Preprint typeset in JHEP style - HYPER VERSION

Deepen in the Schrödinger invariant and Logarithmic sectors of higher-curvature gravity

Eloy Ayón–Beato

Departamento de Física, CINVESTAV–IPN, Apdo. Postal 14–740, 07000, México D.F., México.
E-mail: ayon-beato-at-fis.cinvestav.mx

Gaston Giribet

Instituto de Física de Buenos Aires IFIBA - CONICET, Buenos Aires, Argentina.
E-mail: gaston-at-df.uba.ar

Mokhtar Hassaïne

Instituto de Matemática y Física, Universidad de Talca, Casilla 747, Talca, Chile.
Laboratoire de Mathématiques et de Physique Théorique, Université de Tours, Parc de Grandmont, 37200 Tours, France.
E-mail: hassaine-at-inst-mat.utalca.cl

Abstract: The aim of this paper is to explore D-dimensional theories of pure gravity whose space of solutions contains certain class of AdS-waves, including in particular Schrödinger invariant spacetimes. This amounts to consider higher order theories, and the natural case to start with is to analyze generic square-curvature corrections to Einstein-Hilbert action. In this case, the Schrödinger invariant sector in the space of solutions arises for a special relation between the coupling constants appearing in the action. On the other hand, besides the Schrödinger invariant configurations, logarithmic branches similar to those of the so-called Log-gravity are also shown to emerge for another special choice of the coupling constants. Interestingly enough, these Log solutions can be interpreted as the superposition of the massless mode of General Relativity and two scalar modes that saturate the Breitenlohner-Freedman bound (BF) of the AdS space on which they propagate. These solutions are higher-dimensional analogues of those appearing in three-dimensional massive gravities with relaxed AdS_3 asymptotic, which are candidates to be gravity duals for logarithmic CFTs. Other sectors of the space of solutions of higher-curvature theories correspond to oscillatory configurations, which happen to be below the BF bound. Also, there is a fully degenerated sector, for which any wave profile is admitted. We comment on the relation between this degeneracy and the non-renormalization of the dynamical exponent of the Schrödinger spaces. Our analysis also includes more general gravitational actions with non-polynomial corrections consisting of arbitrary functions of the square-curvature invariants. By establishing a correspondence with the quadratic gravity model, the same sectors of solutions are shown to exist for this more general family of theories. We finally consider the parity-violating Chern-Simons modified gravity in four dimensions, for which we derive both the Schrödinger invariant as well as the logarithmic sectors.

Keywords: Gauge-gravity correspondence, Higher-curvature corrections, Log gravity.

This paper is dedicated to the memory of Laurent Houart.

Contents

1. Introduction

The AdS/CFT correspondence [1, 2, 3, 4] provides a dictionary to relate conformal field theories in flat space to higher-dimensional gravitational theories. In this framework, the gravitational description is weakly coupled when the relativistic CFT is strongly coupled, and thus it becomes a promising tool to explore fundamental physics in the non-perturbative regime. However, in spite of the enormous mathematical success of AdS/CFT correspondence, the experimental applications of this idea have somewhat been braked by the fact that only few relativistic conformal field theories at strong coupling are accessible experimentally.

Nevertheless, unlike what happens in the relativistic case, there exists a plenitude of non-relativistic conformal field models that govern physics in different experimentally accessible areas such as condensed matter physics and atomic or nuclear physics. From this perspective, the idea of generalizing AdS/CFT correspondence to the case of non-relativistic conformal field theories has been proposed [5, 6]. In the non-relativistic case, the symmetry group is identified with the conformal extension of the Galilei group, often called the Schrödinger group. This identification is due to the fact that there exist some analogies between the Minkowski conformal algebra and the Schrödinger algebra in one dimension less. This analogy between both algebras has motivated the search of a geometric framework to understand the Schrödinger algebra. One pioneer work in this direction was the one done by Havas and Plebański [7], where they proposed to introduce the Schrödinger symmetry as a sub-group of the infinite-dimensional group of Galilean conformal transformations of flat spacetime. A geometric picture of the Schrödinger algebra and its relations with the conformal algebra have been explained in [8], and this result was used in [9] to derive the invariant Schrödinger metric in the context of the non-relativistic AdS/CFT correspondence. In turn, following the philosophy of the AdS/CFT holographic correspondence, the hope is that backgrounds whose (asymptotic) isometry group agrees with the Schrödinger group would represent the gravity duals of some conformal quantum mechanical systems with applications to condensed matter physics.[1] Owing to this expectation, there has been an increasing interest in constructing solutions of string theory inspired models whose (asymptotic boundary) isometry group is given by the Schrödinger group. For instance, the embedding of such solutions and their analogues at finite temperature in string theory were considered in [12, 13, 14] (see also references thereof). The effects of string inspired higher-curvature corrections of these spaces were first analyzed in [15], where it was shown that such higher order corrections lead to renormalize the dynamical exponent of the dual conformal field theory. Here, we go deeper into the discussion on higher-curvature actions and we argue that Schrödinger invariant backgrounds arise as solutions of different theories of pure gravity in four and higher dimensions. It is relatively easy to show that Schrödinger invariant spacetimes arise as solutions of Einstein gravity with negative cosmological constant, provided the support of matter fields. This can be achieved with the introduction of some reasonable physical source, like a Proca field [5] and/or an Abelian Higgs field in its broken phase [6]. However, solutions enjoying Schrödinger symmetry are also possible in theories of gravity in absence of matter. The simplest example of a *pure* gravity theory for which solutions possessing full Schrödinger symmetry have been found is three–dimensional Topologically Massive Gravity (TMG) with negative cosmological constant [16]. This theory of gravity has attracted considerable attention the last three years, and, in particular, one of the interesting features of TMG is precisely that it admits exact AdS wave solutions [17, 18, 19, 20, 21, 22]; see below for a precise definition of such spacetimes. While for generic values of the topological mass μ and the cosmological constant $\Lambda = -1/l^2$ these AdS wave solutions are only *partially* Schrödinger invariant [2], it turns out that for the special fine tuning $\mu l = 3$ the solution exhibits the *full* Schrödinger symmetry. This fine tuning corresponds to the critical point of the space of parameters at which the warped AdS_3 solution of the theory has isometry group $SL(2,\mathbb{R}) \times U(1)$ with a null $U(1)$ direction [23]. More recently, in Ref. [24], the authors of the present paper have shown the existence of Schrödinger invariant spaces for the so-called New Massive Gravity (NMG) introduced in Ref. [25].

[1] See [10] and [11] for interesting discussions on holography in Schrödinger spaces.

[2] As shown below, this corresponds to the Galilei transformations together with an anisotropic rescaling

Nevertheless, while the Schrödinger invariant solutions of TMG and NMG are interesting in their own right, their relevance concerning the non-relativistic holographic correspondence one is trying to construct is somewhat questionable. This is basically because its non-relativistic dual model would correspond to a theory with zero spatial dimensions. In turn, it is natural to ask for a higher-dimensional extension of this construction: Is there a theory of pure gravity in four (or higher) dimensions that admits solutions with Schrödinger isometry group? Actually, one can answer in the affirmative. One of the ideas of this paper is to look for such theories and go deepen into the Schrödinger invariant sector of pure gravity.

We will argue that it is actually large the class of theories of higher-curvature gravity that admits solutions with Schrödinger isometry group in four and higher dimensions. To construct a theory of pure gravity that admits exact solutions with Schrödinger isometry group in $D > 3$ dimensions, one can follow two different strategies: The first one is resorting to the results of Ref. [15]. There, it was shown that the inclusion of higher-curvature terms in the gravitational action leads to renormalize the dynamical exponent of the Schrödinger symmetric spaces (this exponent is usually denoted by $z = \nu+1$). Then, one could in principle make inverse engineering and use the running equation that relates the coupling constants of the gravitational Lagrangian to the dynamical exponent ν in order to design the model. Here, instead, we will follow a rather different strategy. We will scan a wider class of higher-order gravitational Lagrangians and exhaustively analyze the sectors of solutions that belong to a special type of Siklos spacetimes. This will lead us to find several sectors of the space of solutions including, in particular, Schrödinger spaces, together with logarithmic solutions similar to those found in three dimensions.

The study of the higher-dimensional case starts in Sec. 3 with an analysis of the most general modified theory of gravity with quadratic dependence on the curvature. We find the explicit configurations for generic dimension, which allow us to identify several different sectors, apart from the Schrödinger invariant one. For example, there are critical points in the space of coupling constants for which a full degeneracy arises and the field equations are not only satisfied for any dynamical exponent z but even for any AdS-wave profile function F. The corresponding set of theories includes (but is not limited to) the usual degenerate case of Chern-Simons gravity in $D = 5$. In analogy with the three-dimensional configurations of TMG [17, 20] and NMG [24],[3] we observe also the existence of other critical values of the coupling constants for which Schrödinger invariance is broken by the appearance of logarithmic behaviors associated to the existence of scalar wave modes that saturate the Breitenlohner-Freedman bound (BF) of the AdS_D space on which the wave propagates. Some of these logarithmic decays are candidate to relax the usual AdS boundary conditions [28, 29, 30] in the context of theories with higher-curvature corrections, in perfect analogy with what occurs in three-dimensional massive gravities [31, 32].

In Sec. 4, we further generalize the results of section 3 by studying non-polynomials gravity modifications which depend arbitrarily on the squared-curvature invariants; this is

[3]See Refs. [26, 27] for similar configurations appearing in bi-gravity and Born-Infeld gravity.

done by establishing a correspondence between such a theory and the one with standard quadratic modifications. Since Siklos spacetimes are conformally equivalent to pp-waves, the metric configurations turn out to be persistent when a wide class of higher-curvature corrections are included, provided a suitable redefinition of the parameters. Within this context, and even if the existence of Schrödinger invariant sectors is not specially related to conformal invariance, we find illustrative to explore in Sec. 5 the particular example of adding to the D-dimensional Einstein-Hilbert action the conformally invariant deformation proportional to $(C_{\mu\nu\alpha\beta}C^{\mu\nu\alpha\beta})^{D/4}$, where $C_{\mu\nu\alpha\beta}$ is the Weyl tensor. Interestingly enough, we observe that only in four dimensions the Schrödinger invariant sector, the logarithmic sector, and the sector below the BF bound are genuinely new configurations, while in arbitrary $D > 4$ the resulting solutions turn out to coincide with those of General Relativity. We will also consider a rather different model of higher-curvature gravity in Sec. 6, given by the parity-violating Chern-Simons modification of General Relativity proposed by Jackiw and Pi [33], and we will show that Schrödinger invariant and Log configurations are also admitted as solutions in this theory. Finally, Section 7 is devoted to our conclusions and further prospect. We include an Appendix describing the behavior of higher-order curvature terms in presence of AdS waves.

Note added. When our paper was being prepared for publication, Refs. [34] and [35] appeared, which consider some of the configurations studied in this work, including the relevant ones for the recently proposed critical gravity theories [36, 37].

2. Schrödinger isometry group and the Siklos spacetimes

For the presentation to be self-contained, let us begin by reviewing some aspects of the Schrödinger group. The Schrödinger group has been defined in [38, 39, 40] as the largest group of space-time transformations which leaves the Schrödinger equation for a free particle invariant. Schrödinger invariance has been considered in a wide variety of situations, including celestial mechanics [8], non-relativistic field theory [41, 42], non-relativistic quantum mechanics [43], and hydrodynamics [44, 45, 46, 47, 48]. Mathematical aspects of the Schrödinger symmetry have been analyzed, for instance, in [49]. The Schrödinger group can be viewed as the semi-direct product of the connected static Galilei group together with the $\mathrm{SL}(2,\mathbb{R})$ group, which includes time translation, dilatation, and special conformal transformations. This is an extension of the Lifshitz group, which is also considered in the holographic description of non-relativistic models [50].

As mentioned, the main idea of the proposal for a non-relativistic version of the AdS/CFT correspondence [5, 6] is to consider a metric whose isometry is given by the non-relativistic conformal Schrödinger symmetry. In this context, we will be concerned with the following class of metrics

$$ds^2 = \frac{l^2}{r^2}\Big(- \frac{dt^2}{r^{2\nu}} + 2dtd\xi + dr^2 + d\vec{x}^2\Big), \tag{2.1}$$

where $\vec{x}$ is a d-dimensional vector, l is a constant associated to the curvature of the space, and ν is the so-called dynamical exponent (this is also usually denoted by $z = \nu+1$). For $\nu = 0$, the metric (2.1) corresponds to anti-de Sitter spacetime and enjoys the full relativistic conformal symmetry. For an arbitrary value of the exponent ν, this metric exhibits as isometries the Galilei transformations on the space $(t, \vec{x})$ as well as the dilatations transformations. Indeed, apart from the spacetime translations $(t, \vec{x}) \mapsto (t + b, \vec{x} + \vec{\delta})$ and spatial rotations in the $\vec{x}$-planes, $\vec{x} \mapsto R\,\vec{x}$ with $R \in \mathrm{SO}(d)$, the metric (2.1) is invariant under the Galilean boosts

$$\varphi_{\vec{v}}(t, \xi, r, \vec{x}) = (t, \xi + \vec{v}\cdot\vec{x} - \frac{1}{2}|\vec{v}|^2 t, r, \vec{x} - t\vec{v}), \tag{2.2}$$

as well as under dilations

$$\varphi_a(t, \xi, r, \vec{x}) = (e^{(1+\nu)a}\, t, e^{(1-\nu)a}\, \xi, e^a\, r, e^a\, \vec{x}). \tag{2.3}$$

For the special value $\nu = 1$ of the dynamical exponent, the metric admits additionally a special conformal transformation given by the diffeomorphism

$$\varphi_\kappa(t, \xi, r, \vec{x}) = \left(\frac{t}{1-\kappa t}, \xi - \frac{\kappa(|\vec{x}|^2 + r^2)}{2(1-\kappa t)}, \frac{r}{1-\kappa t}, \frac{\vec{x}}{1-\kappa t}\right), \tag{2.4}$$

and it is only in this case that the metric is said to enjoy the full Schrödinger symmetry.

The symmetry transformations above are generated for each of its infinitesimal parameters ϵ by a Killing vector K in the standard way, namely $\varphi_\epsilon = \varphi_0 + i\epsilon K + O(\epsilon^2)$, which gives rise to the following set of Killing vectors

$$\begin{gathered} M_{ij} = -i(x_i\partial_j - x_j\partial_i), \qquad K_i = -i(x_i\partial_\xi - t\partial_i), \\ P_i = -i\partial_i, \qquad H = -i\partial_t, \qquad N = -i\partial_\xi, \\ D = -i[(1+\nu)t\partial_t + (1-\nu)\xi\partial_\xi + r\partial_r + x^i\partial_i]. \end{gathered} \tag{2.5}$$

For $\nu = 1$ (i.e. $z = 2$), we find the additional special conformal generator

$$C = -i\left(t^2\partial_t - \frac{|\vec{x}|^2 + r^2}{2}\partial_\xi + tr\partial_r + tx^i\partial_i\right). \tag{2.6}$$

These Killing vectors realize the algebra whose non-vanishing commutation relations are

$$\begin{gathered} [M_{ij}, M_{kl}] = i(\delta_{ik}M_{jl} - \delta_{jk}M_{il} + \delta_{il}M_{kj} - \delta_{jl}M_{ki}), \\ [M_{ij}, P_k] = i(\delta_{ik}P_j - \delta_{jk}P_i), \qquad [M_{ij}, K_k] = i(\delta_{ik}K_j - \delta_{jk}K_i), \\ [P_i, K_j] = -i\delta_{ij}N, \qquad [H, K_i] = iP_i, \qquad [D, P_i] = iP_i, \\ [D, K_i] = -i\nu K_i \qquad [D, H] = i(1+\nu)H, \qquad [D, N] = i(1-\nu)N. \end{gathered} \tag{2.7}$$

Besides, when $\nu = 1$,

$$[D, C] = -2iC, \qquad [H, C] = -iD, \tag{2.8}$$

and in this case the algebra corresponds to the Schrödinger algebra.

Notice that the case $\nu = 1$ is special not only because it allows for the additional special conformal generator C, but also because the generator N becomes a central element of the algebra, as it commutes additionally with the dilation generator D in this case.

In what follows, we will refer to the set of transformations involving the Galilei transformations M_{ij}, K_i P_i, H, N and the dilatations D as the *partial* Schrödinger symmetry. In the applications to condensed matter physics, each conformal system turns out to be characterized by the value of ν its symmetry corresponds to. This helps to identify the candidates to be the corresponding gravity duals of the form (2.1). For example, models describing itinerant (anti)ferromagnetic materials are thought to be described by the model with $\nu = 2$ (resp. with $\nu = 1$).

Here, we are mainly interested in the class of metrics admitting the full (or partial) Schrödinger symmetry (2.1). Nevertheless, from the gravity viewpoint, it is interesting to consider first a more general (less symmetric) setting. This is important to understand what is the appropriate setup these configurations arise in. With this motivation, we consider an ansatz of the following form

$$ds^2 = \frac{l^2}{r^2}\left[-F(r)dt^2 + 2dtd\xi + dr^2 + d\vec{x}^2\right], \tag{2.9}$$

where F is the only undetermined structural metric function, and it only depends on the coordinate r. This ansatz corresponds to a particular class of the so-called Siklos spacetimes [51]. To be more precise, Siklos spacetimes correspond to metric (2.9) with a function F that depends on all the variables except the null coordinate ξ. In particular, for a vanishing structural function $F = 0$, we recover the metric of anti-de Sitter space in Poincaré coordinates, while for $F \ll 1$ this metric describes just a perturbation of AdS. In fact, the metric (2.9) and, more generally, the Siklos spacetimes, can also be obtained from the AdS one by a generalized Kerr-Schild transformation (see Appendix). Consequently, they can be though as describing exact gravitational waves propagating along the AdS spacetime [52] (AdS-waves). In fact, they become the particular case, admitting a Killing field, of the more general exact gravitational waves propagating in presence of a cosmological constant originally found by García and Plebański [53].[4]

In the context of higher-order gravity theories it has been observed in many cases that the on-shell profile function F of exact gravitational waves behaves as an exact scalar massive mode [57, 20, 24], since it satisfies a Klein-Gordon equation

$$\Box F = m^2 F, \tag{2.10}$$

for some effective mass m, and where $\Box$ stands for the d'Alambertian operator. For example, in the case of the profile F defining the spacetimes (2.1), the corresponding mass is defined in terms of the dynamical exponent as

$$m^2 = \frac{2\nu(2\nu + D - 1)}{l^2}, \tag{2.11}$$

[4]See Refs. [54, 55, 56] for further generalizations.

and this will allow us to establish a correspondence between many of the gravity configurations we study and exact scalar massive modes propagating on these backgrounds.

3. Square-curvature corrections in arbitrary dimensions

The existence of an abundant Schrödinger symmetric sector in higher-dimensional pure gravity becomes clear from the analysis of Ref. [15] where, besides their general arguments, an explicit example was worked out for a particular five-dimensional quadratic modification to the Einstein-Hilbert action. Here, we provide explicitly the general configurations for arbitrary dimension. In particular, this will lead us to observe the existence of special critical points in the space of coupling constants at which logarithmic dependences that necessarily break Schrödinger symmetry appear. Such logarithmic falling-off that emerges at these special points potentially leads to the definition of weakened asymptotically AdS boundary conditions. The results of this section can be thought of as a higher-dimensional generalization of the recent results of Ref. [24] for the three-dimensional parity-preserving massive gravity introduced in Ref. [25].

Unlike the three or four-dimensional cases, in higher dimensions, three different invariants have to be used to write the most general quadratic action; namely

$$S[g_{\mu\nu}] = \int d^D x\sqrt{-g}\left(R - 2\lambda + \beta_1 R^2 + \beta_2 R_{\alpha\beta}R^{\alpha\beta} + \beta_3 R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\right). \tag{3.1}$$

Here, we denote the cosmological constant by λ for reasons that will become clear later. The constants β_i are the coupling constants for the different curvature square modifications. The field equations obtained by varying the action (3.1) with respect to the metric read

$$\begin{aligned} G_{\mu\nu} + \lambda g_{\mu\nu} + (\beta_2 + 4\beta_3)\,\Box R_{\mu\nu} + \frac{1}{2}(4\beta_1 + \beta_2)\, g_{\mu\nu}\Box R - (2\beta_1 + \beta_2 + 2\beta_3)\,\nabla_\mu\nabla_\nu R \\ + 2\beta_3 R_{\mu\gamma\alpha\beta}R_\nu{}^{\gamma\alpha\beta} + 2(\beta_2 + 2\beta_3)\, R_{\mu\alpha\nu\beta}R^{\alpha\beta} - 4\beta_3 R_{\mu\alpha}R_\nu{}^\alpha + 2\beta_1 R R_{\mu\nu} \\ - \frac{1}{2}\left(\beta_1 R^2 + \beta_2 R_{\alpha\beta}R^{\alpha\beta} + \beta_3 R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\right) g_{\mu\nu} = 0. \end{aligned} \tag{3.2}$$

Before deriving the different class of solutions, we first fix the cosmological constant λ such that the AdS spacetime of radius l is a solution of the equations (3.2). By doing so, we find the following constraint between the cosmological constant λ, the AdS radius l, and the coupling constants β_i

$$\lambda = -\frac{(D-1)(D-2)}{2l^2} + \frac{(D-1)(D-4)}{2l^4}\left[(D-1)(D\beta_1 + \beta_2) + 2\beta_3\right]. \tag{3.3}$$

From this, we notice that only in four dimensions the cosmological constant is related to the AdS radius in the usual way, without involving the couplings β_i.

With the choice (3.3) for the cosmological constant, Einstein equations (3.2) for the AdS waves (2.9) yield the following equations (see details in the Appendix)

$$\Big\{(\beta_2+4\beta_3)\left[r^4F'''' - 2(D-4)r^3F'''\right]$$
$$+\left[l^2-2D(D-1)\beta_1+(D-2)(D-8)\beta_2+4(D-2)(D-5)\beta_3\right]r^2F''$$
$$-(D-2)\left[l^2-2D(D-1)\beta_1-(3D-4)\beta_2-8\beta_3\right]rF'\Big\}\frac{\delta_\mu^t\delta_\nu^t}{2l^2r^2}=0. \qquad (3.4)$$

In what follows, we provide a detailed analysis of the equation (3.4) in order to survey the different solutions exhaustively.

3.1 The second order sector

The first case we will explore is the one for which the fourth-order differential equation (3.4) reduces to a second-order equation. This occurs for the special election $\beta_2=-4\beta_3$, yielding

$$\frac{1}{l^2}\left[l^2-2D(D-1)\beta_1+12(D-2)\beta_3\right]\left[r^2F''-(D-2)rF'\right]=0. \qquad (3.5)$$

It is clear from this equation that, if additionally one chooses $\beta_1=[l^2+12(D-2)\beta_3]/[2D(D-1)]$, a full degeneracy appears and the field equations are satisfied for any profile F; this degenerate class will be analyzed in details in Subsec. 3.4. Then, for $\beta_1\neq[l^2+12(D-2)\beta_3]/[2D(D-1)]$ the resulting equations are of second order as in the Lovelock case [58]; actually, the Lovelock theory which corresponds to $\beta_2=-4\beta_3$ and $\beta_1=\beta_3$, appears as a particular case in this analysis. More precisely, for the above inequality (up to an additive constant that can be removed by coordinate transformations) the solution is given by

$$F(r)=c_0r^{D-1}, \qquad (3.6)$$

and coincides with the solution of General Relativity, i.e. the one with all the constants $\beta_i=0$. This solution can be thought of as the usual General Relativity exact massless scalar mode since the field equation (3.5) is proportional to the wave equation

$$\Box F=0. \qquad (3.7)$$

It is also interesting to note that the solution (3.6) preserves only the *partial* Schrödinger symmetry and not the *full* one. In the next subsection, we shall establish explicitly the existence of a higher order sector enjoying the *full* Schrödinger symmetry.

3.2 The Schrödinger invariant sector

For $\beta_2\neq-4\beta_3$, the field equation (3.4) is a fourth-order Euler differential equation. In the generic case, the space of linearly independent solutions is spanned in power-laws $F\propto r^\alpha$, where the exponents α are the roots of the following fourth-degree characteristic polynomial

$$\alpha(\alpha-D+1)\left[\left(\alpha-\frac{D-1}{2}\right)^2-\frac{(D-1)^2}{4}-\frac{l^2-2(D-1)(D\beta_1+\beta_2)+4(D-4)\beta_3}{\beta_2+4\beta_3}\right]=0. \qquad (3.8)$$

Since the constant solution, i.e. $\alpha = 0$, can be removed by coordinate transformations, the general solution is then given by

$$F(r) = c_0 r^{D-1} + c_+ r^{\alpha_+} + c_- r^{\alpha_-}, \tag{3.9a}$$

where

$$\alpha_\pm = \frac{D-1}{2} \pm \left(\frac{(D-1)^2}{4} + \frac{2(D-1)(D\beta_1+\beta_2) - 4(D-4)\beta_3 - l^2}{\beta_2 + 4\beta_3} \right)^{1/2}, \tag{3.9b}$$

and where c_0 and $c_\pm$ are integrations constants. It is worth noticing that the solutions (3.9) generate an exact scalar massive excitation (2.10) of mass

$$m^2 = \frac{2(D-1)(D\beta_1+\beta_2) - 4(D-4)\beta_3 - l^2}{l^2(\beta_2+4\beta_3)}. \tag{3.10}$$

To be more precise, the solution (3.9) describes the superposition of three exact scalar modes given by the massless mode of General Relativity (3.6) and two other modes generated by the squared modifications and both sharing the same mass (3.10). It is worth pointing out that the solution (3.9) is valid only when the roots (3.9b) are real. This in turn constraints the mass (3.10) to obey strictly the Breitenlohner-Freedman bound associated to the AdS space where the waves are propagating on [59, 60]

$$m^2 > -\frac{(D-1)^2}{4l^2}. \tag{3.11}$$

It is also easy to see that taking any pair of the integrations constants in (3.9) to zero, the isometry group of the resulting background gets enhanced to the *partial* Schrödinger group. The *full* Schrödinger isometry can only be achieved by choosing $c_0 = 0$ and $c_+ = 0$ while the coupling constants must be constrained by

$$\beta_2 = \frac{D(D-1)}{2}\beta_1 - (3D-2)\beta_3 - \frac{l^2}{4}, \tag{3.12}$$

and this value corresponds to a mass (3.10) given by (see also (2.11) for $\nu = 1$)

$$m^2 = \frac{2(D+1)}{l^2}. \tag{3.13}$$

3.3 The Logarithmic sectors

We now turn to the cases for which the roots of the characteristic polynomial (3.8) may have some multiplicities. As it is well-known, for multiple roots the power laws fail to span all the linearly independent solutions to Eq. (3.4) and additional behaviors exhibiting logarithmic dependence typically occur. The existence of such *exact* logarithmic behaviors has been established for TMG in Refs. [17, 20]. Moreover, some of the logarithmic configurations of TMG have been shown to be compatibles with some relaxed AdS asymptotic [31, 32] and define a sector of the theory currently known as the Log Gravity sector [61]; see also [62].

The relevance of this sector is that, in three dimensions, it may be holographically dual at the quantum level to a Logarithmic CFT [63, 64, 65]. We have shown in [24] that a similar *exact* Log sector exists also within the context of NMG.[5]

The first source of multiplicity appears when the roots (3.9b) become one single root, which occurs for

$$\beta_2=\frac{4\left[l^2-2D(D-1)\beta_1-(D^2-6D+17)\beta_3\right]}{(D+7)(D-1)}, \tag{3.14}$$

and the related double multiplicity solution turns to be

$$F(r)=c_0r^{D-1}+r^{\frac{D-1}{2}}\left(c_1\ln r+c_2\right). \tag{3.15}$$

For $c_1=0$, two residual sectors having the *partial* Schrödinger symmetry still remain, taking either $c_0=0$ or $c_2=0$. In contrast, the *full* Schrödinger symmetry is forbidden for all sectors of the solution.

The global interpretation of the solution (3.15) is also of interest. The configuration represents the superposition of the massless mode of GR plus two additional exact scalar modes saturating the BF bound of the AdS space where the waves are propagating on [59, 60], since the related profile satisfies

$$\Box F=m_{\rm BF}^2F,\qquad m_{\rm BF}^2\equiv-\frac{(D-1)^2}{4l^2}. \tag{3.16}$$

Notice that this value for the mass is included in the range (3.10) with β_2 given by (3.14).

The other multiplicities may appear when one of the two generic roots (3.9b) either vanishes or takes the value $D-1$. In fact, these two possibilities occur simultaneously and in this case the coupling constants must be restricted as follows

$$\beta_2=\frac{l^2-2D(D-1)\beta_1+4(D-4)\beta_3}{2(D-1)}. \tag{3.17}$$

The solution with simultaneous double multiplicity is then of the form

$$F(r)=c_0r^{D-1}+\left(c_1r^{D-1}+c_2\right)\ln r. \tag{3.18}$$

At this point of the space of parameters there is no sector compatible with the *partial* Schrödinger invariance except the trivial mode of General Relativity, i.e. $c_1=c_2=0$. In addition of being incompatible with the Schrödinger symmetry, the modes generated by these higher-curvature modifications can not be interpreted as exact scalar modes as they do not satisfy the Klein-Gordon equation. However, interesting enough, the profile (3.18) can be understood as a "local" superposition

$$F=r^{\frac{D-1}{2}}\left[\frac{1}{r^{\frac{D-1}{2}}}F|_{c_2=0}\right]+\frac{1}{r^{\frac{D-1}{2}}}\left[r^{\frac{D-1}{2}}F|_{c_0=0,c_1=0}\right], \tag{3.19}$$

of exact massive scalar modes saturating the BF bound [those between brackets].

[5]See Refs. [66, 67] for the linearized case and Refs. [26, 27] for the case of other theories as bi-gravity and Born-Infeld gravity. Log gravity was also studied in higher dimensions recently [68].

3.4 Below the Breitenlohner-Freedman bound

As it was mentioned in Subsec. 3.2 the generic solution (3.9) is only valid for mass values above the Breitenlohner-Freedman bound [59, 60], which constraints the scalar modes propagating on AdS. However, the modes we consider here are of gravitational nature and there are it a priori no reasons for which they must obey the celebrated BF bound. Hence, the sector having a mass below the Breitenlohner-Freedman bound (3.16),

$$m^2 < m_{\rm BF}^2, \tag{3.20}$$

might be considered as well, as it has been done in [69] in the context of non-relativistic holographic correspondence. In this case, the roots (3.9b) take complex conjugate values and the solution acquires the following oscillatory behavior

$$F(r) = c_0 r^{D-1} + r^{\frac{D-1}{2}} \left[c_1 \sin\left(l\sqrt{m_{\rm BF}^2 - m^2}\ln r \right) + c_2 \cos\left(l\sqrt{m_{\rm BF}^2 - m^2}\ln r \right) \right], \tag{3.21}$$

where m^2 is again given by Eq. (3.10).

3.5 The degenerate sector

Now, let us discuss the degeneracy in space of solutions and its relation with the non-renormalization of the dynamical exponent ν. As it was previously mentioned, it is remarkable that when the coupling constants are tied in the following manner

$$\beta_1 = \frac{l^2 + 12(D-2)\beta_3}{2D(D-1)}, \qquad \beta_2 = -4\beta_3, \tag{3.22}$$

the metrics (2.9) solves the equations of motion (3.2) for any wave profile $F(r)$. In particular, solutions with *full* Schrödinger symmetry are admitted in this case. The specific theories allowing this kind of degeneracy are described by the following Lagrangian

$$R - 2\lambda + \beta_3 \mathcal{L}_{\rm GB} + \frac{l^2 - 2(D-3)(D-4)\beta_3}{2D(D-1)} R^2, \tag{3.23}$$

where $\mathcal{L}_{\rm GB} = R^2 - 4R_{\alpha\beta}R^{\alpha\beta} + R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ is the usual quadratic Gauss-Bonnet Lagrangian, and the cosmological constant is fixed as

$$\lambda = -\frac{(D-1)}{4l^2}\left(D - \frac{4(D-3)(D-4)}{l^2}\beta_3 \right). \tag{3.24}$$

There are two interesting cases included within this family of degenerate theories. The simplest one is achieved for $\beta_3 = 0$ and is described by

$$R - 2\lambda - \frac{1}{8\lambda} R^2. \tag{3.25}$$

Such fine-tuning in the coupling constant of the quadratic term is crucial in order to obtain *full* Schrödinger invariant configurations and more general profiles; for any other coupling constant the theory belongs to the ones considered in Subsec. 3.1 and the *full* Schrödinger invariance is forbidden. It is interesting to note that the Lagrangian (3.25) is the precise combination that allows the existence of Lifshitz black holes [70, 71] for the gravity theory with R^2-corrections. It is also remarkable that for such a fine-tuning of the coefficients, the Lagrangian, which adopts the $f(R)$-form, can not be reduced to a scalar-tensor theory through the standard frame-changing trick [71].

Another interesting case occurs for

$$\beta_3 = \frac{l^2}{2(D-3)(D-4)},$$

and in this case the Lagrangian becomes

$$R - 2\lambda - \frac{(D-1)(D-2)}{8(D-3)(D-4)\lambda}\mathcal{L}_{\mathrm{GB}}. \tag{3.26}$$

This last case includes the point of the space of parameters where the Einstein-Gauss-Bonnet gravity coincides with the Chern-Simons gravity in $D = 5$ [72], that is $\beta_1 = -\beta_2/4 = \beta_3 = -3/(4\lambda)$. At this point, the theory exhibits local gauge invariance under the AdS group $\mathrm{SO}(4,2)$. A particular feature of Chern-Simons (CS) gravity is that its space of solutions is highly degenerate, and thus it is not necessarily surprising that all the metrics (2.9) solve the equations of motion.[6] It is worth pointing out that the degeneracy that appears at $\beta_1 = -\beta_2/4 = \beta_3 = -3/(4\lambda)$ in $D = 5$ is exactly what happens in the case $c = 0$ and $a\Lambda = 3/4$ of Ref. [15] (see Eq. (2.25) therein), where degenerate solutions arise.

We have confirmed by explicit calculation that a similar feature is found in arbitrary number of dimensions as long as one choose the coupling constants for the Lovelock Lagrangian to exhibit local gauge invariance. More precisely, if one considers the theory defined by the action

$$\int d^D x\sqrt{-g}\sum_{n=0}^{[\frac{D-1}{2}]}\frac{\beta_n}{2^n n!}\delta^{\mu_1}_{[\sigma_1}\delta^{\nu_1}_{\rho_1}\dots\delta^{\mu_n}_{\sigma_n}\delta^{\nu_n}_{\rho_n]}\prod_{r=1}^{n}R^{\sigma_r\rho_r}{}_{\mu_r\nu_r}, \tag{3.27}$$

there always exists a special choice of coupling constants β_1, β_2, β_3, $\dots$, such that the above Lagrangian can be written as a Chern-Simons form in odd dimensions [72] and as a Pfaffian form in even dimensions. For the theories defined with such fine tuning, it turns out that the metric (2.1) is admitted as solution for arbitrary ν. In the language of [15] this would mean that, if such a precise tuning between coupling constants β_i is considered, the dynamical exponent $z = \nu + 1$ does not get renormalized. Here, we point out that this non-renormalization is associated to the enhancement of local (AdS) symmetry at the *Chern-Simons point*. The question remains whether all the cases (3.22) exhibit enhancement of symmetry that permits to explain the non-renormalization of the dynamical exponent in a natural way. To answer this question one has to be reminded of the fact that the enhancement of symmetry that occurs at the Chern-Simons point may also explain the existence of other

[6] In fact, a similar behavior occurs for statics configurations [73].

degenerate points in the moduli space. That is, the Schrödinger invariant backgrounds are such that changes in the values of β_i may be absorbed in a redefinition of the parameters. In turn, the family of theories that admit solutions with arbitrary ν would be parameterized by those changes of the couplings β_i that induce renormalization of l leaving ν unchanged.

It is worth mentioning that the AdS wave-like solutions we found here also appear in other theories with higher-curvature (and not only square-curvature) terms. A particular case is the family of theories considered in [74]. In the next section we will consider another type of theories which consists of non-polynomial corrections of the invariants.

4. Non-polynomial corrections

In this section we will extend our previous results by considering a gravity theory including modifications that are more general than the quadratic curvature terms discussed above. In fact, we will consider the most general action depending on the three curvature invariants R, $R_{\alpha\beta}R^{\alpha\beta}$ and $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, namely[7]

$$S[g_{\mu\nu}] = \int d^D x \sqrt{-g} \left[R - 2\tilde{\lambda} + f\left(R, R_{\alpha\beta}R^{\alpha\beta}, R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} \right) \right], \tag{4.1}$$

where f is a smooth function of the three quadratic curvature invariants.

Now, let us denote the cosmological constant by $\tilde{\lambda}$ to emphasize the difference with the previous case (3.1). The field equations obtained by varying the action (4.1) give rise to fourth order equations[8]

$$\begin{aligned} G_{\mu\nu} + \tilde{\lambda} g_{\mu\nu} - \frac{f}{2} g_{\mu\nu} + \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu + R_{\mu\nu} \right) f_1 + \Box\left(f_2 R_{\mu\nu} \right) + 2 f_2 R_{\mu\alpha\nu\beta} R^{\alpha\beta} \\ + \frac{1}{2} g_{\mu\nu} \nabla_\alpha \left(2R^{\alpha\beta}\nabla_\beta f_2 + f_2 \nabla^\alpha R \right) - \nabla_{(\mu} \left(2R_{\nu)}{}^{\alpha}\nabla_\alpha f_2 + f_2 \nabla_{\nu)} R \right) \\ + 2 f_3 \left(2\Box R_{\mu\nu} - \nabla_\mu\nabla_\nu R + R_{\mu\gamma\alpha\beta} R_\nu{}^{\gamma\alpha\beta} + 2 R_{\mu\alpha\nu\beta}R^{\alpha\beta} - 2 R_{\mu\alpha} R_\nu{}^\alpha \right) \\ + 4 \left(R_{\mu\alpha\nu\beta}\nabla^\beta + 2\nabla_\alpha R_{\mu\nu} - 2\nabla_{(\mu} R_{\nu)\alpha} \right) \nabla^\alpha f_3 = 0, \end{aligned} \tag{4.2}$$

where

$$f_1 \equiv \frac{\partial f}{\partial R}, \qquad f_2 \equiv \frac{\partial f}{\partial R_{\alpha\beta}R^{\alpha\beta}}, \qquad f_3 \equiv \frac{\partial f}{\partial R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}}. \tag{4.3}$$

We will now argue that in the particular case of AdS-waves (2.9) the previous equations can be translated into the equations arising from a square-curvature modification (3.2) as studied in the previous section. Notice that the curvature invariants of an AdS-wave in any dimension

[7] It can be seen that similar results are obtained if one allows for more general modifications, like Lagrangians with functions $f(X)$ of other invariants like $X = R_{\alpha\mu\beta\nu}R^{\alpha\beta}R^{\mu\nu}$.

[8] It is also possible to consider gravity actions depending on invariants constructed with orther-kth derivatives of the curvature. In this case the resulting equations would be of order $2(k+2)$.

are independent of the specific profile function F and thus are specified by the AdS space constant invariants

$$R=-\frac{D(D-1)}{l^2},\qquad R_{\alpha\beta}R^{\alpha\beta}=\frac{D(D-1)^2}{l^4},\qquad R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}=\frac{2D(D-1)}{l^4}. \tag{4.4}$$

This in turn implies that both the function f and its derivatives f_i, $i=1,2,3$, evaluated on AdS-waves solutions (conformally pp-waves) become constants denoted by

$$\tilde{f}\equiv f|_{ds^2_{\rm AdS}}\,,\qquad \tilde{f}_i\equiv f_i|_{ds^2_{\rm AdS}}\,. \tag{4.5}$$

These two properties imply that the resulting equations coincide exactly with those of the square-modified gravity (3.2) with the following identifications

$$\beta_1=\frac{\tilde{f}_1}{2R}=-\frac{l^2\tilde{f}_1}{2D(D-1)},\qquad \beta_2=\tilde{f}_2,\qquad \beta_3=\tilde{f}_3, \tag{4.6}$$

while the cosmological constant is given by

$$\lambda=\tilde{\lambda}-\frac{\tilde{f}}{2}-\frac{D(D-1)}{4l^2}\left\{\tilde{f}_1-\frac{2}{l^2}\left[(D-1)\tilde{f}_2+2\tilde{f}_3\right]\right\}. \tag{4.7}$$

The relations above establish a correspondence between the configurations of square-modified gravity and those considered in (4.1). Starting from this observation, we can easily summarize some properties relative to this generic theory of modified gravity. The first observation is that an AdS space of radius l is a vacuum configuration of the generalized modified gravity (4.1) if the cosmological constant is constrained to be

$$\tilde{\lambda}=-\frac{(D-1)(D-2)}{2l^2}+\frac{\tilde{f}}{2}+\frac{(D-1)}{l^2}\left\{\tilde{f}_1-\frac{2}{l^2}\left[(D-1)\tilde{f}_2+2\tilde{f}_3\right]\right\}. \tag{4.8}$$

Being the cosmological constant fixed by (4.8), we can continue our analysis following the lines of the previous section. For example, the second-order sector of this theory is found for

$$\tilde{f}_1\neq-1-\frac{12(D-2)\tilde{f}_3}{l^2},\qquad \tilde{f}_2=-4\tilde{f}_3, \tag{4.9}$$

while the full degeneracy is obtained for

$$\tilde{f}_1=-1-\frac{12(D-2)\tilde{f}_3}{l^2},\qquad \tilde{f}_2=-4\tilde{f}_3. \tag{4.10}$$

In contrast, if $\tilde{f}_2\neq-4\tilde{f}_3$, the field equations are of fourth order, and the generic configurations turn out to be the mode superposition (3.9), where the roots defining the power-law massive modes are now given by

$$\alpha_{\pm}=\frac{D-1}{2}\pm\left(\frac{(D-1)^2}{4}+\frac{-l^2(1+\tilde{f}_1)+2(D-1)\tilde{f}_2-4(D-4)\tilde{f}_3}{\tilde{f}_2+4\tilde{f}_3}\right)^{1/2}, \tag{4.11}$$

and the mass associated to the scalar excitation F reads

$$m^2 = \frac{-l^2(1+\tilde{f}_1) + 2(D-1)\tilde{f}_2 - 4(D-4)\tilde{f}_3}{l^2(\tilde{f}_2 + 4\tilde{f}_3)}. \tag{4.12}$$

These configurations contain sub-sectors which have *partial* Schrödinger symmetry, exhibiting the *full* symmetry only for

$$\tilde{f}_2 = -\frac{l^2}{4}(1+\tilde{f}_1) - (3D-2)\tilde{f}_3, \tag{4.13}$$

with the corresponding mass (3.13) in this case.

The logarithmic sector which includes the modes saturating the BF bound (3.15) appears for a generic gravity modification satisfying

$$\tilde{f}_2 = \frac{l^2(1+\tilde{f}_1) - (D^2 - 6D + 17)\tilde{f}_3}{(D+7)(D-1)}. \tag{4.14}$$

The other Log sector allowing a "local" superposition of modes saturating the BF bound [see Eqs. (3.18) and (3.19)] is possible if

$$\tilde{f}_2 = \frac{l^2(1+\tilde{f}_1) + 4(D-4)\tilde{f}_3}{2(D-1)}. \tag{4.15}$$

5. Conformally invariant corrections

Even though the existence of Schrödinger invariant solutions is not particularly attached to the conformal invariance of the higher-curvature corrections, we find interesting to investigate the particular case of the Einstein gravity supplemented by the conformal Weyl Lagrangian in D-dimensions; namely

$$S[g_{\mu\nu}] = \int d^D x \sqrt{-g} \left[R - 2\Lambda + \frac{1}{2\alpha_w} \left(C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu} \right)^{D/4} \right], \tag{5.1}$$

where α_w is a coupling constant and $C_{\alpha\beta\mu\nu}$ is the Weyl tensor, whose quadratic contraction reads

$$C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu} = \frac{2}{(D-1)(D-2)} R^2 - \frac{4}{D-2} R_{\alpha\beta} R^{\alpha\beta} + R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu}. \tag{5.2}$$

It is easy to verify that the term in the action (5.1) that involves the contraction of the quadratic Weyl tensor is conformally invariant.

Using the notation introduced in the previous section [see Eq. (4.3)], we obtain

$$\begin{aligned} f_1 &= \frac{DR}{2(D-1)(D-2)\alpha_w} \left(C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu} \right)^{(D-4)/4}, \\ f_2 &= -\frac{D}{2(D-2)\alpha_w} \left(C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu} \right)^{(D-4)/4}, \\ f_3 &= \frac{D}{8\alpha_w} \left(C_{\alpha\beta\mu\nu} C^{\alpha\beta\mu\nu} \right)^{(D-4)/4}. \end{aligned} \tag{5.3}$$

For dimensions $D > 4$, the above scalars f_i are proportional to a positive power of the square of the Weyl tensor. The related constants $\tilde{f}$ and $\tilde{f}_i$, being obtained by evaluating the scalars in their AdS values (4.5), vanish since AdS space is conformally flat and hence the Weyl tensor vanishes identically. This means that, in what regards to the AdS-wave solutions, the conformally invariant modification of Einstein gravity gives no correction for dimensions $D > 4$. This case is described by the branch (4.9) and the resulting configuration is the one of General Relativity (3.6) which yields only the *partial* Schrödinger symmetry. In four dimensions the situation is quite different since the resulting theory (5.1)-(5.2) corresponds to the square-curvature corrections that we have considered in Sec. 3 with

$$\beta_1 = \frac{1}{6\alpha_w}, \qquad \beta_2 = -\frac{1}{\alpha_w}, \qquad \beta_3 = \frac{1}{2\alpha_w}.$$

As a consequence, except for the second-order and degenerate sectors which require $\beta_2 = -4\beta_3$, all the other sectors of solutions described in Sec. 3 are present for this four-dimensional Einstein-Weyl gravity theory.

6. Parity-violating Chern-Simons modification

Now, let us move to study another interesting example of gravitational model in four dimensions. This is the Jackiw-Pi theory [33], usually referred to as the Chern-Simons modified gravity in four dimensions. This model is defined by supplementing the Einstein-Hilbert action with a different (parity violating) quadratic term in the curvature, yielding the total action

$$\hat{S}[g_{\mu\nu}] = \int d^4x\sqrt{-g}\left(R - 2\Lambda + \frac{\theta}{4}\,{}^*R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}\right). \tag{6.1}$$

Here θ is a local Lagrange multiplier that couples to the Pontryagin density ${}^*R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, constructed via the dual curvature tensor ${}^*R^{\alpha}{}_{\beta}{}^{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma\mu\nu}R^{\alpha}{}_{\beta\rho\sigma}$, where $\eta_{\rho\sigma\mu\nu}$ is the volume 4-form.[9] The coupling is such that θ has dimensions of [length]^2. Note that the action (6.1) has to be distinguished from the Chern-Simons gravitational theories of Ref. [72], which exist in odd dimensions and correspond to a particular case of Lovelock Lagrangian [58].

The inclusion of the non-dynamical field θ comes from the fact that the Pontryagin form ${}^*R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ is a total derivative. As a consequence, the variation with respect to the metric brings an additional piece to the Einstein equations, which corresponds to a four-dimensional version of the Cotton tensor whose definition obviously depends on θ. Since the field θ can be fixed arbitrarily, the diffeomorphism invariance is broken. Nevertheless, the conservation of the equations of motion makes diffeomorphism symmetry to be restored dynamically. That is, the consistency of the theory imposes that only geometries with vanishing Pontryagin form are allowed as solutions. The same conclusion is obtained by considering the non-dynamical field θ as a Lagrange multiplier; see Ref. [75] for a careful digression on this point.

[9] Here $\eta_{t\xi rx} = \sqrt{-g}$ ($\eta^{t\xi rx} = -1/\sqrt{-g}$).

The equations of motion derived from the Jackiw-Pi action (6.1) then take the form

$$C_{\mu\nu} + G_{\mu\nu} - \frac{3}{l^2} g_{\mu\nu} = 0, \tag{6.2}$$

where, as mentioned above, $C_{\mu\nu}$ is a sort of generalization of the three-dimensional Cotton tensor that appears in the equations of motion of TMG, given by[10]

$$C^{\mu\nu} = \nabla_\alpha \left(\nabla_\beta \theta \, {}^*R^{\alpha(\mu|\beta|\nu)} \right). \tag{6.3}$$

The conservation of the Einstein equations yields the additional constraint

$$\nabla^\mu C_{\mu\nu} = \frac{1}{8} \, {}^*R_{\alpha\beta\rho\sigma} R^{\alpha\beta\rho\sigma} \nabla_\nu \theta = 0, \tag{6.4}$$

which imposes that all solutions of the Jackiw-Pi theory (6.1) must obey ${}^*R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = 0$. It is worth noticing that all Siklos spacetimes, being AdS waves, satisfy this necessary condition as they have vanishing Pontryagin invariant. In fact, we have ${}^*C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = 0$ which in turn implies that ${}^*R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = 0$ since both invariants agree (for a detailed analysis, see the first Appendix of Ref. [77]). This is a promising scenario for the search of Schrödinger symmetric configurations. Indeed, we will now show that the Jackiw-Pi theory (6.1) admits solutions with Schrödinger symmetry. In order to realize this task, the first step is to make an "educated" choice for the breaking of diffeomorphism invariance by considering[11]

$$\theta = \frac{1}{\alpha_p} \frac{x}{r}, \tag{6.5}$$

where α_p is a coupling constant with dimensions of $[\text{length}]^{-2}$. This choice reduces the variational Cotton-like tensor (6.3) to the single expression

$$C_{\mu\nu} = -\frac{r^2}{2\alpha_p l^2} \left(\frac{F''}{r} \right)' \delta^t_\mu \delta^t_\nu. \tag{6.6}$$

Taking all this into account, the Jackiw-Pi equations (6.2) become

$$-\frac{r^2}{2\alpha_p l^2} \left[\frac{1}{r^{1-\alpha_p l^2}} \left(\frac{F'}{r^{\alpha_p l^2}} \right)' \right]' \delta^t_\mu \delta^t_\nu = 0. \tag{6.7}$$

For generic values of the coupling constant α_p, with $\alpha_p \neq -1/l^2$ and $\alpha_p \neq 2/l^2$, the solution of this equation is given by

$$F(r) = c_0 r^3 + c_1 r^{1+\alpha_p l^2}, \tag{6.8}$$

where we have discarded the additive constant removable through a coordinate transformation. On the other hand, for the critical values $\alpha_p = -1/l^2$ and $\alpha_p = 2/l^2$, we obtain the following logarithmic branches

$$F(r) = c_0 r^3 + c_1 \ln r, \tag{6.9}$$

$$F(r) = c_0 r^3 + c_1 r^3 \ln r, \tag{6.10}$$

[10] There are other generalizations of the Cotton tensor in higher dimensions, see e.g. Ref. [76].

[11] It is possible to show that there exists an infinite family of elections compatible with the existence of Schrödinger symmetry, and we are just presenting the simplest one.

respectively. Solutions (6.9) and (6.10) enjoy the partial Schrödinger symmetry for $c_1 = 0$ while the generic solution (6.8) is also partially Schrödinger invariant as long as one of the two constants is set to zero. The analogy with the theories studied previously is complete since the generic solution (6.8) have the *full* Schrödinger symmetry for the special fine tuning

$$\alpha_p = -\frac{3}{l^2}.$$

There also exists a close relation to TMG: Identifying the three-dimensional topological mass in term of the four-dimensional coupling constant as $\mu = \alpha_p l$,[12] both the generic solution (6.8) and the critical one (6.9) can be represented for $c_0 = 0$ as a warped product having as base the corresponding TMG configurations of Refs. [17, 18, 19, 20, 21, 22] with a real line fiber generated by the spatial direction along the coordinate x; namely

$$ds^2_{\mathrm{JP}} = ds^2_{\mathrm{TMG}} + \frac{l^2}{r^2} dx^2. \tag{6.11}$$

The other critical solution (6.10) which corresponds to $\alpha_p = 2/l^2$, does not allow such a simple representation in term of the remaining three-dimensional critical TMG solution with topological mass $\mu = 1/l$. However, since this three-dimensional case turns out to be a consistent asymptotically AdS configuration in TMG despite its weakened logarithmic decay [31, 32], it would be very interesting to check if the above critical solution (6.10) for $\alpha_p = 2/l^2$ is also an asymptotically AdS solution of the Jackiw-Pi theory, relaxing the standard asymptotic conditions known for General Relativity [28].

7. Conclusions

In this paper, we have been concerned with a special class of Siklos spacetimes that contains the Schrödinger invariant metrics as particular cases. Our main purpose was to identify pure gravity theories that exhibit solutions of this special kind. We began by considering the Einstein gravity augmented by square-curvature corrections in arbitrary dimensions D. In this case, we have shown that for a particular choice of the coupling constants Schrödinger invariant metrics are allowed as solutions, while other choice yields to a sector of solutions with logarithmic falling-off. We have also observed the existence of a degenerate sector whose space of solutions contains all these particular Siklos spacetimes, without restricting the profile function F that appears in the metric. We discussed the relation of this degeneracy to the non-renormalization of the dynamical exponent $z = \nu + 1$ observed at special points of the moduli space. We further extended our results to a larger set of gravity theories, whose Lagrangians are given by arbitrary functions of the square-curvature invariants. This was achieved by establishing a correspondence with square-curvature models discussed first. All the sectors studied for the square-curvature actions were shown to also appear in this more general class of models. As a special example, we considered the theory defined by

[12]Notice we are using here the definition of the topological mass for example of Ref. [32]

adding to the Einstein-Hilbert action non-polynomial conformally invariant corrections in arbitrary dimension D. In this case, we observed that only in $D=4$, which corresponds to a particular case of square-curvature action, this theory presents genuinely new features, as for the case of higher dimensions the type of solution we discuss simply reduces to the solutions of D-dimensional General Relativity. Finally, we have also analyzed a completely different higher-order theory of gravity, the so-called Chern-Simons modification of four-dimensional General Relativity proposed by Jackiw and Pi. This model involves a non-dynamical field that plays the role of a Lagrange multiplier which forces the Pontryagin density to vanish. We have shown that for some precise choices of the non-dynamical field, the parity-violating Chern-Simons modification of General Relativity exhibits both the Schrödinger invariant and the logarithmic sectors as exact solutions.

In two recent papers, [71] and [78], we have shown that the Einstein gravity together with appropriated square-curvature corrections in arbitrary dimensions D admits black holes configurations that asymptote the Lifshitz spacetimes [50]. A natural continuation of the work would be finding black holes with Schrödinger asymptotic for (some of) the higher order gravity theories considered in this paper. It would be interesting to obtain black hole configurations which asymptote the metrics (2.1) for arbitrary $\nu \neq 1$ and, consequently, to get black holes with partial Schrödinger asymptotic. This question is of physical relevance as the black hole solutions that have been derived so far in the context of string theory and through the null Melvin twist (see [14] and [12]) possess the full Schrödinger symmetry asymptotically by construction and hence correspond only to the class of solutions with $\nu=1$. This prevents us from the possibility to use an holographic description for the finite temperature effects of condensed matter systems having $\nu \neq 1$.

Finally, let us conclude with few words about some classes of metrics that depends on time, and therefore the time translation is not longer an isometry, while the dilatation (2.3) or the special conformal transformations (2.4) still act as isometries [79]. A particular class of such metrics can be written in the following form

$$ds^2 = \frac{l^2}{r^2}\left[-G\left(\frac{t}{r^{1+\nu}}\right)\frac{dt^2}{r^{2\nu}} + 2dtd\xi + dr^2 + d\vec{x}^2\right], \tag{7.1}$$

where G is an arbitrary function of the argument $t/r^{1+\nu}$. In the sectors of solutions studied in this paper, the solutions depend on arbitrary constants denoted by c_i or $c_\pm$. However, it is easy to see that these constants can be replaced by arbitrary functions of the time t. As a consequence, all the power law solutions derived previously of the form $F(r) = c\, r^\alpha$ can be extended to $F(t,r) = c(t)\, r^\alpha$ where $c(t)$ is an arbitrary function of t. Therefore, choosing $c(t) = \tilde{c}/t^{\frac{2\nu+\alpha}{1+\nu}}$, where $\tilde{c}$ is a constant, makes the solution dilatation invariant for $\nu \neq 1$, while for $\nu = 1$ the solution admits the special conformal transformation as isometry but not the dilatation.

Acknowledgments

Conversations and correspondence with M. Aiello, A. Garbarz, A. García, J. Gomis, A.

Maloney, and J. Oliva are acknowledged. This work has been partially supported by grant 1090368 from FONDECYT, by grants PICT, PIP and UBACyT from ANPCyT, CONICET and UBA, and by grant 82443 from CONACyT.

A. AdS-waves and higher order terms

Let us consider an AdS-wave of the type (2.9). It is worth noticing that the following formulae is valid even when the profile has a more general dependence $F = F(u, r, \vec{x})$. We use the null geodesic vector $k^\mu \partial_\mu = (r/l)\partial_\xi$ that allows the reinterpretation of these backgrounds as generalized Kerr-Schild transformations of AdS

$$g_{\mu\nu} = g_{\mu\nu}^{\text{AdS}} - F k_\mu k_\nu. \tag{A.1}$$

We start with the Ricci tensor, which for a D-dimensional AdS-wave is written as

$$R_{\mu\nu} = -\frac{(D-1)}{l^2} g_{\mu\nu} + \frac{1}{2} k_\mu k_\nu \Box F, \tag{A.2}$$

yielding the scalar curvature $R = -D(D-1)/l^2$, exactly the same as for AdS space. This gives the Einstein tensor

$$G_{\mu\nu} = \frac{(D-1)(D-2)}{2l^2} g_{\mu\nu} + \frac{1}{2} k_\mu k_\nu \Box F, \tag{A.3}$$

and the squared-curvature combinations

$$RR_{\mu\nu} = \frac{D(D-1)^2}{l^4} g_{\mu\nu} - \frac{D(D-1)}{2l^2} k_\mu k_\nu \Box F, \tag{A.4}$$

$$R_{\mu\alpha} R_\nu{}^\alpha = \frac{(D-1)^2}{l^4} g_{\mu\nu} - \frac{(D-1)}{l^2} k_\mu k_\nu \Box F. \tag{A.5}$$

The following squared-curvature combinations involve explicitly the Riemann tensor

$$R_{\mu\alpha\nu\beta} R^{\alpha\beta} = \frac{(D-1)^2}{l^4} g_{\mu\nu} - \frac{(D-2)}{2l^2} k_\mu k_\nu \Box F, \tag{A.6}$$

$$R_{\mu\gamma\alpha\beta} R_\nu{}^{\gamma\alpha\beta} = \frac{2(D-1)}{l^4} g_{\mu\nu} - \frac{2}{l^2} k_\mu k_\nu \Box F. \tag{A.7}$$

Using the expression for the Ricci tensor (A.2), together with the null and geodesic properties of k^μ, it is not hard to verify that

$$\Box R_{\mu\nu} = \frac{1}{2} k_\mu k_\nu \Box \left(\Box - \frac{2}{l^2} \right) F. \tag{A.8}$$

If we denote by $K_{\mu\nu}$ the modification to the Einstein equations (3.2) coming from the squared curvatures, the expressions above allow to write this tensor as

$$K_{\mu\nu} = -\frac{(D-1)(D-4)}{2l^2} \left[(D-1)(D\beta_1 + \beta_2) + 2\beta_3 \right] g_{\mu\nu}$$
$$+ \frac{1}{2} k_\mu k_\nu \Box \left\{ (\beta_2 + 4\beta_3)\Box - \frac{2}{l^2} \left[(D-1)(D\beta_1 + \beta_2) - 2(D-4)\beta_3 \right] \right\} F.$$

Now the Einstein equations (3.2) become

$$\left\{\lambda+\frac{(D-1)(D-2)}{2l^2}-\frac{(D-1)(D-4)}{2l^4}\left[(D-1)\left(D\beta_1+\beta_2\right)+2\beta_3\right]\right\}g_{\mu\nu}$$
$$+\frac{1}{2}k_\mu k_\nu\Box\left\{(\beta_2+4\beta_3)\Box-\frac{1}{l^2}\left[2(D-1)(D\beta_1+\beta_2)-4(D-4)\beta_3-l^2\right]\right\}F=0,$$

from which it follows that the cosmological constant must be chosen as in Eq. (3.3). Since the d'Alembertian of any function $\Phi=\Phi(r)$ depending only in the front-wave coordinate r becomes

$$\Box\Phi=\frac{1}{l^2}\left[r^2\Phi''-(D-2)r\Phi'\right], \tag{A.9}$$

we obtain finally equation (3.4).

References

[1] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998).

[2] S. Gubser, I. Klebanov and A. Polyakov, Phys. Lett. **B428**, 105 (1998).

[3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. **323**, 183 (2000).

[5] D. T. Son, Phys. Rev. D **78**, 046003 (2008).

[6] K. Balasubramanian and J. McGreevy, Phys. Rev. Lett. **101**, 061601 (2008).

[7] P. Havas and J. Plebański, Jour. Math. Phys **19**, 482 (1978).

[8] C. Duval, G. W. Gibbons and P. Horvathy, Phys. Rev. D **43**, 3907 (1991).

[9] C. Duval, M. Hassaine and P. A. Horvathy, Annals Phys. **324**, 1158 (2009).

[10] M. Guica, K. Skenderis, M. Taylor and B. van Rees, JHEP **1102** (2011) 056.

[11] K. Balasubramanian and J. McGreevy, JHEP **1101** (2011) 137.

[12] C. P. Herzog, M. Rangamani, S. F. Ross, JHEP **0811**, 080 (2008).

[13] J. Maldacena, D. Martelli, Y. Tachikawa, JHEP **0810**, 072 (2008).

[14] A. Adams, K. Balasubramanian, J. McGreevy, JHEP **0811**, 059 (2008).

[15] A. Adams, A. Maloney, A. Sinha and S. E. Vazquez, JHEP **0903**, 097 (2009).

[16] S. Deser, R. Jackiw and S. Templeton, Annals Phys. **140**, 372 (1982) [Erratum-ibid. **185**, 406 (1988)]; Phys. Rev. Lett. **48**, 975 (1982).

[17] E. Ayón-Beato and M. Hassaine, Annals Phys. **317**, 175 (2005).

[18] T. Dereli and O. Sarioglu, *Topologically massive gravity and black holes in three dimensions*, arXiv:gr-qc/0009082.

[19] S. Olmez, O. Sarioglu and B. Tekin, Class. Quant. Grav. **22**, 4355 (2005).

[20] E. Ayón-Beato and M. Hassaine, Phys. Rev. D **73**, 104001 (2006).

[21] S. Carlip, S. Deser, A. Waldron and D. K. Wise, Phys. Lett. B **666**, 272 (2008).

[22] G. W. Gibbons, C. N. Pope and E. Sezgin, Class. Quant. Grav. **25**, 205005 (2008).

[23] D. Anninos, W. Li, M. Padi, W. Song and A. Strominger, JHEP **0903**, 130 (2009).

[24] E. Ayón-Beato, G. Giribet and M. Hassaine, JHEP **0905**, 029 (2009).

[25] E. A. Bergshoeff, O. Hohm and P. K. Townsend, Phys. Rev. Lett. **102** (2009) 201301.

[26] H. R. Afshar, M. Alishahiha, A. Naseh, Phys. Rev. **D81**, 044029 (2010).

[27] M. Alishahiha, A. Naseh, H. Soltanpanahi, Phys. Rev. **D82**, 024042 (2010).

[28] M. Henneaux and C. Teitelboim, Commun. Math. Phys. **98**, 391 (1985).

[29] J. D. Brown and M. Henneaux, Commun. Math. Phys. **104**, 207 (1986).

[30] M. Henneaux, in *Rome 1985, Proceedings, General Relativity, Pt. B*, ps. 959-966.

[31] D. Grumiller, N. Johansson, JHEP **0807**, 134 (2008).

[32] M. Henneaux, C. Martinez, R. Troncoso, Phys. Rev. **D79**, 081502 (2009); **D82**, 064038 (2010).

[33] R. Jackiw, S. Y. Pi, Phys. Rev. **D68**, 104012 (2003).

[34] M. Alishahiha, R. Fareghbal, *D-Dimensional Log Gravity*, arXiv:1101.5891.

[35] I. Gullu, M. Gurses, T. C. Sisman and B. Tekin, *AdS Waves as Exact Solutions to Quadratic Gravity*, arXiv:1102.1921.

[36] H. Lu and C. N. Pope, *Critical Gravity in Four Dimensions*, arXiv:1101.1971.

[37] S. Deser, H. Liu, H. Lu, C. N. Pope, T. C. Sisman and B. Tekin, *Critical Points of D-Dimensional Extended Gravities*, arXiv:1101.4009.

[38] R. Jackiw, Phys. Today, **25**, 23 (1972).

[39] U. Niederer, Helv. Phys. Acta **45**, 802-810 (1972).

[40] C. R. Hagen, Phys. Rev. **D5**, 377-388 (1972).

[41] R. Jackiw, Ann. of Phys. **201**, 83 (1990).

[42] R. Jackiw and S. Y. Pi, Phys. Rev. D **42**, 3500 (1990) [Erratum-ibid. D **48**, 3929 (1993)].

[43] P. K. Ghosh, Phys. Rev. A **65**, 012103 (2002).

[44] M. Hassaine and P. A. Horvathy, Annals Phys. **282**, 218 (2000).

[45] L. O'Raifeartaigh and V. V. Sreedhar, Annals Phys. **293**, 215 (2001).

[46] M. Hassaine and P. A. Horvathy, Phys. Lett. A **279**, 215 (2001).

[47] T. S. Nyawelo, J. W. van Holten and S. G. Nibbelink, Phys. Rev. D **68**, 125006 (2003).

[48] M. d. Montigny, F. C. Khanna and A. E. Santana, J. Phys. A **36**, 2009 (2003).

[49] M. Henkel and J. Unterberger, Nucl. Phys. B **660**, 407 (2003).

[50] S. Kachru, X. Liu and M. Mulligan, Phys. Rev. D **78**, 106005 (2008).

[51] S.T.C. Siklos, in *Galaxies, axisymmetric systems and relativity*, ed. M.A.H. MacCallum (Cambridge Univ. Press, Cambridge 1985).

[52] J. Podolsky, Class. Quant. Grav. **15**, 719 (1998).

[53] A. García and J. Plebański, J. Math. Phys. **22**, 2655 (1981).

[54] H. Salazar, A. García, and J. Plebański, J. Math. Phys. **24**, 2191 (1983).

[55] A. García, Nuovo Cim. B **78**, 255 (1983).

[56] I. Ozsvath, I. Robinson, and K. Rozga, J. Math. Phys. **26**, 1755 (1985).

[57] E. Ayón-Beato and M. Hassaine, Phys. Rev. D **71**, 084004 (2005).

[58] D. Lovelock, J. Math. Phys. **12**, 498 (1971).

[59] P. Breitenlohner and D. Z. Freedman, Phys. Lett. B **115**, 197 (1982); Annals Phys. **144**, 249 (1982).

[60] L. Mezincescu and P. K. Townsend, Annals Phys. **160**, 406 (1985).

[61] A. Maloney, W. Song and A. Strominger, Phys. Rev. D **81**, 064007 (2010).

[62] G. Giribet, M. Kleban and M. Porrati, JHEP **0810** (2008) 045.

[63] D. Grumiller, I. Sachs, JHEP **1003**, 012 (2010).

[64] D. Grumiller, O. Hohm, Phys. Lett. **B686**, 264-267 (2010).

[65] D. Grumiller, N. Johansson, J. Phys. Conf. Ser. **222**, 012047 (2010).

[66] Y. Liu and Y. W. Sun, JHEP **0905**, 039 (2009)

[67] Y. Liu and Y. W. Sun, Phys. Rev. D **79**, 126001 (2009)

[68] E. Bergshoeff, O. Hohm, J. Rosseel and P. Townsend, *Modes of Log Gravity*, arXiv:1102.4091.

[69] S. Moroz, Phys. Rev. D **81**, 066002 (2010)

[70] R. G. Cai, Y. Liu, Y. W. Sun, JHEP **0910**, 080 (2009).

[71] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, JHEP **1004**, 030 (2010).

[72] J. Zanelli, *Lecture notes on Chern-Simons (super-)gravities*, 2nd Ed. (2008), arXiv:hep-th/0502193.

[73] G. Dotti, J. Oliva, R. Troncoso, Phys. Rev. **D75**, 024002 (2007).

[74] J. Oliva and S. Ray, Class. Quant. Grav. **27** (2010) 225002; Phys. Rev. **D82** (2010) 124030.

[75] R. Jackiw, *Lorentz Violation in a Diffeomorphism-Invariant Theory,* arXiv:0709.2348.

[76] A. Garcia, F. W. Hehl, C. Heinicke and A. Macias, Class. Quant. Grav. **21** (2004) 1099.

[77] D. Grumiller and N. Yúñes, Phys. Rev. D **77**, 044015 (2008).

[78] E. Ayon-Beato, A. Garbarz, G. Giribet and M. Hassaine, Phys. Rev. D **80**, 104029 (2009).

[79] Y. Nakayama, JHEP **1004**, 102 (2010).