Casimir force for absorbing media in an open quantum system framework: Scalar model

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In this article we compute the Casimir force between two finite-width mirrors at finite temperature, working in a simplified model in $1 + 1$ dimensions. The mirrors, considered as dissipative media, are modeled by a continuous set of harmonic oscillators which in turn are coupled to an external environment at thermal equilibrium. The calculation of the Casimir force is performed in the framework of the theory of open quantum systems. It is shown that the Casimir interaction has two different contributions: the usual radiation pressure from the vacuum, which is obtained for ideal mirrors without dissipation or losses, and a Langevin force associated with the noise induced by the interaction between dielectric atoms in the slabs and the thermal bath. Both contributions to the Casimir force are needed in order to reproduce the analogous Lifshitz formula in $1 + 1$ dimensions. We also discuss the relationship between the electromagnetic properties of the mirrors and the spectral density of the environment.

I. INTRODUCTION

Given the precision that has been recently achieved in the measurement of the Casimir forces [1], the use of realistic models for the description of the media that constitute the mirrors is an unavoidable step for the improvement of Casimir energy calculations, which is needed for comparison with the experimental data. Moreover, from a conceptual point of view, the theoretical calculations for mirrors with general electromagnetic properties, including absorption, is not a completely settled issue [2–4]. Since dissipative effects imply the possibility of energy interchange between different parts of the full system (mirrors, vacuum field, and environment), the theory of open quantum systems [5] is the natural approach to clarify the role of dissipation in Casimir physics. Indeed, in this framework, dissipation and noise appears in the effective theory of the relevant degrees of freedom (the electromagnetic field) after integration of the matter and other environmental degrees of freedom.

Dielectric slabs are, in general, nonlinear, inhomogeneous, dispersive, and also dissipative media. These aspects render difficult the quantization of a field when they all have to be taken into account simultaneously. There are different approaches to address this problem. On the one hand, one can use a phenomenological description based on the macroscopic electromagnetic properties of the materials. The quantization can be performed starting from the macroscopic Maxwell equations and including noise terms to account for absorption [6]. In this approach, a canonical quantization scheme is not possible, unless one couples the electromagnetic field to a reservoir (see [3]) following the standard route to include dissipation in simple quantum mechanical systems. Another possibility is to establish a first-principles model in which

the slabs are described through their microscopic degrees of freedom, which are coupled to the electromagnetic field. In this kind of model, losses are also incorporated by considering a thermal bath to allow for the possibility of the absorption of light. There is a large body of literature on the quantization of the electromagnetic field in dielectrics. Regarding microscopic models, the fully canonical quantization of the electromagnetic field in dispersive and lossy dielectrics has been performed by Huttner and Barnett (HB) [7]. In the HB model, the electromagnetic field is coupled to matter (the polarization field), and the matter is coupled to a reservoir that is included into the model to describe the losses. In the context of the theory of open quantum systems, one can think of the HB model as a composite system in which the relevant degrees of freedom belong to two subsystems (the electromagnetic field and the matter), and the matter degrees of freedom are in turn coupled to an environment (the thermal reservoir). The indirect coupling between the electromagnetic field and the thermal reservoir is responsible for the losses. As we will comment below, this will be our starting point to compute the Casimir force between absorbing media.

Regarding the Casimir force, the celebrated Lifshitz formula [8] describes the forces between dielectrics in terms of their macroscopic electromagnetic properties. The original derivation of this very general formula is based on a macroscopic approach, starting from stochastic Maxwell equations and using thermodynamical properties for the stochastic fields. As pointed out in several papers, the connection between this approach and an approach based on a fully quantized model is not completely clear. Moreover, some doubts have been raised about the applicability of the Lifshitz formula to lossy dielectrics [2–4].

The first calculation of the Casimir force between two absorbing slabs using a microscopic approach is, to our knowledge, due to Kupiszewska [9], who modeled dielectric atoms as a set of harmonic oscillators coupled to an environment with $T = 0$ and in which the atoms can dissipate energy. In that work, a scalar field in $1 + 1$ dimensions was considered,

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and all the environmental effects were described through a dissipative constant and a Langevin force. In the context of open quantum systems, this is tantamount to considering an Ohmic environment. The force between slabs was then obtained in terms of the reflection coefficients associated with the slabs, which are described by a complex dielectric function. This result was rederived using a Green-function method for quantizing the macroscopic field in absorbing systems in one-dimension (1D) in conjunction with the scattering matrix approach [10]. This was also extended to two identical absorbing superlattices [11]. Esquivel-Sirvent *et al.* demonstrated an alternative Green-function approach that makes the quantization of the field within the slabs unnecessary and calculated the Casimir force in an asymmetric configuration [12] which was earlier considered only in the lossless case [13]. In Ref. [14], the Casimir force was calculated in a lossless dispersive layer of an otherwise absorbing multilayer by using the macroscopic field operators considered by [15]. In the series of works [4,16,17], Rosa *et al.* considered the evaluation of the electromagnetic energy density in the presence of an absorbing and dissipative dielectric, using the HB model for $T = 0$ and constant dissipation. In particular, in the recent Ref. [17], they obtained the force density associated with spatial variations of the permittivity from which, in principle, one could obtain the Casimir force between slabs.

In this paper we will follow a program similar to that of Ref. [9], generalizing it by considering a general and well-defined open quantum system. We will work with a simplified model analogous to the HB model, assuming that the dielectric atoms in the slabs are quantum Brownian particles and that they are subjected to fluctuations (noise) and dissipation due to the coupling to an external thermal environment. We will keep generality in the type of spectral density to specify the bath to which the atoms are coupled, generalizing the constant dissipation model used in Ref. [9]. Indeed, after integration of the environmental degrees of freedom, it will be possible to obtain the dissipation and noise kernels that modify the unitary equation of motion of the dielectric atoms. As we will see, general non-Ohmic environments do not provide constant dissipation coefficients in the equation of motion of the Brownian particles, even at high temperatures. Moreover, the spectral density of the environment determines the electromagnetic properties of the mirrors and therefore have a direct influence on the Casimir force.

In addition to the conceptual issues described above, there are additional motivations to consider detailed microscopic models of the Casimir force, in particular, the controversy about its temperature dependence. Assuming simple phenomenological descriptions of the materials based on the plasma or Drude models, the theoretical predictions for the Casimir force are different due to the contribution (or not) of the transverse electric (TE) zero mode [18]. At small distances a such that $aT \ll 1$, the differences are not too large, and the experimental results by Decca *et al.* [19] seem to be well described by the plasma model. However, at large distances $aT \gg 1$, the theoretical predictions differ by a factor of two. The Casimir force at such large distances have been recently measured [20],

and the results are compatible with the Drude model after taking into account the interactions due to patch potentials on the surfaces of the conductors (some authors disagree with the evaluation of the effects of the patch potentials; see [21]). In any case, these controversies show that more detailed microscopic models are necessary to clarify the situation. For example, considering that the slabs contain classical or quantum nonrelativistic charges interacting via the static Coulomb potential, the result for the large-distance limit agrees with that of the Drude model [22].

Another motivation for considering the Casimir forces in the framework of open quantum systems is the possibility of analyzing nonequilibrium effects, such as the Casimir force between objects at different temperatures [23] and the power of heat transfer between them [24], including the time-dependent evolution until reaching a stationary situation.

This paper is organized as follows: In the next section we present the model, the Heisenberg equations of motion for the different operators, and the vacuum and Langevin contributions to the field operator. In Sec. III we study in depth the relationship between the microscopic model and the macroscopic electromagnetic properties of the mirrors. Section IV is dedicated to the evaluation of the Casimir force. After adding the vacuum and Langevin contributions, we show that the total force is given by a Lifshitz-like formula, where the reflection coefficients of the slabs depend on the properties of the atoms and the environment considered in the model. In Sec. V we comment on the relationship of the open quantum systems approach developed in this paper and the Euclidean computation of the Casimir force. We summarize our findings in Sec. VI. The appendices contain some details of the calculations.

II. THE MODEL

A. Lagrangian density

With the aim of including effects of dissipation and noise in the calculation of Casimir force, we will use the theory of open quantum systems, having in mind the paradigmatic example of quantum Brownian motion (QBM) [5].

The model consists of a system composed of two parts: a massless scalar field and dielectric slabs which, in turn, are described by their internal degrees of freedom (a set of harmonic oscillators). Both subsystems conform a composite system which is coupled to a second set of harmonic oscillators that plays the role of an external environment or thermal bath. For simplicity we will work in $1 + 1$ dimensions. In our toy model the massless field represents the electromagnetic field, and the first set of harmonic oscillators directly coupled to the scalar field represents the atoms in the slab.

Considering the usual interaction term between the electromagnetic field and the ordinary matter, the coupling between the field and the atoms in the slab will be taken as a current-type coupling, where the field couples to the velocity of the atoms. The coupling constant for this interaction is the electric charge e. We will also assume that there is no direct coupling between the field and the thermal bath. The Lagrangian density

is therefore given by

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_\phi + \mathcal{L}_S + \mathcal{L}_{\phi-S} + \mathcal{L}_B + \mathcal{L}_{S-B} \\
&= \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + 4\pi\eta\left(\frac{1}{2}m\dot{r}^2(x;t) - \frac{1}{2}m\omega_0^2 r^2(x;t)\right) \\
&\quad + 4\pi\eta e\phi(x;t)\dot{r}(x;t) \\
&\quad + 4\pi\eta\sum_n\left(\frac{1}{2}m_n\dot{q}_n^2(x;t) - \frac{1}{2}m_n\omega_n^2 q_n^2(x;t)\right) \\
&\quad - 4\pi\eta\sum_n\lambda_n q_n(x;t)r(x;t),
\end{aligned} \tag{1}
$$

where we have stressed the fact that r and q_n are also functions of the position (i.e., each atom interacts with a thermal bath placed at the same position). We have denoted by η the density of atoms in both slabs. The constants λ_n are the coupling constants between the atoms and the bath oscillators. It is implicitly understood that Eq. (1) represents the Lagrangian density inside the plates, while outside the plates the Lagrangian is given by the free-field one. The configuration of the slabs, of thickness d and separated by a distance a, is shown in Fig. 1.

The quantization of the theory is straightforward. It should be noted that the full Hilbert space of the model H, where the quantization is performed, is not only the field Hilbert space H_ϕ (as is considered in others works where the field is the only relevant degree of freedom), but also includes the Hilbert spaces of the atoms H_S and the bath oscillators H_B, in such a way that $H = H_\phi \otimes H_S \otimes H_B$. We will assume, as frequently done in the context of QBM, that for $t < 0$ the three parts of the systems are uncorrelated and not interacting. Interactions are turned on at $t = 0$. Therefore, the initial conditions for the operators $\hat{\phi}$ and $\hat{r}$ must be given in terms of operators acting in each part of the Hilbert space. The interactions will cause initial operators to become operators over the whole space H. The initial density matrix of the total system is of the form

$$
\hat{\rho}(0) = \hat{\rho}_\phi(0) \otimes \hat{\rho}_S(0) \otimes \hat{\rho}_B. \tag{2}
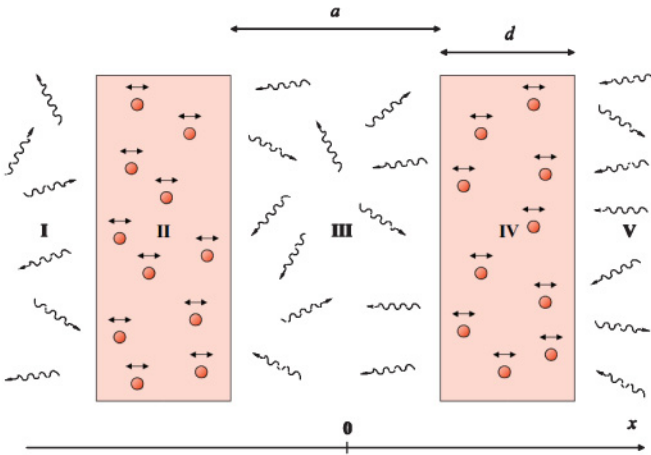$$

FIG. 1. (Color online) Plate configuration defining the five regions in space. The plates are formed by oscillators which are in contact with thermal environments. The entire system is in thermal equilibrium at a temperature T. In this work we consider a $1+1$ version of this configuration.

Since we are interested in the steady state ($t \to +\infty$), we will assume thermal equilibrium at temperature $T = 1/\beta$ between the three parts. Each density matrix in Eq. (2) will be taken as thermal type (we have set $\hbar = k_B = c = 1$).

B. Heisenberg equations of motion

Starting from the Lagrangian (1), it is easy to derive the Heisenberg equations of motion for the different operators. They are are given by

$$
\hat{p}_n = m_n\dot{\hat{q}}_n, \tag{3}
$$
$$
\dot{\hat{p}}_n = -m_n\omega_n^2\hat{q}_n + \lambda_n\hat{r}, \tag{4}
$$
$$
\hat{p} = m\dot{\hat{r}} + e\hat{\phi}, \tag{5}
$$
$$
\dot{\hat{p}} = -m\omega_0^2\hat{r} + \sum_n\lambda_n\hat{q}_n, \tag{6}
$$
$$
\Box\hat{\phi} = 4\pi\eta e\dot{\hat{r}}, \tag{7}
$$

where the operators $\hat{p}$ and $\hat{p}_n$ are the conjugate momentum operators associated with the operators $\hat{r}$ and $\hat{q}_n$, respectively. Substituting Eq. (3) into Eq. (4), we get

$$
m_n\ddot{\hat{q}}_n + m_n\omega_n^2\hat{q}_n - \lambda_n\hat{r} = 0. \tag{8}
$$

As usual in the context of QBM, we solve the equations for the operators $\hat{q}_n$ taking $\hat{r}$ as a source, and substitute the solutions into Eq. (6). In this way, the microscopic degrees of freedom in the slabs satisfy a Langevin-like equation of the form

$$
\dot{\hat{p}} = -m\omega_0^2\hat{r} - m\frac{d}{dt}\int_0^t d\tau\,\gamma(t-\tau)\hat{r}(x;\tau) + \hat{F}(x;t), \tag{9}
$$

where the damping kernel γ and the stochastic force operator $\hat{F}$ are the same as those of QBM (see [5] for a general and complete view). They are given by

$$
\gamma(t) = \frac{2}{m}\int_0^{+\infty} d\omega\frac{J(\omega)}{\omega}\cos(\omega t), \tag{10}
$$
$$
\hat{F}(t) = \sum_n\frac{\lambda_n}{\sqrt{2m_n\omega_n}}\left(e^{-i\omega_n t}\hat{b}_n + e^{i\omega_n t}\hat{b}_n^\dagger\right). \tag{11}
$$

Here, $\hat{b}_n$ and $\hat{b}_n^\dagger$ are the annihilation and creation operators associated with $\hat{q}_n$, and $J(\omega)$ is the spectral density that characterizes the environment. This function gives the number of oscillators in each frequency for given values of the coupling constants λ_n:

$$
J(\omega) = \sum_n\frac{\lambda_n^2}{2m_n\omega_n}\delta(\omega - \omega_n). \tag{12}
$$

In order to obtain a true irreversible dynamics, we introduce a continuous distribution of bath modes that replaces the spectral density by a smooth function of the frequency ω of the bath modes. Different functions will describe different types of environments. Physically, the thermal bath has a finite number of oscillators in a given range of frequencies. Then, a cutoff function must be introduced, containing some characteristic frequency scale Λ. In this case, the spectral density takes the following form:

$$
J(\omega) = \frac{2}{\pi}m\gamma_0\omega\left(\frac{\omega}{\Lambda}\right)^{\alpha-1}f\left(\frac{\omega}{\Lambda}\right). \tag{13}
$$

In the equation above, γ_0 is the relaxation constant of the environment, while f is the frequency cutoff function. The values of α classify the different types of environments: $\alpha = 1$ corresponds to an Ohmic environment (in which there is a dissipative term proportional to the velocity in the Langevin equation for the Brownian particle), while $\alpha < 1$ and $\alpha > 1$ describe sub-Ohmic and supra-Ohmic environments, respectively [5].

At equilibrium, the stochastic force operator in Eq. (11) and the damping kernel γ in Eq. (10) are not independent. The statistical properties of the stochastic force operator are given by the dissipation and noise kernels

$$D(t - t') \equiv i\langle[\widehat{F}(t); \widehat{F}(t')]\rangle = i[\widehat{F}(t); \widehat{F}(t')]$$
$$= -2\int_0^{+\infty} d\omega J(\omega)\sin[\omega(t - t')], \quad (14)$$

$$D_1(t - t') \equiv \langle\{\widehat{F}(t); \widehat{F}(t')\}\rangle$$
$$= 2\int_0^{+\infty} d\omega J(\omega)\coth\left(\frac{\omega}{2T}\right)\cos[\omega(t - t')], \quad (15)$$

which are the formal open quantum systems generalization of the relations employed in Ref. [9] for general environments and arbitrary temperature. Note that only the noise kernel D_1 involves the environmental temperature T as a parameter. Considering Eqs. (10) and (14), it is easy to show that

$$\frac{d}{dt}\gamma(t - s) = -\frac{1}{m}D(t - s), \quad (16)$$

which relates the damping kernel γ to the statistical properties of the stochastic force operator $\widehat{F}$.

All in all, the set of equations to solve now are Eqs. (5), (7), and (9).

It is possible to obtain a formal solution for the operators $\widehat{r}(x; t)$ by considering the field ϕ as a source for the equation. This solution generalizes the crude approximation made in Ref. [9] for the evolution of the microscopic degrees of freedom in the mirrors. It is given by

$$\widehat{r}(x; t) = G_1(t)\widehat{r}(x; 0) + G_2(t)\dot{\widehat{r}}(x; 0)$$
$$+ \frac{1}{m}\int_0^t ds\, G_2(t - s)[\widehat{F}(x; s) - e\widehat{\phi}(x; s)], \quad (17)$$

where $G_{1,2}$ are the Green functions associated with the QBM equation that satisfy

$$G_1(0) = 1, \quad \dot{G}_1(0) = 0, \quad (18)$$
$$G_2(0) = 0, \quad \dot{G}_2(0) = 1, \quad (19)$$

for which, the Laplace transforms are given by

$$\widetilde{G}_1(z) = \frac{z}{z^2 + \omega_0^2 + z\widetilde{\gamma}(z)}, \quad (20)$$

$$\widetilde{G}_2(z) = \frac{1}{z^2 + \omega_0^2 + z\widetilde{\gamma}(z)}, \quad (21)$$

where $\widetilde{\gamma}$ is the Laplace transform of the damping kernel. Note that, given these conditions, one can prove that $G_1(t) = \dot{G}_2(t)$.

Inserting this solution into Eq. (7), we obtain the following equation for the field operator:

$$\Box\widehat{\phi} + \frac{4\pi\eta e^2}{m}\int_0^t G_1(t - \tau)\widehat{\phi}(x; \tau)d\tau$$
$$= 4\pi\eta e\left(\dot{G}_1(t)\widehat{r}(x; 0) + G_1(t)\dot{\widehat{r}}(x; 0)\right.$$
$$\left. + \frac{1}{m}\int_0^t G_1(t - \tau)\widehat{F}(x; \tau)d\tau\right), \quad (22)$$

subjected to the free-field initial conditions

$$\widehat{\phi}(x; 0) = \int dk\left(\frac{1}{\omega_k}\right)^{\frac{1}{2}}(\widehat{a}_k e^{ikx} + \widehat{a}_k^\dagger e^{-ikx}), \quad (23)$$

$$\dot{\widehat{\phi}}(x; 0) = \int dk\left(\frac{1}{\omega_k}\right)^{\frac{1}{2}}(-i\omega_k\widehat{a}_k e^{ikx} + i\omega_k\widehat{a}_k^\dagger e^{-ikx}), \quad (24)$$

where $\widehat{a}_k$ and $\widehat{a}_k^\dagger$ are the annihilation and creation operators for the free field, and $\omega_k = |k|$. The boundary conditions are the continuity of the field and its spatial derivative at the interface points.

We will compute the Casimir force from the xx-component of the energy-momentum tensor

$$\widehat{T}_{xx}(x; t) = \frac{1}{2}[(\partial_0\widehat{\phi})^2 + (\partial_x\widehat{\phi})^2]. \quad (25)$$

The force is explicitly given by

$$F_C = \langle\widehat{T}_{xx}^{\text{ext}}\rangle - \langle\widehat{T}_{xx}^{\text{int}}\rangle, \quad (26)$$

where the expectation values are taken on the regions outside the planes (regions I or V) and between them (region III), respectively, in a thermal equilibrium situation.

For this calculation, we need the explicit solution for the field equation (22). Considering the properties of the fundamental solutions $G_{1,2}$, the first step is to take the Laplace transform of the equation in order to obtain

$$0 = \frac{\partial^2}{\partial x^2}\widehat{\phi}(x; s) - s^2\left(1 + \frac{4\pi\eta e^2}{m}\widetilde{G}_2(s)\right)\widehat{\phi}(x; s)$$
$$+ s\widehat{\phi}(x; 0) + \dot{\widehat{\phi}}(x; 0)$$
$$+ \frac{4\pi\eta e}{m}\widetilde{G}_1(s)\widehat{F}(x; s) + \frac{4\pi\eta e}{m}\widetilde{G}_1(s)\widehat{p}(x; 0)$$
$$+ 4\pi\eta e\widehat{r}(x; 0)(s\widetilde{G}_1(s) - 1). \quad (27)$$

Since we are interested in the long-time behavior ($t \to +\infty$), we can omit terms containing the positions $\widehat{r}$ and momenta $\widehat{p}$ of the oscillators at $t = 0$. This assumption is well justified for $t \gg 1/\gamma_0$, the scale associated with the damping or relaxation time of the environment. Therefore, the equation for the field can be approximated by

$$\frac{\partial^2}{\partial x^2}\widehat{\phi}(x; s) - s^2\left(1 + \frac{4\pi\eta e^2}{m}\widetilde{G}_2(s)\right)\widehat{\phi}(x; s)$$
$$= -s\widehat{\phi}(x; 0) - \dot{\widehat{\phi}}(x; 0) - \frac{4\pi\eta e}{m}\widetilde{G}_1(s)\widehat{F}(x; s). \quad (28)$$

This is the equation for the Laplace transform of the field, with the initial conditions and the Laplace transform of the

stochastic Langevin force as sources. For simplicity, the spatial dependence of the matter terms was omitted, but it is important to remember that this expression is valid for the points inside the plates. Note also that, because there is no contribution from the atoms to the sources, the field operator $\widehat{\phi}$ acts on H_S as an identity.

We propose a solution of the form $\widehat{\phi}(x;t) = \widehat{\phi}_V(x;t) + \widehat{\phi}_L(x;t)$, where two contributions are distinguished: the vacuum contribution $\widehat{\phi}_V(x;t)$, which results from the modified field modes, and the Langevin contribution $\widehat{\phi}_L(x;t)$, which depends linearly on the Langevin forces. Each contribution satisfies

$$\frac{\partial^2}{\partial x^2}\widehat{\phi}_V(x;s) - s^2\left(1 + \frac{4\pi\eta e^2}{m}\widetilde{G}_2(s)\right)\widehat{\phi}_V(x;s)$$
$$= -s\widehat{\phi}(x;0) - \dot{\widehat{\phi}}(x;0), \qquad (29)$$

$$\frac{\partial^2}{\partial x^2}\widehat{\phi}_L(x;s) - s^2\left(1 + \frac{4\pi\eta e^2}{m}\widetilde{G}_2(s)\right)\widehat{\phi}_L(x;s)$$
$$= -\frac{4\pi\eta e}{m}\widetilde{G}_1(s)\widehat{F}(x;s). \qquad (30)$$

It is worth noting that the first equation presents only operators acting on H_ϕ, thus the associated field contribution is an operator on that space. In the same way, the second equation depends only on operators acting on H_B, and therefore the Langevin contribution acts nontrivially only there. Taking all this into account, the field operator reads

$$\widehat{\phi}(x;t) = \widehat{\phi}_V(x;t) \otimes \mathbb{I}_S \otimes \mathbb{I}_B + \mathbb{I}_\phi \otimes \mathbb{I}_S \otimes \widehat{\phi}_L(x;t). \qquad (31)$$

In summary, the atoms act like a *bridge* between the field and the thermal bath, making no contributions to the total field. The problem, then, has been completely separated into two parts that will be computed in the next subsection.

C. Vacuum and Langevin contributions

We will now solve the two independent equations (29) and (30). For the vacuum contribution $\widehat{\phi}_V$, the solution for $t \to +\infty$ is assumed to have the form

$$\widehat{\phi}_V(x;t) = \left[\int_0^{+\infty} + \int_{-\infty}^0\right]\frac{dk}{2\pi}\left(\frac{\pi}{\omega_k}\right)^{\frac{1}{2}}$$
$$\times \left(\widehat{a}_k e^{-i\omega_k t}f_k(x) + \widehat{a}_k^\dagger e^{i\omega_k t}f_k^*(x)\right), \qquad (32)$$

where the first integral comprises the waves going from left to right, and the second one comprises the waves going from right to left. The mode functions $f_k(x)$ satisfy the equation

$$\frac{d^2}{dx^2}f_k(x) + \omega_k^2 n^2(\omega_k)f_k(x) = 0. \qquad (33)$$

The refractive index $n(\omega_k)$ is given by

$$n^2(\omega_k) = 1 + \frac{4\pi\eta e^2}{m}\widetilde{G}_2(-i\omega_k)$$
$$= 1 + \frac{\omega_P^2}{\omega_0^2 - \omega_k^2 - i\omega_k\widetilde{\gamma}(-i\omega_k)}, \qquad (34)$$

where $\omega_P^2 = \frac{4\pi\eta e^2}{m}$ is the plasma frequency.

It is worth noting that Eq. (33) is of the same form as the equation for the modes in a nonabsorbing dielectric medium

[25], except that, in this case, the refractive index is frequency-dependent. On the other hand, Eq. (34) depends on the Fourier transform of the damping kernel, which is associated with a general environment. There is no assumption either about the spectral density of the environment or about the value of the equilibrium temperature. In the next section we will analyze in more detail the relation between the electromagnetic response of the medium and the spectral density.

Equation (33) for the modes can be solved by imposing continuity of the mode functions f_k at all interfaces. The calculation is long but straightforward and is described in Appendix A.

In order to obtain the Langevin contribution, we must solve Eq. (30). Since the interaction begins at $t = 0$, the equations for the Laplace and Fourier transforms for this contribution are identical and related by $s = -ik$. Thus, the long-time solution can be written as $\widehat{\phi}_L(x;t) = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}\widehat{\phi}_L(x;k)e^{-ikt}$, where $\widehat{\phi}_L(x;k)$ satisfies

$$\frac{\partial^2}{\partial x^2}\widehat{\phi}_L(x;k) + k^2\widehat{\phi}_L(x;k) = 0, \qquad (35)$$

in regions I, III, and V, and

$$\frac{\partial^2}{\partial x^2}\widehat{\phi}_L(x;k) + k^2 n^2(k)\widehat{\phi}_L(x;k)$$
$$= -\frac{4\pi\eta e}{m}\frac{ik}{\left[k_0^2 - k^2 - ik\widetilde{\gamma}(-ik)\right]}\widehat{F}(x;k), \qquad (36)$$

in regions II and IV. Here, $\widehat{F}(x;k)$ is the Fourier transform of the stochastic force operator. The explicit solution is presented in Appendix B.

In Sec. IV we will use the vacuum and Langevin contributions to the field operator in order to obtain the Casimir force between slabs. Before doing that, we describe in more detail the relation between the macroscopic electromagnetic properties of the slabs and the microscopic model.

III. GENERALIZED PERMITTIVITY FROM OPEN QUANTUM SYSTEM

With the aim of checking if our model is physically consistent, we analyze the properties of the refraction index given in Eq. (34). Considering that $\epsilon(\omega) = n^2(\omega)$ is the permittivity of the material plates, we have

$$\epsilon(\omega) - 1 = \frac{\omega_P^2}{\omega_0^2 - \omega^2 - i\omega\widetilde{\gamma}(-i\omega)}. \qquad (37)$$

We can define the susceptibility kernel $\chi(\tau)$ for the model as [26]

$$\chi(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}[\epsilon(\omega) - 1]e^{-i\omega\tau}d\omega$$
$$= \frac{\omega_P^2}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-i\omega\tau}}{\omega_0^2 - \omega^2 - i\omega\widetilde{\gamma}(-i\omega)}d\omega. \qquad (38)$$

In principle, this integral can be evaluated by contour integration. Inversely, the permittivity can be expressed in terms of

$\chi(\tau)$ as

$$\epsilon(\omega) = 1 + \int_{-\infty}^{+\infty} \chi(\tau)e^{i\omega\tau}d\tau, \qquad (39)$$

which can be viewed as a representation of $\epsilon(\omega)$ in the complex ω-plane. The permittivity is well defined when $\chi(\tau)$ is finite for all τ and $\chi(\tau) \to 0$ as $\tau \to \pm\infty$, and its analytical properties can be studied directly from this expression.

All properties of the permittivity and susceptibility functions are strongly dependent on the Laplace transform of the damping kernel $\widetilde{\gamma}(ik)$, which in turn depends on the spectral density of the environment. After taking the Laplace transform of Eq. (10), we obtain

$$\widetilde{\gamma}(s) = \frac{2}{m}\int_0^{+\infty} d\omega \frac{J(\omega)}{\omega}\frac{s}{(s^2+\omega^2)}. \qquad (40)$$

As already described, a physical spectral density must incorporate a cutoff function. One could use a sharp cutoff or, alternatively, choose a continuous cutoff function that approaches zero rapidly for frequencies greater than the cutoff frequency Λ, ensuring the convergence of the integral.

The first alternative, although simpler, makes the function $\widetilde{\gamma}(ik)$ not well defined in the complex plane. The second alternative solves this problem and allows the use of the residue theorem to evaluate the integral. Inserting Eq. (13) into Eq. (40), we obtain

$$\widetilde{\gamma}(s) = \frac{4\gamma_0 s}{\pi\Lambda^{\alpha-1}}\int_0^{+\infty} d\omega \frac{\omega^{\alpha-1}}{(s^2+\omega^2)}f\left(\frac{\omega}{\Lambda}\right). \qquad (41)$$

In order to apply the residue theorem, the integrand must be holomorphic on the superior complex half plane, except at a finite number of points which are not on the real axis. Thus, different results are obtained considering distributions with or without poles on the superior half plane. For an Ohmic environment ($\alpha = 1$) and no cutoff function it is easy to see that $\widetilde{\gamma}(s) = 2\gamma_0$.

In the case of distributions without poles (for example, a Gaussian cutoff function), we have

$$\widetilde{\gamma}_{\text{NP}}(-ik) = \frac{\pi}{mk}J(k) \equiv \widetilde{\gamma}_1(k), \qquad (42)$$

where the subscript NP denotes the fact that the distribution has no poles. The resulting function is real and even in the variable k.

On the other hand, the distributions usually considered in the literature have poles on $\pm i\Lambda$ (for example, a Lorentzian

distribution). In these cases, for odd ω^α [with $\alpha < 4$ to maintain the convergence in Eq.(41)], we get

$$\widetilde{\gamma}_P(-ik) = \frac{\pi}{mk}J_\Lambda(k) + i(-1)^{\frac{\alpha-1}{2}}\frac{\pi}{mk}\left(-\frac{k}{\Lambda}\right)^{\alpha-1}J_{-k}(\Lambda), \qquad (43)$$

where the subscripts on J denote the location of the pole. Although the resulting function is complex, the second equality in Eq. (42) remains valid.

Taking into account the above properties of the damping kernel, we now continue analyzing the properties of the permittivity and susceptibility functions. As a particular example, in the Drude model one has $\widetilde{\gamma}(-i\omega) \equiv \gamma_0$. Therefore, the denominator in Eq. (38) has two poles, both in the lower-half ω plane. Thus, as expected from a physical point of view, the susceptibility kernel shows a causal behavior, since it vanishes for $\tau < 0$. The analyticity of $\epsilon(\omega)$ in the upper-half ω plane allows the use of Cauchy's theorem, resulting in the well-known Kramers-Kronig relations for the real and imaginary part of the permittivity function $\epsilon(\omega)$.

In our more general case, the physical properties of $\epsilon(\omega)$ are determined by the function $\widetilde{\gamma}(-i\omega)$. This dissipation function is given by the theory of open quantum systems through Eqs. (42) and (43) and depends on the chosen cutoff function.

Let us first consider, for simplicity, the case in which the cutoff function has no poles, which is represented by the Eq. (42). For a given spectral density, the denominator in Eq. (38) reads

$$D_{\text{NP}}^{(\alpha)}(\omega) = \omega_0^2 - \omega^2 - i2\gamma_0\omega\left(\frac{\omega}{\Lambda}\right)^{\alpha-1}f\left(\frac{\omega}{\Lambda}\right). \qquad (44)$$

If we choose an Ohmic environment ($\alpha = 1$) and no cutoff function (which is equivalent to putting $f \equiv 1$), we reobtain the Drude model (if $\omega_0 = 0$) or the one-resonance model (when $\omega_0 \neq 0$). In principle, we could consider other values of α while keeping $f \equiv 1$. In this case, ω^α should be an odd function. For example, $\alpha = 3$ gives an ill-defined pole configuration, since one of the poles lies on the upper-half ω plane, breaking the analyticity of the integrand in Eq. (38) and resulting in a noncausal susceptibility, which turns out to be unphysical.

Therefore, we see that, for this supra-Ohmic environment, the use of a cutoff is unavoidable. We may use an analytical cutoff (like a Gaussian function), or a Lorentzian cutoff function. The first alternative leads to a denominator $D_{\text{NP}}^{(\alpha)}$ whose zeros cannot be obtained analytically. The second alternative, valid for $\alpha < 4$ such that ω^α is an odd function, leads to a denominator

$$D_P^{(\alpha)}(\omega) = \frac{(\Lambda^2+\omega^2)(\omega_0^2-\omega^2) + 2\gamma_0\Lambda^{3-\alpha}\omega^2[(-1)^{\frac{\alpha-1}{2}}\Lambda^{\alpha-2} - i\omega^{\alpha-2}]}{(\Lambda^2+\omega^2)}, \qquad (45)$$

which for $\alpha = 3$ gives

$$D_P^{(3)}(\omega) = \frac{\Lambda\omega_0^2 - i\omega_0^2\omega - (2\gamma_0+\Lambda)\omega^2 + i\omega^3}{\Lambda - i\omega}. \qquad (46)$$

We denote the zeros of $D_P^{(3)}$ as $\omega_i = \omega_0 x_i$ (with $i = 1, 2, 3$). The three roots turn out to be located in the lower-half ω plane, which ensures the causality property. Also, one of the roots is purely imaginary ($x_1 = -x_1^* = -i|x_1|$) and the two others

have the same (negative) imaginary part but opposite real parts ($x_3 = -x_2^*$). Thus, the susceptibility kernel reads

$$\chi_P^{(3)}(\tau) = -\left(\frac{\omega_P}{\omega_0}\right)^2 \left\{ \frac{(\Lambda - \omega_0|x_1|)}{(x_1 - x_2)(x_1 + x_2^*)} e^{-\omega_0|x_1|\tau} \right.$$
$$\left. + 2\mathrm{Re}\left[\frac{(\Lambda - i\omega_0 x_2)}{(x_2 - x_1)(x_2 + x_2^*)} e^{-i\omega_0 x_2 \tau} \right] \right\} \theta(\tau), \quad (47)$$

where it is clear that it is a causal real function and, due to the negativity of the imaginary part of the roots x_i, we have $\chi_P^{(3)}(\tau) \to 0$ for $\tau \to +\infty$, as expected. We also have that $\chi_P^{(3)}(0) = 0$ but $\chi_P^{(3)'}(0) \neq 0$, and therefore the asymptotic expression found in [26] still remains valid as well as the Kramers-Kronig relations.

It is worth noting that an Ohmic environment ($\alpha = 1$) can be also studied with a cutoff function, obtaining similar results.

All in all, we have shown that our model is physically consistent and generalizes previous results for the permittivity of absorbing media, including as a particular case the Drude model. Plasma-like models do not contain dissipation and can be obtained by taking $\gamma_0 = 0$, which corresponds to no coupling between the system and the bath.

IV. CASIMIR FORCE

A. Energy-momentum tensor and the different contributions to Casimir force

Once we have determined the two contributions to the field, we proceed to compute the Casimir force between the plates, as given by Eq. (26). For this purpose, it is necessary to compute the expectation value of $\widehat{T}_{xx}$ which is given by Eq. (25).

Considering the vacuum and Langevin contributions according to Eq. (31), we have

$$\widehat{T}_{xx}(x;t) = \tfrac{1}{2}\{[\partial_0(\widehat{\phi}_V + \widehat{\phi}_L)]^2 + [\partial_x(\widehat{\phi}_V + \widehat{\phi}_L)]^2\}$$
$$= \widehat{T}_{xx}^V \otimes \mathbb{I}_S \otimes \mathbb{I}_B + \mathbb{I}_\phi \otimes \mathbb{I}_S \otimes \widehat{T}_{xx}^L$$
$$+ (\partial_x \widehat{\phi}_V) \otimes \mathbb{I}_S \otimes (\partial_x \widehat{\phi}_L) + (\partial_t \widehat{\phi}_V) \otimes \mathbb{I}_S \otimes (\partial_t \widehat{\phi}_L).$$
$$(48)$$

It is worth remarking that there are cross terms which act over two parts of the total Hilbert space.

Because we are interested in the steady state of the system, which is assumed to be at thermal equilibrium, each part of the total density matrix is represented by a thermal-type density matrix. On the other hand, both field contributions $\widehat{\phi}_V$ and $\widehat{\phi}_L$ are linear on the annihilation and creation operators of their respective parts of the total Hilbert space. Thus, the cross terms do not contribute to the force in the case of thermal equilibrium. Then, the problem is reduced to computing the expectation values over thermal states of the operators $\widehat{T}_{xx}^V$ and $\widehat{T}_{xx}^L$; namely,

$$\mathfrak{f} = \langle \widehat{T}_{xx} \rangle = \mathrm{Tr}_\phi(\widehat{\rho}_\phi \widehat{T}_{xx}^V) + \mathrm{Tr}_B(\widehat{\rho}_B \widehat{T}_{xx}^L) = \mathfrak{f}_V + \mathfrak{f}_L. \quad (49)$$

Thus, the Casimir force also has two contributions:

$$F_C = \mathfrak{f}_I - \mathfrak{f}_{III} = (\mathfrak{f}_I^V + \mathfrak{f}_I^L) - (\mathfrak{f}_{III}^V + \mathfrak{f}_{III}^L) = F_C^V + F_C^L.$$
$$(50)$$

B. Vacuum Casimir force

For the vacuum contribution, $\widehat{T}_{xx}^V$ is quadratic in the annihilation and creation operators. Thus, in order compute the expectation value over the thermal state, we need to evaluate the expectation values of the products of the annihilation and creation operators. These are given by the known expressions

$$\langle \widehat{a}_k \widehat{a}_{k'} \rangle = \langle \widehat{a}_k^\dagger \widehat{a}_{k'}^\dagger \rangle = 0, \quad (51)$$
$$\langle \widehat{a}_k \widehat{a}_{k'}^\dagger \rangle = \delta(k - k')[1 + N(\omega_k)], \quad (52)$$
$$\langle \widehat{a}_k^\dagger \widehat{a}_{k'} \rangle = \delta(k - k')N(\omega_k), \quad (53)$$

where $N(\omega_k) = 1/(e^{\beta\omega_k} - 1)$.

Taking into account Eq. (32), we have, in region l

$$\mathfrak{f}_l^V(x) = \mathrm{Tr}_\phi(\widehat{\rho}_\phi \widehat{T}_{xx}^{V,l}) = \frac{1}{4}\left[\int_0^{+\infty} + \int_{-\infty}^0 \right] \frac{dk}{2\pi} \coth\left(\frac{\beta\omega_k}{2}\right)$$
$$\times \left(\omega_k |f_k^l(x)|^2 + \frac{1}{\omega_k}\left|\frac{df_k^l}{dx}\right|^2 \right), \quad (54)$$

which is identical to the expression for a nonabsorbing medium except that, in this case, there is a thermal factor $\coth(\beta\omega_k/2)$ related to the temperature of the field.

Using the solutions for the modes functions f_k^l in regions I and III (see Appendix A), the vacuum Casimir force is given by

$$F_C^V = \mathfrak{f}_I^V - \mathfrak{f}_{III}^V = \frac{1}{2}\int_0^{+\infty} \frac{dk}{2\pi} k \coth\left(\frac{\beta\omega_k}{2}\right)$$
$$\times [1 + |R_k|^2 + |T_k|^2 - 2(|C_k|^2 + |D_k|^2)]. \quad (55)$$

The coefficients R_k, T_k, C_k, and D_k are given explicitly in Appendix A. It is worth noting the appearance of a thermal global factor in the last expression, which comes from the field's thermal state at temperature T, based on the equilibrium assumption.

C. Langevin Casimir force

For computing the Langevin contribution to the force, it is necessary to know the expectation value (over the bath's thermal state) of the force operator, where binary products are evaluated at different frequencies.

For any time-dependent Hermitian operator, the expectation value evaluated at different times corresponds to the correlation function of the operator. This matches with half of the anticommutator expectation value at different times. Thus, making Fourier transforms over both times, we can compute the desired products of the Fourier transform of the force operator at different frequencies.

For the case of thermal equilibrium, the anticommutator expectation value at different times of the force operator is provided by the QBM theory. One can show that it matches the noise kernel $D_1(t - t')$ of Eq. (15). Thus, we obtain

$$\langle \{ \widehat{F}(k); \widehat{F}(k') \} \rangle = J(\omega_k) \coth\left(\frac{\beta\omega_k}{2}\right) \delta(k + k'). \quad (56)$$

Considering that, in our case, the stochastic force operator depends on the position where the atom is located, we finally have

$$\langle \widehat{F}(x;k)\widehat{F}(x';k')\rangle = \delta(x-x')\frac{J(\omega_k)}{2\eta}\coth\left(\frac{\beta\omega_k}{2}\right)\delta(k+k'),$$
(57)

where we have included the atom density η due to dimensional issues. As can be seen, the frequency spectrum is not flat.

Taking all this into account, $\widehat{T}_{xx}^L$ in regions I and III are given by

$$\widehat{T}_{xx}^{L,I}(x;t) = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}\int_{-\infty}^{+\infty}\frac{dk'}{2\pi}(-kk')\widehat{W}_1(k)\widehat{W}_1(k')$$
$$\times e^{-i(k+k')(x+t)},$$
(58)

$$\widehat{T}_{xx}^{L,III}(x;t) = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}\int_{-\infty}^{+\infty}\frac{dk'}{2\pi}(-kk')e^{-i(k+k')t}$$
$$\times (\widehat{W}_2(k)\widehat{W}_2(k')e^{i(k+k')x}$$
$$+ \widehat{W}_3(k)\widehat{W}_3(k')e^{-i(k+k')x}).$$
(59)

The coefficients $\widehat{W}_i(k)$ in these equations are given in Appendix B and are linear functions of the Fourier transform of the stochastic force operator. Taking into account the explicit expressions in Appendix B and Eq. (57), the desired expectation values are

$$\langle \widehat{W}_1(k)\widehat{W}_1(k')\rangle$$
$$= |\mathfrak{W}(k)|^2\frac{2\pi}{m}\frac{J(\omega_k)}{k^2\widetilde{\gamma}_1(k)}e^{-z_2}\coth\left(\frac{\beta\omega_k}{2}\right)\delta(k+k')$$
$$\times \{n_1(1-e^{-z_2})(|t|^2e^{z_2}+|r_n+re^{i2ka}|^2+e^{z_2}|1$$
$$+rr_ne^{i2ka}|^2+|t|^2|r_n|^2)+2n_2\text{Re}[|t|^2i(e^{-iz_1}-1)r_n^*$$
$$+i(1-e^{iz_1})(1+r^*r_n^*e^{-i2ka})(r_n+re^{i2ka})]\},$$
(60)

$$\langle \widehat{W}_2(k)\widehat{W}_2(k')\rangle$$
$$= \langle \widehat{W}_3(k)\widehat{W}_3(k')\rangle = |\mathfrak{W}(k)|^2\frac{2\pi}{m}\frac{J(\omega_k)}{k^2\widetilde{\gamma}_1(k)}(1+|r|^2)$$
$$\times \coth\left(\frac{\beta\omega_k}{2}\right)\delta(k+k')\{n_1(1-e^{-z_2})(1+|r_n|^2e^{-z_2})$$
$$+2n_2e^{-z_2}\text{Re}[i(1-e^{iz_1})r_n]\},$$
(61)

where $n = n_1 + in_2$ [i.e., $n_1 = \text{Re}(n)$ and $n_2 = \text{Im}(n)$], $\widetilde{\gamma}(ik) = \widetilde{\gamma}_1(k) + i\widetilde{\gamma}_2(k)$, $z_1 = 2kn_1d$, and $z_2 = 2kn_2d$. The explicit expressions for the coefficients r, r_n, and t can be found in Appendix A.

Therefore, the Langevin contribution to the force in regions I and III is given by

$$\mathfrak{f}_I^L = \text{Tr}_B\left(\widehat{\rho}_B\widehat{T}_{xx}^{L,I}\right) = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}|\mathfrak{W}(k)|^2\frac{2\pi}{m}\frac{J(\omega_k)}{\widetilde{\gamma}_1(k)}e^{-z_2}$$
$$\times \coth\left(\frac{\beta\omega_k}{2}\right)\{n_1(1-e^{-z_2})(|t|^2e^{z_2}+|r_n+re^{i2ka}|^2$$
$$+e^{z_2}|1+rr_ne^{i2ka}|^2+|t|^2|r_n|^2)$$
$$+2n_2\text{Re}[|t|^2i(e^{-iz_1}-1)r_n^*+i(1-e^{iz_1})$$
$$\times(1+r^*r_n^*e^{-i2ka})(r_n+re^{i2ka})]\},$$
(62)

$$\mathfrak{f}_{III}^L = \text{Tr}_B(\widehat{\rho}_B\widehat{T}_{xx}^{L,III}) = \int_{-\infty}^{+\infty}\frac{dk}{2\pi}|\mathfrak{W}(k)|^2\frac{4\pi}{m}\frac{J(\omega_k)}{\widetilde{\gamma}_1(k)}$$
$$\times \coth\left(\frac{\beta\omega_k}{2}\right)(1+|r|^2)\{n_1(1-e^{-z_2})(1+|r_n|^2e^{-z_2})$$
$$+2n_2e^{-z_2}\text{Re}[i(1-e^{iz_1})r_n]\}.$$
(63)

Taking advantage that the integration is over all the values of k, the fact that the change $k \leftrightarrow -k$ is equivalent to complex conjugation, and the second equality of Eq. (42), we obtain

$$\mathfrak{f}_I^L = \int_0^{+\infty}\frac{dk}{2\pi}\frac{k}{2}\frac{|n+1|^2}{|n|^2}\frac{8|t|^2n_1e^{-z_2}}{|1-r^2e^{i2ka}|^2}\coth\left(\frac{\beta k}{2}\right)(1-e^{-z_2})$$
$$\times(|t|^2e^{z_2}+|r_n+re^{i2ka}|^2+e^{z_2}|1+rr_ne^{i2ka}|^2+|t|^2|r_n|^2),$$
(64)

$$\mathfrak{f}_{III}^L = \int_0^{+\infty}\frac{dk}{2\pi}\frac{k}{2}\frac{|n+1|^2}{|n|^2}\frac{16|t|^2n_1}{|1-r^2e^{i2ka}|^2}\coth\left(\frac{\beta k}{2}\right)(1-e^{-z_2})$$
$$\times(1+|r|^2)(1+|r_n|^2e^{-z_2}).$$
(65)

Note that the presence of the thermal factor $\coth(\frac{\beta k}{2})$ is in agreement with the null temperature result obtained in other works for an Ohmic environment [9], since when $T \to 0$, $\coth(\frac{\beta k}{2}) \to 1$.

Finally, the Langevin contribution for the Casimir force is

$$F_C^L = \mathfrak{f}_I^L - \mathfrak{f}_{III}^L = \int_0^{+\infty}\frac{dk}{2\pi}\frac{k}{2}\frac{|n+1|^2}{|n|^2}\frac{8|t|^2n_1}{|1-r^2e^{i2ka}|^2}$$
$$\times \coth\left(\frac{\beta k}{2}\right)(1-e^{-z_2})[|t|^2+|r_n+re^{i2ka}|^2e^{-z_2}$$
$$+|1+rr_ne^{i2ka}|^2+|t|^2|r_n|^2e^{-z_2}-2(1+|r|^2)$$
$$\times(1+|r_n|^2e^{-z_2})].$$
(66)

It should be noted that here also appears a global thermal factor, as in the vacuum case, but this comes from the bath's temperature while in the vacuum case comes from the field's equilibrium temperature.

Note also that F_C^L vanishes when there is no coupling to an environment, since in this case the refraction index is real and therefore $z_2 = 0$.

D. Total Casimir force

The total Casimir force is determined from the expressions (50), (55), and (66). The resulting force can be written in a very compact form. On the one hand, it can be proven that the total free energy in region I (outside the plates) coincides with that for the free field at temperature T; that is,

$$\mathfrak{f}_I = \mathfrak{f}_I^V + \mathfrak{f}_I^L = \int_0^{+\infty}\frac{dk}{2\pi}k\coth\left(\frac{\beta k}{2}\right),$$
(67)

which is expectable due to translational invariance outside the plates and our assumption of thermal equilibrium. On the other hand, in region III we have

$$\mathfrak{f}_{III} = \mathfrak{f}_{III}^V + \mathfrak{f}_{III}^L = \int_0^{+\infty}\frac{dk}{2\pi}k\coth\left(\frac{\beta k}{2}\right)\frac{(1-|r|^4)}{|1-r^2e^{i2ka}|^2}.$$
(68)

Therefore, the total Casimir force is finally written as

$$F_{\text{C}}[a] = \mathfrak{f}_I - \mathfrak{f}_{III} = \int_0^{+\infty} \frac{dk}{2\pi} k \coth\left(\frac{\beta k}{2}\right)$$
$$\times \left(1 - \frac{1 - |r|^4}{|1 - r^2 e^{i2ka}|^2}\right), \tag{69}$$

or equivalently

$$F_{\text{C}}[a] = -\frac{1}{\pi}\text{Re}\left[\int_0^{+\infty} dk k \coth\left(\frac{\beta k}{2}\right) \frac{r^2 e^{i2ka}}{1 - r^2 e^{i2ka}}\right]. \tag{70}$$

This expression is formally identical to the case of a dissipative material (Ohmic environment) at zero temperature found in previous works [9,25]. However, it contains two generalizations: On the one hand, temperature has been included in the formalism in a natural way by means of the open quantum systems theory. On the other hand, the equation is valid for a general environment: the refraction index is in general complex and is dependent on the function $\widetilde{\gamma}(ik)$, which comes from the interactions at the microscopic level. The associated permittivity depends on the type of environment and reproduces known results (as the Drude model) as particular examples.

E. Convergence and limits

In this section we study some properties and limits of our final result Eq. (70). Let us first study the convergence of this expression. In general, the Casimir force calculations involve several regularization methods to achieve a finite result. A usual approach is to introduce a high-frequency cutoff in order to take into account the fact that real materials are transparent at high frequencies. This characteristic is already incorporated in our model. Indeed, the complex refraction index n includes all the environment properties which produce dissipation and noise. For large values of k, taking into account Eq. (34), one can check that

$$n_1 \to 1 - \frac{2\pi\eta e^2}{m}\frac{1}{k^2}, \tag{71}$$

$$n_2 \to \frac{2\pi\eta e^2}{m}\frac{\widetilde{\gamma}_1(k)}{k^3}. \tag{72}$$

Then, in the same limit, the reflection coefficient r behaves as

$$r \to \frac{\pi\eta e^2}{m}\frac{1}{k^2}(1 - e^{i2kd}), \tag{73}$$

and in consequence the integrand of Eq. (70) is $O(k^{-3})$ for large values of k. Thus, the convergence is ensured when $k \to +\infty$, regardless of the temperature and the type of environment considered.

On the other hand, Eq. (70) contains as particular cases some known results. The nonabsorbing-medium case can be easily obtained by setting the relaxation constant $\gamma_0 = 0$ in all the expressions. This makes the refraction index real, which cancels the Langevin contribution in Eq. (66) since the factor $1 - e^{-z_2} \to 0$. Thus, only the vacuum contribution in Eq. (55) survives but with a real refraction index [25].

Another important limit is the well-known Lifshitz formula. In the original work [8], Lifshitz considered semispaces separated by a finite distance. Therefore, one should take the limit of large thickness ($d \to +\infty$), in which r should be replaced by r_n. After a rotation to the imaginary-frequency axis, Eq. (70) becomes, in the $T = 0$ case,

$$F_{\text{C}}[a] = \frac{1}{\pi}\int_0^{+\infty} ds s \frac{r_n^2(is)e^{-2sa}}{1 - r_n^2(is)e^{-2sa}}, \tag{74}$$

which is of the form of the Lifshitz formula for a scalar field in $1 + 1$ dimensions. In the case $T \neq 0$, we must take into account that $\coth(\frac{\beta k}{2})$ has poles on the imaginary axis at the Matsubara frequencies $2\pi i\beta j = i\xi_j, j = 0, 1, 2, \ldots$. Therefore, the integration path can be rotated to the imaginary axis, but must be deformed to avoid the poles. This is a standard procedure that converts the integral over frequencies into the Matsubara sum

$$F_{\text{C}}[a] = 2T\sum_{j\geqslant 1}\xi_j \frac{r_n^2(i\xi_j)e^{-2\xi_j a}}{1 - r_n^2(i\xi_j)e^{-2\xi_j a}}, \tag{75}$$

which is the standard expression for Lifshitz formula at $T \neq 0$.

It is interesting to remark that, in our simplified $1 + 1$ model, there is no discontinuity in the transition between the Drude and plasma models. The Drude model is recovered, in the Ohmic-environment case, by setting $\omega_0 = 0$ (i.e., free particles instead of harmonic oscillators for modeling the dielectric atoms) and, once this limit is taken, the plasma model corresponds to the particular case $\gamma_0 = 0$ (no coupling to the environment). The reflection coefficient $r_n \to 1$ in the zero-frequency limit, for any value of ω_{P} and γ_0, even setting $\gamma_0 = 0$ from the beginning. In the case of absence of coupling to an environment, of course one must assume thermal equilibrium. Analogies between thermodynamics of a free Brownian particle and that of an electromagnetic field between two mirrors of finite conductivity have been studied in Ref. [27].

V. CONNECTION WITH EUCLIDEAN FORMALISM

Given that we are assuming thermal equilibrium between the different parts of the system, the results presented in this paper for the Casimir force could be derived following a functional approach in Euclidean space, as described for instance in Refs. [28,29]. We mention briefly the relation between both approaches.

As is well known, in $1 + 1$ dimensions the free energy $\mathcal{E}$ for a quantum system in thermal equilibrium at temperature T can be computed as

$$\mathcal{E} = -T\ln\frac{\mathcal{Z}(a)}{\mathcal{Z}(\infty)}, \tag{76}$$

where $\mathcal{Z}(a)$ is the partition function when the plates are separated by a distance a. The partition function can be represented by the functional integral

$$\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}r \mathcal{D}q_n e^{-S_E}, \tag{77}$$

where S_E is the Euclidean action for the full system. The integration is performed by imposing periodic boundary conditions on the temporal coordinate.

After integrating the matter and bath degrees of freedom it is possible to find an effective action for the scalar field of the form

$$S_{\text{eff}} = \int d^2x \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \int d^2x \int d^2x' V(x,x') \phi(x) \phi(x'),$$
(78)

and the vacuum-persistence amplitude reads

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S_{\text{eff}}}.$$
(79)

The effective action for the scalar field is quadratic because of the linear coupling we are choosing for the interaction between the vacuum field and the matter degrees of freedom. The potential $V(x,x')$ is different from zero only inside the plates and, as in our model the field $\phi(x,t)$ interacts only with the atom at x, it can be shown that the potential is of the form $V(x,t,x',t) = \delta(x-x')\lambda(t-t')$. The function $\lambda(t-t')$ encodes the information about the interaction of the vacuum field with the matter degrees of freedom, and also of the influence of the thermal bath. Its Fourier transform is related to the reflection coefficient of the slab.

Formally, the vacuum persistence amplitude is given by the functional determinant

$$\mathcal{Z} = (\det[-\Box + V])^{-\frac{1}{2}}.$$
(80)

An explicit evaluation of this determinant leads to the Lifshitz formula [29]. So when considering thermal equilibrium, one has an alternative route to the evaluation of the Casimir force, even when the field is coupled to other degrees of freedom. However, this Euclidean functional approach would not be adequate to compute the force for other initial states or, in general, in nonequilibrium situations. In these cases, the use of the theory of open quantum systems as described in this paper is unavoidable.

VI. CONCLUSIONS

In this paper we have presented a derivation of the Casimir force between two absorbing slabs in the framework of the theory of open quantum systems. We worked with a simplified model of a scalar vacuum field in $1+1$ dimensions. In order to describe the interaction of the vacuum field with the mirrors, we considered a model analogous to the HB model for QED, where the matter degrees of freedom are described by a continuous set of harmonic oscillators, which are coupled not only to the vacuum field but also to a thermal bath that accounts for dissipative effects.

Following a standard procedure in the theory of open quantum systems, we showed that the field operator satisfies the modified Klein Gordon equation (22). This is a nonlocal Langevin equation, which describes the interaction of the vacuum field with the matter degrees of freedom and the effects of the thermal bath on its dynamics (the environment is indirectly coupled to the quantum field through the matter). This equation is similar to the equation that describes a Brownian particle coupled to an environment. Both the noise and dissipation are determined by the properties of the environment.

The field operator that solves this "Klein-Gordon–Langevin" equation can be written as the sum of two terms: a vacuum contribution and a Langevin contribution. The same happens with the associated energy-momentum tensor, and therefore we have a similar decomposition for the Casimir force between slabs. The final result for the total Casimir force is equivalent to a $1+1$ version of the Lifshitz formula, expressed in terms of the reflection coefficients associated with the slabs. Therefore, we have presented, in this simplified model, a first-principles derivation of Lifshitz formula in the framework of quantum open systems.

The present work is closely related to Ref. [9], which has been improved upon and generalized in several directions. Indeed, that work assumes the simplest forms for noise and dissipation (Ohmic environment and constant dissipation, respectively), without specifying the properties of the environment. Moreover, it is doubtful whether a general non-Ohmic environment can produce such effects at $T = 0$. Here we worked at $T \neq 0$ and considered very general environments. We also linked the properties of the environment with the macroscopic electromagnetic properties of the mirrors. There is also a close relationship with the recent work [17], where the authors computed the force density associated with spatial variations of the permittivity. As compared with ours, in this reference the authors considered the more realistic case of a $3+1$ dimensional electromagnetic field, but only in the particular case of $T = 0$ and constant dissipation. Moreover, they did not consider the presence of boundaries as we did here, which allowed us to compute explicitly the Casimir force between slabs and to obtain Lifshitz formula.

In order to apply the open quantum systems approach to a realistic calculation of the Casimir force, we should generalize our results to a $3+1$ model with the electromagnetic field. Although technically more complex, we do not expect conceptual complications in doing so. Regarding the long-standing controversy about the temperature corrections to the Casimir force, a crucial point is the behavior of the quantity

$$\lim_{\zeta \to 0} \zeta^2 [\epsilon(i\zeta) - 1],$$

which vanishes for the Drude model and is different from zero for the plasma model, producing in the latter case an additional contribution to the force coming from the TE zero mode. In the kind of microscopic models considered here, the TE zero mode is suppressed as long as $\omega_0 \neq 0$, as can be seen from Eq. (37).

Finally, in $3+1$ dimensional models one could consider more general initial states and/or nonequilibrium situations. We hope to address this issue in a forthcoming presentation.

APPENDIX A: BOUNDARY CONDITIONS AND SOLUTIONS FOR VACUUM CONTRIBUTION

In this appendix we present the explicit form of the solutions of Eq. (33). In each region we have

$$f_k^I(x) = e^{ikx} + R_k e^{-ikx}, \tag{A1}$$

$$f_k^{II}(x) = A_k e^{iknx} + B_k e^{-iknx}, \tag{A2}$$

$$f_k^{III}(x) = C_k e^{ikx} + D_k e^{-ikx}, \tag{A3}$$

$$f_k^{IV}(x) = E_k e^{iknx} + F_k e^{-iknx}, \tag{A4}$$

$$f_k^V(x) = T_k e^{ikx}. \tag{A5}$$

The different coefficients can be obtained by imposing continuity of the mode functions and their derivatives at the interfaces. They are given by

$$R_k = e^{-ika}\left(r + \frac{t^2 r e^{i2ka}}{(1 - r^2 e^{i2ka})}\right), \tag{A6}$$

$$A_k = e^{-ik\frac{a}{2}} e^{ikn\frac{a}{2}} \frac{t(1 + r_n r e^{i2ka})}{t_n(1 - r^2 e^{i2ka})}, \tag{A7}$$

$$B_k = e^{-ik\frac{a}{2}} e^{ikn\frac{a}{2}} \frac{t(r_n + r e^{i2ka})}{t_n(1 - r^2 e^{i2ka})}, \tag{A8}$$

$$C_k = \frac{t}{(1 - r^2 e^{i2ka})}, \tag{A9}$$

$$D_k = \frac{t r e^{ika}}{(1 - r^2 e^{i2ka})}, \tag{A10}$$

$$E_k = \frac{t^2}{t_n(1 - r^2 e^{i2ka})} e^{ik\frac{a}{2}} e^{ikd} e^{-ikn\frac{a}{2}} e^{-iknd}, \tag{A11}$$

$$F_k = \frac{t^2 r_n}{t_n(1 - r^2 e^{i2ka})} e^{ik\frac{a}{2}} e^{ikd} e^{ikn\frac{a}{2}} e^{iknd}, \tag{A12}$$

$$T_k = \frac{t^2}{(1 - r^2 e^{i2ka})}, \tag{A13}$$

for $k > 0$ (while for $k < 0$ the order of the solutions must be reversed and the refractive index and the coefficients should be conjugated), where $r = [r_n(e^{i2knd} - 1)]/(1 - r_n^2 e^{i2knd})$ and $t = \frac{4n}{(n+1)^2}(e^{iknd}e^{-ikd})/(1 - r_n^2 e^{i2knd})$ are the reflection and transmission coefficients for one plate, while $r_n = \frac{n-1}{n+1}$ and $t_n = \frac{2n}{n+1}$ are those for an interface. The coefficients R_k and T_k can be interpreted as the reflection and transmission coefficients of the two-plate configuration. However, it should be noted that, due to the presence of absorption, $|r|^2 + |t|^2 \neq 1$ and $|R_k|^2 + |T_k|^2 \neq 1$ in this case.

APPENDIX B: BOUNDARY CONDITIONS AND SOLUTIONS FOR LANGEVIN CONTRIBUTION

In this appendix we solve Eqs. (35) and (36). Due to the presence of a source in regions II and IV [see Eq. (36)], the solutions will have to parts: one associated with the homogeneous equation and other related directly to the source. Therefore, the solutions are

$$\widehat{\phi}_L^I(x;k) = \widehat{W}_1(k)e^{-ikx}, \tag{B1}$$

$$\widehat{\phi}_L^{II}(x;k) = \widehat{U}_1(k)e^{iknx} + \widehat{U}_2(k)e^{-iknx} + \widehat{A}_1(x;k)e^{iknx} + \widehat{A}_2(x;k)e^{-iknx}, \tag{B2}$$

$$\widehat{\phi}_L^{III}(x;k) = \widehat{W}_2(k)e^{ikx} + \widehat{W}_3(k)e^{-ikx}, \tag{B3}$$

$$\widehat{\phi}_L^{IV}(x;k) = \widehat{V}_1(k)e^{iknx} + \widehat{V}_2(k)e^{-iknx} + \widehat{B}_1(x;k)e^{iknx} + \widehat{B}_2(x;k)e^{-iknx}, \tag{B4}$$

$$\widehat{\phi}_L^V(x;k) = \widehat{W}_4(k)e^{ikx}, \tag{B5}$$

with

$$\widehat{A}_1(x;k) = \frac{1}{2n}\int_{-d-\frac{a}{2}}^x \widehat{G}(x';k)e^{-iknx'}dx', \tag{B6}$$

$$\widehat{A}_2(x;k) = -\frac{1}{2n}\int_{-d-\frac{a}{2}}^x \widehat{G}(x';k)e^{iknx'}dx', \tag{B7}$$

$$\widehat{B}_1(x;k) = \frac{1}{2n}\int_{\frac{a}{2}}^x \widehat{G}(x';k)e^{-iknx'}dx', \tag{B8}$$

$$\widehat{B}_2(x;k) = -\frac{1}{2n}\int_{\frac{a}{2}}^x \widehat{G}(x';k)e^{-iknx'}dx', \tag{B9}$$

where, for simplicity, we write $\widehat{G}(x;k) = \frac{4\pi\eta e}{m}\widehat{F}(x;k)/[k_0^2 - k^2 - ik\widetilde{\gamma}(-ik)]$.

The coefficients $\widehat{W}_l(k)$, $\widehat{U}_l(k)$, and $\widehat{V}_l(k)$ are obtained by means of the appropriate boundary conditions. Thus, they are given by

$$\widehat{W}_1(k) = \mathfrak{W}(k)e^{iknd}e^{-ik(a+d)}[\widehat{K}(1 + rr_n e^{i2ka}) + \widehat{L}(r_n + re^{i2ka}) + \widehat{M}te^{ikd}e^{ik(a-nd)} + \widehat{N}tr_n e^{ikd}e^{ik(a+nd)}], \tag{B10}$$

$$\widehat{W}_2(k) = \mathfrak{W}(k)(\widehat{K}r_n e^{i2knd} + \widehat{L} + \widehat{M}re^{ika} + \widehat{N}rr_n e^{ik(a+2nd)}), \tag{B11}$$

$$\widehat{W}_3(k) = \mathfrak{W}(k)(\widehat{K}rr_n e^{ik(a+2nd)} + \widehat{L}re^{ika} + \widehat{M} + \widehat{N}r_n e^{i2knd}), \tag{B12}$$

$$\widehat{U}_1 = \frac{r_n}{t_n}e^{ik(n+1)(\frac{a}{2}+d)}\widehat{W}_1, \tag{B13}$$

$$\widehat{U}_2 = \frac{1}{t_n}e^{ik(1-n)(\frac{a}{2}+d)}\widehat{W}_1, \tag{B14}$$

$$\widehat{V}_1 = \frac{1}{t_n}e^{ik(1-n)\frac{a}{2}}(\widehat{W}_2 + r_n e^{-ika}\widehat{W}_3), \tag{B15}$$

$$\widehat{V}_2 = \frac{1}{t_n}e^{ik(n-1)\frac{a}{2}}(r_n e^{ika}\widehat{W}_2 + \widehat{W}_3), \tag{B16}$$

$$\widehat{W}_4 = e^{ik(n-1)d}\left(\widehat{W}_2 + r_n e^{-ika}\widehat{W}_3 + \frac{e^{-ik\frac{a}{2}}}{(n+1)}\widehat{N}\right), \tag{B17}$$

where

$$\widehat{K} = e^{ikn\frac{a}{2}}\int_{-d-\frac{a}{2}}^{-\frac{a}{2}} \widehat{G}(x;k)e^{iknx}dx, \tag{B18}$$

$$\widehat{L} = e^{-ikn\frac{a}{2}}\int_{-d-\frac{a}{2}}^{-\frac{a}{2}} \widehat{G}(x;k)e^{-iknx}dx, \tag{B19}$$

$$\widehat{M} = e^{-ikn\frac{a}{2}}\int_{\frac{a}{2}}^{d+\frac{a}{2}} \widehat{G}(x;k)e^{iknx}dx, \tag{B20}$$

$$\widehat{N} = e^{ikn\frac{a}{2}} \int_{\frac{a}{2}}^{d+\frac{a}{2}} \widehat{G}(x;k)e^{-iknx}dx, \qquad \text{(B21)}$$

$$\mathfrak{W}(k) = \frac{we^{ik\frac{a}{2}}}{2n(1 - r^2 e^{i2ka})} \quad \text{with} \quad w = \frac{2n}{(n+1)(1 - r_n^2 e^{i2knd})}. \tag{B22}$$

The Langevin contribution is evaluated in the five regions. Since $\widehat{K}, \widehat{L}, \widehat{M}$, and $\widehat{N}$ depend linearly on the Fourier transform of the stochastic force operator, it should be noted that the coefficients also depend on the same way. In fact, since the stochastic force operator depends linearly on the bath's annihilation and creation operators, the Langevin contribution depend in that way too.

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