Applied Mathematical Modelling 50 (2017) 585–603

Contents lists available at ScienceDirect

Applied Mathematical Modelling

journal homepage: www.elsevier.com/locate/apm

An anisotropic continuum model for traffic assignment in mixed transportation networks

Víctor H. Cortínez [a,b,c,*], Patricia N. Dominguez [a,b]

[a] Centro de Investigaciones en Mecánica Teórica y Aplicada, Universidad Tecnológica Nacional, 11 de abril 445, Bahía Blanca, Argentina
[b] Departamento de Ingeniería, Universidad Nacional del Sur., Av. Alem 1253, Bahía Blanca, Argentina
[c] Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), Argentina

ARTICLE INFO

Article history:
Received 16 April 2016
Revised 24 May 2017
Accepted 5 June 2017
Available online 15 June 2017

Keywords:
Traffic assignment
Two-dimensional anisotropic continuum model
Finite element method
Urban traffic congestion

ABSTRACT

This work deals with a two-dimensional continuum model for the problem of congested traffic assignment in an urban transportation system consisting of a set of freeways superimposed over a dense street network. The formulation leads to a system of non-linear differential equations whose unknowns are given by the travel times from arbitrary points of the network to the corresponding destinations. The governing equations are appropriately solved by means of the Finite Element Method. Then, traffic flow on every link of the network can be obtained. Numerical examples are given in order to demonstrate the efficiency of the developed model.

1. Introduction

Urban traffic congestion is one of the most important problems in modern cities. Moreover, motor vehicles constitute the main sources of air and noise pollution.

The development of cities requires frequent modifications of the traffic network. In order to evaluate the effect of these changes, the prediction of traffic patterns is of great importance. To fulfill this objective, mathematical modeling is used to estimate traffic flows and travel times between different points of the city. This problem, known as “static traffic assignment”, can be formulated by means of the Wardrop’s first principle [1]. According to this concept, a user chooses the route toward a given destination with the aim of minimizing his (her) own travel time (at least for frequently performed trips). This principle leads to an equivalent optimization problem developed by Beckmann et al. [2]. It constitutes the base for a discrete model where each link of the network is considered individually and the demand is assumed to be concentrated at fictitious points denominated “centroids”. Link flows constitute the unknowns of the problem. A detailed explanation of this formulation along with solution algorithms may be found in Sheffi’s work [3]. This and other similar approaches have long been used by several authors for problems of design and management of transportation networks [4–7]. The discrete approach allows to carry out a very detailed representation of the network and, accordingly, the same degree of detail may be obtained for link flows and travel times. However, this comprehensive description becomes a disadvantage in very large networks, since computation time increases drastically with the number of nodes and links. For this reason, an extensive research activity has been directed to the development of more efficient methods to analyze this problem [8–10].

* Corresponding author at: Centro de Investigaciones en Mecánica Teórica y Aplicada, Universidad Tecnológica Nacional, 11 de abril 445, Bahía Blanca, Argentina.

E-mail addresses: vcortine@frbb.utn.edu.ar (V.H. Cortínez), pdoming@uns.edu.ar (P.N. Dominguez).

http://dx.doi.org/10.1016/j.apm.2017.06.004

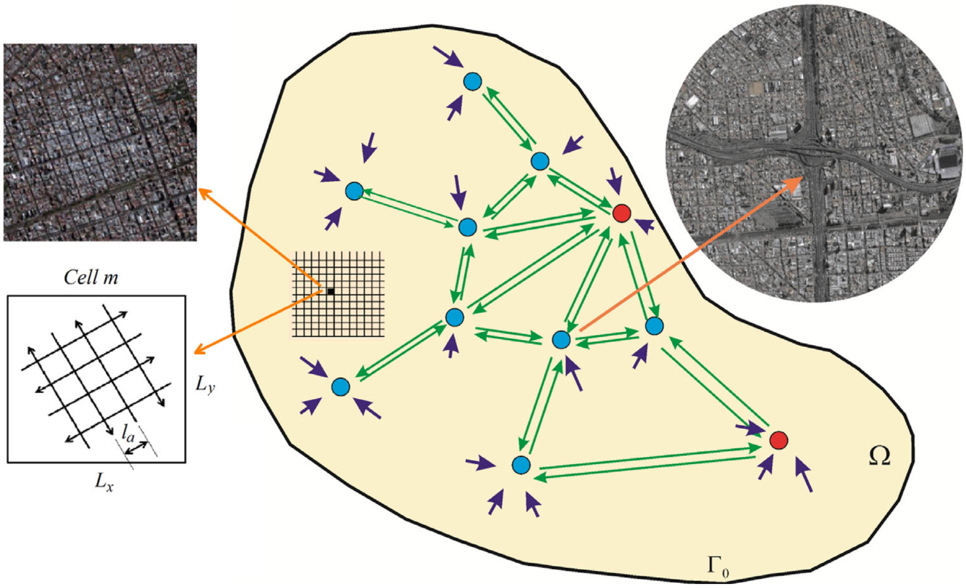

Fig. 1. Scheme of the urban traffic network.

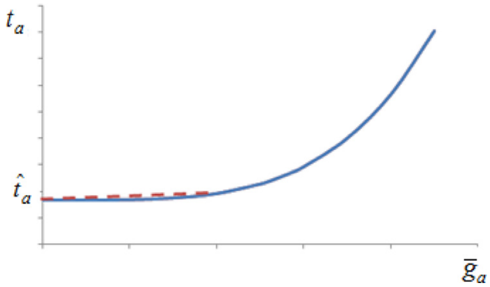

Fig. 2. Link cost function.

A different and less used approach is the continuum formulation of the assignment problem. According to this methodology, a dense network of streets is approximated by means of a two-dimensional continuous medium. The fundamental idea is that variations in the traffic pattern between adjacent regions are small when compared to differences in the entire system. Accordingly, continuous functions may be used to represent network characteristics such as travel costs and flows.

This formulation leads to a system of partial differential equations [11], the solution of which can be obtained analytically for simple cases or numerically for more complex situations.

In their excellent review, Ho and Wong [12] have indicated the following advantages of the continuum approach over the discrete one for very dense networks: (a) it can reduce the size of the problem, since the number of variables in the continuum model depends on the numerical method adopted for solving the problem but not on the quantity of nodes of the network itself; (b) it can be applied without the need of a detailed construction of the network graph; (c) it can offer a quick view of the impact produced by a change in the network, and (d) it makes unnecessary certain non-real hypotheses established for the discrete model such as the artificial definition of "centroids". These advantages are important in the initial stage of planning and regional studies of great scale. In the following, a brief description of the state of the art is presented. Different application possibilities of this approach can be found in the above mentioned Ref. [12].

Beckmann [13] introduced a continuum approach to analyze the problem of finding the most cost-effective commodity flow with continuously distributed production over a geographical zone. He suggested this problem could be formulated by means of a minimization program constrained by continuity equations.

In subsequent years, continuum models, similar to that developed by Beckmann, were applied to cities with simple shapes for which analytical solutions were obtained. The circular configuration, where the movements are constrained to the radial and circumferential directions, is one of the most analyzed in the literature (Lam and Newell [14], D'Este [15] and Wong [16] among others). The case of rectangular cities was analyzed by Dafermos [17] and Vaughan and Doyle [18].

For arbitrary shape cities, appropriate numerical methods were employed, such as the Finite Element Method (FEM), in order to overcome the difficulty in obtaining analytical solutions.

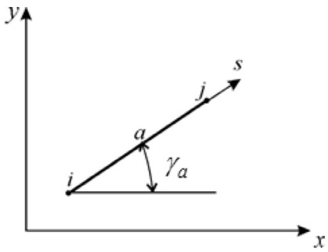

Fig. 3. Scheme of an arbitrary link *a*.

Taguchi and Iri [19] were the first authors to apply FEM for solving the continuum model of transportation networks with arbitrary configuration. They explored three typical problems using three different objective functions: maximum flow, minimum travel distance and minimum total travel cost.

Later, Sasaki et al. [11] developed a continuum formulation of the user equilibrium problem similar to the Beckmann discrete model [2]. They started with a continuum version of the Beckmann variational principle (primal problem) in which the unknown is given by the flow vector. Then, they obtained the corresponding dual formulation in terms of travel time (or potential function) between an arbitrary point of the network and the city centre. This problem was solved by FEM.

The continuum model has been extended to more complex situations using the mixed FEM formulation proposed by Wong et al. [20], in which traffic flow and total travel time are considered to be variationally independent variables. This approach has been generalized to consider multiple user classes [21], distribution-assignment [22,23], toll [24,25] and housing allocation problems [26]. An interesting contribution was proposed by Xu and He [27], who used the continuum model to determine the best location of urban routes in the planning process. For the solution of the problem, they applied FEM along with the Evolutive Structural Optimization method (ESO).

Recently, an interesting calibration procedure for the local travel cost function, adopted in the continuum models previously mentioned, was proposed by Wong and Wong [28–30].

Moreover, it is relevant to mention the concept of macroscopic fundamental diagram (MFD), used in traffic congestion dynamics, because it has certain similarity with the continuum approach considered above. In fact, it is a network-level description that avoids the analysis of individual links. MFD is an observation-based tool that provides a uni-modal, low scatter relationship between network vehicle accumulation and outflow in homogeneous urban areas (MFD can also be expressed as accumulation vs space-mean flow). As a remarkable aspect, MFD can consider over congested flows. This new approach is very useful in the design of control strategies for decreasing congestion in large urban networks [31,32].

The static assignment problem in a mixed transportation system, consisting of major freeways and dense surface streets, was performed by several authors [33–36].

Yang et al. [33] have analyzed a monocentric city considering the freeway links in a discrete form and the street network by means of the continuum approach previously developed by Sasaki et al. [11]. Robusté et al. [34] employed a similar formulation, although the resultant optimization problem was solved by means of Genetic Algorithms. Wong et al. [35] proposed an improved approach in which the problem was reformulated as a simultaneous optimization model. This problem was solved by using a fixed point algorithm based on a sensitivity analysis of both the discrete and continuous systems. In that work, the continuum network was analyzed by means of a mixed variational formulation in which the unknowns are given by flows and travel times simultaneously. Recently, Du et al. [36] proposed an extension of the aforementioned work in order to take into account several destination points.

However, some important aspects were not considered in the continuum models cited previously. First, travel cost functions were formulated directly for the continuum case. Accordingly, there is no evident relationship between the discrete and continuum formulations. Unfortunately, in this way, the experience obtained in the development and calibration of link travel time expressions cannot be exploited. Moreover, the local travel cost used in previous continuum models is isotropic. In fact, it depends on the flow intensity but not on the direction. This hypothesis leads to isotropic relationships between the local flow vector and the gradient of the potential function. However, the discrete transportation network could show impedance properties dependent on the direction (for example, in the case of an urban cell having streets in N-S direction with higher capacities than those of streets in E-W direction). This fact should be reflected in the continuum model by means of an anisotropic relationship between the flow vector and the gradient of the potential function in a similar form to the generalized Darcy's filtration law in porous media. This concept of network anisotropy should not be confused with the anisotropy related to traffic dynamics models [37,38].

Previous continuum approaches could present a difficulty to determine individual link flows. This is not a disadvantage when the objective is to obtain general traffic flow patterns, especially when the details of the network structure are not known. However, there are situations where the topological and geometrical features of the network are known, at least in an approximate manner. For instance, this may be the case when the network is existing and the aim is to redesign a part of it. In this case, it could be interesting to recover individual link flows. These magnitudes are useful to estimate emission rates of noise and air pollutants, since the corresponding formulas depend on such flows. Moreover, the knowledge of link flows could be useful for the design of traffic signals in a localized zone.

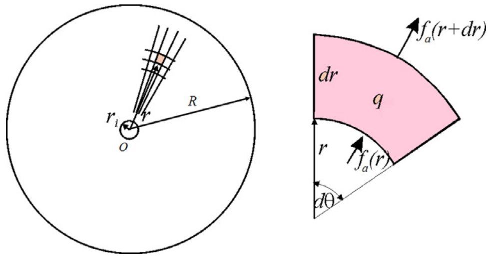

Fig. 4. Radial city. Differential element for the model in cylindrical coordinates.

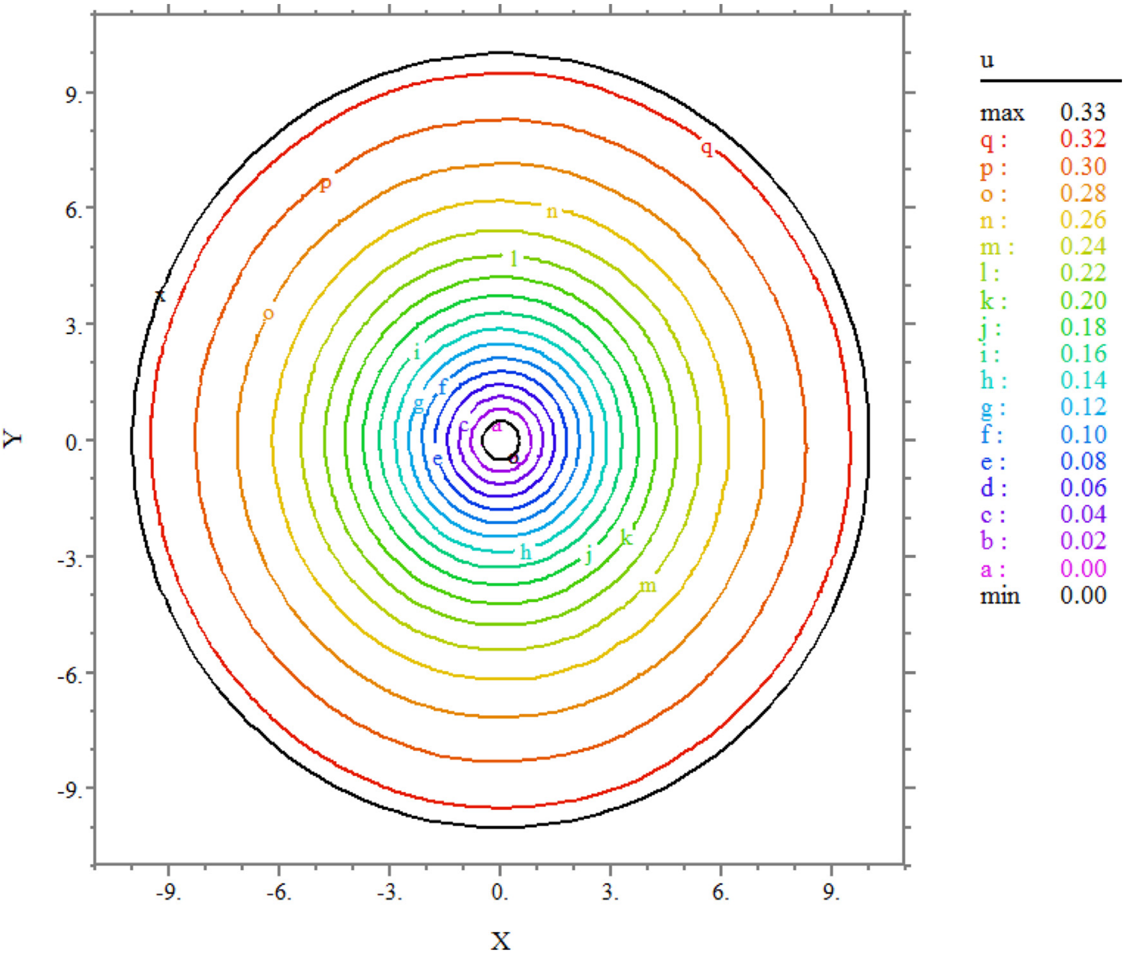

Fig. 5. Radial city. Contour plot of travel time to the centre (hours).

Two recent works, partially considering some of these aspects, are those by Saumtally et al. [39] and Cortínez and Dominguez [40].

Saumtally et al. [39] developed a local travel cost function for a two dimensional anisotropic continuum. Based on this function, they proposed a Beckmann-like variational formulation and used the Uzawa algorithm for the numerical solution. They applied this methodology to analyze congestion traffic patterns in the city of Paris.

Cortínez and Dominguez [40] developed a continuum approximation of the discrete transportation network by using a homogenization procedure. The objective of the homogenization techniques, that were successfully employed in different areas of science [41–43], is to obtain an equivalent continuous representation of a discrete medium (or of an heterogeneous continuous medium) preserving information about the properties of the original one.

In order to do this, the static assignment problem for the discrete transportation network was derived from the Beckmann dual formulation, where the fundamental unknown corresponds to the travel time between an arbitrary point in the network and the city centre. The applied homogenization methodology led to a diffusion-like anisotropic non-linear differ-

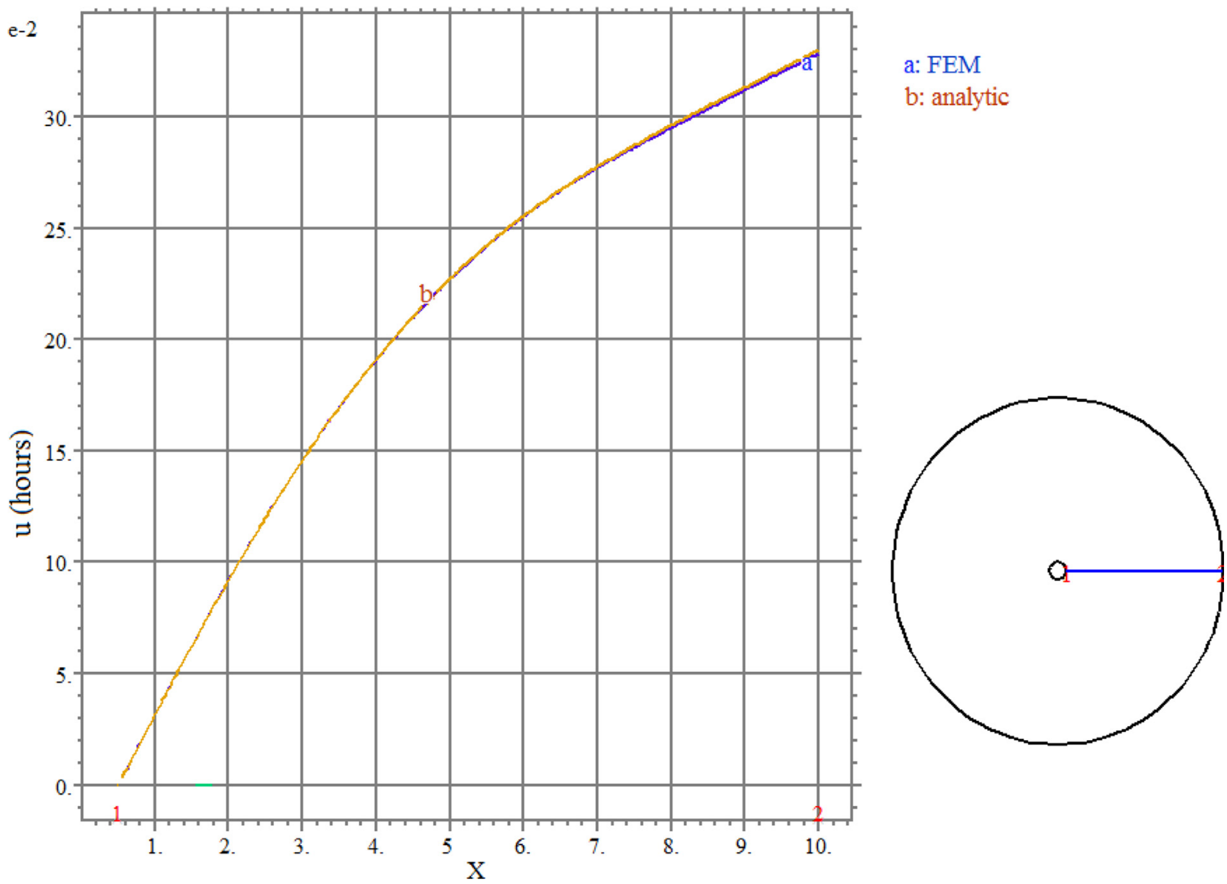

Fig. 6. Radial city. Travel time to the centre (hours).

ential equation. Then, the approach was generalized to consider polycentric cities. The governing equations were solved by means of FEM. The adopted derivation approach allows to demonstrate the straightforward relationship between the discrete and continuous formulations. This fact makes it possible to compare numerical results obtained by means of both models for the same transportation network. As shown by the numerical examples, the continuum model yields results comparable with those given by the discrete model, although with a lower computational cost.

In this paper, a generalization of the previous research by the authors [40] is proposed in order to consider a mixed transportation system constituted by a set of freeways superimposed over a dense network of surface streets. Moreover, a polycentric city with different user classes is taken into account.

The present model begins with the formulation of the assignment problem for a discrete network from three fundamental assumptions: (a) link travel time is an increasing function of the corresponding flow; (b) the Wardrop's first principle may be established by postulating the existence of a potential function representing the travel time from a given point of the network to the destination centre; and (c) the number of vehicles at each point of the network is a conservative quantity. Such an approach leads to an algebraic non-linear equation system whose unknowns are the values of the potential function at each node. It is possible to demonstrate that the aforementioned equation system is equivalent to the node-link Beckmann formulation [44,45].

Then, the continuum approximation (homogenization) of the dense street network is performed by means of the application of the Galerkin method [46]. Finally, variational calculus leads to the governing differential equations. They are diffusion-like equations in a non-linear anisotropic medium, although presenting multipoint constraints corresponding to the interaction between freeways and surface streets.

It is of interest to highlight that the present procedure considers explicitly the correspondence between the discrete and continuum models and the possible anisotropy of the network. This formulation allows one to determine the link flows after the governing equations have been solved.

Unlike other approaches, in this model, the mixed transportation network is completely solved by means of FEM. Some numerical examples are developed to demonstrate the efficiency of the proposed model.

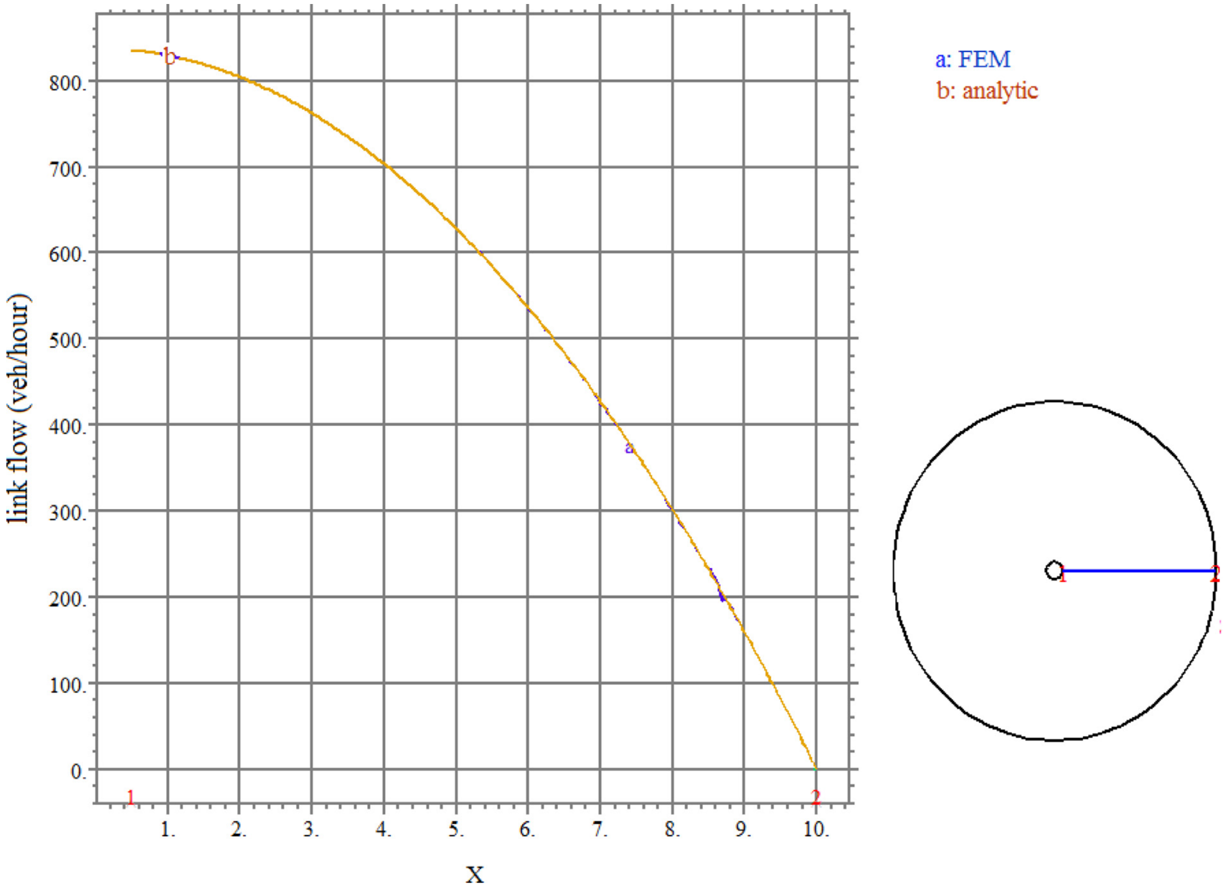

Fig. 7. Radial city. Link flow(veh/h).

In Section 2, the discrete formulation of traffic assignment is presented; in Section 3, the new continuum model for a mixed transportation system is developed; in Section 4, the FEM numerical solution for this problem is commented; in Section 5, some numerical examples are given and, finally, in Section 6, the conclusions are presented.

2. Traffic assignment problem: discrete formulation

A traffic urban network during the rush hour, when users travel from their homes distributed all over the city toward certain destination points d ($d = 1, 2, \ldots, N_d$), is considered. The scheme corresponding to a city having a surface Ω and a perimeter Γ_0 is shown in Fig. 1. The analyzed transportation system consists of two types of route: (a) a dense network of surface streets, and (b) a superimposed set of freeways. The freeways are connected to destination centres and surface streets by means of interchange points (ramps). Moreover, for the street network, it is assumed that for each street in one direction, another one exists in the opposite direction.

The traffic assignment problem consists in determining the flow $\bar{g}_a$ (veh/h) on each link a ($a = 1, 2, \ldots, N_a$) of the network from the knowledge of the destination locations, the characteristics of the network and the travel demand Q_j^d (veh/h) at each node j for users going to a destination d.

The mathematical formulation of such a problem is based on the three fundamental concepts described below.

2.1. Link congestion function

The congestion effect on every link a can be measured by means of the corresponding travel time t_a that is empirically determined as an increasing function of the link flow $\bar{g}_a$ (veh/h). The corresponding analytical expression adopts the following general form[3] (see Fig. 2)

$$t_a = t_a(\bar{g}_a) \tag{1}$$

In the literature, such expression is known as cost function. Inverting expression (1), one obtains:

$$\bar{g}_a = \bar{g}_a(t_a) \tag{2}$$

In this work, expression (2) is denominated "congestion function".

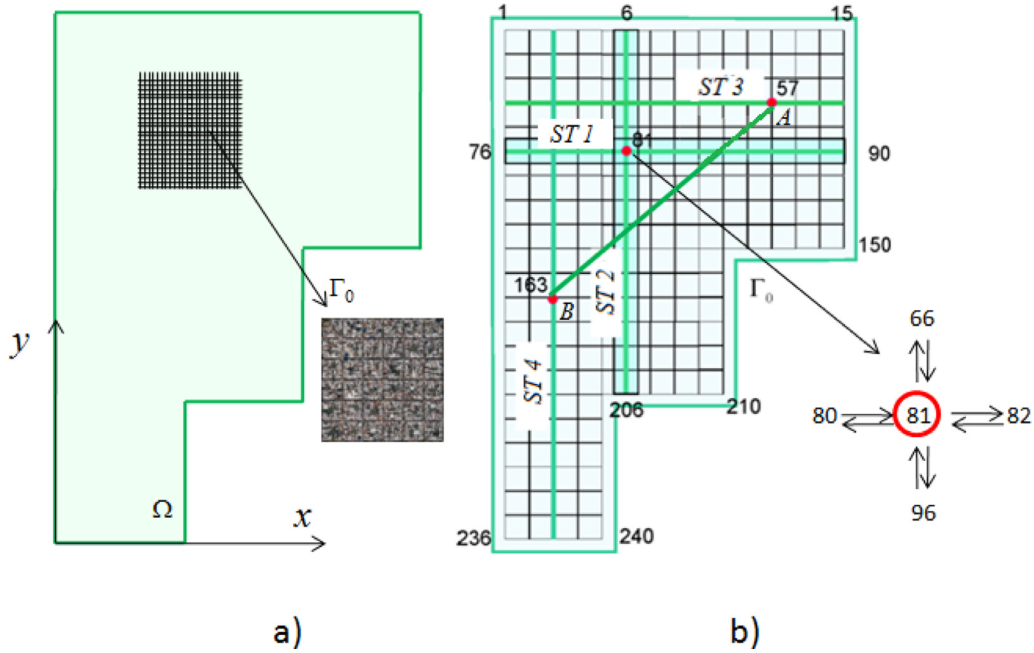

Fig. 8. (a) Analyzed transportation network. (b) Discrete model scheme.

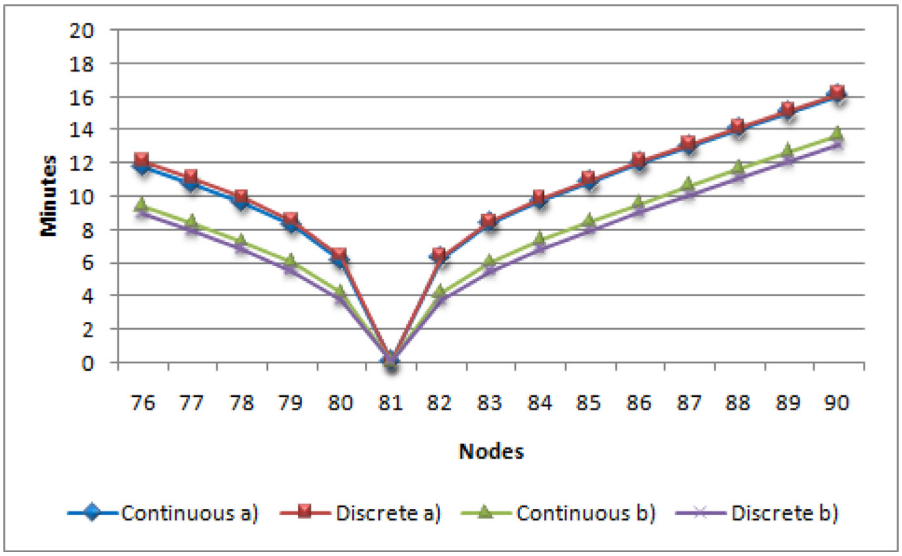

Fig. 9. Travel time to the centre on *ST*1 street. Cases (a) and (b).

The total link flow is the sum of vehicles travelling to a destination *d*, denoted by g_a^d, and those vehicles going to the rest of destinations, denoted by g_{aRes}^d. This latter is denominated residual flow with respect to destination *d*. Thus, the link flow toward a destination *d* can be expressed by means of the congestion function and the residual flow as follows:

$$g_a^d = \bar{g}_a(t_a) - g_{aRes}^d \tag{3}$$

There exist empirical formulas to express (1) in an explicit form. In particular, in this work, a modification of the well-known BPR formula [47] is adopted which allows one to write the congestion function in the following form:

$$\bar{g}_a = R_a \tau_a^{\beta_a} \tag{4}$$

where:

$$\tau_a = t_a - \hat{t}_a \tag{5}$$

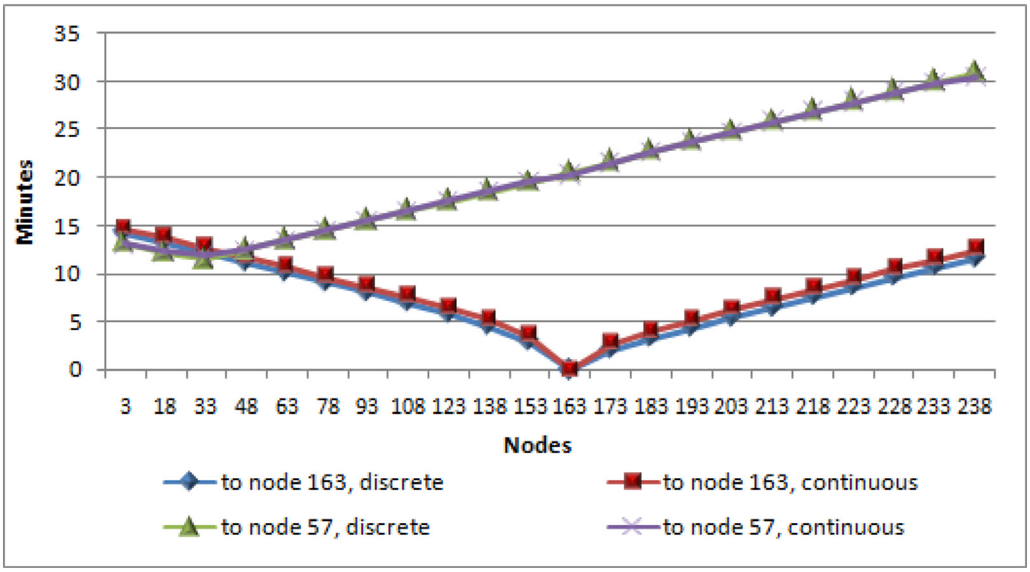

Fig. 10. Travel times to nodes 57 and 163 on *ST*4 street. Case (c).

$$\beta_a = \begin{cases} 1 & \tau_a < \tau_{1a} \\ 1/\varphi_a & \tau_a \geq \tau_{1a} \end{cases} \tag{6}$$

$$R_a = \begin{cases} \dfrac{C_a}{\left(\alpha_a \hat{t}_a\right)^{1/\varphi_a}} \tau_{1a}^{(1/\varphi_a - 1)} & \tau_a < \tau_{1a} \\ \dfrac{C_a}{\left(\alpha_a \hat{t}_a\right)^{1/\varphi_a}} & \tau_a \geq \tau_{1a} \end{cases} \tag{7}$$

$\hat{t}_a$ corresponds to the free-flow travel time for link *a*, C_a is the effective capacity of link *a*, α_a and φ_a are empirical coefficients depending of the link characteristics, and τ_{1a} is a constant adopted to be equal to $\kappa\alpha_a\hat{t}_a$ (κ is a value between 0.1 and 1).

The performed modification consists in using a straight line (the dashed line in Fig. 2) for $\tau \leq \tau_{1a}$, in order to avoid numerical problems without altering the results from the practical point of view, as will be seen below.

2.2. User equilibrium

Wardrop's first principle [1], also known as "user equilibrium principle", states that each driver selects the route to a given destination in order to minimize his (her) own travel time. In other words, the user travel time between a given point in the network and the corresponding destination is the same (and the minimum possible) for any route really used. Accordingly, in equilibrium conditions, the travel time is assumed to be a function of the user location but not of the employed route. This potential function can be expressed in the following form:

$$u^d = u^d(x, y) \tag{8}$$

where, by definition, $u^d(x_d, y_d) = 0$, being x_d and y_d coordinates of the destination *d*. Thus, the link travel time t_a^d for a driver travelling toward a destination *d* is equal to the difference of the values adopted by the potential function at the initial and end points of the link (*i* and *j* respectively):

$$t_a^d = u_i^d - u_j^d \tag{9}$$

where $u_i^d = u^d(x_i, y_i)$ and in the same form u_j^d (Fig. 3). As can be observed, t_a^d is the travel time on link *a* obtained from the potential function corresponding to a destination *d*.

2.3. Conservation of vehicles

The number of vehicles at every node is a conservative quantity. This is also valid for vehicles travelling toward a particular destination *d*. Hence, it is possible to formulate the following continuity equation for every node of the network:

$$\sum_{i=1}^{M_j} g_a^d \lambda_{ij} + Q_j^d = 0 \qquad j = 1,\ 2,\ \ldots, N_n \tag{10}$$

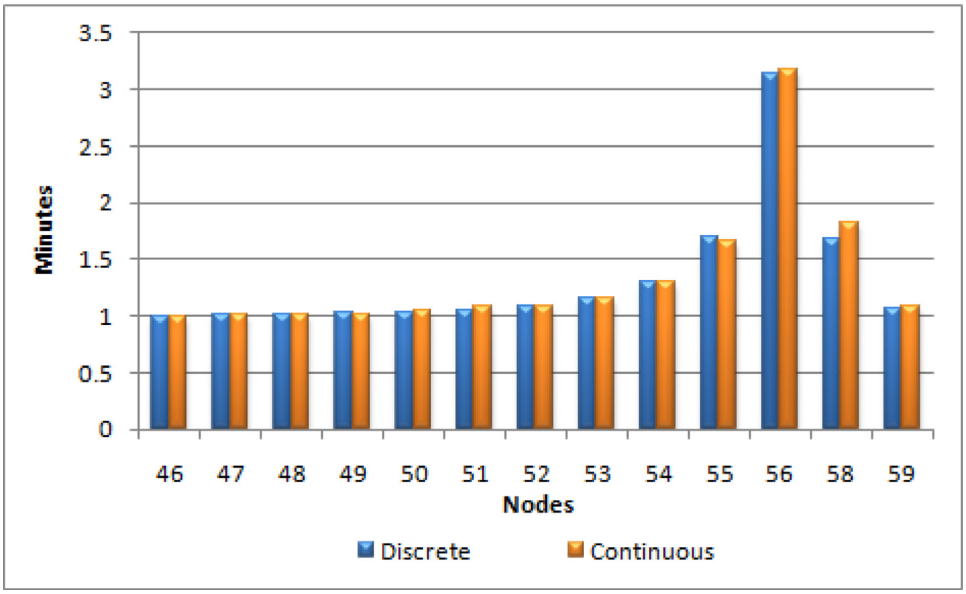

Fig. 11. Link travel times on *ST*3 street. Case (c).

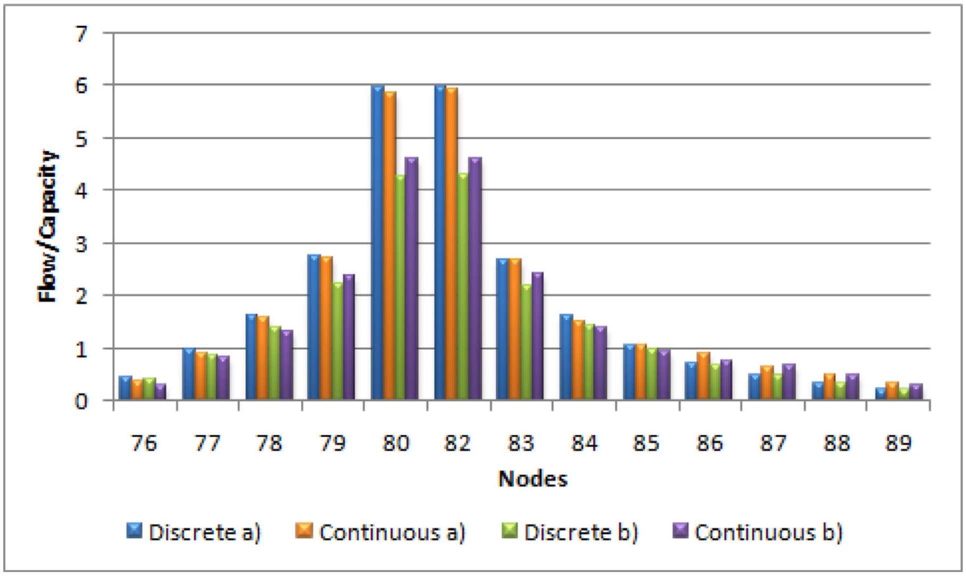

Fig. 12. Flow/capacity ratios for links on *ST*1 street. Cases (a) and (b).

where λ_{ij} is 1 if the traffic is directed from i to j, and -1 in the opposite direction, M_j corresponds to the number of nodes adjacent to node j and N_n is the number of nodes. It should be observed that a system of equations of the form (10) exists for every destination d ($d = 1, 2, \ldots, N_d$). This fact will be the same for each of the equations that follow.

It is important to remark that the system of Eqs. (1), (3), (9) and (10) is equivalent to the node-link formulation of the "user equilibrium" ([44] Section 2.2; [45])

It is useful to re-write the above system by separating the flows using the surface streets from those using the set of freeways. In order to achieve this objective, G_A^d denotes the flow on freeway link A and U_J^d the travel time from the ramp J to the destination d for those vehicles using the freeway network. Lower case letters are kept for denoting flows on the surface streets. Moreover, $\bar{Q}_j^d$ is the interchange flow between the node J of the freeway and the corresponding node j of the surface streets, and is positive when traffic enters the freeway and negative in the opposite direction. Also, Q_J^{*d} is defined as the external traffic flow entering the city directly by the freeway network and travelling toward destination d. With this

notation, the above system can be expressed as follows:

$$\sum_{i=1}^{M_j} g_a^d \lambda_{ij} + Q_j^d - \bar{Q}_j^d = 0 \qquad j = 1,\ 2,\ \ldots, N_s \tag{11}$$

$$\bar{Q}_j^d = \begin{cases} -\left(\sum\limits_{I=1}^{M_J} G_A^d \lambda_{IJ} + Q_J^{*d}\right) & if \quad \mathbf{x_j} = \mathbf{x_J} \\ 0 & if \quad \mathbf{x_j} \neq \mathbf{x_J} \end{cases} \quad J = 1, 2, \ldots, N_h \tag{12}$$

where $\mathbf{x_j}$ and $\mathbf{x_J}$ are the position vectors corresponding to street network nodes and ramps respectively, M_J represents the number of nodes of freeway links adjacent to the ramp J. N_h is the number of ramps and N_s is the number of nodes corresponding to the set of streets. It should be observed that according to Wardrop's principle [1], the travel time from a given point of the network is the same using either the freeways set or the street network. Accordingly:

$$u^d(x_j, y_j) = U_J^d \quad if \quad \mathbf{x_j} = \mathbf{x_J} \quad J = 1, 2, \ldots, N_h \tag{13}$$

By substituting expression (9) into expressions (3), (4) and (5) and the result into the system (12), one can write formally the latter in the following form:

$$\bar{Q}_j^d = \bar{Q}_j^d(U_I^d, U_J^d, Q_j^{*d}) \tag{14}$$

where U_I^d corresponds to the values of the potential function at the ramps I adjacent to J. This last expression shows clearly the dependence of the interchange flows respect to the values of the potential function at different ramps. The system is completed by adding the following equation given by the definition of total travel time:

$$u^d(x_d, y_d) = 0 \tag{15}$$

Substituting expression (14) into (11) and taking into account the definitions (3)–(7), a system of non-linear algebraic equations is obtained, the unknowns of which are given by u_i^d $(i = 1, 2, \ldots, N_s)$.

3. Continuum model for traffic assignment

It is possible to transform the system (11), (15) corresponding to the dense street network into variational equations by means of the Galerkin Method [46]. Accordingly, each of the equations (11) is multiplied by the virtual variation δu_j^d of the potential function corresponding to a destination d. Summing the equations over all nodes of the street network (for every destination d), the following expression can be obtained:

$$\sum_{j=1}^{N_s} \sum_{i=1}^{M_j} g_a^d \lambda_{ij} \delta u_j^d + \sum_{j=1}^{N} \left(Q_j^d - \bar{Q}_j^d\right) \delta u_j^d = 0 \tag{16}$$

where it has been assumed $\delta u_j^d(x_d,\ y_d) = 0$.

Dividing the dense network into M rectangular cells of side lengths L_x and L_y, as shown in Fig. 1, it is possible to re-write expression (16) in the following form:

$$\sum_{m=1}^{M} \left(-\sum_{a\in m} g_a^d \delta t_a^d + \sum_{j\in m} \left(Q_j^d - \bar{Q}_j^d\right) \delta u_j^d \right) = 0 \tag{17}$$

where t_a^d given by expression (9) along with the definitions of λ_{ij} have been used. In (17), the sums are first performed for the links inside the cell and then, the contributions of all the cells are considered.

Assuming that u^d is a continuous function of geographical coordinates, x and y, and every link length l_a is very small in comparison with the total dimensions of the city, the link travel time may be approximated in the following form (Fig. 3):

$$t_a^d = u_i^d - u_j^d \approx -l_a \frac{\partial u^d}{\partial s} = -l_a(u_x^d \cos\gamma_a + u_y^d \sin\gamma_a) \tag{18}$$

In this last expression, s is a link spatial coordinate in the traffic direction, γ_a is the angle between the s and x directions, $u_x^d = \frac{\partial u^d}{\partial x}$ and $u_y^d = \frac{\partial u^d}{\partial y}$. Moreover, it is reasonable to assume that the partial derivatives of u_d do not change within the cells because of their small dimensions.

From (18) one can obtain $\delta t_a^d = -l_a\left(\cos\gamma_a \delta u_x^d + \sin\gamma_a \delta u_y^d\right)$. Substituting this variation into expression (17), multiplying and dividing by the cell area $L_x L_y$ and considering the limit when the latter tends to be a differential area, one obtains the next integral expression:

$$\int_\Omega \left(f_x^d \delta u_x^d + f_y^d \delta u_y^d + \left(q^d - \bar{q}^d\right)\delta u^d\right) d\Omega = 0 \tag{19}$$

where:

$$
\begin{aligned}
f_x^d &= \sum_{a\in m} \frac{g_a^d l_a c_a}{L_x L_y}\, \xi_a^d \\
f_y^d &= \sum_{a\in m} \frac{g_a^d l_a s_a}{L_x L_y}\, \xi_a^d \\
q^d &= \frac{Q_j^d}{L_x L_y} \qquad \bar{q}^d = \frac{\bar{Q}_j^d}{L_x L_y} \qquad c_a = \cos\gamma_a \quad s_a = \sin\gamma_a \\
\xi_a^d &= \begin{cases} 1 & if\ t_a^d > 0 \\ 0 & if\ t_a^d \le 0 \end{cases}
\end{aligned}
\tag{20}
$$

As observed, f_x^d and f_y^d are components of the continuous traffic flow vector (veh/h/km). It should be noted that expression (18) has meaning only if $t_a^d \ge 0$ for the considered link. That is to say, the potential function u^d must decrease in the traffic direction. For a link in which this is not the case, the corresponding flow g_a^d must be null (vehicles cannot go the wrong way). In order to consider this fact, the coefficients ξ_a^d were introduced for every link in the above expressions.

Substituting expression (18) into (4) and the result into the two first expressions (20), the continuous traffic flow components can be written in terms of u^d as follows:

$$
\begin{cases}
f_x^d = -D_{xx}^d \dfrac{\partial u^d}{\partial x} - D_{xy}^d \dfrac{\partial u^d}{\partial y} - P_x^d \\
f_y^d = -D_{xy}^d \dfrac{\partial u^d}{\partial x} - D_{yy}^d \dfrac{\partial u^d}{\partial y} - P_y^d
\end{cases}
\tag{21}
$$

where:

$$
\begin{aligned}
D_{xx}^d &= \sum_{a\in m} \frac{l_a^2 c_a^2 R_a \left(\tau_a^d\right)^{\beta-1}}{L_x L_y}\, \xi_a^d \\
D_{xy}^d &= \sum_{a\in m} \frac{l_a^2 c_a s_a R_a \left(\tau_a^d\right)^{\beta-1}}{L_x L_y}\, \xi_a^d \\
D_{yy}^d &= \sum_{a\in m} \frac{l_a^2 s_a^2 R_a \left(\tau_a^d\right)^{\beta-1}}{L_x L_y}\, \xi_a^d \\
P_x^d &= \sum_{a\in m} \frac{l_a c_a R_a \left(\tau_a^d\right)^{\beta-1}}{L_x L_y} \hat{t}_a \xi_a^d + \sum_{a\in m} \frac{l_a c_a}{L_x L_y} g_{aRes}^d \xi_a^d \\
P_y^d &= \sum_{a\in m} \frac{l_a s_a R_a \left(\tau_a^d\right)^{\beta-1}}{L_x L_y} \hat{t}_a \xi_a^d + \sum_{a\in m} \frac{l_a s_a}{L_x L_y} g_{aRes}^d \xi_a^d
\end{aligned}
\tag{22}
$$

here τ_a^d is obtained by substituting t_a^d, given by Eq. (18), into (5).

It is important to note that the coefficients (22) are not undetermined for low flows ($\tau < \tau_1$) due to the modification introduced into the BPR formula shown in expressions (4)–(7). In this way, the equations remain well-defined.

The relationship between the continuous traffic flow and the potential gradient expressed by means of Eq. (21) corresponds to an anisotropic medium. In order to show this fact more clearly, the present theoretical relationship is compared to that utilized in previous continuum models in Appendix A.

By applying variational calculus to the equation (19), the following differential system, for each destination d, can be obtained:

$$
\begin{cases}
-\left(\dfrac{\partial f_x^d}{\partial x} + \dfrac{\partial f_y^d}{\partial y}\right) + q^d - \bar{q}^d = 0 \\[2ex]
\qquad f_x^d n_x + f_y^d n_y\big|_{\Gamma_0} = 0
\end{cases}
\tag{23}
$$

where n_x and n_y are components of the unit normal vector of the perimeter Γ_0. Therefore, the governing equations of the assignment problem are obtained by substituting (21) and (22) into (23). The resulting equations along with expression

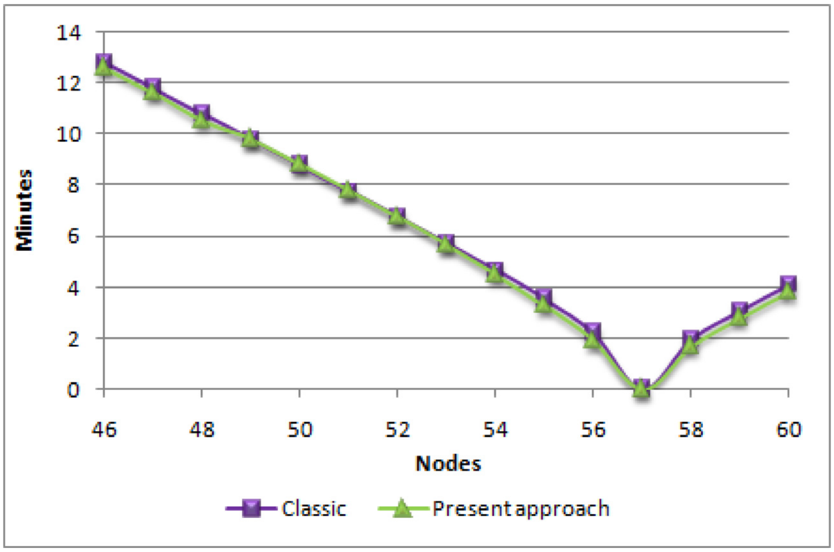

Fig. 13. Travel time to node 57 on *ST*3 street. Case (d).

Table 1
Comparison of results for the freeway links.

	Travel time (minutes)		Link flow (veh/h)	
	B to A	A to B	B to A	A to B
Classic	8.38	7.55	12,206	8926
Present approach	8.28	7.56	12,039	8936
Error(%)	−1.22	0.08	0.11	−1.37

(15) constitute the governing system that can be written in the form:

$$\begin{cases} \dfrac{\partial}{\partial x}\left(D^d_{xx}\dfrac{\partial u^d}{\partial x} + D^d_{xy}\dfrac{\partial u^d}{\partial y}\right) + \dfrac{\partial}{\partial y}\left(D^d_{xy}\dfrac{\partial u^d}{\partial x} + D^d_{yy}\dfrac{\partial u^d}{\partial y}\right) + q^d_T = 0 \\ -\left(\left(D^d_{xx}\dfrac{\partial u^d}{\partial x} + D^d_{xy}\dfrac{\partial u^d}{\partial y}\right)n_x + \left(D^d_{xy}\dfrac{\partial u^d}{\partial x} + D^d_{yy}\dfrac{\partial u^d}{\partial y}\right)n_y\right)\Big|_{\Gamma_0} = \left(P^d_x n_x + P^d_y n_y\right)\big|_{\Gamma_0} \\ u^d(x_d, y_d) = 0 \qquad\qquad d = 1, 2, \ldots, N_d \end{cases} \tag{24}$$

where:

$$q^d_T = q^d - \bar{Q}^d_j \delta(\vec{x} - \vec{x}_j) + \frac{\partial P^d_x}{\partial x} + \frac{\partial P^d_y}{\partial y} \tag{25}$$

It should be noted that these governing equations have multipoint relationships. In fact, $\bar{Q}^d_j$ in expression (25) depends on the values of u^d at the different ramps of the freeway network and is given by expression (14).

The unknowns of the system (24) are given by the N_d potential functions $u^d(x, y)$. It is a non-linear system of diffusion-like equations in which the coefficients D^d_{xx}, D^d_{xy}, D^d_{yy}, as well as the functions P^d_x, P^d_y depend on the unknowns.

If the network consists only of streets, without freeways, $\bar{Q}^d_j$ will be null and the above equation system will be reduced to that developed by the authors in Ref. [40].

The developed equations can be generalized in order to take into account multiple user classes. In this case, each user class is identified by means of its link travel time as a function of the corresponding total link flow. In this study, this function is given by the modified BPR formula explained in Section 2, although considering different values of α_a, φ_a and $\hat{t}_a$ for each class. In this way, following a derivation similar to that presented above, an equation system of the form (24) will be obtained, for each destination and user class, having as unknowns the functions $u^{d,\,c}$ with $d = 1, 2, \ldots, N_d$, $c = 1, 2, \ldots, N_c$, being N_c the number of classes.

Link travel times can be determined by means of expression (18) after the potential functions $u^{d,\,c}$ have been obtained by solving the system (24). It is clear that, in equilibrium conditions, the same values of t^d_a will be obtained for a given link by using any of the functions u^d (while $t^d_a \neq 0$). Finally, link flows can be determined from the knowledge of link travel times by using expressions (3) and (4).

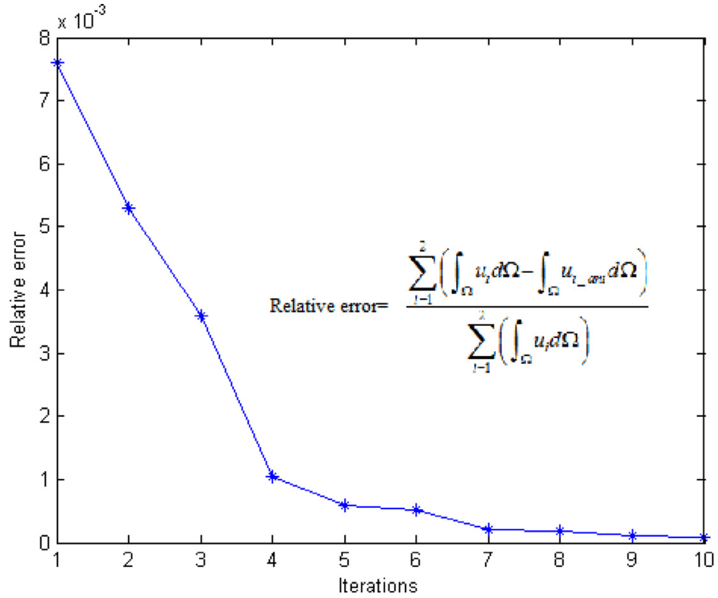

Fig. 14. Convergence study of the iterative procedure. Case (d).

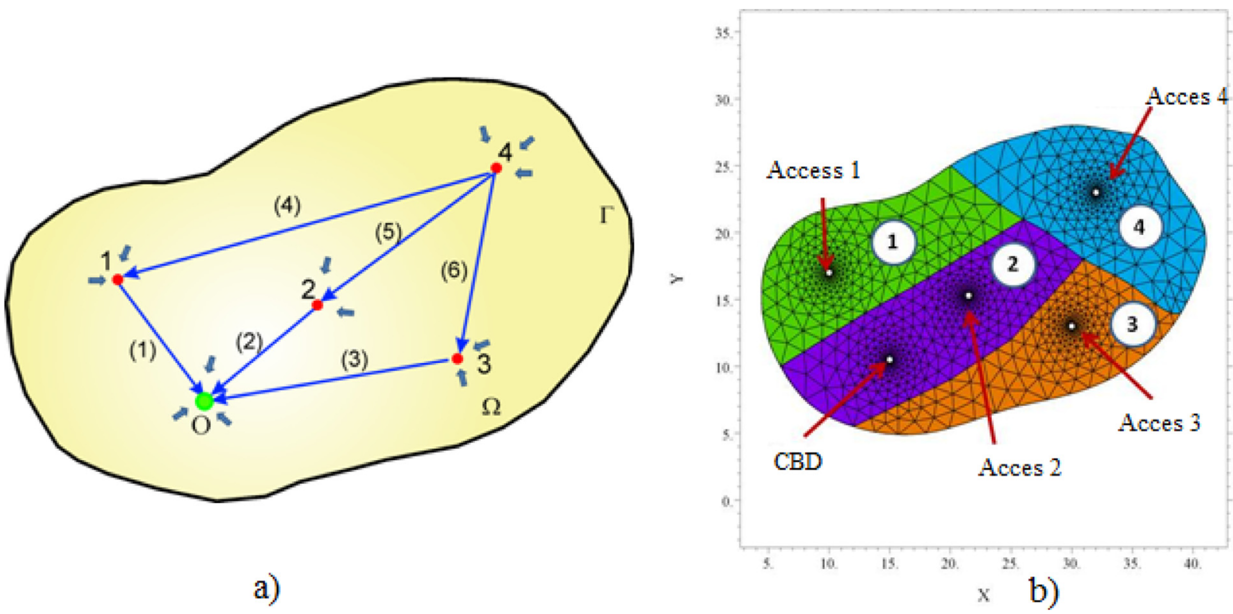

Fig. 15. (a) Transportation network with freeways; (b) city zones and FEM mesh.

4. Computational solution by means of the Finite Element Method (FEM)

The governing differential equation system with multipoint relationships expressed by Eq. (24) can be solved by means of several appropriate numerical approaches such as finite differences, differential quadrature, weighted residual methods, etc. In this work, FEM is used by means of the software FlexPDE [48].

In this computational program, the mathematical problem is defined by writing the corresponding governing differential equations and boundary conditions in a text file (descriptor), employing a simple notation similar to that commonly used in mathematics.

Internally, the program builds the pertinent variational equations, by using the Galerkin's procedure, and then performs the corresponding discretization with an automatic mesh generator. In addition, it has several strategies that users can select for solving non-linear boundary and/or initial value problems.

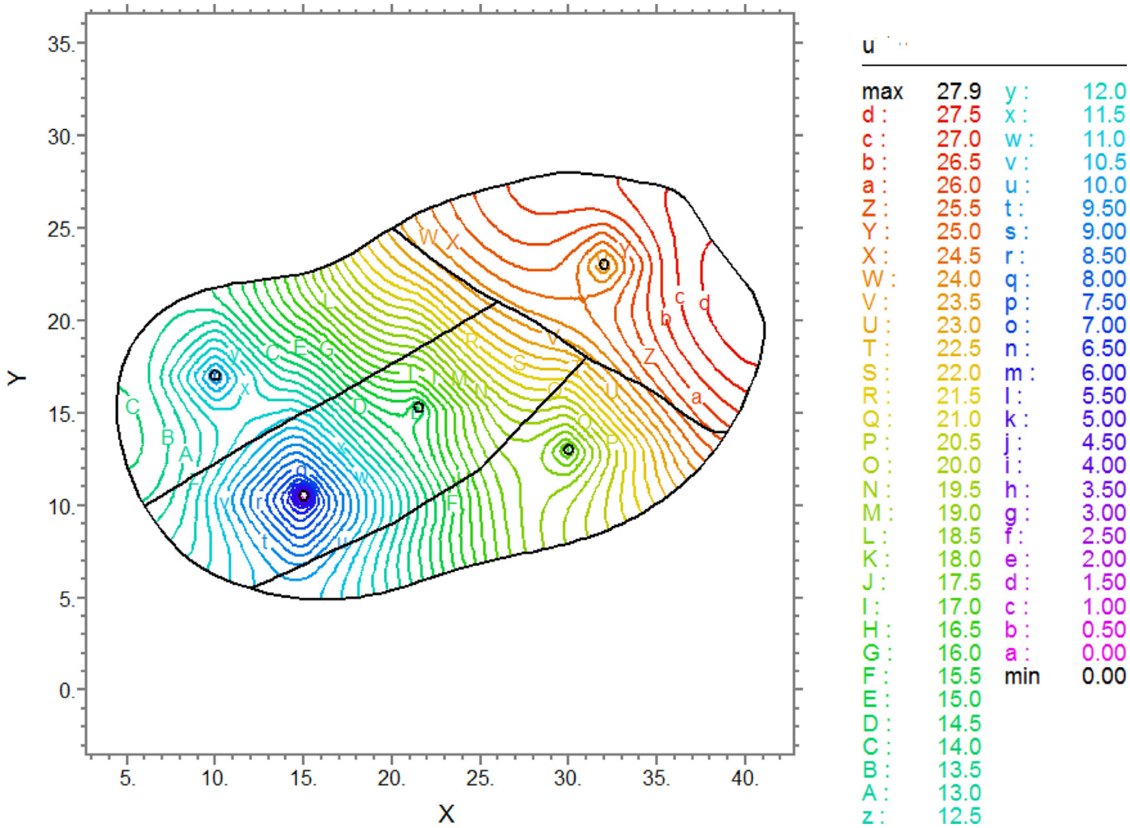

Fig. 16. Contour plot of the total travel time (minutes) to the city centre.

Therefore, for the present problem, the descriptor should contain the definitions of the network characteristics such as free-flow link travel times, link capacities, street directions and the expressions of the diffusion coefficients (22). Then, the system (24) must be written. Moreover, the geometry, boundary conditions and the characteristics of the graphical outputs of results should be specified.

As an initial approximation, the governing equations are solved by taking constant values for the diffusion coefficients (negligible congestion: $\tau \leq \tau_{1a}$). Starting from this initial solution and applying a Newton–Raphson method, the problem may be solved for a monocentric city.

For the case of two or more destinations, an iterative procedure is implemented. This consists in solving the equation corresponding to one destination at a time, while keeping the flows corresponding to the other centres as constant values. As convergence criterion of this iterative algorithm, the expression $\frac{\sum_{i=1}^{N_d} \left(\int_\Omega u_i d\Omega - \int_\Omega u_{i\_ant} d\Omega\right)}{\sum_{i=1}^{N_d} \left(\int_\Omega u_i d\Omega\right)}$ in two successive iterations, has to be less than certain pre-defined tolerance. Several details are taken into account to prevent numerical difficulties. For example, in order to avoid singularities, ramps are defined as small areas.

5. Numerical examples

5.1. Example 1: circular city

The objective of the present example is to show the efficiency of FEM for solving the governing equations (24). If this system is expressed in cylindrical coordinates, it is possible to obtain an analytical solution for the particular case of a circular city, without freeways, presenting radial symmetry. That is to say, vehicles travel toward the central point using radial streets. Such a solution is shown in Appendix B. However, this problem in cartesian coordinates is very complex and should be treated in a numerical form. Thus, it is possible to analyze the accuracy of the present numerical solution by means of the comparison between the FEM and analytical results [49].

Therefore, it is considered a circular city of radius R (Fig. 4) in which q_a vehicles per hour and per unit surface are generated with destination at centre O. It is assumed that flow across the external boundary is null ($f_a = 0$ at $r = R$). The following numerical data are adopted: R=10 km, q = 80 veh/h/km^2, link length l_a=0.2 km, capacity of radial streets C_a

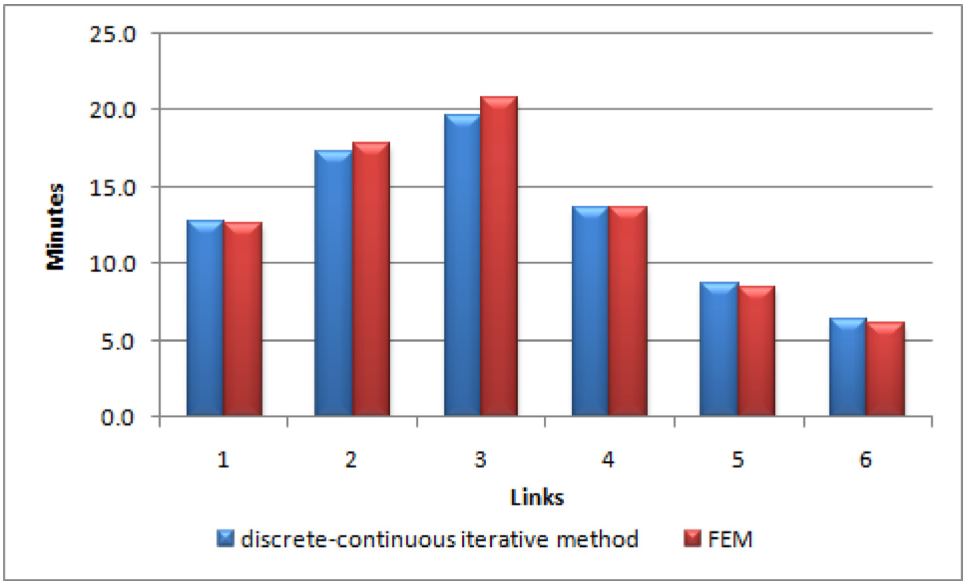

Fig. 17. Freeway link travel times.

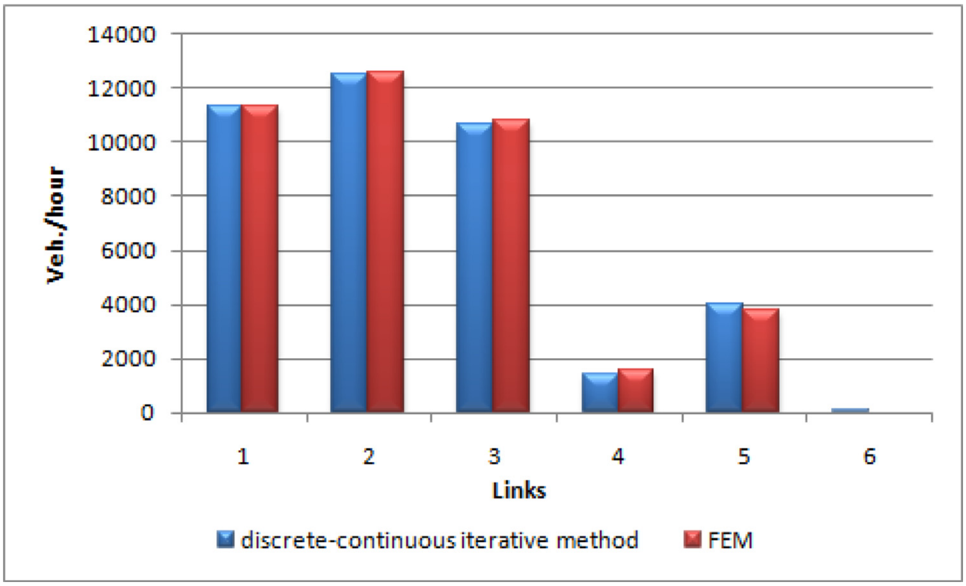

Fig. 18. Freeway link flows.

=400 veh/h and maximum velocity 60 km/h (corresponding to the free-flow travel time $\hat{t}_a = 0.00\hat{3}$ h). For the congestion function, the following values are adopted: $\alpha_a = 0.15$ and $\varphi_a = 4$.

A contour plot of the total travel time obtained by means of FEM is shown in Fig. 5. As observed, the numerical results reproduce correctly a radial symmetric pattern.

In Figs. 6 and 7 comparisons are shown between the FEM and analytical results for the total travel time u and the link flow g_a, respectively. As appreciated, numerical and analytical results are practically indistinguishable, thus demonstrating the very good accuracy of the FEM solution.

5.2. Example 2: urban transportation network with two freeway links

This example is performed to compare the present continuum theory with the classical discrete Beckmann's formulation. The optimization problem corresponding to the latter is solved by means of the Frank–Wolfe method [50]. Fig. 8a shows the considered city. It has an area of approximately 240 km^2 and is limited by an external boundary Γ_0 across of which

vehicles cannot flow. The trips are assumed to be generated uniformly over the domain. The street network presents two perpendicular directions having link lengths of 100 m. Moreover, there are two freeway links joining two points of the city (A and B) in both directions.

In order to build the continuum model for the dense street network, cells of 0.8x0.8 km^2 are adopted. For this reason, taking into account the uniform distribution of streets within each cell, 32 links for each direction are considered for the determination of the coefficients given by Eq. (22). The streets are supposed to coincide with x or y axes, that is to say, γ_a can adopt the following values: 0°, 90°, 180° or 270°.

To apply the traditional discrete model, a simplified scheme of the street network is defined as a set of 240 nodes and 886 equivalent arcs of 1 km length. Each arc of the discrete model represents a set of 5 parallel streets in the same direction. Accordingly, the capacity of each equivalent arc corresponds to the addition of the capacities of the pertinent streets ($5C_a$). Fig. 8b shows the equivalent discrete model under analysis where four paths are identified for future reference: ST_1, ST_2, ST_3 and ST_4.

Four particular cases are considered:

a) *Monocentric transportation network having identical link characteristics.*
A uniform travel demand $q = 300$ veh/h/km^2 with destination at node 81 (see Fig. 8) is considered. The freeways are not taken into account. The streets characteristics are identical in all directions: the maximum velocity is 60 km/h, the street capacity is C_a=600 veh/h and the link travel time is given by the BPR function with $\alpha_a = 0.15$ and $\varphi_a = 2$.
b) *Monocentric transportation network having different link characteristics.*
The data are the same as in the above case, except for the capacities of the streets constituting the paths $ST1$ and $ST2$ directed to the centre that, in this case, adopt values 40% higher than those corresponding to the rest of the streets.
c) *Street network with two destinations.*
The data are the same as in case (a) but now, half of the demand has as destination the point A (node 57) and the other half the point B (node 163).
d) *A two destination network constituted by streets and two freeway links.*
The data are the same as in case (c) but now two freeway links are included, one of them from point A (node 57) to point B (node 163) and the other in the opposite direction (Fig. 8).

In Figs. 9–14, some results for the four analyzed cases are shown. In Fig. 9, one can observe, for cases (a) and (b), the travel times between each node of $ST1$ and the centre (node 81). The mean difference between the results of both approaches is of the order of 4%.

Fig. 10 shows, for case (c), the travel times between each node of $ST4$ and the centres A (node 57) and B (node 163). Mean differences between both approaches are of approximately 4%.

Fig. 11 shows, for case (c), the link travel times corresponding to $ST3$. The mean difference between values corresponding to both approaches is less than 2%.

In Fig. 12, flow/capacity ratios corresponding to the streets constituting path $ST1$ are shown for cases (a) and (b). As can be seen, the difference for the most congested links is less than 1%. In the last two figures, node numbers on the horizontal axis correspond to the origins of each one of the considered links.

In Fig. 13, travel times from each node of $ST3$ to point A, for case (d), are shown. The values obtained by means of both approaches are practically identical. For the same case, in Table 1, freeway link flows and the corresponding travel times are shown. As can be seen, results obtained by means of the present approach show very small differences with respect to the discrete model values.

It is interesting to analyze the convergence of the iterative procedure for solving a polycentric network as explained in Section 4. Thus, the relative error evolution with the number of iterations is shown in Fig. 14 for case (d). As observed, after seven iterations the relative error becomes negligible.

The computation times to solve the analyzed problems by means of the traditional discrete model were approximately of 1 h [49], while the continuum model required 6 s, 7 s, 72 s and 120 s for cases (a), (b), (c) and (d), respectively.

These calculation times do not constitute a measure totally objective because they can vary when using different computers and software. However, they could be considered as approximate indicators showing the important reduction of computational burden obtained with the continuum model in comparison with the discrete one.

It is interesting to note that, unlike discrete models, continuum formulation does not require larger computational times when the number of links is increased. In fact, in the continuum approach, FEM discretization is independent of the real structure of the network.

5.3. Example 3: urban transportation network constituted by streets and several freeway links

The above example has demonstrated the good accuracy of the continuum model respect to the traditional discrete formulation when considering a mixed transportation network. The aim of the present example is to compare the proposed FEM solution against an iterative method consisting in combining the FEM solution for the street network and the Frank–Wolfe methodology for the freeways.

In this last methodology, the FEM solution of the continuum problem provides the discrete formulation with approximations to the travel demands at the ramps. Then, after solving the discrete model by means of the Frank–Wolfe method,

travel times at the ramps are determined and taken as boundary conditions for the continuum model. Iteration between both models is performed up to convergence. The purpose for analyzing such a methodology is due to the fact that a similar approach was previously presented by Wong et al. [35] and Du et al. [36], although by employing a different continuum model.

The explained approaches are applied to the city shown in Fig. 15 that has an approximate area of 600 km^2, where all the trips are directed to destination O. Moreover, there are no trips across the boundary of the city. The freeway network comprises links (1) to (6) and four access ramps numbered 1 to 4.

The city is divided into 4 zones having different travel demands: 100 veh/h/km^2, 80 veh/h/km^2, 130 veh/h/km^2 and 70 veh/h/km^2 for the zones 1, 2, 3 and 4, respectively.

For freeways, the link travel time function is assumed to be $t_A = \hat{t}_A(1 + 0.85(\bar{G}_A/C_A)^5)$ and for streets $t_a = \hat{t}_a\left(1 + 0.15(\bar{g}_a/C_a)^4\right)$. Free flow travel times $\hat{t}_A$ and $\hat{t}_a$ correspond to maximum velocities of 100 km/h for freeways and 60 km/h for streets. The capacities of freeway links are 10,000 veh/h and 6000 veh/h for links 1–3 and 4–6, respectively. The capacity of every surface street is 600 veh/h.

In Fig. 16, a contour plot of the total travel time (minutes) obtained by means of FEM is shown. In Figs. 17 and 18, link travel times for each one of the freeway links and the corresponding flows obtained by means of both approaches are compared. The mean difference is of the order of 3%.

6. Conclusions

In this paper, a continuum model for the study of congested urban traffic in systems constituted by a set of freeways superimposed over a dense network of streets has been proposed.

The theoretical model is based on three concepts: (a) the quantitative definition of the congestion at every link by means of a cost function (travel time increases with flow); (b) the Wardrop's first principle defined here by postulating the existence of a potential function for each destination that represents the travel time between an arbitrary point of the network and the corresponding destination; and (c) the conservation of vehicles at each node of the transportation network. Unlike previous works, the present model is not based on any variational principle of abstract meaning.

The theory is first developed for a discrete system. Then, the continuum approximation is obtained by means of the Galerkin's method considering the limit when link lengths tend to zero. The theory leads to a system of non-linear diffusion-like differential equations with multipoint relationships. This governing system may be conveniently solved by means of several numerical approaches. FEM is employed in the present work. The theoretical model can consider multiple destinations and user classes and takes into account the possible anisotropic characteristics of the surface street network.

As a remarkable aspect, the present approach is obtained from the discrete formulation, thus allowing the comparison between both models. The numerical examples show that the present methodology could be useful for the analysis of transportation systems involving a high number of streets. Certainly, the numerical results are comparable with those obtained by means of the classical discrete method, although by requiring less computational time. This is due to the fact that FEM numerical discretization does not need to be refined with the number of links. Moreover, the accuracy of the model tends to improve with the increase of the number of streets, since the network becomes more similar to a two-dimensional continuum.

In order to apply the present methodology, one needs to know the characteristics of network structure at least in an approximate form. Therefore, this model may be useful to redesign pre-existent networks. The possibility to recover link flows is, according to the authors' knowledge, a novelty with respect to previously developed continuum models. However, it has to be noted that a high heterogeneity of the characteristics of adjacent links would cause a decrease in accuracy.

This methodology differs from previous continuum models for analyzing a mixed transportation system in the form of the continuum representation for the surface streets. When the features of the street network are approximately known, the present methodology could be considered as an interesting alternative. If this is not the case, the use of previous continuum models may be advantageous. Another difference with respect to the cited approaches is the fact that the present model is entirely solved by FEM. Therefore, the methodology could be completely implemented in the same commercial FEM software. Moreover, traffic-related pollution problems could even be analyzed with the same program.

It is important to realize that the present approach is an approximation to the discrete formulation of the static assignment problem and, accordingly, shares the same limitations. In particular, static equilibrium approach shows a decreased accuracy when many links of the network become oversaturated. In this case, an improved description would be obtained by using a dynamic approach.

The developed model could be useful for the optimal design of urban transportation networks. This aspect will be presented in the future.

Acknowledgments

The present study is part of the projects "Mathematical models in environmental engineering: optimal design and control" (CIMTA, Universidad Tecnológica Nacional) and "Optimization problems in Civil Engineering (Dto. de Ingeniería, Universidad Nacional del Sur). The authors acknowledge the support of the mentioned institutions and CONICET.

Appendix A. Flow-potential gradient relationship

The most common form to express this relationship in a continuum medium is the following:

$$\mathbf{f} = \mathbf{K}\nabla u \tag{A.1}$$

This kind of relationship appears in several problems of mathematical physics such as heat conduction and flow in porous media. In this last context, it is known as Darcy's law.

It is possible to obtain expressions of the form (A.1) equivalent to (21)–(22). In order to do this, components f_x^d and f_x^d given by Eq. (20) are multiplied and divided by t_a^d. Substituting (18) into the resultant expressions allows one to obtain:

$$\begin{cases} f_x^d = -K_{xx}^d \frac{\partial u^d}{\partial x} - K_{xy}^d \frac{\partial u^d}{\partial y} \\ f_y^d = -K_{xy}^d \frac{\partial u^d}{\partial x} - K_{yy}^d \frac{\partial u^d}{\partial y} \end{cases} \tag{A.2}$$

where the conductivity tensor is given by:

$$\begin{aligned} K_{xx}^d &= \sum_{a\in m} \frac{g_a^d \xi_a^d}{t_a^d} \frac{l_a^2 \cos^2 \gamma_a}{L_x L_y} \\ K_{yy}^d &= \sum_{a\in m} \frac{g_a^d \xi_a^d}{t_a^d} \frac{l_a^2 \sin^2 \gamma_a}{L_x L_y} \\ K_{xy}^d &= \sum_{a\in m} \frac{g_a^d \xi_a^d}{t_a^d} \frac{l_a^2 \cos \gamma_a \sin \gamma_a}{L_x L_y} \end{aligned} \tag{A.3}$$

It should be to noted that, in this case, $K_{xx}^d \neq K_{yy}^d \neq K_{xy}^d$, thus defining an anisotropic medium.

In previous continuum models, expressions of the form (A.2) were obtained with the following definition of the conductivity tensor [36]:

$$K_{xx}^d = K_{yy}^d = \frac{\left|\mathbf{f}^d\right|}{\alpha + \beta \sum_{r=1}^{ND} \left|\mathbf{f}^r\right|} \quad K_{xy}^d = 0 \tag{A.4}$$

These last expressions correspond to an isotropic medium.

It should be remarked that expressions (A.3) or (A.4) give very small values for low flows. This aspect causes numerical problems.

In order to avoid this fact, in this paper the alternative flow relationship (21) has been used because its coefficients D_{xx}^d, D_{xy}^d, D_{yy}^d, as well as the functions P_x^d, P_y^d, given by Eq. (22) keep determined even for low flows.

Appendix B. Analytical solution for a radially symmetric circular city

For the analyzed case, the system (23) may be written in cylindrical coordinates as follows:

$$-\frac{1}{r}\frac{d(r f_r)}{dr} + q = 0 \tag{B.1}$$

with $f_r(R) = 0$.

The solution of this equation is given by:

$$f_r = \frac{q}{2}\left(\frac{r^2 - R^2}{r}\right) \tag{B.2}$$

Taking into account that the city has M radial streets, link flow may be obtained from:

$$|g_a| = \frac{2\pi r}{M}|f_r| \tag{B.3}$$

Substituting (B.2) into (B.3), one obtains:

$$|g_a| = \frac{q\pi}{M}\left(R^2 - r^2\right) \tag{B.4}$$

Inserting this last expression into (4) and taking into account (18) one can obtain, after rearrangement, the following expression:

$$\frac{du}{dr} = \frac{1}{\hat{v}_a} + \frac{\alpha_a}{\hat{v}_a}\kappa^{\beta-1/\varphi}\left(\frac{q\pi\left(R^2 - r^2\right)}{MC}\right)^{1/\beta} \tag{B.5}$$

where $\hat{v}_a$ ($=l_a/\hat{t}_a$) is the free-flow velocity. Expression (B.5) is easily integrated in a numerical form in order to determine $u(r)$.

References

[1] J.G. Wardrop, Some Theoretical Aspects of Road Traffic Research, Institution of Civil Engineers, 1952, pp. 325–378.
[2] M.J. Beckmann, C. Mc Guire, C. Weinstein, Studies in Economics of Transportation, Yale University Press, New Haven, Connecticut, 1956.
[3] Y. Sheffi, Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods, Prentice-Hall, Inc., Englewood Cliffs, New Jersey 07632, 1984.
[4] H. Poorzahedy, O.M. Rouhani, Hybrid meta-heuristic algorithms for solving network design problem, Eur. J. Oper. Res. 182 (2007) 578–596.
[5] P.N. Dominguez, M.C. Vidal, V.H. Cortínez, Optimal design of urban transportation networks considering environmental aspects (in Spanish), Mecánica Computacional (Asociación Argentina de Mecánica Computacional) XXVIII (2009) 2599–2624.
[6] G.M. Wang, Z.Y. Gao, M. Xu, An MPEC formulation and its cutting constraints algorithm for continuous network design problem with multi-user classes, Appl. Math. Model. 36 (2014) 1846–1858.
[7] S.W. Chiou, A non-smooth model for signalized road network design problems, Appl. Math. Model. 32 (2008) 1179–1190.
[8] H. Bar-Gera, D. Boyce, Origin-based algorithms for combined travel forecasting models, Transp. Res. Part B 37 (2003) 405–422.
[9] D.H. Kang, Multi-commodity flow estimation with partial counts on selected links Ph.D. thesis, Texas A&M University, USA, 2005.
[10] M. Xu, A. Chen, Y. Qu, Z. Gao, A semismooth newton method for traffic equilibrium problem with a general nonadditive route cost, Appl. Math. Model. 35 (2011) 3048–3062.
[11] T. Sasaki, Y. Iida, H. Yang, User equilibrium traffic assignment by continuum approximation of network flow, in: Proceedings of the 11th International Symposium of Transportation and Traffic Theory, Yokohama, Japan, 1990, pp. 233–252.
[12] H.W. Ho, S.C. Wong, Two-dimensional continuum modeling approach to transportation problems, J. Transp. Syst. Eng. Inf. Technol. 6 (2006) 53–72.
[13] M. Beckmann, A continuous model of transportation, Econometrica 20 (1952) 643–660.
[14] T.N. Lam, G.F. Newell, Flow dependent traffic assignment on a circular city, Transp. Sci. (1967) 318–361.
[15] G. D'Este, Trip assignment to radial major roads, Transp. Res. Part B 19 (1987) 227–237.
[16] S.C. Wong, An alternative formulation of D'Este's trip assignment model, Transp. Res. Part B 28 (1994) 187–196.
[17] S. Dafermos, Continuum modeling of transportation network, Transp. Res. Part B 14 (1980) 295–301.
[18] R.J. Vaughan, E. Doyle, Rectangular routing in Smeed's city, Transp. Res. Part B 13 (1979) 181–182.
[19] A. Taguchi, M. Iri, Continnum approximation to dense networks and its application to the analysis of urban road networks, Math. Program. Study 20 (1982) 178–217.
[20] S.C. Wong, C.K. Lee, C.O. Tong, Finite element solution for the continuum traffic equilibrium problem, Int. J. Numer. Methods Eng. 43 (1998) 1253–1273.
[21] H.W. Ho, S.C. Wong, B.P.Y. Loo, A continuous traffic equilibrium model with multiple user classes, J. East. Asia Soc. Transp. Stud. 5 (2003) 2831–2845.
[22] K. Wong, S.C. Wong, J. Wu, H. Yang, W. Lam, A combined distribution hierarchical mode choice and assignment network model with multiple user and mode classes, in: Lee D.-H. (Ed.), Urban and Regional Transportation Modeling: Essays in Honor of David Boyce, Edward Elgar Publishing Inc., Northampton, U.S.A., 2003.
[23] H.W. Ho, S.C. Wong, B.P.Y. Loo, Combined distribution and assignment model for a continuum traffic equilibrium problem with multiple user classes, Transp. Res. Part B 40 (2006) 633–650.
[24] H.W. Ho, S.C. Wong, H. Yang, B.P.Y. Loo, Cordon-based congestion pricing in a continuum traffic equilibrium system, Transp. Res. Part A 39 (2005) 813–834.
[25] H.W. Ho, S.C. Wong, T.D. Hau, A multi-class congestion-pricing problem in a continuum transportation system, J. East. Asia Soc. Transp. Stud. 7 (2007) 238–253.
[26] H.W. Ho, S.C. Wong, Housing allocation problem in a continuum transportation system, Transportmetrica 3 (2007) 21–39.
[27] S. Xu, Y. He, Model and algorithm for initial route planning, in: Workshop on Power Electronics and Intelligent Transportation System, 2008.
[28] W. Wong, S.C. Wong, Biased standard error estimations in transport model calibration due to heteroscedasticity arising from the variability of linear data projection, Transp. Res. Part B 88 (2016) 72–92.
[29] W. Wong, S.C. Wong, Network topological effects on the macroscopic bureau of public roads function, Transportmetrica A 12 (3) (2016) 272–296.
[30] W. Wong, S.C. Wong, Systematic bias in transport model calibration arising from the variability of linear data projection, Transp. Res. Part B 75 (2015) 1–18.
[31] N. Geroliminis, C.F. Daganzo, Existence on urban-scale macroscopic fundamental diagrams: some experimental findings, Transp. Res. Part B 42 (2008) 759–770.
[32] K. Aboudolas, N. Geroliminis, Perimeter and boundary flow control in multi-reservoir heterogeneous networks, Transp. Res. Part B 55 (2013) 265–281.
[33] H. Yang, S. Yagar, Y. Iida, Traffic assignment in a congested discrete/continuous transportation system, Transp. Res. Part B 28 (1994) 161–174.
[34] F. Robusté, E. Oñate, J.M. Ramos, A hybrid discrete-continuum model for traffic assignment by means of genetic algorithm (in spanish), in: IX Congreso Panamericano de Ingeniería de Tránsito y Transporte, La Habana, Cuba, 1996.
[35] S.C. Wong, Y.C. Du, H.W. Ho, L.J. Sun, A simultaneous optimization formulation discrete/continuous transportation system, Transp. Res. Record 1857 (2003) 11–20.
[36] Y.C. Du, S.C. Wong, L.J. Sun, A multi-commodity discrete/continuous model for a traffic equilibrium system, Transportmetrica A: Transp. Sci. 12 (2016) 249–271.
[37] A.K. Gupta, V.K. Katiyar, A new multiclass continuum model for traffic flow, Transportmetrica 3 (2007) 73–85.
[38] A.K. Gupta, V.K. Katiyar, A new anisotropic continuum model for traffic flow, Physica A 368 (2006) 551–559.
[39] T. Saumtally, J.P. Lebacque, H.H. Salem, Static traffic assignment with side constraints in a dense orthotropic network, Procedia - Social Behav. Sci. 20 (2011) 465–474.
[40] V.H. Cortínez, P.N. Dominguez, An anisotropic diffusion model for the study of the urban traffic (in spanish), Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería. 29 (1) (2013) 1–11.
[41] K. Attipou, S. Nezamabadi, E.M. Daya, H. Zahrouni, A multiscale approach for the vibration analysis of heterogeneous materials: application to passive damping, J. Sound Vibr. 332 (2013) 725–739.
[42] S. Gonella, M. Ruzzene, Homogenization of vibrating periodic lattice structures, Appl. Math. Model. 32 (2008) 459–482.
[43] A.K. Noor, W.C. Russell, Anisotropic continuum models for beamlike lattice trusses, Comput. Methods Appl. Mech. Eng. 57 (1986) 257–277.
[44] M. Patriksson, The Traffic Assignment Problem: Models and Methods, Dover Publications, 2015.
[45] M.C. Ferris, A. Meeraus, T.F. Rutherford, Computing Wardropian equilibria in a complementarity framework, Optim. Methods Softw. 10 (1999) 669–685.
[46] K. Washizu, Variational Methods in Elasticity and Plasticity, 2nd, Pergamon Press, 1975.
[47] U.S. Bureau of Public Roads, Traffic Assignment Manual. U.S. Department of Commerce. Washington D.C., 1964.
[48] FlexPDE, V.6, PDE Solutions Inc., 2010, (9408 E. Holman Rd. Spokane Valley, WA 99206).http://www.pdesolutions.com.
[49] P.N. Dominguez, A new continuum model of traffic assignment for optimal design of urban transportation networks (in spanish) Ph.D. thesis, Universidad Nacional del Sur, Bahía Blanca, Argentina, 2013.
[50] M. Frank, P. Wolfe, An algorithm for quadratic programming, Nav. Res. Logist. Q. (1956) 3:95–110.