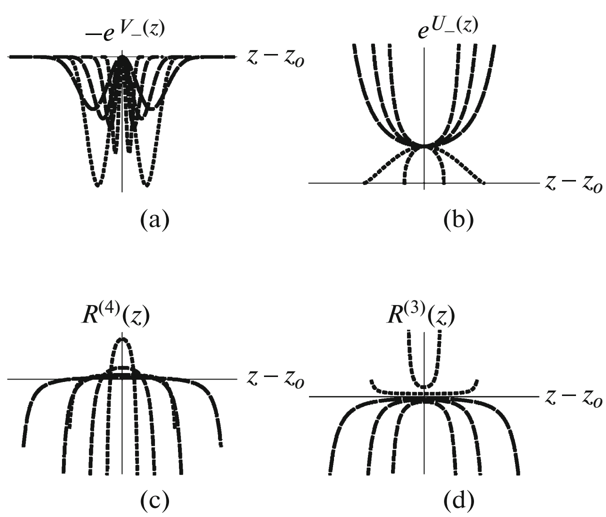

Fig. 6. Plots of the lapse function $g_{00} = -e^{V_-(z)}$ (a) and the spatial volume function $g_{xx} = g_{yy} = e^{U_-(z)}$ (b), space-time curvature $R^{(4)}(z)$ (c) and space curvature $R^{(3)}(z)$ (d) for $\Lambda_W > 0$ and different values of λ. Notice the change in the behavior of the lapse function at $\lambda = 1$ in figure (b).

identification of the two membrane-like singularities as a single membrane at the center of space.

In conclusion, the solutions of the $+$ branch consist in two point-like singularities separated by a finite distance. On the other hand, the solutions of the $-$ branch consist at $\lambda < 1$ in two pointlike singularities, and at $\lambda > 1$ in two parallel membranes or a single membrane in a compact topology.

Plots of the solutions with the $+$ sign with the corresponding space-time and space curvatures for different values of λ can be seen in Fig. 5. Similar plots for the solutions with the $-$ sign can be seen in Fig. 6.

3. DISCUSSION

We have explored membrane solutions in the Hořava nonrelativistic theory of gravity when the detailed balance condition is satisfied. We have found that at arbitrary values of the parameter $\lambda > 1/3$ there exist branches of membrane solutions.

In the particular case $\lambda = 1$ there is a single branch that corresponds to a $\mathbb{Z}_2$-symmetric space-time. In the case of a negative cosmological constant, the space-time and spatial curvatures have a peak at $z = z_o$, which allows us to identify the solution as a membrane sitting at the center of space. The curvatures become proportional to Λ_W at large distances, where the metric corresponds to an AdS space-time. On the other hand, in the case of positive cosmological constant, the space has boundaries at $|z - z_o| = 2\pi/(3\sqrt{\Lambda_W})$ beyond which the metric becomes complex. The space-time and space curvatures diverge at the boundaries. The spatial volume of the slices containing the singularities vanishes, which implies that the singularities are pointlike. On the other hand, both curvatures are finite and positive in the intermediate space.

At generic values of λ in the range $1/3 < \lambda \neq 1$, two branches appear that correspond to $\mathbb{Z}_2$-symmetric solutions. In the case of a negative cosmological constant, the space is unbounded, and the space curvature asymptotes a constant value proportional to Λ_W. The same is true for the space-time curvature, but with a proportionality factor that is a function of λ. The asymptotic metric corresponds to a Lifshitz space-time. Again, the curvatures being finitely peaked at the center of space, we interpret the solution as representing a membrane sitting there. The curvatures at the center depend on the value of λ. On the other hand, in the case of a positive cosmological constant, the space is bounded, and the curvatures are singular at the boundaries. The spatial volume of those boundaries vanishes for $\lambda < 1$ in the $-$ branch and for all values of λ in the $+$ branch, implying that the singularities are pointlike. Conversely, at $\lambda > 1$ in the $-$ branch the area of the slices containing the singularities diverges, allowing their interpretation as membranes.

It should be kept in mind that our solutions are purely gravitational solitons since no additional matter terms have been added to the Hořava action. Nevertheless, our nomenclature was inspired from the point of view of a low-energy observer, according to which the dynamics of gravity is totally covariant and given by Einstein's theory. He/she would necessarily interpret our gravitational domain wall as originated from some kind of membrane-like matter sources.

At any value of λ, there exists an additional branch. It is degenerate, in the sense that the lapse function is completely undetermined by the equations of motion. This behavior has been reported before in the case of spherically symmetric [10] and warped BTZ string [18] solutions and was related to the detailed balance condition [10].

It is interesting to note that the solutions behave analytically in the parameter λ at all values of $\lambda \neq 1$. The solutions with the $+$ sign approach the regular $\lambda = 1$ branch in the limit $\lambda \to 1$, while the solution with the $-$ sign is not analytic in that limit.

As possible continuations of this work, it may be of interest to study solutions where the detailed balance condition is softly broken, or where the action is extended to the so called "healthy version" of Hořava gravity. Moreover, a study of stability under perturbations may be interesting, as well as the inclusion of a Lifshitz scalar field with a symmetry breaking potential that could provide a topologically conserved charge.

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