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Nonlinear dynamic systems design based on the optimization of the domain of attraction

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ABSTRACT

In this paper an optimization-based methodology for the design of the operating equilibrium of a nonlinear dynamic system based on a measure of the extension of its domain of attraction is proposed. The approach consists in maximizing the radius of a ball in the state space contained in the region of negative definiteness of the time derivative of a quadratic Lyapunov function, using a two level optimization strategy.

A deterministic global optimization problem is solved at the inner level to ensure proper estimation of the domain of attraction for each feasible realization of the design variables which are optimized at the outer level. In order to cope with the non-differentiable nature of the inner problem, a stochastic algorithm is applied to manipulate the design variables at the outer level.

The methodology is applied to several examples to illustrate different aspects of the approach.

1. Introduction

The Domain of Attraction (DOA) of an asymptotically stable equilibrium point of a dynamic system is the portion of the state space where trajectories that converge to such equilibrium point originate. Some knowledge of its size and shape is usually required for the proper planning of the operation of the nonlinear system [1]. However, for the general case, the DOA is a complicated set that does not admit analytical representation.

While the estimation of the DOA of a given asymptotically stable equilibrium has been largely addressed, the design of the nonlinear system with a specific focus on the extension of the DOA has been less studied. This is indeed a challenging problem since it is related to other two important underlying problems: the "design-for-stability-problem" and the "DOA estimation problem".

The design-for-stability problem essentially seeks to find an operating equilibrium point which is asymptotically stable while optimizing some appropriate objective function, typically economical. Such an important problem has been addressed with different approaches in many disciplines. In [2] an iterative strategy to bound the eigenvalues of the Jacobian matrix was applied to the synthesis of reactor networks. In [3] different problems of the mechanical engineering discipline were addressed through an eigenvalue optimization technique which relies on an interior point/logarithmic barrier transformation approach. Monningmann and Marquardt [4] presented a steady state design methodology which addressed the stability issue making use of bifurcation theory elements. The proposed approach was applied to a very rich continuous polymerization process. In [5] eigenvalue optimization techniques which makes use of standard NLP solvers to constraint the real part of the eigenvalues were proposed and applied in [6] to chemical and biochemical engineering

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systems. The common framework of the above work is the optimization of a certain ''cost'' objective function while guaranteeing local dynamic asymptotic stability of the equilibrium at the solution.

Regarding the estimation of DOAs many techniques have been proposed so far. The interested reader is referred to [7] for a clear classification of the techniques and a review of classic work on the topic. In particular, the Lyapunov stability theory provides a methodology whose rationale is to approximate the DOA by a level set of a Lyapunov function. Usually, polynomial type Lyapunov functions are adopted. Basically, the idea is to find the largest level set of the Lyapunov function fully contained in the region of negative definiteness of its time derivative.

In general, Lyapunov based estimations are sensibly smaller than the actual domains. Typically, infinite size (open) DOAs are approximated by finite size (closed) level sets. Except for very special cases, it is in general very difficult to completely describe open DOAs by level sets of Lyapunov functions. However, Lyapunov based estimations usually provide good representation in a neighborhood of the boundary of the actual DOA. Such approximation is usually valuable since if the equilibrium is close to this boundary, loss of stability might occur in the face of even modest disturbances.

Among the available techniques, in a seminal work [8] introduced the Maximal Lyapunov functions which are of rational type instead of polynomials. The denominator has a ''blow up'' effect near some of the boundaries of the DOA, making that their level sets closely represent the region of stability in that part of the states space and in some cases reproduce the exact DOA. The authors also proposed a recursive algorithm to simultaneously construct the maximal function and compute the estimation of the domain. Chesi et al. [9] used Linear Matrix Inequalities (LMI) optimization to address the estimation of DOAs of dynamic systems based on level sets of Lyapunov functions. In [10] a methodology was proposed that approximates the DOA as the union of an infinite number of level sets of Lyapunov functions instead of only one as is usually done. In [11] a strategy for estimating DOAs of non-polynomial systems was proposed. Formulations that make use of results on deterministic global optimization based on the theory of moments were also proposed [12,13]. The technique allows the identification of the best possible level set of a rational Lyapunov function that constitutes an estimation of the DOA for nonlinear dynamic systems of polynomial type. In [14] an extension of the approach proposed in [13] was presented which applied global optimization solvers of the branch and bound type to address general nonlinear systems.

The ''DOA-enlargement-problem'' closely related to the ''DOA-estimation-problem'', has to do with the optimization of the estimation of the DOA according to some criterion. In this case the equilibrium point is known to be asymptotically stable in the analyzed range of optimization variables, meaning that no bifurcation occurs as the variables change. Tan [15], addressed the enlargement of the DOA using ''sum of squares programming'' applied to polynomial dynamic systems. Hashemzadeh and Yazdanpanah [16] developed an approach to enlarge the DOA of a nonlinear affine system based on the Zubov theorem. Gonzalez and Odloak [17] and Limon et al. [18] presented methods for enlarging the DOA of nonlinear systems under model predictive control schemes.

In this contribution a design problem is formulated in order to find an asymptotically stable equilibrium point whose DOA is the largest in some sense. Estimation of DOAs based on appropriate level sets of Lyapunov functions are adopted. In order to address the stability issue, the eigenvalues of the Jacobean matrix of the dynamic system are forced to belong to the left half of the complex space, meaning that they are required to be strictly negative in real part.

On the other hand, the problem of identification of the best level set of a Lyapunov function which approximates the DOA of a given equilibrium point is a global optimization problem itself [13]. Such an ''inner'' sub-problem is therefore embedded in the ''outer'' design optimization problem where the design variables are manipulated in order to find the largest possible estimation. Since explicit ''global optimality conditions'' of the inner sub-problem are in general not available to be included as constraints in the ''outer'' problem, a ''two level'' approach is proposed.

The remaining of the paper is organized as follow. Section 2 introduces basic definitions and theorems used along the paper. In Section 3 some important topics are further developed before presenting the proposed approach in Section 4. In Section 5 the developed methodology is applied to some illustrative examples. A conclusions section closes the paper.

2. Background definitions and theorems

In this section, the basic definitions and theorems required to support the proposed contribution are introduced. All of these can be found in classic texts on nonlinear systems analysis such as [1,19,20].

Consider the following autonomous nonlinear dynamic system:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x} \in \Re^n, \quad \mathbf{x}(t_0) = \mathbf{x}_0 \tag{1}$$

being $\mathbf{x} = \mathbf{x}^*$, an asymptotically stable equilibrium point of (1).

Definition 2.1 (*Equilibrium Point*)**.** A point $\mathbf{x}^* \in \Re^n$ is called an equilibrium point of system (1) if $\mathbf{f}(\mathbf{x}^*) = \mathbf{0}$. The equilibrium points of system (1) correspond to the intersection of the nullclines of the system, meaning the curves given by $\mathbf{f}(\mathbf{x}) = \mathbf{0}$.

Remark 2.1. In the sequel, we assume without loss of generality, that the equilibrium point under study coincides with the origin of the state space of $\Re^n$, ($\mathbf{x}^* = \mathbf{0}$).

Definition 2.2 (*Asymptotic Stability*)**.** Let $\mathbf{x}(t, \mathbf{x}_0)$ denote the trajectory initiated at state $\mathbf{x}_0$ in time t_0. Equilibrium $\mathbf{x}^* = \mathbf{0}$ of system (1) is asymptotically stable if there exists a $\eta > 0$ such that:

$$\lim_{t\to\infty} \mathbf{x}(t, \mathbf{x}_0) = \mathbf{0}, \quad \text{whenever } \|\mathbf{x}_0\| < \eta.$$

Lyapunov stability theory provides a tool to assess the stability of equilibrium points by means of the so-called Lyapunov functions.

Definition 2.3 (*Positive and Negative Definite Functions*)**.** A continuously differentiable real-valued function $\varphi(\mathbf{x})$ defined on a domain $R(\mathbf{0}) \subseteq \Re^n$ containing point $\mathbf{x} = \mathbf{0}$ is called positive definite if the following conditions hold:

- $\varphi(\mathbf{0}) = 0$
- $\varphi(\mathbf{x}) > 0 \; \forall \mathbf{x} \in \{R(\mathbf{0}) \setminus \mathbf{0}\}$.

Function $\varphi(\mathbf{x})$ is negative definite if $-\varphi(\mathbf{x})$ is positive definite.

Remark 2.2. In the remaining, the symbol $\succ 0$ ($\prec 0$) is used to denote positive (negative) definiteness of functions.

Lyapunov stability theory provides the basis of a family of techniques for the estimation of regions of asymptotic stability whose rationale is to approximate the DOA($\mathbf{0}$) by a level set of a Lyapunov function of the equilibrium point.

Definition 2.4 (*Lyapunov Function*)**.** Let $V(\mathbf{x})$ be a continuously differentiable real-valued function defined on a domain $D \subseteq \Re^n$ containing the equilibrium $\mathbf{x} = \mathbf{0}$. Function $V(\mathbf{x})$ is a Lyapunov function of equilibrium $\mathbf{x} = \mathbf{0}$ of system (1) if the following conditions hold:

- $V(\mathbf{x})$ is positive definite on $R(\mathbf{0})$
- The time derivative of $V(\mathbf{x})$, $\frac{dV(\mathbf{x})}{dt} = [\nabla V(\mathbf{x})]^{\mathrm{T}} \mathbf{f}(\mathbf{x})$, is negative definite on $R(\mathbf{0})$.

Theorem 2.1 (*Asymptotic Stability in the Lyapunov Sense*)**.** *If there exists a Lyapunov function $V(\mathbf{x})$ for equilibrium point $\mathbf{x} = \mathbf{0}$ of system (1), then $\mathbf{x} = \mathbf{x}^* = \mathbf{0}$ is asymptotically stable.*

Definition 2.5 (*Domain of Attraction*)**.** The DOA of the equilibrium point $\mathbf{x} = \mathbf{0}$ is given by:

$$\mathrm{DOA}\,(\mathbf{0}) = \{\mathbf{x}_0 \in \Re^{\mathrm{n}} : \lim_{t\to\infty} \mathbf{x}\,(t, \mathbf{x}_0) \to \mathbf{0}\}. \tag{2}$$

Theorem 2.2 (*Estimation of the Domain of Attraction*)**.** *Let $V(\mathbf{x})$ be a Lyapunov function for the equilibrium $\mathbf{x} = \mathbf{0}$ of system (1). Consider that $\mathrm{d}V(\mathbf{x})/\mathrm{d}t$ is negative definite in the region:*

$$S(\mathbf{0}) = \{\mathbf{x} : V(\mathbf{x}) \le c, c > 0\}. \tag{3}$$

Then, every trajectory initiated within region $S(\mathbf{0})$ tends to $\mathbf{x} = \mathbf{0}$ as time tends to infinity.

Theorem 2.3 (*Jacobean Matrix Spectrum*)**.** *If the equilibrium $\mathbf{x} = \mathbf{0}$ of system (1) is asymptotically stable, then the real part of the eigenvalues of the corresponding Jacobean matrix, $\mathbf{A}$, are strictly negative.*

Theorem 2.4 (*Lyapunov Identity*)**.** *If the equilibrium $\mathbf{x} = \mathbf{0}$ of system (1) is asymptotically stable, then there exists a Lyapunov function of the quadratic type, $V(\mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x}$, where $\mathbf{P}$ is a positive definite matrix which can be calculated from the so-called Lyapunov identity:*

$$\mathbf{A}^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}. \tag{4}$$

A common choice is to set $\mathbf{Q} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix.

Theorem 2.5. *Consider the following representation of system (1): $\mathbf{f}(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{f}_1(\mathbf{x})$, where $\mathbf{f}_1(\mathbf{x})$ comprises the nonlinear part of function $\mathbf{f}(\mathbf{x})$. It can be shown that if the following condition holds, [19]:*

$$\frac{\|\mathbf{f}_1(\mathbf{x})\|}{\|\mathbf{x}\|} \le \frac{\lambda_{\min}(\mathbf{Q})}{2\lambda_{\max}(\mathbf{P})}, \quad \forall \mathbf{x} \in B_r \tag{5}$$

$V(\mathbf{x})$ and its time derivatives are positive and negative definite, respectively, within the ball B_r of radius r. It is clear that the larger the ratio $\lambda_{\min}(\mathbf{Q})/2(\lambda_{\max}(\mathbf{P}))$, the larger the possible choice of r.

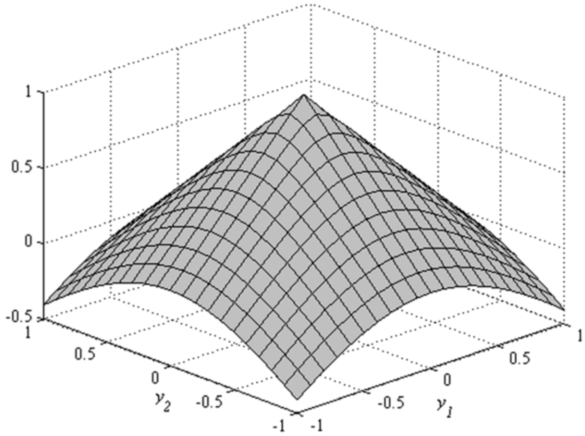


Fig. 1. Non-differentiability of the maximum eigenvalue of Overton's example.

3. Preliminary remarks on nonlinear systems design and analysis

In this section we discuss general aspects of the two problems underlying the design of stable nonlinear systems: the "design for stability problem" and the "DOA estimation problem".

Eigenvalue optimization

Local stability of dynamic systems is usually studied in terms of eigenvalues. Therefore, the design for stability problem can be formulated as an eigenvalue optimization problem [3]. As pointed out by [2] there exists the impossibility of obtaining mathematical expressions for the eigenvalues of larger than 4 by 4 systems. This makes it impossible to include eigenvalues within the optimization model in a straightforward manner as objectives and constraints. Furthermore, even in the cases where analytical expressions can be obtained, their usual high complexity and non-convexity make difficult to standard NLP solvers to cope with them. Besides this issue, a critical difficulty in eigenvalue optimization problems is the potential coalescence of eigenvalues [21]. The eigenvalues of a matrix with differentiable elements (smooth in the optimization variables) are themselves non-differentiable (non-smooth) at the points where coalescence occurs. It is also frequent that the optimization objective tends to make the eigenvalues coalesce at the solutions. The following classic example illustrates this point. Consider the matrix:

$$\mathbf{A}(\mathbf{y}) = \begin{bmatrix} 1+y_1 & y_2 \\ y_2 & 1-y_1 \end{bmatrix}$$

whose eigenvalues are:

$$1 \pm \sqrt{y_1^2 + y_2^2}.$$

It can be seen that that the minimum eigenvalue is maximized by $y_1 = y_2 = 0$. Clearly the minimum eigenvalue is not a smooth function in such a point as can be concluded from Fig. 1. In order to cope with such non-differentiability it is necessary to use specialized optimization methods when eigenvalues are present.

DOA estimation problem

Regarding the "DOA estimation problem" a methodology based on Lyapunov stability results is introduced in the following. Consider that a Lyapunov function $V(\mathbf{x})$ is given, meaning that (Definition 2.3):

$$\begin{aligned} & V(\mathbf{x}) \succ 0 \quad \text{in } R(\mathbf{0}) \\ & \frac{dV(\mathbf{x})}{dt} \prec 0 \quad \text{in } R(\mathbf{0}). \end{aligned} \tag{6}$$

According to (3) the larger the level set value c, the better the estimation of DOA($\mathbf{0}$). The calculation of the maximum level set of the Lyapunov function which is still an estimation of DOA($\mathbf{0}$) can be obtained by solving a problem whose pseudo-optimization formulation is as follows:

$$\max_{c,\mathbf{x}} c$$

$$\text{s.t.} \quad \{\mathbf{x} \text{ belong to level set } V(\mathbf{x}) - c = 0\} \tag{7a}$$

$$\{\mathbf{x} \text{ is any point belonging to region } S(\mathbf{0}) \text{ contained in } R(\mathbf{0})\}. \tag{7b}$$

The idea behind problem (7) is to find the maximum level set of $V(\mathbf{x})$ (7a) which is fully contained in the region of negative definiteness of $dV(\mathbf{x})/dt$ (7b). The desired solution of problem (7) is a single point in the state space, which corresponds to a tangencial contact of level sets $V(\mathbf{x}) = c$ and $dV(\mathbf{x})/dt = 0$.

Table 1
Data and results for example.

x_{1s} x_{2s}	$\lambda_1(\mathbf{A})$ $\lambda_2(\mathbf{A})$	x_1 x_2	$\lambda_1(\mathbf{P})$ $\lambda_2(\mathbf{P})$	r
0.0 0.0	$-0.5000 + 0.8660i$ $-0.5000 + 0.8660i$	0.0 0.2763	0.6910 1.8090	0.2763

The largest level set that verifies constraints (7a) and (7b) turns to be the smallest hyper-sphere contained in $dV(\mathbf{x})/dt = 0$. Any larger estimation would fail to ensure simultaneously positive definiteness of function $V(\mathbf{x})$ and negative definiteness of its time derivative [14].

In [13,22], rational type Lyapunov functions were used in the context of problem (7).

If quadratic type Lyapunov functions are adopted (Theorem 2.4), problem (7) can be reformulated as follows by using the results of Theorem 2.5:

$$\min_{r,\mathbf{x},\mathbf{P},\mathbf{Q}} \; r$$

$$\text{s.t.} \quad \|x\| - r = 0 \tag{8a}$$

$$\mathbf{A}(\mathbf{x})^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A}(\mathbf{x}) = -\mathbf{Q} \tag{8b}$$

$$\frac{\|\mathbf{f}_1(\mathbf{x})\|}{\|\mathbf{x}\|} - \frac{\lambda_{\min}(\mathbf{Q})}{2\lambda_{\max}(\mathbf{P})} = 0 \tag{8c}$$

$$r > 0.$$

The desired solution of problem (8) is also a single point in the state space, which corresponds to a contact of the ball B_r of radius r and the surface $\frac{\|\mathbf{f}_1(\mathbf{x})\|}{\|\mathbf{x}\|} - \frac{\lambda_{\min}(\mathbf{Q})}{2\lambda_{\max}(\mathbf{P})} = 0$.

The rationale behind problem (8) is that within ball B_r it is always possible to inscribe an elliptic invariant set which is an estimation of the actual DOA.

It should be noted that (8) is a nonlinear optimization model and therefore it may have many local solutions. In order to avoid dummy solutions, problem (8) has to be solved to global optimality. Global optimality condition is ensured in this work by solving the NLP problem (8) with state of the art global optimization software. In particular the GAMS platform [23] with the global optimization solver BARON [24] is adopted. BARON implements a deterministic global optimization algorithm of the branch and bound type, which guarantees to provide the global optima under fairly general assumptions. For a complete presentation of the theory behind the BARON solver see [25].

In the following example it is graphically shown the solution of problem (8) for a simple nonlinear dynamic system [19]:

$$\begin{aligned} \frac{dx_1}{dt} &= -x_1 + x_2 + x_1 x_2 \\ \frac{dx_2}{dt} &= -x_1 + x_2^2. \end{aligned} \tag{9}$$

The identity matrix $\mathbf{I}$ is adopted as the $\mathbf{Q}$ matrix in the Lyapunov equation in (8). There exists only one equilibrium point, the origin of coordinates, which is stable as can be inferred from the inspection of the eigenvalues in Table 1. In Fig. 2 it is shown the nullclines (dotted lines) of system (9). The ball B_r (solid line) is also shown together with surface $\frac{\|\mathbf{f}_1(\mathbf{x})\|}{\|\mathbf{x}\|} - \frac{1}{2\lambda_{\max}(\mathbf{P})} = 0$ (dashed line). With the small circle it is indicated the intersection point between the level sets of interest also reported in Table 1 together with the value of the corresponding radius.

Global optimization

In the following section it will be shown that the previous optimization problem (8) is solved to global optimality at an "inner" level of the proposed design problem. The design problem itself might be considered therefore as the outer level of a bi-level optimization problem. Bi-level optimization is a very challenging discipline and many approaches have been proposed to address such problems [26].

A classic methodology is to formulate de Karush–Kuhn–Tucker (KKT) optimality conditions of the inner sub-problem and to embed it as a set of algebraic nonlinear constraints in the outer optimization model. However, the KKT set is verified by every stationary point of the optimization model, meaning all local minima and maxima altogether with the saddle points [27]. Therefore it is not straightforward how to identify the global minimum among all the possible the solutions of the KKT set. In other words, the KKT conditions do not provide the "global optimality" conditions required by problem (8).

If the inner problem is simple enough, other intuitive approach is to obtain analytical expressions of its solutions parameterized in the design variables. However, it will be shown through a simple example that, in general, the solutions of global optimization problems are non-differentiable on the design parameters. Consider the following global optimization problem:

$$\begin{aligned} &\max_{d,x} \; x^2 \\ &\text{s.t.} \quad x^2 - dx - 1 = 0 \end{aligned} \tag{10}$$

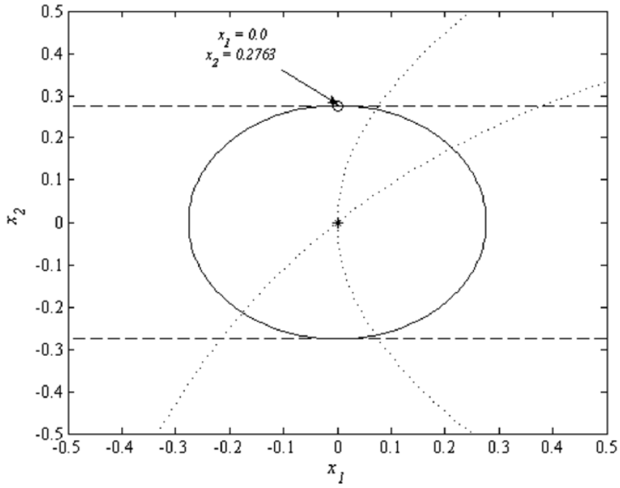


Fig. 2. Results for system (9). Ball B_r (solid), constraint (8c) (dashed).

where d is the design variable of the outer problem and x the optimization variable of the inner problem itself. The parameterized solution of this problem is: $\max\left\{\left(\frac{d+\sqrt{d^2+4}}{2}\right)^2, \left(\frac{d-\sqrt{d^2+4}}{2}\right)^2\right\}$. Both branches of the solution are plotted in Fig. 3. The solution of problem (10) are the upper parts of each branch. It can be seen that the solution is clearly non-differentiable at $d = 0$.

4. New design formulation for nonlinear systems based on DOAs

While for the general case the DOA of a stable equilibrium of a nonlinear system is a set of complex shape and possible infinite size, estimations based on quadratic Lyapunov functions are always hyper-ellipses. Such ellipses are contained in the ball B_r previously defined. The radius r of such a ball is used in this work as a merit function for the design optimization problem.

Based on the above considerations the problem of design of the nonlinear dynamic system in order to optimize a measure of the extension of the DOA can be posed as follows:

$$\max_{r,\mathbf{x_s},\mathbf{d}} r$$

$$\text{s.t.}\quad \mathbf{f}(\mathbf{x_s},\mathbf{d}) = \mathbf{0} \tag{11a}$$

$$\mathrm{Re}\{\lambda_i[\mathbf{A}(\mathbf{x_s},\mathbf{d})]\} < 0 \quad i = 1,\ldots,n \tag{11b}$$

$$\left\{\begin{array}{ll} & \min\limits_{r,\mathbf{x},\mathbf{P},\mathbf{Q}} r \\ \text{s.t.} & \|\mathbf{x}\| - r = 0 \\ & \mathbf{A}(\mathbf{x_s})^{\mathrm{T}}\mathbf{P} + \mathbf{P}\mathbf{A}(\mathbf{x_s}) = -\mathbf{Q} \\ & \dfrac{\|\mathbf{f}_1(\mathbf{x})\|}{\|\mathbf{x}\|} - \dfrac{\lambda_{\min}(\mathbf{Q})}{2\lambda_{\max}(\mathbf{P})} = 0 \\ & r > 0 \end{array}\right\} \tag{11c}$$

$$\mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U$$

$$\mathbf{x_s}^L \le \mathbf{x} \le \mathbf{x_s}^U.$$

Eq. (11a) is the dynamic system in steady state version where $\mathbf{x}_s$ is the equilibrium point and $\mathbf{d}$ is the vector of design variables. Constraint (11b) stands for the asymptotic stability constraint of the equilibrium point $\mathbf{x}_s$ by imposing that the real part of the eigenvalues of the Jacobean matrix $\mathbf{A}(\mathbf{d},\mathbf{x}_s)$ be strictly negative. Constraint (11c) is an optimization problem for the calculation problem of the radius of ball B_r (8) as described in Section 3. It should be noted that each equilibrium $\mathbf{x}^* = \mathbf{x}_s$ is moved into the origin $\mathbf{x} = \mathbf{0}$ in order to solve sub-problem (11c). Altogether formulation (11) is a bi-level nonlinear optimization problem.

The objective function to be maximized is the radius of ball B_r, namely the distance between the stable equilibrium $\mathbf{x}_s$ and the point in the state space, $\mathbf{x}$, where the surface of the ball B_r and the surface defined by $\frac{\|\mathbf{f}_1(\mathbf{x})\|}{\|\mathbf{x}\|} - \frac{\lambda_{\min}(\mathbf{Q})}{2\lambda_{\max}(\mathbf{P})} = 0$ intersects.

As discussed in Section 3, constraint (11b) involving the eigenvalues of a matrix does not possess explicit expressions to be written down for the general case. Moreover, eigenvalue constraints usually introduces non-differentiability issues due to the effect of coalescence [21].

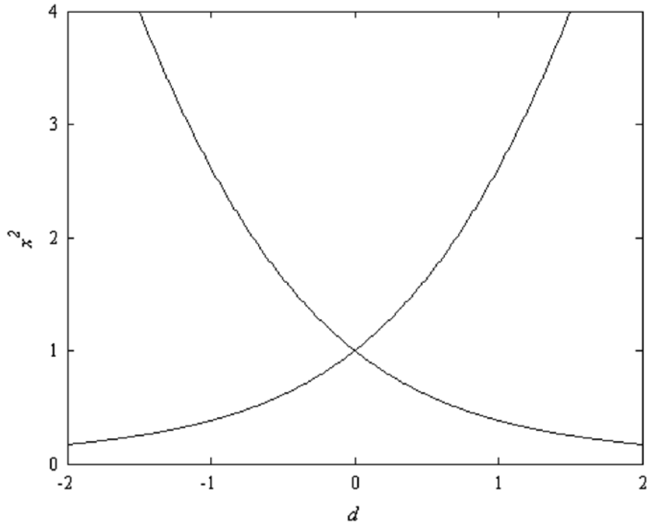


Fig. 3. Branches of the solution of problem (10).

Step 1 **-** Generate an initial population for design variables **d**
Step 2 **-** For each member of the population calculate the equilibrium $\mathbf{x}_s$
Step 3 **-** Evaluate merit function r as follows
- a. If $\mathbf{x}_s$ does not exist then set $r = 0$
- b. If $\mathbf{x}_s$ does exist then evaluate the eigenvalues of Jacobean matrix $\mathbf{A(d, x_s)}$
  - i. If the real part of some of the eigenvalues is positive then set $r = 0$
  - ii. If the real part of all the eigenvalues is negative then solve problem (11c) to evaluate radius r

Step 4 **-** Check the stopping criterion
- c. If the stopping criterion is not verified then generate a new population based on the value of the objective function of the individuals and return to Step 2.
- d. If the stopping criterion is verified then terminate

Fig. 4. Pseudo code for the resolution procedure of problem (11).

On the other hand, constraint (11c) is an optimization problem itself which has to be solved to global optimality since local solutions are not appropriate. Note that because the inner problem is an NLP, embedding its solution into the outer problem as their KKT conditions is not conclusive and the inclusion of parameterized solutions, if possible, might introduce non-differentiability (Section 3).

In order to deal with these issues in (11b) and (11c) a two level solution strategy is proposed. In the "outer level" a stochastic "derivative free" optimization engine is adopted to explore the design variables space. The stochastic engine is based on the evolution of a population of solutions according to a set of rules. For each individual of the population of design variables, an equilibrium point $\mathbf{x}_s$ is calculated from (11a) and its feasibility with respect to constraint (11b) verified. If the equilibrium is asymptotically stable, problem (11c) is solved at the "inner level" to find the ball which verifies (5) and its radius is returned as the objective function value. A deterministic solver of the branch and bound type is adopted in this step. The pseudo-code of Fig. 4 summarizes the proposed procedure.

If no equilibriums exist or they are unstable, then no DOAs exist and their radiuses are set to zero (steps 3a and 3b–i). If stable equilibriums exist their merit function assumes the actual value of the radiuses (step 3b–ii). In this way individuals with the largest radiuses are favored in the search.

There exist several very well developed stochastic techniques for nonlinear optimization such as Genetic Algorithms [28], Differential Evolution [29] and Particle Swarm Optimization [30]. It is beyond of the scope of this contribution to present a discussion of benefits and disadvantages of the different techniques. In general, with the appropriate parameter tuning, such algorithms use to perform well in the exploration of unconstrained complex spaces which may be multi-modal and discontinuous. In this contribution a standard implementation of a genetic algorithm is adopted [31]. It should be mentioned that a major drawback of stochastic solvers is the lack of a general technique for the handling of equality and inequality constraints other than upper and lower bounds on the variables. However, in this application not such constraints exist since the search space is only box constrained in problem (11).

Once a population of individuals in the design space is generated, the merit function, namely the radius of the ball B_r has to be calculated for each member. The first step is therefore to evaluate the corresponding equilibrium point by solving Eq. (11a). Since (11a) is a nonlinear system one of the following situations may happen for a certain realization of vector **d**:

Table 2
Data and results for Example 1.

Case	b	x_{1s} x_{2s}	$\lambda_1(\mathbf{A})$ $\lambda_2(\mathbf{A})$	x_1 x_2	$\lambda_1(\mathbf{P})$ $\lambda_2(\mathbf{P})$	r	δ
Base	1.0	0.0 0.0	−0.3820 −2.6180	−0.2950 0.0	0.1910 1.3090	0.2950	0.8740
Optimal	0.1	0.0 0.0	−0.9084 −2.0916	−0.5188 0.0	0.2260 0.6345	0.5188	1.3665

(i) No equilibrium point exists.
(ii) Only one equilibrium point exists.
(iii) Two or more equilibriums exist (steady state multiplicity).

Therefore Step 2 of the pseudo-code of Fig. 4 has to do with the calculation of all the solutions of a system of nonlinear equations. If no equilibrium exists or only one equilibrium point exists, the evaluation of the merit function can be straightforwardly implemented as described in Step 3. A problem arises in the case where two or more equilibrium points exist. In such a situation it might happen for example that one of the equilibriums is a promising, stable, large radius candidate $(\mathbf{d}, \mathbf{x}_s^1)$, while the other is perhaps an unstable solution $(\mathbf{d}, \mathbf{x}_s^2)$. In order to cope with such situations it is necessary to design some mechanism to be able to preserve the good couple $(\mathbf{d}, \mathbf{x}_s^1)$ and prune the less promising $(\mathbf{d}, \mathbf{x}_s^2)$. The following approach is adopted in this work.

A standard Newton-type algorithm is used to solve system (11a) in Step 2 of the procedure. Such algorithms require a starting point, $\mathbf{x}_0$, to initiate the search. The solution found is therefore highly dependant on this initialization. In order to avoid the necessity of finding all the roots for a given $\mathbf{d}$ and then keeping track of the most promising ones, the design vector $\mathbf{d}$ is enlarged with the initialization vector, $\mathbf{x}_0$. Therefore the space $[\mathbf{d}, \mathbf{x}_0]$ is the one explored at the outer level with the stochastic engine. In this way, information on the equilibrium point is simultaneously considered in the search together with the design variables vector.

Finally, from an implementation point of view it should be mentioned that the strict negative and strict positive constraints in (11b) and (11c) respectively are handled through a constant ε which assumes a small positive value ($\mathrm{Re}(\lambda_i) \leq -\varepsilon$ and $r \geq \varepsilon$ where $\varepsilon \approx 0.0001$).

5. Examples

In the following, a number of examples are developed in order to illustrate different aspects of the proposed methodology. In all cases the identity matrix $\mathbf{I}$ is adopted as the $\mathbf{Q}$ matrix in the Lyapunov equation.

Example 1. Consider system:

$$\begin{aligned} \frac{dx_1}{dt} &= -x_1 + x_2 \\ \frac{dx_2}{dt} &= bx_1 - 2x_2 - x_1^2 + x_1^3. \end{aligned} \tag{12}$$

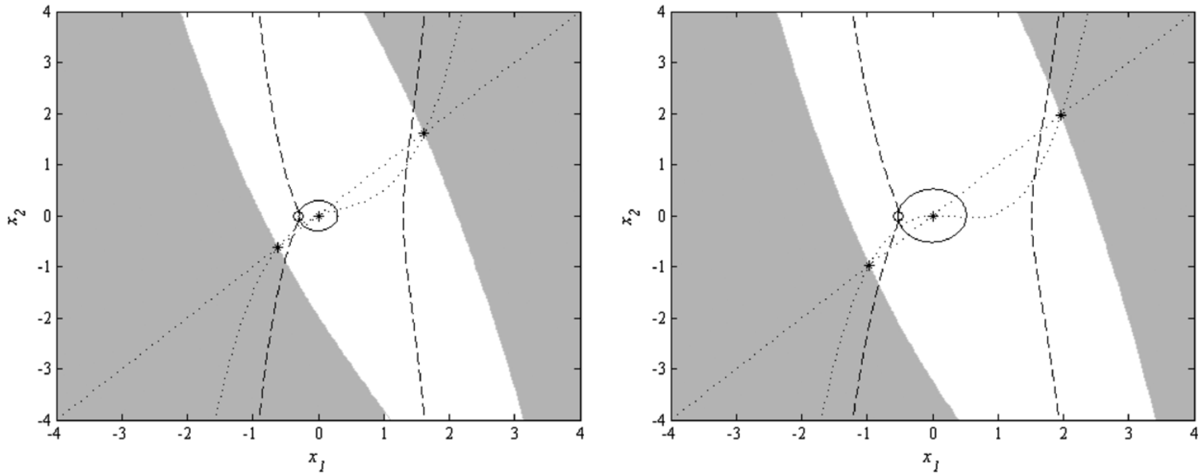
For the purposes of this study parameter b will be considered as the design variable, able to vary within the range [0.1, 2.5]. The analyzed equilibrium is the (0, 0) which is known to be stable within the whole range of variable b. For comparison purposes the optimal solution will be analyzed against a base case where the design variable is arbitrarily fixed, $b = 1.0$.

For the base case, the radius of ball B_r is calculated by solving problem (8). Basic data and results are provided in Table 2. In Fig. 5(a) are shown the nullclines (dotted lines) of system (12). The three equilibrium points of this system can be appreciated: the origin of coordinates, which is the stable one and two other equilibrium points which are saddle nodes (unstable). The ball B_r (solid line) is also shown together with surface $\frac{\|\mathbf{f}_1(\mathbf{x})\|}{\|\mathbf{x}\|} - \frac{1}{2\lambda_{\max}(\mathbf{P})} = 0$ (dashed line). With the small circle is indicated the intersection point between the level sets of interest.

For the optimal case, variable b is allowed to vary within its range in problem (11). Results are shown in Table 2 and Fig. 5(b). It can be seen that the radius of the ball in the optimal case is larger than the one in the base case as expected.

Moreover, the real DOA, (white region) in the optimal case is larger than the one corresponding to the base case. In this particular example the boundaries of the DOAs are the stable manifolds of the saddle nodes. It is evident from the comparison that the effect of a larger radius is to "push away" the unstable equilibriums.

In order to provide a quantitative measure of the effective enlargement of the DOA is defined parameter δ which, for this specific problem, represents the distance between the equilibrium point of interest and the closest saddle node. It can be seen that this measure is increased by 56.35% due to the effect of the optimization regarding the base case (Table 2).



(a) Base case. (b) Optimal case.

Fig. 5. Results for Example 1. Ball B_r (solid), constraint (8c) (dashed).

Table 3
Data and results for Example 2.

Case	b	x_{1s} x_{2s}	$\lambda_1(\mathbf{A})$ $\lambda_2(\mathbf{A})$	x_1 x_2	$\lambda_1(\mathbf{P})$ $\lambda_2(\mathbf{P})$	r	δ
Base	0.3	0.0 0.0	$-0.2500 + 0.6554i$ $-0.2500 + 0.6554i$	-0.2277 0.2277	1.1583 7.3181	0.3221	5.6663
Optimal	0.1	0.0 0.0	$-0.2500 + 0.6554i$ $-0.2500 + 0.6554i$	-0.6832 0.6832	1.1583 7.3181	0.9662	16.9990

Example 2. Consider system:

$$\begin{aligned}\frac{dx_1}{dt} &= -0.84x_1 - 1.44x_2 - bx_1x_2 \\ \frac{dx_2}{dt} &= 0.54x_1 + 0.34x_2 + bx_1x_2.\end{aligned} \tag{13}$$

Parameter b will be considered as the design variable, able to vary within the range [0.1, 0.9]. The analyzed equilibrium is the (0, 0) which is known to be stable within the whole range of variable b. For comparison purposes the optimal solution will be analyzed against a base case where the design variable is arbitrarily fixed at 0.3.

Basic data and results are provided in Table 3. In Fig. 6(a) are shown the nullclines (dotted lines). Two equilibrium points can be identified, the stable origin and a saddle whose stable manifold is the boundary of the actual DOA of the origin. The corresponding ball B_r is shown in solid line together with level set $\frac{\|\mathbf{f}_1(\mathbf{x})\|}{\|\mathbf{x}\|} - \frac{1}{2\lambda_{\max}(\mathbf{P})} = 0$ (dashed line). With the small circle is indicated the intersection point between the level sets of interest.

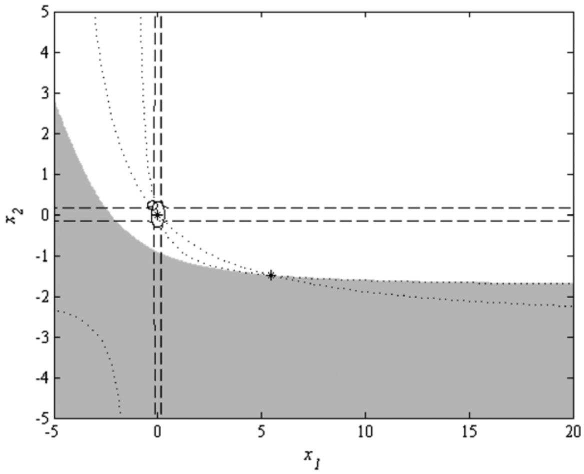
For the optimal case, variable b is allowed to vary within its range in problem (11). Results are shown in Table 3 and Fig. 6(b). It can be seen that the radius of the ball B_r in the optimal case is larger than the one to the base case as expected. The enlargement of the DOA is, as in the previous example, due to the "push away" effect on a saddle node and therefore on its stable manifold which is the actual boundary of the DOA.

For comparison purposes parameter δ is defined again as the distance between the equilibrium point of interest and the closest saddle node (Table 3). It can be seen that this measure is increased by 200% due to the effect of the optimization.

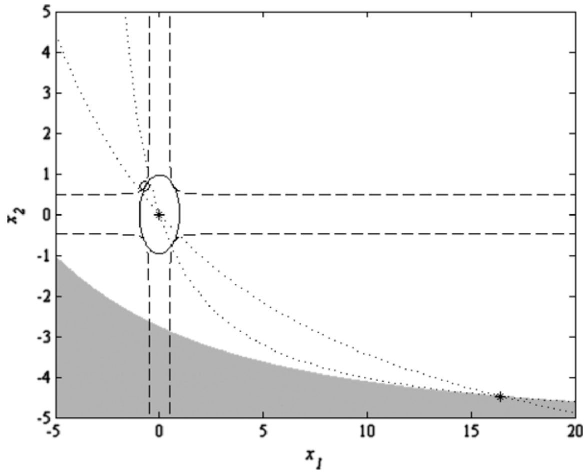
Example 3. In this third example the proposed methodology is applied to the *van der Pol* oscillator:

$$\begin{aligned}\frac{dx_1}{dt} &= -x_2 \\ \frac{dx_2}{dt} &= x_1 - x_2 + (1 - 0.2b)x_1^2x_2.\end{aligned} \tag{14}$$

Parameter b is the design variable, able to vary within the range [−1, 1]. The analyzed and unique equilibrium is the (0, 0) which is known to be stable within the whole range of variable of b. The DOA of the stable equilibrium is bounded by



(a) Base case.

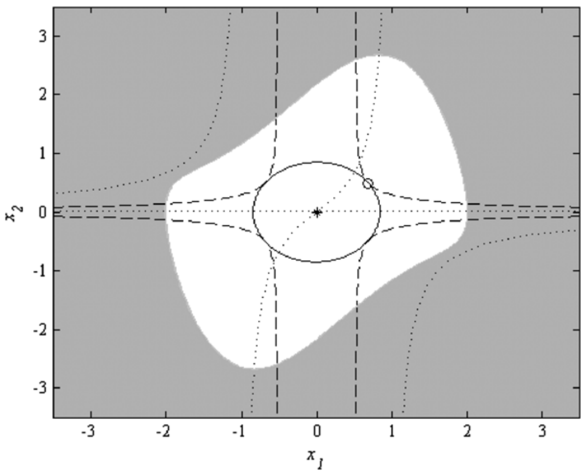


(b) Optimal case.

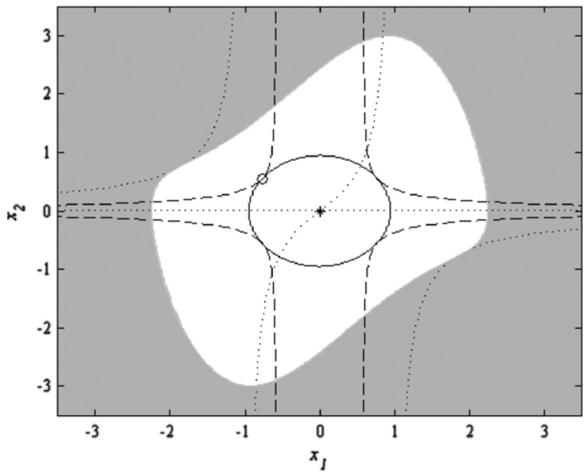
Fig. 6. Results for Example 2. Ball B_r (solid), constraint (8c) (dashed).

Table 4
Data and results for Example 3.

Case	b	x_{1s} x_{2s}	$\lambda_1(A)$ $\lambda_2(A)$	x_1 x_2	$\lambda_1(P)$ $\lambda_2(P)$	r	δ
Base	0	0.0 0.0	$-0.5000 + 0.8660i$ $-0.5000 + 0.8660i$	0.6919 0.4892	0.6910 1.8090	0.8474	1.5402
Optimal	1	0.0 0.0	$-0.5000 + 0.8660i$ $-0.5000 + 0.8660i$	−0.7736 −0.5470	0.6910 1.8090	0.9474	1.7029



(a) Base case.



(b) Optimal case.

Fig. 7. Results for Example 3. Ball B_r (solid), constraint (8c) (dashed).

a limit cycle. For comparison purposes the optimal solution will be analyzed against a base case where the design variable is arbitrarily fixed at value 0.

Data and results for the base case are provided in Table 4 and Fig. 7(a). The nullclines (dotted lines) demonstrate the existence of a unique equilibrium. As in the previous examples it is shown the ball B_r in solid line together the level set $\frac{\|\mathbf{f}_1(\mathbf{x})\|}{\|\mathbf{x}\|} - \frac{1}{2\lambda_{\max}(\mathbf{P})} = 0$ in dashed line. With the small circle it is indicated the intersection point between the level sets of interest.

For the optimal case, variable b is allowed to vary within its range in problem (11). Results are shown in Table 4 and Fig. 7(b). It can be seen that the radius of the ball in the optimal case is larger than the one in the base case as well as the real DOAs (white regions).

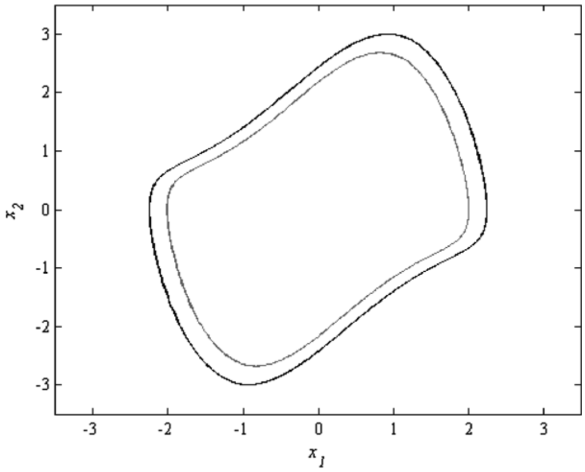


Fig. 8. Limit cycles for Example 3. Base case (gray) and optimized (black).

Table 5
Data and results for Example 4.

b	x_{1s} x_{2s}	$\lambda_1(\mathbf{A})$ $\lambda_2(\mathbf{A})$	x_1 x_2	$\lambda_1(\mathbf{P})$ $\lambda_2(\mathbf{P})$	r	δ
0	0.5236 0.0000	$-0.5000 + 1.2174i$ $-0.5000 - 1.2174i$	1.0364 0.0000	0.7013 1.7421	0.5128	2.6180
0.15	0.4823 0.0723	$-0.5750 + 1.2614i$ $-0.5750 - 1.2614i$	1.0134 0.0769	0.6289 1.4084	0.5311	2.8833
0.25	0.4586 0.1146	$-0.6250 + 1.2856i$ $-0.6250 - 1.2856i$	0.9977 0.1230	0.5896 1.2437	0.5391	3.1115

For comparison purposes both DOAs are shown simultaneously in Fig. 8. In this case, the effect of optimizing the radius of the ball is to "push away" a limit cycle which is the actual boundary of the DOA of the origin. In order to quantify the improvement, parameter δ is defined in this case as the radius of the largest ball that can be fully inscribed within the limit cycle (Table 4). Such a measure was increased by 10.56% due to the effect of the optimization.

Example 4. Consider the following dynamic system, which is a modification of the system presented in [32].

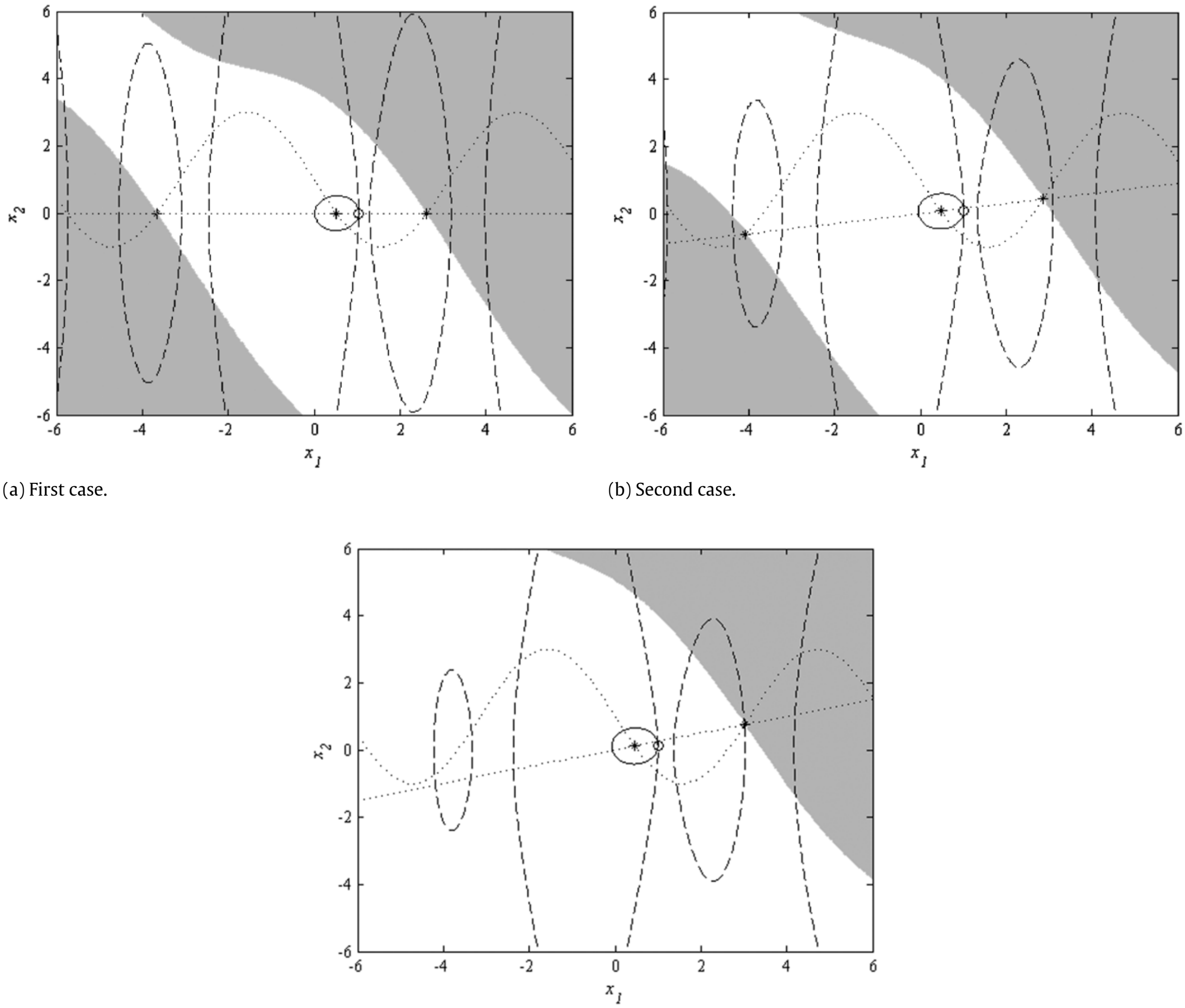
$$\begin{aligned}\frac{dx_1}{dt} &= x_2 - bx_1 \\ \frac{dx_2}{dt} &= x_2 - 2\sin(x_1) + 1.\end{aligned} \tag{15}$$

Differently from the previous examples, the topology for this dynamic system can be modified with parameter b. In other words equilibrium points can appear or disappear with the design variable and the position of the stable equilibrium can also change.

Parameter b is initially set to zero (Table 5). The analyzed asymptotically stable equilibrium point is (0.5236, 0). In Fig. 8(a) is shown the stable equilibrium surrounded by two saddles whose stable manifolds delimit the DOA of the stable point (intersections of dotted lines). As in the previous examples the solid line represents the ball B_r and the dashed line constraint (5).

As a second case, problem (11) was solved considering that variable b is able to vary within the range [0, 1.5]. The optimal solution corresponds to parameter b equal to 1.5 (Table 5). The topology of the system is the same than in the previous case since the two saddles are preserved although there was a shift in the position of the three equilibriums of interest (Fig. 9(b)). The "push away" effect on the saddles produces an effective enlargement of the DOA of the equilibrium of interest. Parameter δ is defined again as the distance between the equilibrium point of interest and the closest saddle node. This measure is increased by 10.13% due to the effect of the optimization regarding the first case (Table 5).

Finally, a third case is considered where variable b is allowed to vary within range [0, 2.5]. The optimal solution corresponds to the parameter b equal to 2.5 (Table 5). In this case, the topology of the system changes because only two equilibrium points survived: the stable one and one saddle. The real DOA is larger than in cases (a) and (b) due to a more intense manifestation of the "push away" effect (Fig. 9(c)). Parameter δ was increased by 18.85% regarding the first case and by 7.91% regarding the second due to the effect of the optimization (Table 5).



(a) First case.

(b) Second case.

(c) Third case.

Fig. 9. Results for Example 4. Ball B_r (solid), constraint (8c) (dashed).

Example 5. Consider an isothermal nonlinear continuous fermenter [33] with constant volume and physical–chemical properties:

$$\frac{dX}{dt} = \mu(S)X - \frac{XF}{V_f} \tag{16a}$$

$$\frac{dS}{dt} = -\frac{\mu(S)X}{Y} + \frac{\left(S_f - S\right)F}{V_f} \tag{16b}$$

where,

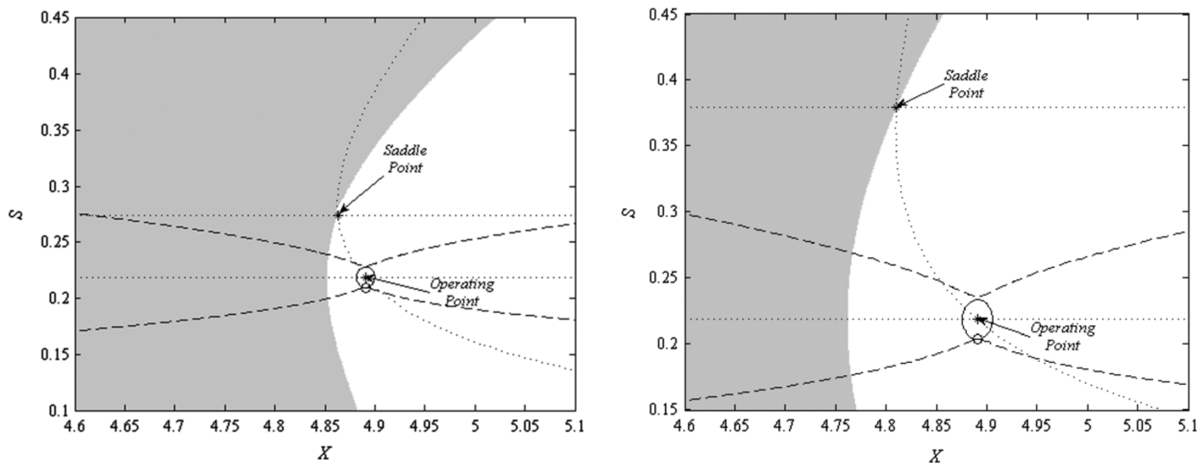
$$\mu(S) = \mu_{\max}\frac{S}{K_2S^2 + S + K_1}. \tag{17}$$

The states and parameters of the model are given in Table 6. Eq. (16a) originate from the biomass balance, while Eq. (16b) results from the substrate's mass balance. Both balances are coupled by the nonlinear growth rate function (17) which is the main source of non-linearity and uncertainty in this model.

The operating point is stable, but it is very close to saddle point ($X = 4.8627, S = 0.2744$) whose stable manifold delimits the DOA. Therefore, although the stability region is infinite (open) the equilibrium is very close to the boundary of its domain of attraction and modest disturbances in certain directions may drive the system away from the stability region (Fig. 10(a)).

Table 6
Variables and parameters of the fermentation process model (16)–(17).

X	Biomass concentration	4.8907 (steady state)	(g/l)
S	Substrate concentration	0.2187 (steady state)	(g/l)
F	Feed flow rate	3.2089	(l/h)
V_f	Volume	4	(l)
S_f	Substrate feed concentration	10	(g/l)
Y	Yield coefficient	0.5	(–)
$\mu_{\max}$	Maximal growth rate	1	(l/h)
K_1	Saturation parameter	0.03	(g/l)
K_2	Inhibition parameter	0.5	(l/g)



(a) Open loop.

(b) Closed loop.

Fig. 10. Results for Example 5. Ball B_r (solid), constraint (8c) (dashed).

Table 7
Data and results for Example 5.

Case	k_X k_S	X_s S_s	$\lambda_1(\mathbf{A})$ $\lambda_2(\mathbf{A})$	x_1 x_2	$\lambda_1(\mathbf{P})$ $\lambda_2(\mathbf{P})$	r	δ
Open loop	0 0	4.9807 0.2187	−0.8022 −0.8034	0.0 9.112×10^{-3}	0.3497 2.8405	9.112×10^{-3}	0.0624
Closed loop	0 −0.35	4.9807 0.2187	−0.8022 −1.6580	0.0 −0.0149	0.2329 1.5968	0.0149	0.3547

The solution of problem (11) for this case is reported in Table 7 (open loop) and the corresponding B_r shown in Fig. 10(a) (solid line) along with the corresponding level set for constraint (5) (dashed line).

In order to improve the stability features of this system a simple proportional feedback control law on both states is applied. The feed flow rate F is the manipulated variable:

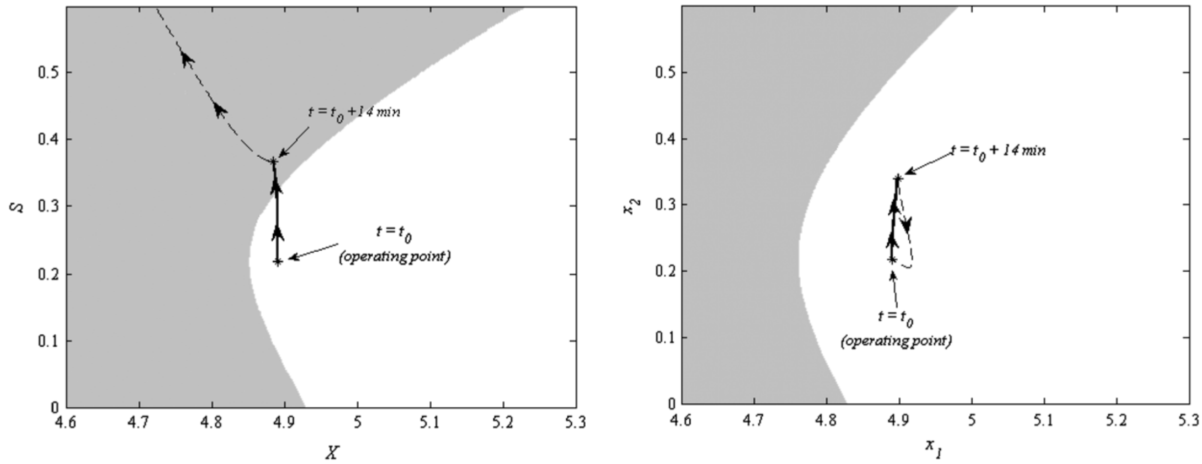
$$F = k_X\,(X - X_{ss}) + k_S\,(S - S_{ss}) + F_{sp}. \tag{18}$$

The proportional control constants k_X and k_S may vary within the range [−0.35, 0]. The objective is to find an optimal combination k_X–k_S such that the domain of attraction of the operating point of the closed loop system is larger than that corresponding to the open loop one. For this purpose problem (11) is solved for the closed loop model (16)–(18).

From Fig. 10(b) it can be observed that the saddle node and its stable manifold are farther than in the closed loop case due to the ''push away'' effect of the increased ball B_r. The corresponding ball and level set $\frac{\|\mathbf{f}_1(\mathbf{x})\|}{\|\mathbf{x}\|} - \frac{1}{2\lambda_{\max}(\mathbf{P})} = 0$ are also shown for the controlled case in Fig. 10(b) in solid and dashed lines respectively.

As in the previous examples parameter δ is also reported for this case showing an improvement in the size of the DOA of about 468.42% due to the effect of the optimization (Table 7).

In order to illustrate the dynamic behavior of this system, consider the following scenario. A 5% step increase in the feed substrate concentration occurs at time t_0. Under such a disturbance the system starts to evolve towards some new operating condition. After 14 min, the disturbance is removed and the feed substrate concentration returns to its original value. As shown in Fig. 11(a) the open loop system is not able to return to the original operating point because, after 14 min, the trajectory has left its DOA (solid arrow). From the state reached, the system evolves (slashed arrow) towards the undesirable washout condition ($X = 0$, $S = 10$). Under feedback control, on the other hand (Fig. 11(b)), the system is able to return



(a) Open loop. (b) Closed loop.

Fig. 11. Dynamic simulation of the fermentation process.

to the original operating point once the disturbance disappears, because the state reached after 14 min, still belongs to its DOA.

6. Conclusions

A design approach has been proposed to simultaneously ensure asymptotic stability and an optimum domain of attraction of the resulting equilibrium point in a certain sense. Specifically, the proposed merit function to be maximized was the radius of a ball in the state space within which negative definiteness of the time derivative of a quadratic type Lyapunov function can be ensured. Within such a ball it is possible to inscribe an invariant elliptic set which can be considered an estimation of the DOA of the equilibrium.

DOAs of stable equilibriums of nonlinear systems are usually complex shaped sets which can be hardly described precisely. However, the enlargement of the radius of the described ball has a "push away" effect on the actual boundaries of the DOA no matter their shape and localization. Such boundaries are usually the stable manifolds of saddle nodes and limit cycles. Therefore, as demonstrated through several examples, the effect of seeking an optimized radius is a net increase in the actual region of asymptotic stability.

From a mathematical point of view, the resulting is a bi-level optimization problem with non-differentiable inner sub problems. In order to address such potential non-differentiability, a stochastic (derivative free) genetic algorithm was adopted in the outer level to explore the design space.

To ensure asymptotic stability, the spectrum of each potential equilibrium point is first checked to belong to the negative part of the complex space. Then, the radius of the ball is calculated for each surviving candidate by solving a global optimization sub-problem and its value is returned as the merit function for the outer problem. In this way the evolution of the population is encouraged in the direction of stable, large radius, equilibrium points.

Besides asymptotic stability and large domains of attraction, real systems and processes usually require the optimization of economic objectives as well. Future work should address such multi objective optimization problems with application to larger systems in the state and parameter spaces. For example, the design for improved DOAs might be of practical interest for the operation of electrical networks in the face of short circuits [1]. Applications to chemical engineering systems, for instance to investigate the run-away phenomenon in non-isothermal reactor design [34] can be conceived as well.

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